On the Chow Ring of the Stack of truncated Barsotti-Tate Groups and of the Classifying Space of some Chevalley Groups

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Abstract

Let p be a prime. We compute the Chow ring of the stack of truncated displays and investigate the pull-back morphism of the truncated display functor. From this we can determine the Chow ring of the stack of truncated Barsotti-Tate groups over a field of characteristic p up to p-torsion.
Moreover, we compute the Chow ring of the classifying space of some Chevalley groups $G(\mathbb{F}_q)$, q being a power of p, when considered as a finite algebraic group over a field of characteristic p. Using specialization from characteristic 0 to characteristic p we also obtain results over the complex numbers.

Sei p eine Primzahl. Wir bestimmen den Chowring des Stacks der abgeschnittenen Displays. Weiter untersuchen wir die Pull-Back Abbildung des abgeschnittenen Display Funktors. Dies liefert den Chow Ring des Stacks der abgeschnittenen Barsotti-Tate Gruppen über einem Körper der Charakteristik p bis auf p-Torsion.
Sei q eine Potenz von p. Dann berechnen wir außerdem den Chow Ring des klassifizierenden Raumes einiger Chevalley Gruppen $G(\mathbb{F}_q)$ aufgefasst als endliche algebraische Gruppe über einem Körper der Charactersitik p. Durch Spezialisierung von Charakteristik 0 zu Charakteristik p erhalten wir auch Resultate über den komplexen Zahlen.

Contents

Introduction

In [EG] Edidin and Graham develop an equivariant intersection theory for actions of linear algebraic groups G on algebraic spaces X. For such G-spaces they define G-equivariant Chow groups $A_*^G(X)$ generalizing Totaros' defintion of the G-equivariant Chow ring of a point in [To]. These equivariant Chow groups have all functorial properties of ordinary Chow groups. Moreover, they are an invariant of the corresponding quotient stack $[X/G]$, i.e. they are independent of the choice of a presentation. Hence they can be used to define the integral Chow group of the quotient stack $[X/G]$. Let us write $CH_G^*(X)$ for the equivariant Chow group of X graded by codimension. If X is smooth $CH_G^*(X)$ carries a ring structure which makes it into a commutative graded ring that is naturally isomorphic to the operational Chow ring $A^*([X/G])$ of $[X/G]$. In the special case $X = \operatorname{Spec} k$ one obtains for $[X/G]$ the classifying space BG of G.
Therefore equivariant intersection theory is useful for computing Chow groups of quotient stacks. In [EG] Edidin and Graham used their theory to compute the Chow ring of the stacks $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ of elliptic curves. In an Appendix to the same paper Angelo Vistoli computed the Chow ring of $\mathscr{M}_2$. In the later paper [EF] Edidin and Fulghesu computed the integral Chow ring of the stack of hyperelliptic curves of even genus. In [To] Totaro computed the Chow ring of the classifying space of the classical groups and he treated the case of some finite abstract groups, including the symmetric groups.
In this thesis we investigate the Chow ring of the stack of truncated Barsotti-Tate groups and of the classifying space of some Chevalley groups. In both cases the computation can be reduced to the situation of Proposition B below.

The Chow Ring of the Classifying Space of some Chevalley Groups. Let p be a prime and q be a power of p. In [Gu] Guillot computes the mod l Chow ring of the classifying space of $\mathrm{GL}_n(\mathbb{F}_q)$, when considered as a finite algebraic group over the complex numbers, for odd primes l different from p. If $\mathbb{F}_q$ contains the l^b-th roots of unity for some integer b he also computes the mod l^b Chow ring. Using a different strategy we determine $CH^*B(\mathrm{GL}_n(\mathbb{F}_q)_\mathbb{C})$ after inverting p or $2p$ depending on whether $q \equiv 1$ mod 4 or not. We also obtain results for the groups $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.
To explain our strategy let us consider more generally the case of Chevalley groups, i.e. the finite groups $G(\mathbb{F}_q)$, where G is a connected split reductive group scheme over $\mathbb{Z}$. We then look at the specialization map

$$\sigma\colon CH^*BG(\mathbb{F}_q)_\mathbb{C} \to CH^*BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}.$$

Proposition A. (i) *The specialization map for the classical groups* $\mathrm{GL}_n(\mathbb{F}_q)$*,* $\mathrm{Sp}_{2m}(\mathbb{F}_q)$*,* $\mathrm{O}_n(\mathbb{F}_q)$ *and* $\mathrm{SO}_n(\mathbb{F}_q)$ *becomes injective after inverting* $2p$*.*

(ii) *If* $q \equiv 1$ *mod* 4 *the specialization map for* $\mathrm{GL}_n(\mathbb{F}_q)$*,* $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ *and* $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ *becomes injective after inverting* p*.*

(iii) *If S denotes the product of p and all prime divisor of $q-1$ the specialization map for $\mathrm{SL}_n(\mathbb{F}_q)$ becomes injective after inverting S.*

In the proof it suffices to see that the specialization map is injective for the respective l-Sylow subgroups of $G(\mathbb{F}_q)$ by the usual transfer argument. These l-Sylow subgroups are known to be products of iterated wreath products $(\mathbb{Z}/l)^{\wr i} \wr (\mathbb{Z}/l^b)$ by work of Weir ([Wei]). The Chow ring of the classifying space of this kind of iterated wreath products is computed over the complex numbers by Totaro ([To]), using the cycle map to Borel-Moore homology. In Section 2.3 we carry his proof over to the case of positive characteristic by using etale homology instead of Borel-Moore homology. It is an interesting question whether the specialization map is always injective (an isomorphism) after inverting p for an arbitrary finite abstract group; see Section 2.5.
As it turns out the computation of $CH^*BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$ is much simpler than the computation of $CH^*BG(\mathbb{F}_q)_{\mathbb{C}}$, because we have another presentation of the stack $BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$. Namely, it follows from a theorem of Lang-Steinberg that $BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$ is canonical isomorphic to the quotient stack $[G_{\bar{\mathbb{F}}_p}/G_{\bar{\mathbb{F}}_p}]$, where the action is given by conjugation with the q-th power Frobenius (Corollary 2.1.7). This case is dealt with in Chapter 3 and our result is the following.

Proposition B. *Let G be a connected split reductive group over $\mathbb{F}_q$ with split maximal torus T. We write $S = \mathrm{Sym}(\hat{T}) = A^*_T$ and $S_+ = A^{\geq 1}_T$. If σ denotes the q-th power Frobenius, we have a natural action of σ on S, that we will also denote by σ.*
Let $P \supset T$ be a parabolic subgroup with Levi component L and consider the action of L on G by σ-conjugation. If $W_G = W(G,T)$ and $W_L = W(L,T)$ denote the respective Weyl groups we have

$$A^*_L(G)_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/(S_+^{W_G}).$$

If G and L are both special we have

$$A^*_L(G) = S^{W_L}/(f - \sigma f \mid f \in S_+^{W_G}).$$

We recall that an algebraic group G is called special, if every G-torsor is locally trivial for the Zariski toplogy. Since GL_n, Sp_{2m} and SL_n are special we obtain a complete description of $CH^*BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$ in these cases and in particular we see that these Chow rings are generated by Chern classes of representations. Using the theory of Brauer lifts we see that the specialization map for these groups is surjective. We thus obtain the following result.

Theorem A. *Let S be the product of p and the primes that divide $q-1$. Then the* *following equations hold.*

$$
\begin{aligned}
&CH^*B(\mathrm{GL}_n(\mathbb{F}_q)_{\mathbb{C}})_{2p}\\
&\qquad = \mathbb{Z}[(2p)^{-1}][c_1,\dots,c_n]/((q-1)c_1,(q^2-1)c_2,\dots,(q^n-1)c_n)\\
&CH^*B(\mathrm{Sp}_{2m}(\mathbb{F}_q)_{\mathbb{C}})_{2p}\\
&\qquad = \mathbb{Z}[(2p)^{-1}][c_2,c_4,\dots,c_{2m}]/((q^2-1)c_2,(q^4-1)c_4,\dots,(q^{2m}-1)c_{2m}),\\
&CH^*B(\mathrm{SL}_n(\mathbb{F}_q)_{\mathbb{C}})_S\\
&\qquad = \mathbb{Z}[S^{-1}][c_2,c_3,\dots,c_n]/((q^2-1)c_2,(q^3-1)c_3,\dots,(q^n-1)c_n),
\end{aligned}
$$

where c_i denotes the i-th Chern class of the Brauer lift of the canonical representation of the respective groups.
If $q \equiv 1$ mod 4 it suffices to invert p in the first and second equation.

The Chow Ring of the Stack of truncated Barsotti-Tate Groups. Let us fix a field k of characteristic $p > 0$. Although the stack BT_n^h over k of truncated Barsotti-Tate groups of constant height h has a natural presentation $[Y_n^h/\,\mathrm{GL}_h]$ as a quotient stack with Y_n^h being quasi-affine and smooth (cf. [We]), it seems unlikely that this presentation can be used directly to compute the Chow ring due to the complicated nature of Y_n^h. Hence one either has to find a simpler presentation that we do not know of, or relate the stack of truncated Barsotti-Tate groups to a stack whose Chow ring is easier to compute, but still closely related to the Chow ring of BT_n.
Our choice for this stack is the stack $\mathcal{D}\mathrm{isp}_n$ of truncated displays introduced in [La]. Displays were first introduced in [Zi] to provide a Dieudonne theory that is valid not only over perfect fields but more generally over $\mathbb{F}_p$-algebras or p-adic rings. In Cartier theory a display over a p-adic ring R encodes the structure equations of a Cartier module of a formal p-divisble group over R and is given by an invertible matrix with entries in witt ring $W(R)$, if a basis of the Cartier module is fixed.
Using crystalline Dieudonne theory one can associate to every p-divisible group a display yielding a morphism $\phi\colon BT \to \mathcal{D}\mathrm{isp}$ from the stack of Barsotti-Tate groups to the stack of displays. While displays are given by invertible matrices over $W(R)$, a truncated display is given by an invertible matrix over the truncated Witt ring $W_n(R)$, and the morphism ϕ induces a morphism $\phi_n\colon BT_n \to \mathcal{D}\mathrm{isp}_n$. This morphism is a smooth morphism of smooth algebraic stacks over k and an equivalence on geometric points. This is the main result in [La].

Theorem B. *The pull-back $\phi_n^*\colon A^*(\mathcal{D}isp_n) \to A^*(BT_n)$ is injective and an isomorphism after inverting p.*

Let us sketch the proof. Consider a field L and a morphism $\operatorname{Spec} L \to BT_n$. After base change to a finite field extension of p-power degree the fiber $\phi_n^{-1}(\operatorname{Spec} L)$ is equal to the classifying space of an infinitesimal group scheme necessarily of p-power degree. For an appropiate notion of higher Chow groups it follows that the pull-back $A_*(\operatorname{Spec} L, m) \to A_*(\phi_n^{-1}(\operatorname{Spec} L), m)$ becomes an isomorphism after

inverting p. Using the long localization exact sequence the theorem follows from a limit argument and noetherian induction similar to that in [Qu2, Prop. 4.1]. The injectivity assertion follows since $A^*(\mathcal{D}\mathrm{isp}_n)$ is p-torsion free.
Thus to compute the Chow ring of BT_n at least up to p-torsion it suffices to compute the Chow ring of $\mathcal{D}\mathrm{isp}_n$, which is a much simpler task due to the simpler presentation as a quotient stack. More precisely, if $\mathcal{D}\mathrm{isp}_n^{h,d}$ denotes the open and closed substack in $\mathcal{D}\mathrm{isp}_n$ of truncated displays with constant dimension d and height h we have

$$\mathcal{D}\mathrm{isp}_n^{h,d} = [\mathrm{GL}_h(W_n(\cdot))/G_n^{h,d}],$$

where $G_n^{h,d}$ is an extension of $\mathrm{GL}_d \times \mathrm{GL}_{h-d}$ by a unipotent group. Moreover, we have the following theorem.

Theorem C. *The pull-back* $\tau_n^*\colon A^*(\mathcal{D}isp_1) \to A^*(\mathcal{D}isp_n)$ *of the truncation map* $\tau_n\colon \mathcal{D}isp_n \to \mathcal{D}isp_1$ *is an isomorphism.*

This follows easily from the factorization

$$[\mathrm{GL}_h(W_n(\cdot))/G_n^{h,d}] \to [GL_h/G_n^{h,d}] \to [GL_h/G_1^{h,d}]$$

of τ_n and the fact that the first map is an affine bundle and that $G_n^{h,d}$ is an extension of $G_1^{h,d}$ by a unipotent group. Hence it suffices to compute $A^*(\mathcal{D}\mathrm{isp}_1)$. In this case $G_1^{h,d}$ is a split extension of $\mathrm{GL}_d \times \mathrm{GL}_{h-d}$ by a unipotent group and the induced action of $\mathrm{GL}_d \times \mathrm{GL}_{h-d}$ on GL_h is given by σ-conjugation, where σ denotes the Frobenius. Using Proposition B we thus obtain the following result for the Chow ring of $\mathcal{D}\mathrm{isp}_1$.

Theorem D. *The following equation holds*

$$\begin{aligned} A^*(\mathcal{D}isp_1^{h,d}) &= A^*_{\mathrm{GL}_{h-d}\times\mathrm{GL}_d}(\mathrm{GL}_h) \\ &= \mathbb{Z}[t_1,\ldots,t_n]^{S_{h-d}\times S_d}/((p-1)c_1,\ldots,(p^h-1)c_h), \end{aligned}$$

where $c_1,\ldots,c_h$ *are the elementary symmetric polynomials in the variables* $t_1,\ldots,t_h$.

Moreover, $t_1,\ldots,t_d$ resp. $t_{d+1},\ldots,t_h$ are the Chern roots of the vector bundle $\mathcal{L}ie$ resp. ${}^t\mathcal{L}ie^\vee$ over $\mathcal{D}\mathrm{isp}_1^{h,d}$ of rank d resp. $h-d$. We refer to Definition 4.1.5 in the main text for the precise definition of $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$.
It follows that the $\mathbb{Q}$-vectorspace $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})_\mathbb{Q}$ is finite dimensional of dimension $\binom{h}{d}$, which also equals the number of isomorphism classes of truncated displays of level 1 with height h and dimension d over an algebraically closed field. We show that a basis is given by the cycles of the closures of the respective EO-Strata. We prove this fact in greater generality for the stack of G-zips in Section 4.4. In this section we will also compute the Chow ring of the stack of G-zips for a connected Frobenius zip datum. As in the case of displays the computation can be reduced to the situation of Proposition B. In fact, truncated displays of level 1 are a special case of G-zips.
Now by the above results we gain the following information on the Chow ring of the stack of truncated Barsotti-Tate groups.

Theorem E. (i) *We have*

$$A^*(BT_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1,\dots,t_h]^{S_d\times S_{h-d}}/((p-1)c_1,\dots,(p^h-1)c_h),$$

where c_i denotes the i-th elementary symmetric polynomial in the variables $t_1,\dots,t_h$ and $t_1,\dots,t_d$ resp. $t_{d+1},\dots,t_h$ are the Chern roots of $\mathcal{L}ie$ resp. ${}^t\mathcal{L}ie^\vee$.

(ii) $\dim_{\mathbb{Q}} A^*(BT_n^{h,d})_{\mathbb{Q}} = \binom{h}{d}$ *and a basis is given by the cyclces of the closures of the EO-Strata.*

(iii)

$$(\operatorname{Pic} BT_n^{h,d})_p = \begin{cases} \mathbb{Z}[p^{-1}]/(p-1) & \text{if } d=0,h \\ \mathbb{Z}[p^{-1}]\times\mathbb{Z}[p^{-1}]/(p-1) & \text{else,}\end{cases}$$

where the generator for the free resp. torsion part is $\det(\mathcal{L}ie)$ *resp.* $\det(\mathcal{L}ie\otimes {}^t\mathcal{L}ie^\vee)$.

It would be interesting to know if the Chow ring of BT_n has p-torsion, and more specifically if the Picard group of BT_n has p-torsion. However, since ϕ_n^* is injective and the Chow ring of $\mathcal{D}\mathrm{isp}_n$ is p-torsion free, p-torsion in the Chow ring of BT_n cannot be constructed using displays.

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Terminology and Notation. In the whole thesis p will be a fixed prime number. In Chapter 2 we denote by l a prime different from p. The letter k denotes an arbitrary field. In Chapter 4 we assume k to be of characteristic p. Every scheme over k is assumed to be of finite type. Algebraic groups are affine smooth group schemes over k. The character group of an algebraic group G will be denoted by $\hat{G}$. A representation V of an algebraic group G is supposed to be finite dimensional and rational, i.e. $G\to \mathrm{GL}(V)$ is a homomorphism of algebraic groups. If X is a scheme $A^*(X)$ will always denote the operational Chow ring of X ([Fu, Chapter 17]). $A_*(X)$ resp. $CH^*(X)$ will be the Chow group of X graded by dimension resp. codimension. If X is of pure dimension n we have $CH^{n-i}(X)=A_i(X)$ and if X is smooth there exists a natural isomorphism $A^*(X)\cong CH^*(X)$ of graded rings.
If X is an algebraic space over k with an action of an algebraic group G we will refer to X as a G-space. We write $[X/G]$ for the corresponding quotient stack. If G acts freely on X, i.e. the stabilizer of every point is trivial, then $[X/G]$ is an algebraic space. In this case we will write X/G instead of $[X/G]$ and call $X\to X/G$ the principal bundle quotient of X with structure group G.

1 Equivariant Intersection Theory

1.1 Unipotent Groups

Definition 1.1.1. *We call an algebraic group G unipotent if G admits a filtration $G = G_0 \supset G_1 \supset \ldots \supset G_e = \{1\}$ by subgroups such that G_i is normal in G_{i-1} with quotient isomorphic to $\mathbb{G}_a$.*

Remark 1.1.2. We remark that this definition differs from that given in [SGA3, Exposé XVII], where the quotients are assumed to be isomorphic to subgroups of $\mathbb{G}_a$. In characteristic zero $\mathbb{G}_a$ only has trivial subgroups but in positive characteristic there are also the subgroups $\mathbb{Z}/p\mathbb{Z}$ and α_p. Moreover, in [SGA3] the filtration accuring in the definition is defined only over $\bar{k}$ and not necessarily over k. Hence unipotent groups in our sense are smooth, connected and split unipotent groups in the sense of [SGA3].

We remind the that an algebraic group G is called special, if every principal G-bundle is locally trivial for the Zariski topology. Let us recall the following fact.

Lemma 1.1.3. *Let*

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

be an exact sequence of algebraic groups. If G_1 and G_3 are unipotent resp. special so is G_2.

Proof. The assertion in the unipotent case is clear. So let us assume that G_1 and G_3 are special and consider a principal G_2-bundle $X \to S$. We have that $X/G_1 \to S$ is a principal G_3-bundle, thus locally trivial for the Zariski topology. We may assume it is trivial. If we chose a section of $X/G_1 \to S$ we can form $X_1 = S \times_{X/G_1} X \to S$. This is a G_1-torsor with $X = G_2 \times^{G_1} X_1 \to S$, i.e. X has a reduction of the structure group to G_1. Since $X_1 \to S$ is locally trivial for the Zariski topology so is $X \to S$. □

Proposition 1.1.4. *Unipotent groups are special.*

Proof. In view of the previous lemma it suffices to show that $\mathbb{G}_a$ is special, but this is well known ([Se2]). □

Proposition 1.1.5. *Let U be unipotent and $P \to X$ be a U-torsor. If X is affine, then P is trivial.*

Proof. Principal U-bundles over X are classified by $\check{H}^1(X_{fl}, U)$. Since U is special we have $\check{H}^1(X_{fl}, U) = \check{H}^1(X_{zar}, U)$. But since X is affine we have $\check{H}^1(X_{zar}, \mathbb{G}_a) = 0$. Using the exact sequence of pointed sets in cohomology it follows $\check{H}^1(X_{fl}, U) = 0$. □

Corollary 1.1.6. *Let U be unipotent of dimension n. The underlying scheme of U is isomorphic to an affine space of dimension n. In particular, every principal U-bundle is an affine bundle.*

Proof. The assertion follows immediately from the preceding propositions and the fact that for a subgroup U' of U the map $U \to U/U'$ is a U'-torsor. □

1.2 Equivariant Chow Groups

In this section we recall the definition of equivariant Chow groups and the results of [EG] we shall need.
Consider an algebraic group G over k. By [EG, Lemma 9] we can find a representation V of G, and an open subset U in V such that the complement of U has arbitrary high codimension, and such that the principal bundle quotient U/G exists in the category of schemes. If X is an algebraic space on which G acts then G acts diagonally on $X \times U$ and we will denote the principal bundle quotient $(X \times U)/G$ by X_G.

Convention 1.2.1. *We call a pair (V, U) consisting of a G-representation V and an open subset U a good pair for G if G acts freely on U, i.e. the stabilizer of every point is trivial. Sometimes we will call the quotient $X_G = (X \times U)/G$ a mixed space for the G-space X. If (V, U) is a good pair for G with $\operatorname{codim}(U^c, V) > i$ we will also call $(X \times U)/G$ an approximation of $[X/G]$ up to codimension i.*

Definition 1.2.2. *If X has dimension n the i-th equivariant Chow group $A_i^G(X)$ is defined in the following way. Chose a good pair (V, U) for G such that the complement of U has codimension greater than $n - i$. Then one defines*

$$A_i^G(X) = A_{i+l-g}(X_G),$$

where l denotes the dimension of V and g is the dimension of G. The definition is independent of the choice of the pair (V, U) as long as $\operatorname{codim}(U^c, V) > n - i$ holds ([EG, Definition-Proposition 1]).

Remark 1.2.3. We remark that in general X_G is only an algebraic space even if X is a scheme. However, the definition of Chow groups for schemes can be carried over immediately to the case of algebraic spaces, so that one has Chow groups for algebraic spaces with the same functorial properties as in the case of schemes. In particular, we have an operational Chow ring $A^*(X)$ for algebraic spaces X defined in the same way as in [Fu, Chapter 17], i.e. an element $c \in A^i(X)$ is defined to be a collection of morphisms

$$c(Y \to X) \colon A_*(Y) \to A_{*-i}(Y)$$

for each algebraic space Y over X that are compatible with flat pull-back, proper push-forward and Gysin homomorphisms. For more details on this subject we refer

to the discussion in [EG] at the end of Section 6.1.
However, there are conditions on X and G so that a mixed space X_G exists in the category of schemes. See [EG, Proposition 23] and Lemma 1.3.4 below. In all our applications the conditions of Lemma 1.3.4 will be satisfied, so that in Chapter 2,3 and 4 algebraic spaces will not appear.

Let $f\colon X \to Y$ be a G-equivariant map of schemes or algebraic spaces. Assume that f has one of the properties proper, flat, smooth, regular embedding or l.c.i. then the induced map $f_G\colon X_G \to Y_G$ on the mixed spaces has the same property by [EG, Proposition 2]. It follows that equiviariant Chow groups have the same functorial properties as ordinary Chow groups.
Again we then have an operational equivariant Chow ring $A_G^*(X)$ ([EG, Section 2.6]), i.e. an element $c \in A_G^i(X)$ consists of operations $c(Y \to X)\colon A_*^G(Y) \to A_{*-i}^G(Y)$ for each G-equivariant map $Y \to X$ that are compatible with flat pull-back, proper push-forward and Gysin homomorphisms.
We will denote by $CH_G^*(X)$ the G-equivariant Chow group of X graded by codimension. Note if X is a pure dimensional G-scheme and (V, U) a good pair for G with $\operatorname{codim}(U^c, V) > i$ then

$$CH_G^j(X) = CH^j((X \times U)/G)$$

for all $j \leq i$. This motivates the term "approximation of $[X/G]$ up to codimension i" in Convention 1.2.1.
If X is smooth then $CH_G^*(X)$ carries a ring structure which makes it into a commutative graded ring with unit element. Moreover, there is a natural isomorphism $A_G^*(X) \cong CH_G^*(X)$ of graded rings ([EG, Proposition 4]). The next proposition shows that the equivariant Chow group $A_*^G(X)$ is an invariant of the corresponding quotient stack $[X/G]$, which will enable us to define the Chow group of a quotient stack.

Proposition 1.2.4. *Let X be a G-space and Y be an H-space such that $[X/G] \cong [Y/H]$. Then $A_{i+g}^G(X) = A_{i+h}^H(Y)$, where $g = \dim G$ and $h = \dim H$.*

Proof. Let (V_1, U_1) resp. (V_2, U_2) be a good pair for G resp. H of dimension l_1 resp. l_2. The fiber product $Z = X_G \times_{[X/G]} Y_H$ is an algebraic space which is open in a vector bundle over Y_H resp. X_H of rank l_1 resp. l_2 and we may assume that its complement is of arbitrary high codimension. We therefore obtain

$$A_{i+g}^G(X) = A_{i+l_1}(X_G) = A_{i+l_1+l_2}(Z) = A_{i+l_2}(Y_H) = A_{i+h}^H(Y).$$

□

Definition 1.2.5. *For a G-space X the i-th equivariant Chow group of $[X/G]$ is defined to be*

$$A_i([X/G]) = A_{i+g}^G(X),$$

where $g = \dim G$.

Proposition 1.2.6. *Let $[X/G]$ be a smooth quotienstack. Then there is an isomorphism $A^*([X/G]) \cong A^*_G(X)$ of graded rings. Moreover, it holds $A^1_G(X) = \mathrm{Pic}([X/G]) = \mathrm{Pic}^G(X)$.*

Proof. We recall the easy proof of the first statement. For the second part we refer to the proof of Proposition 18 in [EG].
There is always a natural map $A^*_G(X) \to A^*([X/G])$ defined in the following way. Let $c \in A^i_G(X)$ and $T \to [X/G]$ be a morphism from an algebraic space T to $[X/G]$. This morphism corresponds to a principal G-bundle $B \to T$ and a G-equivariant morphism $B \to X$. By definition c gives a map $c(B \to X)\colon A^G_*(B) \to A^G_{*-i}(B)$. Since $A^G_{*+g}(B) = A_*(T)$ we obtain an operational class in $A^i([X/G])$.
When X is smooth we have $A^G_{\dim X-i}(X) \cong A^i_G(X)$ and one can easily give an inverse to this map by sending $c \in A^i([X/G])$ to $c(X_G \to [X/G]) \cap [X]_G \in A^G_{\dim X-i}(X)$. □

Before giving the important examples we state and prove some useful facts about equivariant Chow rings.

We can consider any representation V of G as a G-equivariant vector bundle $\mathrm{Spec}(\mathrm{Sym}\, V^*)$ over k. Hence V has Chern classes $c_i^G(V)$, or sometimes just $c_i(V)$ when the context is clear, which we consider as elements of $A^*_G = A^*_G(\mathrm{Spec}\, k)$. Via the pull-back $A^*_G \to A^*_G(X)$ we can also consider them as elements of $A^*_G(X)$ for any G-space X. If T is a split torus in G the T-module V decomposes into a sum $V = V_{\chi_1} \oplus \ldots \oplus V_{\chi_r}$ of 1-dimensional Eigenspaces V_{χ_i} corresponding to characters χ_i of T. It follows from the Whitney Sum Formula that the Chern polynomial of V in the indeterminant t can be written as

$$c^T(V)(t) = \prod_{i=1}^{r}(1 + c_1^T(\chi_i)t)$$

when considered as an element of $A^*_T[t]$. The $c_1^T(\chi_i)$ are called the Chern roots of V. Any symmetric polynomial in the Chern roots lies in the image of the restriction map $A^*_G \to A^*_T$. If G is special reductive with split maximal torus T we will see in Section 1.9 that this map is injective with image being the invariants of A^*_T under the action of the Weyl group of (G, T).

For every principal G-bundle $X \to Y$, there is a natural action of the character group of G on $A_*(Y)$ defined in the following way. If λ is a character of G we consider the action of G on $\mathbb{A}^1$ via λ and the given action on X. This makes $\mathbb{A}^1 \times X$ into a G-equivariant line bundle over X, thus inducing a line bundle over Y that we will denote by $\mathcal{L}_\lambda$. Its first Chern class $c_1(\lambda) := c_1(\mathcal{L}_\lambda)$ then acts on $A_*(Y)$.
Equivalently we may view λ as a 1-dimensional representation of G, thus inducing an element $c_1(\lambda)$ in A^1_G as explained above. Via the pull-back $A^*_G \to A^*_G(X) = A^*(Y)$ this element then acts on $A_*(Y)$.

Lemma 1.2.7. *Let T be a split torus and $X \to Y$ be a principal T-bundle. Then pull-back induces an isomorphism*

$$A_*(X) \cong A_*(Y)/(\hat{T} A_*(Y)),$$

where $\hat{T}$ denotes the character group of T.

Proof. We first consider the case $T = \mathbb{G}_m$. Let $\mathcal{L} = (X \times \mathbb{A}^1)/\mathbb{G}_m$ be the corresponding line bundle over Y. Then X is the complement of the zero section $s \colon Y \to \mathcal{L}$ in $\mathcal{L}$. The zero section embeds Y as a Cartier Divisor in $\mathcal{L}$. In particular, s is a regular embedding of codimension 1 and furthermore the normal bundle $N_Y \mathcal{L}$ of this embedding is isomorphic to $\mathcal{L}$. The self intersection formula then states $s^* s_*(\alpha) = c_1(\mathcal{L}) \cap \alpha$ for all $\alpha \in A_*(Y)$. Hence we have the following commutative diagram, where the first row is the localization exact sequence.

$$\begin{array}{ccccccc} A_*(Y) & \xrightarrow{s_*} & A_*(\mathcal{L}) & \longrightarrow & A_*(X) & \longrightarrow & 0 \\ & \searrow^{c_1(\mathcal{L})} & \cong \uparrow \pi^* & \nearrow_{\eta^*} & & & \\ & & A_{*-1}(Y) & & & & \end{array}$$

Here η denotes the restriction of the projection $\pi \colon \mathcal{L} \to Y$ to X (See also [Fu, Example 2.6.3]). From this the lemma follows in the case $T = \mathbb{G}_m$.
For general T we write $T = T' \times \mathbb{G}_m$ where T' is a split torus of smaller dimension. We can then write $X \to Y$ as the composition $X \to X/T' \to Y$, where the first map is a T'-bundle and the second map is a $\mathbb{G}_m$-bundle. The general case thus follows by induction on the dimension of T. □

Proposition 1.2.8. *Consider an exact sequence*

$$0 \longrightarrow K \longrightarrow G \longrightarrow T \longrightarrow 0$$

of algebraic groups and assume T is a split torus. Let X be a G-space. Then $\hat{T}$ acts via $\hat{T} \to \hat{G}$ on $A_*([X/G])$ and pull-back induces an isomorphism

$$A_*([X/K]) \cong A_*([X/G])/(\hat{T} A_*([X/G])).$$

Proof. Choosing a good pair (V, U) for G the morphism $(X \times U)/K \to (X \times U)/G$ is a principal T-bundle and the proposition follows from the previous lemma. □

Remark 1.2.9. Let T be a split torus and X be a T-space. The above proposition then implies $A_*(X) = A_*^T(X)/(\hat{T} A_*^T(X))$. This result can also be found in [Br, Corollary 2.3] but with a different proof.

Another useful lemma regards the special case when G is a split extension by a unipotent group, i.e. there is a split exact sequence

$$0 \longrightarrow U \longrightarrow G \longrightarrow N \longrightarrow 0$$

of algebraic groups, where U is unipotent. It says that one can forget about the unipotent part when computing the Chow groups.

Lemma 1.2.10. *Let G be special and a split extension of an algebraic group N by a unipotent group. Chose a splitting $N \hookrightarrow G$. Then for any G-scheme X the restriction map yields an isomorphism $A_*^G(X) \cong A_*^N(X)$.*

Proof. The natural map $(X \times U)/N \to (X \times U)/G$ is a G/N-bundle which is locally trivial for the Zariski topology since G is special. Now G/N is isomorphic to an affine space and it follows from [Gi, Theorem 8.3], that the flat pull-back of this map induces an isomorphism of Chow groups. □

Remark 1.2.11. Note that an exact sequence as above does not need to be split even if N is reductive. An example is the exact sequence

$$0 \longrightarrow \mathrm{Lie}(\mathrm{SL}_2) \longrightarrow \mathrm{SL}_2(W_2) \longrightarrow \mathrm{SL}_2 \longrightarrow 0,$$

where we consider $\mathrm{Lie}(\mathrm{SL}_2)$ as a vector group over k. For details see Remark 5 in [Mc].
A variant of the above lemma also holds without the split assumption by Corollary 1.5.2. The proof uses higher Chow groups which we will adress in the next section.

Example 1.2.12. (Equivariant Chow Ring of the Classical Groups)
• $G = \mathrm{GL}_n$: For GL_n we can take the representation $V = Mat(n \times p, k)$ with $p > n$ and $U = \{M \in V \mid rkM = n\}$. Then U/G is the Grassmannian $Gr(n,p)$. Let E be the canonical representation of GL_n. Then $(E \times U)/G \to U/G$ is the universal rank n quotient bundle over $Gr(n,p)$ and for p sufficiently large ($p > i + n$) we get that A_G^i is the group of homogeneous symmetric polynomials of degree i in the Chern roots t_k of E. (See [Fu, Chapter 14] for the Chow ring of the Grassmannian.) Therefore

$$A^*_{\mathrm{GL}_n} = \mathbb{Z}[c_1, \dots, c_n]$$

where c_i is of weighted degree i.

• $T = \mathbb{G}_m^n$: Let us consider T as the torus of diagonal matrices in GL_n. We can take $V = \bigoplus_{i=1}^n (k^l)$ with $\mathbb{G}_m^n$-action given by

$$(\lambda_1, \dots, \lambda_n)(x_1, \dots . x_n) = (\lambda_1 x_1, \dots, \lambda_n x_n)$$

and $U = \prod_{i=1}^n k^l - \{0\}$. Then $U/T = (\mathbb{P}^{l-1})^n$. Let again E be the standard representation as above and $E = E_{\chi_1} \oplus \dots \oplus E_{\chi_n}$ the decomposition into 1-dimensional T-Eigenspaces. Here $\chi_i \colon T \to \mathbb{G}_m$ is just the i-th projection. Then $(E_{\chi_i} \times U)/T = pr_i^* \mathcal{O}(1)$ and therefore A_T^i is the group of homogeneous polynomials of degree i in the Chern roots $t_1, \dots, t_n$ of E. Hence

$$A_T^* = \mathbb{Z}[t_1, \dots, t_n]$$

and especially

$$A^*_{\mathrm{GL}_n} = (A_T^*)^{S_n}.$$

• SL_n: We apply Proposition 1.2.8 to the exact sequence

$$0 \longrightarrow \mathrm{SL}_n \longrightarrow \mathrm{GL}_n \xrightarrow{det} \mathbb{G}_m \longrightarrow 0.$$

Let again E denote the canonical representation of GL_n. Then we have $c_1(det) = c_1(\wedge^n E) = c_1(E) \in A^1_{GL_n}$ and therefore

$$A^*_{\mathrm{SL}_n} = \mathbb{Z}[c_2, \dots, c_n].$$

Here $c_1(\wedge^n E) = c_1(E)$ holds by Lemma 1.7.5 below.

• $\mathrm{Sp}_{2n}, \mathrm{O}_n, \mathrm{SO}_n$: The calculations for the other classical groups are slightly more involved and are carried out in [To] and [RV]. We have

$$A^*_{\mathrm{Sp}_{2n}} = \mathbb{Z}[c_2, c_4, \dots, c_{2n}], \quad A^*_{\mathrm{O}_n} = \mathbb{Z}[c_1, \dots, c_n]/(2c_{\text{odd}})$$

and in case that n is odd

$$A^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n]/(2c_{\text{odd}}).$$

There is also a result for even n ([Fi]).

• μ_n: We apply Proposition 1.2.8 to the Kummer sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(\ )^n} \mathbb{G}_m \longrightarrow 0,$$

thus obtaining

$$A^*_{\mu_n} = \mathbb{Z}[t]/(nt).$$

Here t is the first Chern class of the character $\mu_n \hookrightarrow \mathbb{G}_m$.

• $\mathbb{G}_a$: We have $A^*_{\mathbb{G}_a} = \mathbb{Z}$ in degree 0. This follows immediately from Lemma 1.2.10. More generally, we obtain $A^*_U = \mathbb{Z}$ in degree 0 for any unipotent group U.

• $\mathbb{Z}/n\mathbb{Z}$: If k is a field of characteristic not dividing n that contains the n-th roots of unity, then $(\mathbb{Z}/n\mathbb{Z})_k = \mu_n$. If k is a field of characteristic $p > 0$ we use the Artin-Schreier exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

to deduce $A^*_{(\mathbb{Z}/p\mathbb{Z})_k} = A^*_{\mathbb{G}_a} = \mathbb{Z}$ in degree 0. Namely, if (V, U) is a good pair for $\mathbb{G}_a$ then $U/(\mathbb{Z}/p) \to U/\mathbb{G}_a$ is a principal $\mathbb{G}_a$-bundle and hence its pull-back is an isomorphism.
In particular, the equivariant Chow ring of $\mathbb{Z}/n\mathbb{Z}$ considered as an algebraic group over a field k depends on the characteristic of k.

1.3 Higher Equivariant Chow Groups

The reason why we shall need higher Chow groups is that they extend the localization exact sequence to the left. Let us recall Bloch's definition of higher Chow groups in the case of schemes ([Bl]).

Bloch's higher Chow groups. Let Δ^m be the (algebraic) m-simplex

$$\Delta^m = \operatorname{Spec} k[t_0, \ldots, t_m]/(t_0 + \ldots + t_m - 1) \cong \mathbb{A}^m.$$

For an injective and increasing map $\rho\colon \{0, \ldots, m\} \to \{0, \ldots, n\}$ we define the corresponding face map $\tilde{\rho}\colon \Delta^m \to \Delta^n$ via

$$\tilde{\rho}^*(t_i) = \begin{cases} t_j & \text{if } \rho(j) = i, \\ 0 & \text{if } \rho^{-1}(\{i\}) = \emptyset. \end{cases}$$

Let $z_i(X, n)$ be the free abelian group generated by subvarieties V of $X \times \Delta^n$ of dimension $i + n$ meeting all m-faces $F = X \times \Delta^m \subset X \times \Delta^n$ properly, i.e. $\dim(V \cap F) \leq i + m$. For $i = 0, \ldots, n$ let $\delta_i\colon z_*(X, n) \to z_*(X, n-1)$ be the pull-back along the face map given by the inclusion $\{1, \ldots, n-1\} \to \{0, \ldots, n\}$ that leaves out i. We then obtain a chain complex $z_*(X, \cdot)$ with chain maps $\sum_{i=0}^{n} (-1)^i \delta_i\colon z_*(X, n) \to z_*(X, n-1)$. Bloch's higher Chow groups are then defined to be the homology groups of this complex

$$A_*(X, m) = H_m(z_*(X, \cdot)).$$

We remark that for $m = 0$ one gets back the usual Chow group $A_*(X)$ and that $A_i(X, m)$ maybe non-trivial for $-m \leq i \leq \dim X$. The definition of these higher Chow groups also works for algebraic spaces.

In order to define G-equivariant versions $A_*^G(X, m)$ of higher Chow groups we need the homotopy property for the mixed spaces X_G, i.e. the pull-back map

$$A_*(X_G, m) \to A_*(\mathcal{E}, m)$$

for a vector bundle $\mathcal{E}$ over X_G is an isomorphism. This is true for any scheme if $\mathcal{E}$ is trivial by [Bl. Theorem 2.1]. To prove the assertion for arbitrary vector bundles one needs the localization exact sequence of higher Chow groups proved by Bloch in the case of quasi-projective schemes.

Proposition 1.3.1. *Let X be an equidimensional, quasi-projective scheme over k. Let $Y \subset X$ be a closed subscheme and $U = X - Y$. Then the natural map*

$$z_*(X, \cdot)/z_*(Y, \cdot) \to z_*(U, \cdot)$$

is a quasi-isomorphism. Hence there is a long exact sequence of higher Chow groups

$$\begin{aligned} \ldots \to A_*(Y, m) \to A_*(X, m) \to A_*(U, m) \to A_*(Y, m-1) \\ \to \ldots \to A_*(Y) \to A_*(X) \to A_*(U) \to 0. \end{aligned}$$

This long localization exact sequence is compatible with flat pull-back and proper push-forward.

Proof. See [EG, Lemma 4] and [Bl, Theorem 3.1]. □

Corollary 1.3.2. *Let $\mathcal{E} \to X$ be a vector bundle over an equidimensional quasi-projective scheme X. Then the flat pull-back*

$$A_*(X, m) \to A_*(\mathcal{E}, m)$$

is an isomorphism for all m.

Proof. Using noetherian induction and the long localization exact sequence one reduces to the case of a trivial vector bundle and this case is [Bl, Theorem 2.1]. □

Remark 1.3.3. In Corollary 1.4.7 we will prove a stronger version of the above corollary. Namely, the pull-back $f^*\colon A_*(X, m) \to A_*(T, m)$ of a flat map $f\colon T \to X$ is an isomorphism if the fibers of f are affine spaces of some dimension.

In view of the above corollary we will need that we can chose the mixed spaces to be quasi-projective schemes. This holds for example in the following situation.

Lemma 1.3.4. *Let G be an algebraic group and X a normal, quasi-projective G-scheme. Then for any $i > 0$ there is a representation V of G and an invariant open subset $U \subset V$ whose complement has codimension greater than i such that G acts freely on U and the principal bundle quotient $(X \times U)/G$ is a quasi-projective scheme. In other words, the quotient stack $[X/G]$ can be approximated by quasi-projective schemes.*

Proof. Embed G into GL_n for some n. Then there is a representation of GL_n and an open subset U, whose complement has codimension greater than i such that U/GL_n is a Grassmannian (See [EG, Lemma 9] or Example 1.2.12). Since GL_n is special the GL_n/G-bundle $\pi\colon U/G \to U/\mathrm{GL}_n$ is locally trivial for the Zariski topology, and we will first show that π is quasi-projective.
Since GL_n/G is quasi-projective and normal there is an ample GL_n-linearizable line bundle $L \to \mathrm{GL}_n/G$ ([Th, Section 5.7]). Then

$$(U \times L)/\mathrm{GL}_n \to (U \times (\mathrm{GL}_n/G))/\mathrm{GL}_n = U/G$$

is a line bundle relatively ample for π. This shows that π is quasi-projective. The same holds then for U/G. Again by [Th, Section 5.7] there is an ample G-linearizable line bundle on X. The pull-back to $X \times U$ is then relatively ample for the projection $X \times U \to U$. Applying [GIT, Proposition 7.1] to this situation yields the claim. □

Definition 1.3.5. (i) *A pair (V, U) will be called an admissible pair for a G-scheme X if (V, U) is a good pair for G and if the mixed space X_G is quasi-projective and equidimensional over k. X will be called an admissible G-scheme if for any i there is an admissible pair (V, U) for X with $\mathrm{codim}(U^c, V) > i$.*

(ii) *We will say that a stack $\mathscr{X}$ admits an admissible presentation if there exists an admissible G-scheme X such that $\mathscr{X} = [X/G]$.*

(iii) *If X is an admissible G-scheme we define its higher equivariant Chow groups to be*

$$A_i^G(X, m) = A_{i+l-g}(X_G, m),$$

where $g = \dim G$ and X_G is formed from an l-dimensional admissible pair (V, U) such that $\operatorname{codim}(U^c, V) > \dim X + m - i$. The proof that this definition is independent of the choice of the admissible pair (V, U) is the same as for ordinary equivariant Chow groups ([EG, Definition-Proposition 1]) by using Corollary 1.3.2.

Remark 1.3.6. We will frequently encounter the situation of a morphism $T \to X$ of G-schemes such that T is open in a G-equivariant vector bundle over X. We remark that, if X is an admissible G-scheme, so is T. This follows since a vector bundle over a quasi-projective scheme is again quasi-projective.

Remark 1.3.7. We remark that Levine extended Blochs proof of the existence of the long localization exact sequence to all separated schemes of finite type over k ([Le, Theorem 1.7]). Hence for the equivariant higher Chow groups to be well defined it suffices that we can chose the mixed spaces to be separated schemes over k. However, in all applications we have in mind the conditions of Lemma 1.3.4 will be satisfied.

Proposition 1.3.8. *(Localization Sequence,[EG, Prop. 5]) Let X be an equidimensional, quasi-projective and normal G-scheme. Let $Y \subset X$ be a closed invariant subscheme. Write $U = X - Y$. Then there is a long exact sequence of higher equivariant Chow groups*

$$\begin{aligned} \ldots \to A_*^G(Y, m) \to A_*^G(X, m) \to A_*^G(U, m) \to A_*^G(Y, m-1) \\ \to \ldots \to A_*^G(Y) \to A_*^G(X) \to A_*^G(U) \to 0 \end{aligned}$$

This long exact sequence is compatible with flat pull-back and proper push-forward.

The following lemma will allow us to define higher Chow groups for quotient stacks which admit presentations by admissible G-schemes.

Lemma 1.3.9. *Assume that X is an admissible G-scheme and Y is an admissible H-scheme such that $[X/G] = [Y/H]$ as quotient stacks. Then $A_{i+g}^G(X, m) \cong A_{i+h}^H(Y, m)$ where $g = \dim G$ and $h = \dim H$.*

Proof. This is the same proof as in Proposition 1.2.4. □

Definition 1.3.10. *Let $\mathscr{X}$ be a quotient stack that admits a presentation $\mathscr{X} = [X/G]$ by an admissible G-scheme X. We define the higher equivariant Chow groups of $\mathscr{X}$ as*

$$A_*(\mathscr{X}, m) = A_{*+g}^G(X, m)$$

where $g = \dim G$.

Lemma 1.3.11. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a flat map of quotient stacks of relative dimension r. Then there is a flat pull-back map $f^*\colon A_*(\mathscr{Y}) \to A_{*+r}(\mathscr{X})$ between the Chow groups. If $\mathscr{X}$ and $\mathscr{Y}$ admit admissible presentations the same assertion holds for the higher Chow groups.*
Furthermore, if $\mathscr{X}$ and $\mathscr{Y}$ are smooth then under the identification $A_*(\mathscr{X}) = A^*(\mathscr{X})$ the above morphism is just the natural pull-back map between the operational Chow rings.

Proof. Consider presentations $\mathscr{X} = [X/G]$ and $\mathscr{Y} = [Y/H]$. By definition $A_i(\mathscr{X}) = A^G_{i+g}(X)$ with $g = \dim G$ and similar for $A_i(\mathscr{Y})$. Chose a good pair (V_1, U_1) for G and a good pair (V_2, U_2) for H. Let $l_i = \dim V_i$. As usual we will write X_G resp. Y_H for the mixed space $(X \times U_1)/G$ resp. $(Y \times U_2)/H$. Consider the fibersquare

$$\begin{array}{ccccc} Z' & \longrightarrow & Z & \longrightarrow & Y_H \\ \downarrow & & \downarrow & & \downarrow \\ X_G & \longrightarrow & \mathscr{X} & \longrightarrow & \mathscr{Y} \end{array}$$

Then Z' is a bundle over X_G resp. Z with fiber U_2 resp. U_1 and $Z' \to Y_H$ is a flat map of algebraic spaces of relative dimension $l_1 + r$. Hence

$$A_{i+l_1+l_2+r}(Z') = A_{i+l_1+r}(X_G) = A_{i+r}(\mathscr{X})$$

and we define f^* to be the ordinary pull-back of the flat map $Z' \to Y_H$. The exact same construction works for the higher equivariant Chow groups if $\mathscr{X}$ and $\mathscr{Y}$ admit admissible presentations.
For the last part we recall that the isomorphism $A^i(\mathscr{X}) \cong A^G_{\dim X - i}(X)$ maps $c \in A^i(\mathscr{X})$ to $c(X_G \to \mathscr{X}) \cap [X_G] \in A^G_{\dim X-i}(X)$. Thus we need to check the equality

$$f^*(d(Y_H \to \mathscr{Y}) \cap [Y_H]) = d(X_G \to \mathscr{X} \to \mathscr{Y}) \cap [X_G]$$

for $d \in A^i(\mathscr{Y})$. This follows from the compatibility of d with flat pull-backs. □

1.4 Auxiliary Results

Lemma 1.4.1. *Let $f\colon X \to Y$ be a surjective, finite and flat map of degree d. Then the composition*

$$Z_*(Y) \xrightarrow{f^*} Z_*(X) \xrightarrow{f_*} Z_*(Y)$$

is multiplication by d.

Proof. Let $V \subset Y$ be a subvariety of Y. It suffices to see

$$\sum_{V_i} \ell_{\mathcal{O}_{f^{-1}(V),V_i}}(\mathcal{O}_{f^{-1}(V),V_i}) \deg(V_i/V) = \deg f,$$

where the sum goes over the irreducible components of $f^{-1}(V)$. Here $\deg(V_i/V)$ resp. $\ell_{\mathcal{O}_{f^{-1}(V),V_i}}(\mathcal{O}_{f^{-1}(V),V_i})$ denotes the degree of the extension of function fields resp. the length of $\mathcal{O}_{f^{-1}(V),V_i}$ considered as a modul over itself. One easily shows

$$\ell_{\mathcal{O}_{f^{-1}(V),V_i}}(\mathcal{O}_{f^{-1}(V),V_i})\deg(V_i/V) = \ell_{k(\eta)}(\mathcal{O}_{f^{-1}(V),V_i}),$$

where η is the generic point of V. The elements of $f^{-1}(\eta)$ are the generic points η_i of the components V_i of $f^{-1}(V)$ and

$$\deg f = \dim_{k(\eta)} H^0(f^{-1}(\eta), \mathcal{O}_{f^{-1}(\eta)}) = \dim_{k(\eta)} \bigoplus_i \mathcal{O}_{f^{-1}(\eta),\eta_i}.$$

Since $\mathcal{O}_{f^{-1}(V),V_i} = \mathcal{O}_{f^{-1}(\eta),\eta_i}$ the lemma follows. □

Corollary 1.4.2. *Let $f\colon X \to Y$ be a surjective, finite and flat map of degree d. Then the composition*

$$A_*(Y,m) \xrightarrow{f^*} A_*(X,m) \xrightarrow{f_*} A_*(Y,m)$$

is multiplication by d.

Proof. The same proof as in the previous lemma shows that the composition of complexes

$$z_*(Y,\cdot) \xrightarrow{f^*} z_*(X,\cdot) \xrightarrow{f_*} z_*(Y.\cdot)$$

is multiplication by d. □

Corollary 1.4.3. *Let $X \to Y$ be a flat morphism of schemes and $Y' \to Y$ be a finite, flat and surjective map of degree d. Let $X' \to Y'$ be the base change of $X \to Y$ along $Y' \to Y$. Assume the pull-back $A_*(Y',m) \to A_*(X',m)$ becomes an isomorphism after inverting some integer d'. Then the pull-back $A_*(Y,m) \to A_*(X,m)$ is an isomorphism after inverting dd'.*

Proof. The injectivity of the pull-back $A_*(Y,m)_{dd'} \to A_*(X,m)_{dd'}$ follows from the exact diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A_*(Y,m)_{dd'} & \longrightarrow & A_*(Y',m)_{dd'} \\ & & \downarrow & & \cong\downarrow \\ 0 & \longrightarrow & A_*(X,m)_{dd'} & \longrightarrow & A_*(X',m)_{dd'} \end{array}$$

and the surjectivity from the exact diagram

$$\begin{array}{ccccc} A_*(Y',m)_{dd'} & \longrightarrow & A_*(Y,m)_{dd'} & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow & & \\ A_*(X',m)_{dd'} & \longrightarrow & A_*(X,m)_{dd'} & \longrightarrow & 0 \end{array}$$

where the horizonatal maps in the first diagram are induced by pull-back and in the second diagram by push-porward. The commutativity of the second diagram is [Fu, Proposition 1.7]. ☐

The reason why we will frequently encounter finite, flat and surjective maps is due to the following lemma.

Lemma 1.4.4. *Let G be a finite group scheme of degree d over k. Let X be a G-scheme such that the principal bundle quotient X/G exists in the category of schemes, i.e. the quotient stack $[X/G]$ is a scheme. Then the quotient map $X \to X/G$ is finite, flat and surjective of degree d.*

Proof. Since the principal G-bundle $X \to X/G$ is locally trivial for the flat topology the assertion follows from descent theory [SGA1, Exposé VIII, Corollary 5.7]. ☐

Furthermore, we shall make frequent use of the following technical lemma.

Lemma 1.4.5. *Let $T \to X$ be a morphism of quasi-projective schemes over k. We assume that X is equidimensional and that $T \to X$ is flat of relative dimension a. Let $d, i \in \mathbb{Z}$ and for $x \in X$ let $h(x)$ denote the dimension of the closure of $\{x\}$ in X. If the pull-back $A_{i-h(x)}(\operatorname{Spec} k(x), m)_d \to A_{i-h(x)+a}(T_x, m)_d$ is an isomorphism for every $x \in X$ and for any m, then $A_i(X, m)_d \to A_{i+a}(T, m)_d$ is an isomorphism.*

Proof. We follow Quillen's proof of the analogous result in higher K-theory ([Qu2, Prop. 4.1]). First we may assume that X is irreducible for if $X = W_1 \cup \ldots \cup W_r$ is a decomposition into irreducible components we may consider the long localization exact sequence of the pair $(W_1, X - W_1)$. By induction we are thus reduced to the irreducible case. Since the Chow groups only depend on the reduced structure, we may also assume that X is reduced. Let K denote the function field of X. By Lemma 1.4.6 below we have

$$A_{i-n}(\operatorname{Spec} K, m) = \varinjlim_{U} A_i(U, m),$$

$$A_{i-n+a}(T_K, m) = \varinjlim_{U} A_{i+a}(T_U, m),$$

where the limit goes over all non-empty open subsets of X and n denotes the dimension of X. In fact, it suffices to go over all non-empty open subsets with equidimensional complement, since for all non-empty open U in X there exists a non-empty open subset U' contained in U with equidimensional complement. We obtain a commutative diagram

$$\begin{array}{ccccc} A_{i-n}(\operatorname{Spec} K, m+1) & \longrightarrow & \varinjlim_Y A_i(Y, m) & \longrightarrow & A_i(X, m) \\ \downarrow & & \downarrow & & \downarrow \\ A_{i-n+a}(T_K, m+1) & \longrightarrow & \varinjlim_Y A_{i+a}(T_Y, m) & \longrightarrow & A_{i+a}(T, m) \end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & A_{i-n}(\operatorname{Spec} K, m) & \longrightarrow & \varinjlim_Y A_i(Y, m-1) \\
& \downarrow & & \downarrow \\
\longrightarrow & A_{i-n+a}(T_K, m) & \longrightarrow & \varinjlim_Y A_{i+a}(T_Y, m-1)
\end{array}$$

with exact rows, where the limit goes over all proper closed equidimensional subsets of X. After inverting d the first and fourth vertical map become isomorphisms and we conclude by noetherian induction. □

Lemma 1.4.6. (i) *Let K/k be an algebraic field extension and X be a scheme over k. Then there is a natural isomorphism*

$$\varinjlim_L A_i(X_L, m) \cong A_i(X_K, m),$$

where the limes goes over all finite subextensions L over k.

(ii) *Let $T \to X$ be a flat morphism of some relative dimension, where X is an integral scheme. Let n denote the dimension of X and K its function field. Then there is a natural isomorphism*

$$\varinjlim_U A_i(T_U, m) \cong A_{i-n}(T_K).$$

Here the limes goes over all non-empty open subsets of X.

Proof. (i) If V is a scheme over some subextension L over k then the base change V_K to K does not change the dimension. Hence the assignment $[V] \mapsto [V_K]$ defines a map $\varinjlim_L z_i(X_L, \cdot) \to z_i(X_K, \cdot)$ of complexes. This map is in fact an isomorphism since if $V \subset \Delta^m_{X_K}$ is a closed subscheme, then there is a finite extension L of k such that V has a model $\tilde{V}$ over L. The assignment $[V] \mapsto [\tilde{V}]$ then defines an inverse.

(ii) Consider a subvariety $V \subset \Delta^m_{T_U}$ of dimension $i+m$ meeting all faces properly, i.e. $\operatorname{codim}(V \cap F) \geq \operatorname{codim}(V) + \operatorname{codim}(F)$ for every face F of $\Delta^m_{T_U}$. We may assume that the composition $V \hookrightarrow \Delta^m_{T_U} \to U$ is dominant, since otherwise $[V]$ is zero in $\varinjlim_U z_i(T_U, \cdot)$. This means V_K is again a subvariety of $\Delta^m_{T_K}$ with $\operatorname{codim}(V_K) = \operatorname{codim}(V)$ and $\operatorname{codim}(V_K \cap F) \geq \operatorname{codim}(V \cap F)$ for every face F of $\Delta^m_{T_K}$. Hence $[V_K] \in z_{i-n}(T_K, \cdot)$ and the assignment $[V] \mapsto [V_K]$ defines a natural map of complexes $\varinjlim_U z_i(T_U, \cdot) \to z_{i-n}(T_K, \cdot)$. Again we claim that this map is an isomorphism.

For this let $[V = V(I)]$ be an element in $z_{i-n}(T_K, m)$. Let $U \subset X$ be open and affine. We may then consider $\tilde{V} = V(\mathcal{O}_{\Delta^m_{T_U}} \cap I) \subset \Delta^m_U$ and we need to check that $\tilde{V}$ has dimension $i+m$ and intersects all faces properly. But V is the preimage of $\tilde{V}$ under $\Delta^m_{T_K} \to \Delta^m_{T_U}$ and therefore $\operatorname{codim}(\tilde{V}) = \operatorname{codim}(V)$. By shrienking U if necessary we may also assume $\operatorname{codim}(V_K \cap F_K) = \operatorname{codim}(\tilde{V} \cap F)$ for all faces F. Hence the assignment $[V] \mapsto [\tilde{V}]$ defines a map $z_{i-n}(T_K, \cdot) \to \varinjlim_U z_i(T_U, \cdot)$ of complexes that is inverse to the natural map $\varinjlim_U z_i(T_U, \cdot) \to z_{i-n}(T_K, \cdot)$. □

Corollary 1.4.7. *Let $T \to X$ be a flat morphism of quasi-projective schemes over k with fibers being affine spaces of some dimension n. Then the pull-back $A_*(X, m) \to A_{*+n}(T, m)$ is an isomorphism.*

Proof. This is an immediate consequence of Lemma 1.4.5. □

Remark 1.4.8. The assertion of the above corollary in the case $m = 0$ also holds without the quasi-projective assumption. One can use the same proof but using Gillet's higher Chow groups. For his higher Chow groups a long localization exact sequence exists for arbitrary schemes. For details see Chapter 8 in [Gi].

Lemma 1.4.9. *Let K be a unipotent subgroup of an algebraic group G such that the quotient G/K is finite of degree d. Then the pull-back $A^*_G(m) \to A^*_{\{0\}}(m)$ is an isomorphism after inverting d.*

Proof. Let (V, U) be an admissible pair for G. Then $U/K \to U/G$ is a G/K-bundle locally trivial for the flat topology. By assumption on G/K the morphism $U/K \to U/G$ is therefore finite, flat and surjective of degree d. It follows that the pull-back $A_*(U/G, m) \to A_*(U/K, m) \cong A_*(U, m)$ is injective after inverting d. Also for sufficiently high dimension we know that $A_*(\operatorname{Spec} k, m) \to A_*(U, m)$ is surjective. Since we can assume the codimension of U^c in V to be arbitrary high, we obtain the surjectivity of $A^*_G(m) \to A^*_{\{0\}}(m)$. □

Lemma 1.4.10. *Let K/k be a Galois extension with Galois group G and let X be a scheme over k. Then pulling back along $X_K \to X$ induces an isomorphism $A_*(X, m)_{\mathbb{Q}} \cong A_*(X_K, m)^G_{\mathbb{Q}}$. If K/k is a finite Galois extension of degree d it suffices to invert d.*

Proof. We first assume that K/k is finite of degree d. Then on the level of cycles we have an injection $z_*(X, \cdot)_d \hookrightarrow z_*(X_K, \cdot)^G_d$ by Lemma 1.4.1. We claim that this map is also surjective. Let $W \subset X_K \times_K \Delta^r_K$ be a subvariety meeting all faces properly. Let $S \subset G$ be the isotropy group of W. It suffices to see that $\sum_{g \in G/S} [gW]$ lies in $z_*(X, \cdot)_d$. For this consider the closed subscheme $V = \cup_{g \in G/S} gW$ (equipped with the reduced structure). Then V is a G-invariant equidimensional subscheme of $X_K \times_K \Delta^r_K$ that meets all faces properly. Thus it has a model $\tilde{V}$ over k also meeting all faces properly. Finally all components gW have the same multiplicity 1 in the cycle $[V]$ and therefore $\sum_{g \in G/S} [gW] = [\tilde{V}_K]$.
To complete the proof in the finite case it suffices now to note that taking G-invariants is an exact functor on the category of $\mathbb{Z}[\frac{1}{d}]$-modules with G-action, hence $H_i(z_*(X_K, \cdot)^G_d) = H_i(z_*(X_K, \cdot))^G_d$.
The general case follows from the finite case and the fact that $A_*(X_K, m)^G = \varinjlim_{L/k} A_*(X_L, m)^{G(L/k)}$, where the limit goes over all finite Galois subextensions L/k of K. □

1.5 A Pull-Back Lemma

Throughout we consider the situation of an exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

of algebraic groups and an admissible H-scheme X such that the induced G-action on X makes X also into an admissible G-scheme. These conditions are always satisfied if X is quasi-projective and normal by Lemma 1.3.4. We are then interested in properties of the pull-back homomorphism (Lemma 1.3.11)

$$A_*([X/H], m) \to A_*([X/G], m).$$

Proposition 1.5.1. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme. We also assume H to be special.
Let $d \in \mathbb{Z}$ such that $A^*_{A_L}(m) \to A^*_{\{0\}}(m)$ becomes an isomorphism after inverting d for every field extension L of k and every m. Then the pull-back $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting d.

Proof. First note that the natural map $[X/G] \to [X/H]$ is flat of relative dimension $-a$ with $a = \dim A$. We can chose for any $i \in \mathbb{Z}$ an admissible pair (V, U) for the H-action such that $A_{j+l}([(X \times U)/G], m) = A_j([X/G], m)$ and $A_{j+l}((X \times U)/H, m) = A_j([X/H], m)$ for all $j > i$. Here l denotes the dimension of V. Note that $X \times U$ is again an admissible G-scheme (cf. Remark 1.3.6). Replacing X by $X \times U$ we may thus assume that $[X/H]$ is a quasi-projective scheme.
Let now $(X \times U)/G$ be a quasi-projective mixed space for G. Let $\bar{U}$ be the quotient U/A. Then we can identify $(X \times U)/G$ with the quotient $(X \times \bar{U})/H$ and under this identification the map $(X \times U)/G \to X/H$ corresponds to the $\bar{U}$-bundle $(X \times \bar{U})/H \to X/H$. It is Zariksi locally trivial since H is special. We are left to show that the pull-back of this map is an isomorphism after inverting d. This will follow from Lemma 1.4.5 once we have seen that the pull-back $A_{j-h(x)}(\operatorname{Spec} k(x), m)_d \to A_{j-h(x)+l-a}(\bar{U}_{k(x)}, m)_d$ is an isomorphism for every $x \in X/H$. Here $h(x)$ is the dimension of the closure of $\{x\}$ in X/H. Let us write $L = k(x)$. Assuming the codimension of U^c in V to be sufficiently large we obtain by assumption

$$A_{j-h(x)}(\operatorname{Spec} L, m)_d = A_{j-h(x)+l}(U_L, m)_d = A_{j-h(x)+l-a}(\bar{U}_L, m)_d.$$

For this recall $A_{j+l-a}(\bar{U}_L, m) = A_j^{A_L}(m)$ and $A_{j+l}(U_L, m) = A_j^{\{0\}}(m)$. This proves the claim. □

The above proposition applies to the following cases.

Corollary 1.5.2. *In the situation of Proposition 1.5.1 the following assertions hold.*

(i) *If A is unipotent then $A_*([X/H], m) \to A_*([X/G], m)$ is an isomorphism.*

(ii) *If A is finite of degree d then $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting d.*

Proof. The first part follows from Corollary 1.4.7 and the second part follows from Lemma 1.4.9 applied to the case $K = \{0\}$. □

The assumption on H to be special is crucial for the proof of Proposition 1.5.1, since we need to know that the fibers of the $\bar{U}$-bundle $(X \times \bar{U})/H \to X/H$ appearing in the proof are given by $\bar{U}$ in order to apply Lemma 1.4.5. However, we have the following version when H is finite.

Proposition 1.5.3. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme. We assume that H is finite of degree d.
Let $d' \in \mathbb{Z}$ such that $A^*_{A_L}(m) \to A^*_{\{0\}}(m)$ becomes an isomorphism after inverting d' for every field extension L of k and any m. Then the pull-back $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting dd'.

Proof. We argue the same way as in Proposition 1.5.1 and then have to see that the pull-back of $(X \times \bar{U})/H \to X/H$ becomes an isomorphism after inverting dd'. As mentioned earlier we cannot apply Lemma 1.4.5 since the above $\bar{U}$-bundle is not locally trivial for the Zariski topology. Instead it becomes trivial after the finite, flat and surjective base change $X \to X/H$ of degree d, i.e. there is a cartesian diagram

$$\begin{array}{ccc} X \times \bar{U} & \longrightarrow & X \\ \downarrow & & \downarrow \\ (X \times \bar{U})/H & \longrightarrow & X/H. \end{array}$$

The claim thus follows from Corollary 1.4.3. □

Corollary 1.5.4. *In the situation of Proposition 1.5.3 the following assertions hold.*

(i) *If A is unipotent then $A_*([X/H], m)_d \to A_*([X/G], m)_d$ is an isomorphism.*

(ii) *If A is finite of degree d' then $A_*([X/H], m)_{dd'} \to A_*([X/G], m)_{dd'}$ is an isomorphism.*

In the next proposition we proof that the assertion of Proposition 1.5.1 is valid over $\mathbb{Q}$ for arbitrary H.

Proposition 1.5.5. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme.
Assume $A^*_{A_L}(m)_{\mathbb{Q}} \to A^*_{\{0\}}(m)_{\mathbb{Q}}$ is an isomorphism for every field extension L of k and any m. Then the pull-back $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.

Proof. Using the notation of the proof of Proposition 1.5.1 we need to see that the pull-back of the $\bar{U}$-bundle $T := (X \times \bar{U})/H \to X/H$ is an isomorphism over $\mathbb{Q}$. It suffices to see that $A_*(\operatorname{Spec} k(x), m)_{\mathbb{Q}} \to A_*(T_x, m)_{\mathbb{Q}}$ is an isomorphism for $x \in X/H$. The above $\bar{U}$-bundle may not be trivial for the Zariski topology, but we still have $T_{\bar{x}} = \bar{U}_{\bar{x}}$ and therefore $A_*(\operatorname{Spec} k(x)^{sep}, m)_{\mathbb{Q}} \to A_*(T_{\bar{x}}, m)_{\mathbb{Q}}$ is an isomorphism by assumption. The claim then follows from Lemma 1.4.10 and the fact that the Galois action is compatible with pull-back. □

Corollary 1.5.6. *In the situation of Proposition 1.5.5 the following assertions hold.*

(i) *If A is unipotent then $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.*
(ii) *If A is finite then $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.*

Lemma 1.5.7. *Let G be a split extension of an algebraic group H by a unipotent group. Chose a splitting $H \hookrightarrow G$ and let X be a normal, quasi-projective G-scheme. Then the pull-back map*

$$A^G_*(X, m)_{\mathbb{Q}} \to A^H_*(X, m)_{\mathbb{Q}}$$

is an isomorphism. If G is special the above map is an isomorphism over $\mathbb{Z}$.

Proof. Let (V, U) be an admissible pair for the G-action on X. It follows from the proof of Lemma 1.3.4 that (V, U) is then also admissible for the induced H-action. The morphism $(X \times U)/H \to (X \times U)/G$ is a G/H-bundle. If G is special this bundle is locally trivial for the Zariski topology. Hence the lemma follows from Corollary 1.4.7 in the special case and Lemma 1.4.10 and 1.4.5 in the general case. □

1.6 Equivariant Chow Ring of Flag Varieties

We recall that G-space refers to an algebraic space over k that carries the action of an algebraic group G. Let X be a smooth G-space and E be a vector bundle over X of rank e. We denote by $Fl(E) \to X$ the corresponding flag space of E, i.e. the space parametrizing filtrations of E by subbundles with line bundle quotients. We recall the following fact.

Lemma 1.6.1.

$$A^*(Fl(E)) = A^*(X)[t_1, \dots, t_e]/(c_i(t) - c_i(E), i = 1, \dots, e),$$

where $t_1, \dots, t_e$ denote the Chern roots of E and $c_i(t)$ is the i-th elementary symmetric polynomial in the variables $t_1, \dots, t_e$.

Proof. Let $\mathcal{Q}$ denote the universal quotient bundle over $\mathbb{P}(E)$. Then $Fl(Q) = Fl(E)$ and we can write the structure map $Fl(E) \to X$ as the composition $Fl(Q) \to \mathbb{P}(E) \to X$. The lemma then follows easily by induction on the rank of E and the projective bundle theorem ([Fu, Example 8.3.4]). □

Let V be a G-representation. Then the action of G on V induces an action on the projective space $\mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}\, V^*)$ such that the natural morphism $V - \{0\} \to \mathbb{P}(V)$ is a G-equivariant principal $\mathbb{G}_m$-bundle. If E is a G-equivariant vector bundle on a G-space X we also have a natural action of G on $\mathbb{P}(E)$ such that the projection $\mathbb{P}(E) \to X$ is G-equivariant and the same holds for the flag space $Fl(E)$.

Lemma 1.6.2. *Let X be a scheme (or algebraic space) with free G-action and E be a G-equivariant vector bundle over X. Then there exist natural isomorphisms*

$$\mathbb{P}(E/G) \cong \mathbb{P}(E)/G, \quad Fl(E/G) \cong Fl(E)/G.$$

Proof. Let $\pi\colon X \to X/G$ be the quotient map. We have the following isomorphisms of G-spaces

$$\pi^*\mathbb{P}(E/G) = \mathbb{P}(\pi^*E/G) = \mathbb{P}(E) = \pi^*\mathbb{P}(E)/G$$

and similary for the flag space. But since pulling back along π gives a fully faithfull functor from algebraic spaces over X/G to G-spaces over X the lemma follows. □

Lemma 1.6.3. *(Lemma 2.3 in [EF]) Let E be a G-equivariant vector bundle of rank e on a smooth G-space X. Then*

$$A^*_G(\mathbb{P}(E)) = A^*_G(X)[\zeta]/(\zeta^e + C_1\zeta^{e-1} + \ldots + C_e)$$

where $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ and $C_1, \dots, C_e$ are the Chern classes of E.

Proof. We have the cartesian square

$$\begin{array}{ccc} \mathbb{P}(E) \times U & \longrightarrow & X \times U \\ \downarrow & & \downarrow \\ \mathbb{P}(E)_G & \longrightarrow & X_G \end{array}$$

Here $\mathbb{P}(E)_G$ is the projective bundle $\mathbb{P}(E_G)$ of the vector bundle E_G over X_G by the lemma above. The assertion thus follows from the projective bundle theorem for ordinary Chow groups. □

Lemma 1.6.4. *Let E be a G-equivariant vector bundle of rank e on a smooth G-space X. Then*

$$A_G^*(Fl(E)) = A_G^*(X)[t_1, \ldots, t_e]/(c_i(t) - c_i(E), i = 1, \ldots, e),$$

where $t_1, \ldots, t_e$ denote the G-equivariant Chern roots of E and $c_i(t)$ is the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_e$.

Proof. This follows in the same way as above from the case of ordinary Chow groups. □

Remark 1.6.5. For a reminder on G-equivariant Chern roots see also the next section.

We can also consider the equivariant Chow ring of generalized flag spaces: Let G be a connected split reductive group over k. Chose a split maximal torus T of G and a Borel subgroup $B \supset T$. The computation of $A_T^*(G/B)$ is carried out by Brion in [Br]. His result is the following.

Proposition 1.6.6. *Let $S = \mathrm{Sym}(\hat{T}) = A_T^*$ and $W = W(G, T)$ be the Weyl group of G. The multiplication map*

$$S \otimes_{S^W} S \to A_T^*(G/B)$$

is an isomorphism, if G is special. In general it is an isomorphism over $\mathbb{Q}$.

Proof. See [Br, Proposition 6.6]. □

1.7 G-invariant Sections

We start with the following definition.

Definition 1.7.1. *Let X be a G-scheme and E a G-equivariant vector bundle on X. We call a global section $s \in H^0(X, E)$ G-invariant if the corresponding morphism $X \xrightarrow{s} E$ is G-equivariant.*

Remark 1.7.2. A global section s of a G-equivariant vector bundle E is G-invariant if one of the following equivalent conditions hold.

(i) The morphism $\mathbb{A}^1 \times X \to E$ induced by s is a morphism of G-equivariant vector bundles, when $\mathbb{A}^1$ carries the trivial and X the given G-action.

(ii) $p_2^* s$ is mapped to $m^* s$ under the isomorphism $p_2^*(E) \xrightarrow{\cong} m^* E$ of locally free sheaves corresponding to the G-action on E.

This can be easily seen as follows. The commutativity of the induced diagram

$$\begin{array}{ccc} p_2^* E & \xrightarrow{\cong} & m^* E \\ \uparrow & & \uparrow \\ p_2^* \mathcal{O}_X & = & m^* \mathcal{O}_X \end{array}$$

of locally free sheaves on X translates into the commutativity of the diagram

$$\begin{array}{ccc} G \times E & \longrightarrow & E \\ \uparrow & & \uparrow \\ G \times \mathbb{A}^1_X & \longrightarrow & \mathbb{A}^1_X \end{array}$$

of vector bundles over $G \times X$ resp. X where the horizontal maps are the action maps. This shows the equivalence of (i) and (ii). Since $X \xrightarrow{s} E$ is the composite of the unit section $X \to \mathbb{A}^1_X$ and the map $\mathbb{A}^1_X \to E$ induced by s we see that s is G-invariant if and only if (i) holds.
The set of G-invariant sections of E is precisely the image of the pull-back map $H^0([X/G], [E/G]) \to H^0(X, E)$.

For a G-invariant section $s \in H^0(X, E)$ we can form the cartesian square

$$\begin{array}{ccc} Z(s) & \xrightarrow{i} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow[s_E]{} & E \end{array}$$

Here the lower map is the zero section of E which is a G-equivariant regular embedding of codimension $e = \operatorname{rank} E$. Since s is also G-equivariant, we get that $Z(s)$ is G-invariant. We can therefore define a localized G-equivariant top Chern class by

$$\mathbb{Z}(s) = s_E^!([X]_G) \in A^G_{\dim X - e}(Z(s))$$

which has the same properties as in [Fu, Proposition 14.1]. We recall that $s \in H^0(X, E)$ is called a regular section if $i \colon Z(s) \hookrightarrow X$ is a regular embedding of codimension e. In particular, we have

Lemma 1.7.3. *Let X be a G-scheme and E a G-equivariant vector bundle of rank e. If $s \in H^0(X, E)$ is a G-invariant regular section, then the zero scheme $Z(s)$ is G-invariant and we have $[Z(s)]_G = c_e^G(E) \cap [X]_G$ in $A^G_*(X)$.*

We remark that s is regular if X is Cohen-Macaulay and $\operatorname{codim} Z(s) = e$.
As a special case we obtain

Lemma 1.7.4. *(Lemma 2.4 in [EF]) Let T be a torus and V be a T-representation. Let $H \subset \mathbb{P}(V)$ a T-invariant hypersurface defined by a homogeneous form $f \in \operatorname{Sym}^d(V^*)$ that is a T-eigenfunction with eigenvalue $\chi \in \hat{T}$. Then*

$$[H]_T = c_1^T(\mathcal{O}_{\mathbb{P}(V)}(d)) - c_1(\chi)$$

in $A_T^*(\mathbb{P}(V))$.

Proof. Clearly f is a T-invariant regular section of $\mathcal{O}_{\mathbb{P}(V)}(d) \otimes \chi^{-1}$. Hence the assertion follows from the lemma above. $\square$

Equivariant Chern roots Since the flat pull-back $p^*\colon A^G_*(X) \to A^G_*(Fl(E))$ is injective, as follows from Lemma 1.6.4, we have a splitting construction for E, i.e. we can find a G-equivariant flat morphism $f\colon X' \to X$ such that the flat pull-back $f^*\colon A^G_*(X) \to A^G_*(X')$ is injective and f^*E has a filtration by G-invariant subbundles

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{e-1} \subset E_e = f^*E$$

with line bundle quotients $L_i = E_i/E_{i-1}$. We say that the $\alpha_i = c^G_1(L_i)$ are the G-equivariant Chern roots of E. As in the proof of Theorem 3.2 (a) in [Fu] one can show

$$c^G_t(f^*E) = \prod_i (1 + \alpha_i t).$$

If F is another G-equivariant vector bundle on X with rank f and Chern roots β_i the Chern roots of $E \otimes F$ are given by

$$\alpha_i + \beta_j, \quad 1 \le i \le e, \quad 1 \le j \le f.$$

Thus in the same way as for ordinary Chern classes ([Fu, Remark 3.2.3]) we obtain the following equations for G-equivariant Chern classes.

Lemma 1.7.5. *Let X be a G-space and E a G-equivariant vector bundle of rank e with Chern roots $\alpha_1, \ldots, \alpha_e$.*

(i) $c_t(\wedge^p E) = \prod_{i_1 < \ldots < i_p} (1 + (\alpha_{i_1} + \ldots + \alpha_{i_p})t)$*, in particular* $c_1(\wedge^e E) = c_1(E)$.

(ii) *If L is a G-equivariant line bundle* $c_t(E \otimes L) = \sum_{i=0}^e c_t(L)^{e-i} c_i(E) t^i$*, in particular* $c_e(E \otimes L) = \sum_{i=0}^e c_1(L)^i c_{e-i}(E)$.

1.8 Künneth Formula

In this section we investigate the exterior product map

$$A_*(X) \otimes A_*(Y) \to A_*(X \times Y).$$

We list conditions on X, so that the exterior product map is surjective or even an isomorphism for any scheme Y.

Definition 1.8.1. *We say a scheme X has a cellular decomposition if X can be stratified into a finite disjoint union of open subsets of affine spaces. If the cells are in fact affine spaces and not only open subsets, we say that X has a decomposition into affine cells. We say that a G-space X has a cellular decomposition if the cells are G-invariant.*

Examples for schemes with cellular decomposition are the projective space $\mathbb{P}^n$, the Grassmannian and flag varieties. The cells in these examples are in fact affine spaces and not only open subsets.

Lemma 1.8.2. *(cf. [Fu,Example 1.10.2]) If X is a scheme with a cellular decomposition then for all schemes S the exterior product map*

$$\bigoplus_{k+l=m} A_k(X) \otimes A_l(S) \xrightarrow{\times} A_m(X \times S)$$

is surjective.

Proof. Let U be an open cell of X and let Y denote the complement of U. The localization exact sequence then induces a commutative diagram

$$\begin{array}{ccccccc} A_*(Y) \otimes A_*(S) & \longrightarrow & A_*(X) \otimes A_*(S) & \longrightarrow & A_*(U) \otimes A_*(S) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_*(Y \times S) & \longrightarrow & A_*(X \times S) & \longrightarrow & A_*(U \times S) & \longrightarrow & 0 \end{array}$$

with exact rows. Since U is open in affine space we see that the right vertical arrow is surjective and by noetherian induction we may assume that the left vertical arrow is also surjective. An easy diagram chase then shows the assertion. □

The exterior product map is an isomorphism if X belongs to the class of linear schemes as is shown in [To3].

Definition 1.8.3. *([To3, Section 3]) The class of linear schemes is defined inductively in the following way. The affine space of arbitrary dimension is a linear scheme. The complement of an embedding of a linear scheme in affine space is again linear and so is any scheme stratified by a finite disjoint union of linear schemes*

Lemma 1.8.4. *The exterior product map*

$$\bigoplus_{k+l=m} A_k(X) \otimes A_l(Y) \xrightarrow{\times} A_m(X \times Y)$$

is an isomorphism for any scheme Y if X belongs to the class of linear schemes.

Proof. This is Proposition 1 in [To3]. The proof makes use of higher Chow groups. □

The following lemma is an application of the Leray-Hirsch theorem (Lemma 1.9.9) below and gives a criterion in the G-equivariant case.

Lemma 1.8.5. *Assume G is special and X and Y are smooth, proper G-schemes with decomposition into affine cells. Then the exterior product map*

$$A^*_G(X) \otimes_{A^*_G} A^*_G(Y) \to A^*_G(X \times Y)$$

is an isomorphism of A^*_G-algebras.

Proof. If X_i resp. Y_i denote the closure of the cells in X resp. Y the classes $[X_i]$ resp. $[Y_i]$ form a basis of $A^*(X)$ resp. $A^*(Y)$. Consider a good pair (V, U) for G. Applying the Leray-Hirsch theorem to the fibration $(Y \times U)/G \to U/G$ with fiber Y and the fibration $(X \times Y \times U)/G \to (Y \times U)/G$ with fiber X we get isomorphisms

$$A_G^*(X \times Y) \cong A_G^* \otimes A^*(X) \otimes A^*(Y) \cong A_G^*(X) \otimes_{A_G^*} A_G^*(Y)$$

of A_G^*-modules, such that under the above isomorphisms $1 \otimes [X_i] \otimes [Y_j]$ is mapped to $[X_i \times Y_j]_G$ resp. $[X_i]_G \otimes [Y_j]_G$. The lemma follows. □

Before stating the next lemma we briefly recall the notion of wreath products. Let H be a subgroup of the symmetric group S_n and G be an arbitrary group. The wreath product $H \wr G$ is then defined to be the semi-direct product $H \ltimes G^n$, where H acts on G^n by permutation. We will view $\mathbb{Z}/p$ as the subgroup of S_p generated by the cycle $(1\ 2\ \cdots\ p)$.

Lemma 1.8.6. *Let G be an algebraic group and X a scheme considered as a G-scheme with trivial G-action. Then the exterior product map*

$$A_G^* \otimes_{\mathbb{Z}} CH^*(X) \to CH_G^*(X)$$

is an isomorphism in each of the following cases:

(i) *G is the multiplicative group $\mathbb{G}_m$.*

(ii) *G is a finite abelian group of exponent e, e is invertible in k and k contains the e-th roots of unity.*

(iii) *G is an iterated wreath product $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z} \wr \mathbb{G}_m$ over k, p is invertible in k and k contains the p-th roots of unity.*

(iv) *G is an iterated wreath product $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z} \wr A$ where A is a finite abelian group of exponent e. Also, p and e are invertible in k and k contains the p-th and e-th roots of unity.*

In particular, for those G we have isomorphismns $A_G^* \otimes_{\mathbb{Z}} A_H^* \to A_{G \times H}^*$ for any algebraic group H.

Proof. This is Lemma 2.12 in [To2]. The point is that under the above assumptions the classifying space of G can be approximated by smooth linear schemes. The exterior product map is an isomorphism in this case by Lemma 1.8.4. □

1.9 The Restriction Map

Next we want to describe properties of the restriction map $res_T^G \colon A_*^G(X) \to A_*^T(X)$, where T is a split torus in G. This map is defined via flat pull-back of the natural map $X_T \to X_G$ between the mixed spaces. Note that more generally one has a restriction map $res_H^G \colon A_*^G(X) \to A_*^H(X)$ for every subgroup H of G.

Definition 1.9.1. (i) *In the following $A^*(X;\mathbb{Q})$ will denote the operational Chow ring of X consisting of characteristic classes with values in rational Chow groups, i.e. an element $c \in A^*(X;\mathbb{Q})$ assigns to each $T \to X$ a morphism*

$$c(T \to X)\colon A_*(T)_\mathbb{Q} \to A_*(T)_\mathbb{Q}$$

satisfying the usual compatibility conditions ([Fu, Section 17.1]). (Of course, in the above definition one could replace $\mathbb{Q}$ by an arbitrary ring but we will not need this.)

(ii) *A proper map $\pi\colon \tilde{X} \to X$ is called an envelope if for each irreducible subspace $V \subset X$ there exists an irreducible subspace $\tilde{V} \subset \tilde{X}$ sucht π maps $\tilde{V}$ birationally onto V.*

Remark 1.9.2. There is a natural map $A^*(X)_\mathbb{Q} \to A^*(X;\mathbb{Q})$ and this map is an isomorphism if X is smooth. This follows from

$$\begin{array}{ccc} A^*(X)_\mathbb{Q} & \xrightarrow[\cong]{\cap[X]} & A_*(X)_\mathbb{Q} \\ \big\downarrow & & \big\| \\ A^*(X;\mathbb{Q}) & \xrightarrow[\cong]{\cap[X]} & A_*(X)_\mathbb{Q}. \end{array}$$

We recall the following easy lemma.

Lemma 1.9.3. (i) *Let $\pi\colon \tilde{X} \to X$ be a proper surjective map. Then $\pi_*\colon A_*(\tilde{X})_\mathbb{Q} \to A_*(X)_\mathbb{Q}$ is surjective and $\pi^*\colon A^*(X;\mathbb{Q}) \to A^*(\tilde{X};\mathbb{Q})$ is injective.*

(ii) *Let $\pi\colon \tilde{X} \to X$ be a birational envelope. Then $\pi_*\colon A_*(\tilde{X}) \to A_*(X)$ is surjective and $\pi^*\colon A^*(X) \to A^*(\tilde{X})$ is injective.*

Proof. The first part of (i) is [Ki, Proposition 1.3]. The first part of (ii) follows immediately from the definition of an envelope. The second part of (i) and (ii) are formal consequences of their first parts. We only do this for (ii). Let $c \in A^*(X)$ be in the kernel of π^* and consider a morphism $T \to X$. The base change $\pi_T\colon \tilde{T} \to T$ of π along $T \to X$ is again an envelope by [Fu, Lemma 18.3]. Let $a \in A_*(T)$ and chose $\tilde{a} \in A_*(\tilde{T})$ with $(\pi_T)_*(\tilde{a}) = a$. Then

$$\begin{aligned} c(T \to X) \cap a &= (\pi_T)_*(c(\tilde{T} \to X) \cap \tilde{a}) \\ &= (\pi_T)_*((\pi^* c)(\tilde{T} \to \tilde{X}) \cap \tilde{a}) \\ &= 0. \end{aligned}$$

This shows $c(T \to X) = 0$ and hence $c = 0$. □

Lemma 1.9.4. *Let G be a connected reductive group with split maximal torus T and Weyl group $W = W(G,T)$. Let M be smooth and $E \to M$ be a principal G-bundle. Consider a Borel subgroup $B \supset T$. Then W acts on $A^*(E/B)$ and pull-back induces an isomorphism $A^*(M)_\mathbb{Q} \cong A^*(E/B)^W_\mathbb{Q}$.*

Remark 1.9.5. The above lemma is also mentioned (without proof) in [Vi, Section 2.5].

Proof. We identify $W = N_G(T)/T$ and chose $w \in N_G(T)$. Then w induces an automorphism $w\colon E/T \to E/T$. This defines an action of W on $A^*(E/T) = A^*(E/B)$. Since w lies in G the diagram

$$\begin{array}{ccc} E/T & \longrightarrow & E/G = M \\ {\scriptstyle w}\downarrow & \nearrow & \\ E/T & & \end{array}$$

commutes and this implies that the image of the pull-back $A^*(M) \to A^*(E/B)$ lies in $A^*(E/B)^W$. We are left to show that

$$A^*(M)_{\mathbb{Q}} \to A^*(E/B)^W_{\mathbb{Q}}$$

is an isomorphism. Let us first show that $A_*(M)_{\mathbb{Q}} \to A_*(E/B)^W_{\mathbb{Q}}$ is surjective. For this the smoothness assumption on M is not needed. We recall that every G-torsor is locally isotrivial by [Ra, XIV Lemma 1.4]. This means that there exists a covering of M by open subsets U with the property that for each U there is a finite, etale and surjective map $U' \to U$ such that $E_{U'} = E \times_M U' \to U'$ becomes a trivial G-torsor. Let V denote the complement of such an U in M and consider the commutative diagram

$$\begin{array}{ccccccc} A_*(V)_Q & \longrightarrow & A_*(M)_{\mathbb{Q}} & \longrightarrow & A_*(U)_{\mathbb{Q}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_*(E_V/B)^W_{\mathbb{Q}} & \longrightarrow & A_*(E/B)^W_{\mathbb{Q}} & \longrightarrow & A_*(E_U/B)^W_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

with exact rows. An easy diagram chase shows that if the first and last vertical map are surjective so is $A^*(M)_{\mathbb{Q}} \to A^*(E/B)^W_{\mathbb{Q}}$. Using noetherian induction we are thus reduced to the case that there exists a proper surjective map $M' \to M$ such that $E_{M'} \to M'$ is trivial. Since the diagramm

$$\begin{array}{ccc} A_*(M')_{\mathbb{Q}} & \longrightarrow & A_*(E_{M'}/B)^W_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ A_*(M)_{\mathbb{Q}} & \longrightarrow & A_*(E/B)^W_{\mathbb{Q}} \end{array}$$

commutes ([Fu, Proposition 1.7]) and since $A_*(E_{M'}/B)^W_{\mathbb{Q}} \to A_*(E/B)^W_{\mathbb{Q}}$ is surjective by part (i) of the previous lemma we are further reduced to the case of a trivial G-torsor $E = G \times M \to M$. Since G/B has a decomposition into affine cells we obtain in this case $A_*(E/B)_{\mathbb{Q}} = A_*(G/B)_{\mathbb{Q}} \otimes A_*(M)_{\mathbb{Q}}$ by Lemma 1.8.4. From [De, Section 8] we get $A_*(G/B)_{\mathbb{Q}} = S_{\mathbb{Q}}/(S^W_+)$, where $S = \mathrm{Sym}(\hat{T})$ and

S_+^W denotes the submodule generated by homogeneous W-invariant elements of positive degree. Since $(S_{\mathbb{Q}}/(S_+^W))^W = \mathbb{Q}$ we obtain $A_*(E/B)_{\mathbb{Q}}^W = A_*(M)_{\mathbb{Q}}$ as wanted.
By the previous lemma we know that $A^*(M;\mathbb{Q}) \to A^*(E/B;\mathbb{Q})$ is injective but since M (and therefore E) is smooth we obtain the injectivity of $A^*(M)_{\mathbb{Q}} \to A^*(E/B)_{\mathbb{Q}}$. $\square$

Theorem 1.9.6. *Let G be a connected reductive group with split maximal torus T and Weyl group $W = W(G,T)$. Let X be a G-scheme.*

(i) *W acts on $A_*^T(X)$. Furthermore, the restriction morphism $A_*^G(X) \to A_*^T(X)$ induces a map $r\colon A_*^G(X) \to A_*^T(X)^W$.*
(ii) *Assume X is smooth. Then r is an isomorphism after tensoring with $\mathbb{Q}$.*
(iii) *Assume X is smooth and that G is special. Then r is injective. Morever, r is an isomorphism if $A_T^*(X)$ is $\mathbb{Z}$-torsion free (e.g. if $X = \operatorname{Spec} k$).*

Remark 1.9.7. It is claimed in [EG, Proposition 6] that $r\colon A_*^G(X) \to A_*^T(X)^W$ is an isomorphism for arbitrary X if G is special. But according to a footnote in [EF] this is false. Unfortunately, we do not have an example for this.

The assertions (i) and (ii) are immediate consequences of Lemma 1.9.4. Under the assumption that $A_T^*(X)$ is $\mathbb{Z}$-torsion free we will deduce the surjectivity of r from part (ii) by using the argumentation of the proof of Theorem 1 in [EG2]. In this article Edidin and Graham prove the above theorem in the case $X = \operatorname{Spec} k$. Their key ingredient is the following proposition.

Proposition 1.9.8. *([EG2, Proposition 1]) Let $Y \to X$ be a smooth proper Zariski locally-trivial fiber bundle, where X is smooth and whose fiber F has a decomposition into affine cells. Then A^*Y is a free A^*X-module. More precisely, we have a (non-canonical) isomorphism $A^*Y \cong A^*X \otimes A^*F$ of A^*X-modules.*

This proposition is a corollary of an algebraic version of the Leray-Hirsch theorem.

Lemma 1.9.9. *(Leray-Hirsch, [EG2, Lemma 6]) Let $Y \to X$ be a smooth proper Zariski locally-trivial fiber bundle, where X is smooth and whose fiber F has a decomposition into affine cells. Let $\{B_i\} \in A_*(Y)$ be a collection of classes that restrict to a basis of the Chow groups of the fibers. Then $A^*(Y)$ is a free $A^*(X)$-module with basis $\{B_i\}$.*

Proof. (of Theorem 1.9.6) We are left to prove part (iii). To prove the injectivity of r it suffices to show that the pull-back of the morphism $p\colon X_B \to X_G$ between the mixed spaces is injective. But this morphism is a G/B-bundle locally trivial for the Zariski topology since G is special. In particular, p is an envelope and hence $p^*\colon A^*(X_G) \to A^*(X_B)$ is injective by Lemma 1.9.3.

Assume now that $A_T^*(X)$ is $\mathbb{Z}$-torsion free and let $\alpha \in A_T^*(X)^W$. Consider the fiber square

$$\begin{array}{ccc} X_B \times_{X_G} X_B & \xrightarrow{p_1} & X_B \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} \\ X_B & \xrightarrow[p]{} & X_G. \end{array}$$

Then α is lying in the image of p^* if and only if $p_1^*\alpha = p_2^*\alpha$ by [Ki, Theorem 2.3]. We know that $p_1^*\alpha - p_2^*\alpha$ is torsion in $A^*(X_B \times_{X_G} X_B)$ since we already know that r is surjective after tensoring with $\mathbb{Q}$. But by Proposition 1.9.8 we have $A^*(X_B \times_{X_G} X_B) \cong A^*(X_B) \otimes A^*(G/B)$. Since $A^*(X_B)$ and $A^*(G/B)$ are torsion free so is $A^*(X_B \times_{X_G} X_B)$ and the claim follows. □

We state another result in this direction.

Theorem 1.9.10. *Let G be a connected reductive group with split maximal torus T and let X be a G-scheme. Then the map*

$$(A_T^*)_{\mathbb{Q}} \otimes_{(A_G^*)_{\mathbb{Q}}} A_*^G(X)_{\mathbb{Q}} \to A_*^T(X)_{\mathbb{Q}}$$

induced by the restriction map is an isomorphism of $(A_T^*)_{\mathbb{Q}}$-modules. If G is special the same holds over $\mathbb{Z}$.

Proof. This is [Br, Theorem 6.7]. □

Corollary 1.9.11. *The restriction map r is an isomorphism if G is special and $A_*^G(X)$ is a flat A_G^*-module.*

Proof. This follows from the theorem above and the exact sequence

$$0 \longrightarrow A_G^* \longrightarrow A_T^* \longrightarrow \prod_{w \in W} A_T^*$$

of A_G^*-modules. □

In the following special case r becomes an isomorphism after inverting the order of the Weyl group.

Lemma 1.9.12. *Let G be a connected reductive group with split maximal torus T and V be a G-representation. Let $U \subset V$ be open and G-invariant. If G is special then $r\colon A_G^*(U) \to A_T^*(U)^W$ is an isomorphism after inverting ord(W)*

Proof. We have the diagram

$$\begin{array}{ccccc} A_G^*(V) & \longrightarrow & A_G^*(U) & \longrightarrow & 0 \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle r} & & \\ A_T^*(V)^W & \longrightarrow & A_T^*(U)^W & & \end{array}$$

The first vertical map is an isomorphism by Theorem 1.9.6. Note that $A_G^*(V) = A_G^*$ and $A_T^*(V) = A_T^*$ holds. Hence r is surjective if and only if $A_T^*(V)^W \to A_T^*(U)^W$ is surjective which is the case after inverting ord(W). □

We give another useful lemma which is an immediate consequence of the Leray-Hirsch theorem. See [EF, Proposition 2.2].

Lemma 1.9.13. *Let G be connected reductive. Assume G is special and let T in G be a split maximal torus. If X is a smooth G-scheme then $A^*_G(X)$ is (non-canonically) a direct summand in the $A^*_G(X)$-module $A^*_T(X)$.*

Corollary 1.9.14. *Let G be connected reductive. Assume G is special and let T in G be a split maximal torus. Let X be a smooth G-scheme and Y in X a closed G-invariant subscheme. If the image of $CH^*_T(Y) \to CH^*_T(X)$ is generated as a $CH^*_T(X)$-module by elements lying in $CH^*_G(X)$ then the same elements generate the image of $CH^*_G(Y) \to CH^*_G(X)$ as a $CH^*_G(X)$-module.*

Proof. Let U denote the complement of Y. Consider the commutative diagram

$$\begin{array}{ccccccc} CH^*_G(Y) & \longrightarrow & CH^*_G(X) & \longrightarrow & CH^*_G(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ CH^*_T(Y) & \longrightarrow & CH^*_T(X) & \longrightarrow & CH^*_T(U) & \longrightarrow & 0 \end{array}$$

with exact rows. Note that, since the right vertical morphism is also injective, an element of $CH^*_G(X)$ that lies in the image of the map $CH^*_T(Y) \to CH^*_T(X)$ also lies in the image of $CH^*_G(Y) \to CH^*_G(X)$. After choosing a section of $CH^*_G(X) \hookrightarrow CH^*_T(X)$ the proof is straight forward. □

1.10 The Transfer Map

Consider a subgroup H in G of finite index $[G : H]$. By this we mean that G/H is affine and the coordinate ring of G/H is a finite dimensional k-vectorspace of dimension $[G : H]$. The natural map $U/H \to U/G$ between the mixed spaces is then finite, flat and surjective of degree $[G : H]$. The proper push-forward of this map induces an additive map

$$tr^G_H \colon A^i_H \to A^i_G$$

called transfer map.

Lemma 1.10.1. (i) *The transfer map is a homomorphism of A^*_G-modules.*
(ii) *The composition*

$$A^*_G \xrightarrow{res^G_H} A^*_H \xrightarrow{tr^G_H} A^*_G$$

is multiplication with $[G : H]$.

Proof. Going over to the mixed spaces this follows easily from properties of ordinary Chow groups. The first part follows from the projection formula and the second follows since the composition of flat pull-back and proper push-forward of a finite, flat and surjective map is multiplication by the degree of the map. In our case the degree is $[G : H]$. □

We will apply this lemma in the following way.

Corollary 1.10.2. *Let G be a finite group and l a prime number. If P is an l-Sylow subgroup of G then the localized restriction map*

$$res_P^G \colon A_G^* \otimes \mathbb{Z}_{(l)} \to A_P^* \otimes \mathbb{Z}_{(l)}$$

is injective.

Corollary 1.10.3. *Let G be a finite group scheme over k of degree $|G|$. Then*

$$|G| A_G^i = 0$$

for $i > 0$. In particular, $(A_G^*)_{\mathbb{Q}} = \mathbb{Q}$ in degree 0.

Proof. By definition we have $|G| = [G : 1]$. Part (ii) of the above lemma applied to the case $H = \{1\}$ implies $|G| A_G^i = 0$ for $i > 0$ as wanted. Alternatively this follows as a special case of Lemma 1.4.9. □

2 Specialization

2.1 Specialization for Chow Rings

We note that almost all the functorial properties of Chow rings (meaning §1-§6 of [Fu]) remain valid for schemes of finite type over a regular base scheme S. The only exception is the existence of an exterior product map $A_*(X/S) \otimes A_*(Y/S) \to A_*(X \times_S Y/S)$. The reason is that varities over S are not automatically flat. This changes when S is the spectrum of a discrete valuation ring (or more generally a Dedekind domain). In this case every variety over S is either flat or is mapped to the closed point of S. It is then possible to define a product on the level of cycles that passes to rational equivalence ([Fu, Section 20.2]). Section 8 of [Fu] then carries over to smooth schemes X over a discrete valuation ring. In particular, there is an intersection product on $A_*(X/S)$. For more information see [Fu, Section 20].
We now recall the concept of specialization for Chow rings as explained in [Fu, Section 20.4]. Let X be a scheme of finite type over a regular base scheme S. Assume $i \colon S^c \to S$ is a regular embedding of codimension d such that the normal bundle N of S^c in S is trivial. Consider the fibresquare

$$\begin{array}{ccc} X^c & \xrightarrow{i'} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ S^c & \xrightarrow[i]{} & S. \end{array}$$

By [Fu, Corollary 6.3] we have the equation $i^!(i'_*)(\alpha) = c_d(g^*N) \cap \alpha = 0$ for all $\alpha \in A_*(X^c/S^c)$. Writing $X^o = X - X^c$ it follows from the localization exact sequence that there is a unique map $\sigma\colon A_*(X^o/S^o) \to A_*(X^c/S^c)$ such that the diagram

$$\begin{array}{ccc} A_*(X/S) & \xrightarrow{j^*} & A_*(X^o/S^o) \\ {\scriptstyle i^!}\downarrow & \swarrow{\scriptstyle \sigma} & \\ A_*(X^c/S^c) & & \end{array}$$

commutes. Assume S is the spectrum of a discrete valuation ring R with fraction field K and residue field k . If X is smooth over $\operatorname{Spec} R$ then $\sigma_R\colon A_*(X_K) \to A_*(X_k)$ is a ring homomorphism by [Fu, Corollary 20.3].

Remark 2.1.1. (i) We recall the following fact about the refined Gysin homomorphisms $i^!\colon A_*(V) \to A_{*-d}(W)$, where $W \to V$ is the base change of a regular embedding $i\colon Y \to X$ of codimension d along a morphism $V \to X$. Namely, if $W \to V$ is again a regular embedding of codimension d then $i^![V] = [W] \in A_*(W)$. This follows immediately from the definition of $i^!$ ([Fu, Section 6.2]) and the fact that in this case the normal bundle $N_W V$ equals the pull-back of the normal bundle $N_Y X$ to W.
(ii) Let R be a discrete valuation ring with fraction field K and residue field k and denote $i\colon \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$. Let X be of finite type over S. Consider the morphism

$$Z_*(X_K) \to Z_*(X_k), \quad [V] \mapsto [\bar{V}_k],$$

where $\bar{V}$ denotes the closure of V in X. Since $\bar{V}_k \hookrightarrow \bar{V}$ is again a regular embedding of codimension 1 we have $\sigma_R([V]) = i^!([\bar{V}]) = [\bar{V}_k] \in A_*(X_K)$ by part (i), i.e. the specialization map is induced by the above map on the level of cycles. We will therefore denote the map $Z_*(X_K) \to Z_*(X_k)$ also by σ_R.

Similary to ordinary Chow groups equivariant intersection theory remains valid for schemes (or algebraic spaces) that are of finite type over $S = \operatorname{Spec} R$ with R a Dedekind domain: By [EG, Lemma 7] we can find for any affine smooth group scheme G defined over S a finitely generated projective S-module E such that G acts freely on an open subset U of E whose complement has arbitrarily high codimension. For such an U one defines

$$A_i^G(X) = A_{i+l-g}((X \times U)/G)$$

where $l = \dim(U/S)$ and $g = \dim(G/S)$. All the functorial properties of equivariant intersection theory remain valid. In particular, if X is smooth over S there is an intersection product on $A_*^G(X)$.

Remark 2.1.2. Convention 1.2.1 carries over to the situation of an affine smooth group scheme over the spectrum S of a Dedekind domain: A pair (U, E) is called a good pair for G if E is a finitely generated projective S-module and U is an open subset of E on which G acts freely. If X is a G-scheme over S we call the quotient $X_G = (X \times U)/G$ a mixed space for the G-scheme X. If $\operatorname{codim}(U^c, E) > i$ we also call X_G an approximation of the quotient stack $[X/G]$ up to codimension i.

Let us consider now the spectrum S of a discrete valuation ring R with fraction field K and residue field k and a smooth affine group scheme G over S. For every good pair (E, U) of G the above construction gives a homomorphism of graded rings

$$A^*(U_K/G_K) \to A^*(U_k/G_k).$$

Since $\dim U_k^c \leq \dim U^c$ and $\dim U_K^c \leq \dim U^c$ and since we can chose the codimension of U^c to be arbitrarily high these morphisms induce a map

$$\sigma_R \colon A^*_{G_K} \to A^*_{G_k}$$

of graded rings. In particular, there exist such a map for any finite abstract group G viewed as a constant group scheme over S. We will call this map specialization map. We have the following naive criterion for the specialization map to be an isomorphism.

Proposition 2.1.3. *Let G be a finite abstract group viewed as a constant group scheme over a discret valuation ring with quotient field K and residue field k of mixed characteristic. Then the following assertions hold:*

(i) *The specialization map is surjective if $A^*_{G_k}$ is generated as a $\mathbb{Z}$-algebra by Chern classes of representations of G_k.*

(ii) *The specialization map is injective if the same holds true for every Sylow subgroup of G.*

Proof. To prove (i) we use the theory of Brauer lifts ([Se, Chapter 18]). Let $R_K(G)$ resp. $R_k(G)$ denote the representation ring of G_K resp. G_k. Consider the diagram

$$\begin{array}{ccc} R_K(G) & \xrightarrow{d} & R_k(G) \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_i} \\ A^i_{G_K} & \xrightarrow{\sigma} & A^i_{G_k}. \end{array}$$

Here the maps c_i are induced by the i-th Chern class (see Lemma 2.1.4 below) and d is defined as follows. If V is a K-representation of G we chose a G-invariant lattice $\tilde{V}$ of V. The class $[V]$ is then mapped under d to the class $[\tilde{V} \otimes k]$ in $R_k(G)$. This class does not depend on the choice of a lattice ([Se, Section 15.2]). By definition of the specialization map and [Fu, Proposition 6.3] we see that the above diagram is commutative. The map d is surjective by [Se, Section 16.1]. In other words we can lift Chern classes in $A^*_{G_k}$ to Chern classes in $A^*_{G_K}$. Hence the specialization map is surjective if $A^*_{G_k}$ is generated by Chern classes of representations of G_k.

If P is an l-Sylow subgroup of G for some prime l we obtain a diagram

$$\begin{array}{ccc} (A^*_{G_K})_{(l)} & \hookrightarrow & (A^*_{P_K})_{(l)} \\ {\scriptstyle \sigma}\downarrow & & {\scriptstyle \sigma}\downarrow \\ (A^*_{G_k})_{(l)} & \hookrightarrow & (A^*_{P_k})_{(l)} \end{array}$$

where the injectivity of the horizonatal maps follow from Corollary 1.10.2. Moreover, this diagram is commutative by [Fu, Proposition 6.2 (b)]. This proves (ii). □

Lemma 2.1.4. *If G is a finite abstract group and k an arbitrary field, then for any $i \in \mathbb{Z}_{\geq 0}$ there are unique maps $c_i\colon R_k(G) \to A^i_{G_k}$ satisfying the following two properties:*

(i) *For any G_k-representation V one has $c_i([V]) = c_i(V) \in A^i_{G_k}$.*

(ii) $c_i(E_1 + E_2) = \sum_{k+l=i} c_k(E_1)c_l(E_2)$ *for $E_1, E_2 \in R_k(G)$.*

Proof. We note that any virtual representation E of G has a unique expression $E = \sum_{L \in S(G)} \lambda_i[L]$ with $\lambda_i \in \mathbb{Z}$. Here $S(G)$ denotes the set of isomorphism classes of simple representations of G. The properties (i) and (ii) then determine the image $c_i(E)$ of E uniquely. □

Remark 2.1.5. Although condition (i) of the above proposition does hold for many finite groups it does not hold in general. A counter example is given by the symmetric groups ([To, Section 4]).

Computing the equivariant Chow ring of a finite group in characteristic 0 is difficult. Instead one can try to compute its equivariant Chow ring in positive charactersitic and then apply Proposition 2.1.3.

The computation in characteristic p is easier at least for Chevalley groups. Recall that Chevalley groups are the finite groups $G(\mathbb{F}_q)$, where G is a connected, split reductive group scheme over $\mathbb{Z}$. The reason is the following: Let k be a field containing $\mathbb{F}_q$. We will see in the following lemma that there is a canonical 1-isomorphism

$$BG(\mathbb{F}_q)_k \cong [G_k/G_k],$$

where the action of G_k on itself is given by conjugation with the q-th power Frobenius. If G is special (e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}$) we can determine $A^*_{G_k}(G_k)$ and thus also $A^*_{G(\mathbb{F}_q)_k}$ completely. This will be done in Chapter 3.

Lemma 2.1.6. *Let G be a connected algebraic group over an arbitrary field k and $\varphi\colon G \to G$ be an isogeny with only a finite number of fixed points. Consider the G-action on G defined by $g \cdot h = gh\varphi(g)^{-1}$ and let S denote the stabilizer group scheme of the neutral element. Then there is a canonical 1-isomorphism*

$$[G/G] \cong BS.$$

Proof. It suffices to show that the quotient stack $[G/G]$ is a gerbe that has a section over k whose automorphism group is equal to S ([LMB, Lemma 3.21]). Let T be a scheme and B_1, B_2 be two principal G-bundles over T together with G-equivariant maps $B_i \to G$. After replacing T by a suitable covering we may assume that B_1 and B_2 are trivial. The G-equivariant maps $B_i \to G$ are then given by sections $g_i\colon T \to G \times T$ and there exists an isomorphism of principal G-bundles respecting the maps $B_i \to G$ if and only if g_1 and g_2 lie in the same $G(T)$-orbit. This holds

after passing to a suitable covering of T by [Ste, Theorem 10.1] which states that the morphism $G \to G, g \mapsto g\varphi(g)^{-1}$ is faithfully flat. In fact, Steinberg's theorem only states that this map is surjective, but since by assumption φ has only finitely many fixed points all the fibers of the map $g \mapsto g\varphi(g)^{-1}$ have dimension zero and hence this map is flat by the mirracle flatness theorem ([Ma, (21.D) Theorem 51]). This shows that $[G/G]$ is a gerbe.
The section of $[G/G]$ with automorphism group equal to S is given by the trivial G-bundle over $\operatorname{Spec} k$ with G-equivariant map $G \to G, g \mapsto g\varphi(g)^{-1}$. □

Corollary 2.1.7. *Let G be a split reductive group scheme over $\mathbb{Z}$ and let k be a field containing $\mathbb{F}_q$. Let G_k act on itself by conjugation with the q-th power Frobenius. Then there is a canonical 1-isomorphism*

$$BG(\mathbb{F}_q)_k \cong [G_k/G_k].$$

Proof. This follows from the previous lemma applied to the case that φ is the q-th power Frobenius. □

From $\mathbb{Q}_p$ to $\mathbb{C}$. Let K be a finite field extension of $\mathbb{Q}_p$ and R the integral closure of $\mathbb{Z}_p$ in K. Since $\mathbb{Q}_p$ is complete for the p-adic valuation R is again a discrete valuation ring. Also its residue field k is a finite field extension of $\mathbb{F}_p$.

Lemma 2.1.8. *Let K' be another finite extension of $\mathbb{Q}_p$ that contains K. Let R resp. R' be the integral closure of $\mathbb{Z}_p$ in K resp. K' and let k resp. k' denote the residue field. Then the following diagram commutes*

$$\begin{array}{ccc} A^*_{G_{K'}} & \xrightarrow{\sigma_{R'}} & A^*_{G_{k'}} \\ \uparrow & & \uparrow \\ A^*_{G_K} & \xrightarrow{\sigma_R} & A^*_{G_k} \end{array}$$

Here the vertical maps are induced by pull-back.

Proof. We note that R and R' are both finite and free over $\mathbb{Z}_p$. Hence if X is an approximation of $BG_{\mathbb{Z}_p}$ (cf. Remark 2.1.2) then X_R is an approximation of BG_R and similary for R'. Cleary the diagram

$$\begin{array}{ccccc} Z_*(X_{k'}) & \longleftarrow & Z_*(X_{R'}) & \longrightarrow & Z_*(X_{K'}) \\ \uparrow & & \uparrow & & \uparrow \\ Z_*(X_k) & \longleftarrow & Z_*(X_R) & \longrightarrow & Z_*(X_K) \end{array}$$

on the level of cycles commutes. Here all the maps are induced by base change. Passing to cycle classes it thus follows from Remark 2.1.1 (ii) that

$$\begin{array}{ccc} A_*(X_{K'}) & \xrightarrow{\sigma_{R'}} & A_*(X_{k'}) \\ \uparrow & & \uparrow \\ A_*(X_K) & \xrightarrow{\sigma_R} & A_*(X_k) \end{array}$$

commutes. □

In view of the above lemma we can take the direct limit of the specialization maps σ_R yielding a map

$$\varinjlim_K A^*_{G_K} \to \varinjlim_k A^*_{G_k}$$

for every finite group G. Here the limit goes over all finite field extensions K resp. k of $\mathbb{Q}_p$ resp. $\mathbb{F}_p$. Fixing an isomorphism $\mathbb{C} \cong \bar{\mathbb{Q}}_p$ we obtain a homomorphism of graded rings

$$\sigma_G \colon A^*_{G_{\mathbb{C}}} \to A^*_{G_{\bar{\mathbb{F}}_p}}.$$

2.2 Specialization for Etale Cohomology

Etale Homology. For any scheme S over a separably closed field k etale homology is defined as

$$H_i(S, \mathbb{Z}_l) = H^{-i} R\Gamma(S, T_S)$$

for $l \neq \mathrm{char}\, k$, where T_S is the dualizing complex of S. If a denotes the structure map of S then $T_S = Ra^! \mathbb{Z}_l \in D(S, \mathbb{Z}_l)$. Let us recall the properties of etale homology we shall need. For the proof we refer to [LaG].

Proposition 2.2.1. *Let k be a separably closed field and l be a prime different from the characteristic of k. Let X be a scheme over k of dimension d.*

(i) *$H_i(X, \mathbb{Z}_l) = 0$ for $i < 0$ and $i > 2d$ and $H_{2d}(X, \mathbb{Z}_l)$ is freely generated by the irreducible components of X of dimension d.*

(ii) *If X is smooth then $H_i(X, \mathbb{Z}_l) = H^{2d-i}(X, \mathbb{Z}_l)$, where the right hand side denotes the usual l-adic cohomology groups.*

(iii) *(Functoriality) Let $f \colon X \to Y$ be proper resp. flat of relative dimension n. Then f induces an additive push-forward map $f_* \colon H_*(X, \mathbb{Z}_l) \to H_*(Y, \mathbb{Z}_l)$ resp. pull-back map $f^* \colon H_*(Y, \mathbb{Z}_l) \to H_{*+n}(X, \mathbb{Z}_l)$ compatible with composition.*

(iv) *If $f \colon X \to Y$ is finite and locally free of degree n then the composition*

$$H_*(Y, \mathbb{Z}_l) \xrightarrow{f^*} H_*(X, \mathbb{Z}_l) \xrightarrow{f_*} H_*(Y, \mathbb{Z}_l)$$

is multiplication by n.

(v) *(Künneth Formula) If Y is another scheme over k there is an exact sequence of the form*

$$0 \longrightarrow \bigoplus_{r+s=i} H_r(X) \otimes H_s(Y) \longrightarrow H_i(X \times Y)$$

$$\longrightarrow \bigoplus_{r+s=i-1} \mathrm{Tor}_1(H_r(X), H_s(Y)) \longrightarrow 0.$$

(vi) *(Cycle Map) There is an additive cycle map $cl_X \colon A_*(X) \to H_{2*}(X, \mathbb{Z}_l)$ compatible with proper push-forward, flat pull-back and Chern classes. If X is smooth then cl_X defines a morphism of rings.*

Since we do not have a reference for the following lemma, we prove it here.

Lemma 2.2.2. *Let $Z \subset S$ be a closed subscheme and denote by U its complement in S. Denote the inclusion $Z \hookrightarrow S$ resp. $U \hookrightarrow S$ by i resp. j. Then there is a long exact sequence*

$$\ldots \longrightarrow H_{i+1}(U, \mathbb{Z}_l) \longrightarrow H_i(Z, \mathbb{Z}_l) \xrightarrow{i_*} H_i(S, \mathbb{Z}_l) \xrightarrow{j^*} H_i(U, \mathbb{Z}_l) \longrightarrow$$

Proof. If I is an injective object in $\mathrm{Mod}(S, \mathbb{Z}/l^n)$ there is a natural exact sequence

$$0 \longrightarrow i_* i^! I \longrightarrow I \longrightarrow j_* j^* I \longrightarrow 0.$$

This implies that there is an exact triangle in $D(S, \mathbb{Z}_l)$ of the form

$$\begin{array}{ccc} & (Rj_*)j^*T_S & \\ \swarrow & & \nwarrow \\ i_*Ri^!T_S & \longrightarrow & T_S \end{array}$$

Note that $i_*Ri^!T_S = i_*T_Z$ as well as $(Rj_*)j^*T_S = Rj_*T_U$. Applying the functor $R\Gamma(S, \cdot)$ thus yields an exact triangle

$$\begin{array}{ccc} & R\Gamma(U, T_U) & \\ \swarrow & & \nwarrow \\ R\Gamma(Z, T_Z) & \longrightarrow & R\Gamma(S, T_S) \end{array}$$

in $D(\operatorname{Spec} k, \mathbb{Z}_l)$. Taking homology then yields the desired long exact sequence. □

Definition 2.2.3. *Let G be an algebraic group over a separably closed field k and l be a prime different from the characterstic of k. We define the i-th l-adic cohomology group of BG in the following way. Let (V, U) be a good pair for G with $\operatorname{codim}(U^c) \geq (i+1)/2$ then*

$$H^i(BG, \mathbb{Z}_l) = H^i(U/G, \mathbb{Z}_l).$$

By using Proposition 2.2.1 (ii) and Lemma 2.2.2 above one shows in the same way as in the case of Chow groups ([EG, Definition-Proposition 1]), that the above definition is independent of the choice of the pair (V, U) as long as $\operatorname{codim} U^c \geq (i+1)/2$.

Remark 2.2.4. If G is a finite abstract group and l a prime that does not divide the order of G then $H^*(BG_k, \mathbb{Z}_l) = \mathbb{Z}_l$ in degree 0. This follows from the usual transfer argument (cf. Corollary 1.10.3).

Lemma 2.2.5. *Let k be a field of arbitrary characteristic and l a prime different from char k. Assume $(n, \text{char } k) = 1$ then the cycle map*

$$A^*_{\mu_{n,k}} \otimes \mathbb{Z}_l \to H^*(B\mu_{n,k^{sep}}, \mathbb{Z}_l)$$

is an isomorphism.

The above lemma together with Example 1.2.12 yields $H^*(B\mu_{n,k^{sep}}, \mathbb{Z}_l) = \mathbb{Z}_l[t]/(nt)$ where t is the first Chern class of the character $\mu_n \hookrightarrow \mathbb{G}_m$.

Proof. Since $A^*_{\mu_{n,k}} = A^*_{\mu_{n,k^{sep}}}$ we may assume $k = k^{sep}$. The natural map

$$X = (\mathbb{A}^{r+1}_k - \{0\})/\mu_n \to (\mathbb{A}^{r+1}_k - \{0\})/\mathbb{G}_m = \mathbb{P}^r_k$$

is a principal $\mathbb{G}_m$-bundle and the corresponding line bundle is given by $\mathcal{O}_{\mathbb{P}^r}(n)$. In other words μ_n can be approximated by the complement X of the zero section in $\mathcal{O}_{\mathbb{P}^r}(n)$. We will show that the cycle map

$$A^i(X) \otimes \mathbb{Z}_l \to H^{2i}(X, \mathbb{Z}_l)$$

is an isomorphism and $H^i(X, \mathbb{Z}_l) = 0$ for odd i. Consider the diagram

$$\begin{array}{ccccccc}
A^i(\mathbb{P}^r_k) \otimes \mathbb{Z}_l & \longrightarrow & A^{i+1}(\mathcal{O}_{\mathbb{P}^r}(n)) \otimes \mathbb{Z}_l & \longrightarrow & A^{i+1}(X) \otimes \mathbb{Z}_l & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^{2i}(\mathbb{P}^r_k, \mathbb{Z}_l) & \longrightarrow & H^{2i+2}(\mathcal{O}_{\mathbb{P}^r}(n), \mathbb{Z}_l) & \longrightarrow & H^{2i+2}(X, \mathbb{Z}_l) & \longrightarrow & H^{2i+1}(\mathbb{P}^r_k, \mathbb{Z}_l)
\end{array}$$

with exact rows. Here the lower row comes from Lemma 2.2.2. This diagram is commutative since the cycle map is compatible with proper push-forward and flat pull-back by Proposition 2.2.1 (vi). It is well known that the first vertical map is an isomorphism. Hence the second vertical map is also an isomorphism. Since $H^{2i+1}(\mathbb{P}^r_k, \mathbb{Z}_l) = 0$ the first claim follows. For the second claim it suffices to see that the map $H^{2i}(\mathbb{P}^r_k, \mathbb{Z}_l) \to H^{2i+2}(\mathcal{O}_{\mathbb{P}^r}(n), \mathbb{Z}_l)$ is injective. But we know that $A^i(\mathbb{P}^r_k) \otimes \mathbb{Z}_l \to A^{i+1}(\mathcal{O}_{\mathbb{P}^r}(n)) \otimes \mathbb{Z}_l$ is injective since the composition $A^i(\mathbb{P}^r_k) \to A^{i+1}(\mathcal{O}_{\mathbb{P}^r}(n)) \xrightarrow{\cong} A^{i+1}(\mathbb{P}^r_k)$ is capping with $c_1(\mathcal{O}_{\mathbb{P}^r}(n))$ by the self intersection formula (cf. the proof of Lemma 1.2.7) and under the identification $A^i(\mathbb{P}^r_k) = \mathbb{Z} = A^{i+1}(\mathbb{P}^r_k)$ this corresponds to multiplication with n. □

Specialization. Let R be a discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic p and let $X \to \operatorname{Spec} R$ be smooth. We recall the construction of the etale specialization map

$$\sigma_R \colon H^i(X_{\bar{K}}, \mathbb{Z}_l) \to H^i(X_{\bar{k}}, \mathbb{Z}_l)$$

for l a prime different from p that is compatible with the specialization map for the Chow ring under the cycle map, i.e. the diagram

$$\begin{array}{ccc} H^{2i}(X_{\bar{K}}, \mathbb{Z}_l) & \xrightarrow{\sigma_R} & H^{2i}(X_{\bar{k}}, \mathbb{Z}_l) \\ \uparrow & & \uparrow \\ A^i(X_K) & \xrightarrow[\sigma_R]{} & A^i(X_k) \end{array}$$

commutes (cf. [SGA6, Expose 10, 7.14]).

Lemma 2.2.6. *Let A be a normal ring such that $K = Frac(A)$ is separably closed. Let $X \to \operatorname{Spec} A$ be smooth. Then the pull-back morphism $H^i(X, \mathbb{Z}/l^n) \to H^i(X_K, \mathbb{Z}/l^n)$ is an isomorphism.*

Proof. This is more or less a direct consequence of the smooth base change theorem. Consider the cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{g'} & X_K \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a_K} \\ \operatorname{Spec} A & \xleftarrow[g]{} & \operatorname{Spec} K. \end{array}$$

The smooth base change theorem then yields an isomorphism $a^*(R^i g_* \mathbb{Z}/l^n) \cong R^i g'_* \mathbb{Z}/l^n$. Since A is normal we have $g_* \mathbb{Z}/l^n = \mathbb{Z}/l^n$ by [SGA4 Exp. IX, Lemma 2.14.1]. Since K is separably closed g_* is exact, hence $R^i g_* = 0$ for $i > 0$. It follows $g'_* \mathbb{Z}/l^n = \mathbb{Z}/l^n$ and $R^i g'_* \mathbb{Z}/l^n = 0$ for $i > 0$. The Leray spectral sequence $H^p(X, R^q g'_* \mathbb{Z}/l^n) \Rightarrow H^{p+q}(X_K, \mathbb{Z}/l^n)$ for g'_* then yields the claim. $\square$

Let now R be a discrete valuation ring as above. We fix algebraic closures $\bar{K}$ and $\bar{k}$. Let $\tilde{R}$ be the integral closure in $\bar{K}$ of the strict Henselization of R.

Lemma 2.2.7. *$\tilde{R}$ is a normal Henselian local ring with fraction field $\bar{K}$ and residue field $\bar{k}$.*

Proof. Since $\tilde{R}$ is the integral closure of a Henselian local ring, it is also Henselian local. Since k is perfect the residue field of the strict Henselization equals $\bar{k}$. Hence the residue field of $\tilde{R}$ equals $\bar{k}$. $\square$

Definition 2.2.8. *Let R be a discrete valuation ring with fraction field K of characteristic 0 and perfect residue field k of characteristic p. Let $X \to \operatorname{Spec} R$ be smooth. For a prime l different from p we define the specialization map*

$$\sigma_R \colon H^i(X_{\bar{K}}, \mathbb{Z}/l^n) \to H^i(X_{\bar{k}}, \mathbb{Z}/l^n)$$

to be the composition

$$H^i(X_{\bar{K}}, \mathbb{Z}/l^n) \xleftarrow{\cong} H^i(X \otimes_R \tilde{R}, \mathbb{Z}/l^n) \longrightarrow H^i(X_{\bar{k}}, \mathbb{Z}/l^n).$$

Remark 2.2.9. If X is also proper over $\operatorname{Spec} R$ then σ_R is an isomorphism: Let $a\colon X \otimes_R \tilde{R} \to \operatorname{Spec} \tilde{R}$ denote the structure map, then the proper base change theorem yields a canonical isomorphism $(R^i a_* \mathbb{Z}/l^n)_{\bar{x}} = H^i(X_{\bar{k}}, \mathbb{Z}/l^n)$. Here $x \in \operatorname{Spec} \tilde{R}$ denotes the closed point. On the other hand we have $(R^i a_* \mathbb{Z}/l^n)_{\bar{x}} = H^i(X \otimes_R \tilde{R}, \mathbb{Z}/l^n)$ since $\tilde{R}$ is strictly local.

Lemma 2.2.10. *The diagram*

$$\begin{array}{ccc} H^{2c}(X_{\bar{K}}, \mathbb{Z}/l^n) & \xrightarrow{\sigma_R} & H^{2c}(X_{\bar{k}}, \mathbb{Z}/l^n) \\ \uparrow{\scriptstyle cl_{X_K}} & & \uparrow{\scriptstyle cl_{X_k}} \\ Z^c(X_K) & \xrightarrow[\sigma_R]{} & Z^c(X_k) \end{array}$$

commutes. Here $\sigma_R\colon Z^c(X_K) \to Z^c(X_k)$ is the map from Remark 2.1.1 (ii), which induces the specialization map on the Chow groups.

Proof. We need to recall the definition of the cycle map $cl_X\colon Z^c(X) \to H^{2c}(X, \mathbb{Z}/l^n)$ ([SGA4h, Cycle, 2.3]). If $f\colon Y \to S = \operatorname{Spec} R$ is a flat map of finite type and of relative dimension d there is a trace map $Tr_f \in \operatorname{Hom}(R^{2d} f_! \mathbb{Z}/l^n(d), \mathbb{Z}/l^n)$ by [SGA4, Expose XVIII, Theorem 2.9], which is compatible with base change in S. Since $R^k f_! \mathbb{Z}/l^n(d) = 0$ for $k > 2d$ we have natural identifications

$$\begin{aligned} \operatorname{Hom}(R^{2d} f_! \mathbb{Z}/l^n(d), \mathbb{Z}/l^n) &= \operatorname{Hom}(R f_! \mathbb{Z}/l^n(d), \mathbb{Z}/l^n[-2d]) \\ &= \operatorname{Hom}(\mathbb{Z}/l^n, R f^! \mathbb{Z}/l^n(-d)[-2d]) \\ &= H^0(Y, R^{-2d} f^! \mathbb{Z}/n(-d)). \end{aligned}$$

Consider now a cycle $[Y] \in Z^c(X)$. Let

$$f\colon Y \overset{i}{\hookrightarrow} X \xrightarrow{\pi} S$$

denote the structure map. If f is not dominant one defines $cl_X([Y]) = 0$. In the other case f is flat of relative dimension $d = N - c$, where N denotes the relative dimension of $X \to S$. Since π is smooth of relative dimension N we have $R\pi^! \mathbb{Z}/l^n = \mathbb{Z}/l^N(-N)[-2N]$ and from $Rf^! = Ri^! R\pi^!$ it follows

$$H^0(Y, R^{-2d} f^! \mathbb{Z}/n(-d)) = H^0(Y, R^{2c} i^! \mathbb{Z}/l^n(c))$$

By semi-purity [SGA4h, Cycle, 2.2.8] it holds $R^k i^! \mathbb{Z}/l^n = 0$ for $k < 2c$ and from the spectral sequence $H^p(Y, R^q i^! \mathbb{Z}/l^n) \Rightarrow H^{p+q}_Y(X, \mathbb{Z}/l^n)$ we deduce

$$H^0(Y, R^{2c} i^! \mathbb{Z}/l^n(c)) = H^{2c}_Y(X, \mathbb{Z}/l^n(c)).$$

This shows that we may view the trace map Tr_f as an element of $H^{2c}_Y(X, \mathbb{Z}/l^n)$. One then defines $cl_X([Y])$ to be the image of Tr_f under the natural map

$$H^{2c}_Y(X, \mathbb{Z}/l^n) \to H^{2c}(X, \mathbb{Z}/l^n).$$

Consider now a cycle $[V] \in Z^i(X_K)$ and let Y denote the closure of V in X. Since the trace map is compatible with base change we have that the image of $cl_X([Y])$ under $H^{2c}(X, \mathbb{Z}/l^n) \to H^{2c}(X_K, \mathbb{Z}/l^n)$ equals $cl_{X_K}([V])$ and similary the image of $cl_X([Y])$ under $H^{2c}(X, \mathbb{Z}/l^n) \to H^{2c}(X_k, \mathbb{Z}/l^n)$ equals $cl_{X_k}([Y_k])$. Since $[Y_k] = \sigma_R([V])$ the Lemma follows from the definition of the specialization map on etale cohomology. $\square$

Lemma 2.2.11. *Let G be a finite abstract group and $X \to Y$ be a G-torsor over R. If $\sigma_R \colon H^i(X_{\bar{K}}, \mathbb{Z}/l^n) \cong H^i(X_{\bar{k}}, \mathbb{Z}/l^n)$ is an isomorphism for all i, the same is true for $\sigma_R \colon H^i(Y_{\bar{K}}, \mathbb{Z}/l^n) \to H^i(Y_{\bar{k}}, \mathbb{Z}/l^n)$.*

Proof. Consider the Hochschild-Serre spectral sequences

$$H^p(G, H^q(X_{\bar{K}}, \mathbb{Z}/l^n)) \Rightarrow H^{p+q}(Y_{\bar{K}}, \mathbb{Z}/l^n)$$
$$H^p(G, H^q(X_{\bar{k}}, \mathbb{Z}/l^n)) \Rightarrow H^{p+q}(Y_{\bar{k}}, \mathbb{Z}/l^n).$$

Since specialization is compatible with pull-back, the map $\sigma_R \colon H^i(X_{\bar{K}}, \mathbb{Z}/l^n) \to H^i(X_{\bar{k}}, \mathbb{Z}/l^n)$ is in fact an isomorphism of G-modules, thus yielding an isomorphism of spectral sequences compatible with the specialization map for Y. The lemma follows. $\square$

Corollary 2.2.12. *Let G be a finite abstract group. Then the specialization map $H^*(BG_{\bar{K}}, \mathbb{Z}_l) \to H^*(BG_{\bar{k}}, \mathbb{Z}_l)$ is an isomorphism*

Proof. Choose a good pair (E, U) for G_R. Then $\sigma_R \colon H^i(U_{\bar{K}}, \mathbb{Z}_l) \to H^i(U_{\bar{k}}, \mathbb{Z}_l)$ is an isomorphism for all $i < 2\,\mathrm{codim}(E - U)$ by Lemma 2.2.2. Since we can choose this codimension to be arbitrary high the assertion follows from the previous Lemma. $\square$

We shall also have need for the following comparison theorem, whose proof can be found in [SGA 4, XI].

Theorem 2.2.13. *Let $X \to \operatorname{Spec} \mathbb{C}$ be smooth. Then for any finite abelian group M there is a canonical isomorphism*

$$H^i_{sing}(X(\mathbb{C}), M) \cong H^i(X, M).$$

Lemma 2.2.14. *Let X be a topological space and assume that $H_i^{sing}(X, \mathbb{Z})$ and $H_{i-1}^{sing}(X, \mathbb{Z})$ are finitely generated. Then $\varprojlim_n H_i^{sing}(X, \mathbb{Z}/l^n) \cong H_i^{sing}(X, \mathbb{Z}_l)$.*

Proof. By the universal coefficient theorem we have an exact sequence

$$0 \longrightarrow H_i^{sing}(X, \mathbb{Z}) \otimes \mathbb{Z}/l^n \longrightarrow H_i^{sing}(X, \mathbb{Z}/l^n)$$
$$\longrightarrow \mathrm{Tor}_1(H_{i-1}^{sing}(X, \mathbb{Z}), \mathbb{Z}/l^n) \longrightarrow 0.$$

Recall $\mathrm{Tor}_1(H_{i-1}^{sing}(X,\mathbb{Z}),\mathbb{Z}/l^n) = H_{i-1}^{sing}(X,\mathbb{Z})[l^n]$. Clearly the Mittag-Loeffler condition is satisfied for the inverse system $(H_i^{sing}(X,\mathbb{Z})\otimes\mathbb{Z}/l^n)_n$. Taking inverse limits thus yields an exact sequence

$$0 \longrightarrow \varprojlim_n H_i^{sing}(X,\mathbb{Z}) \otimes \mathbb{Z}/l^n \longrightarrow \varprojlim_n H_i^{sing}(X,\mathbb{Z}/l^n)$$

$$\longrightarrow \varprojlim_n H_{i-1}^{sing}(X,\mathbb{Z})[l^n] \longrightarrow 0.$$

By assumption we have

$$\varprojlim_n H_i^{sing}(X,\mathbb{Z}) \otimes \mathbb{Z}/l^n = H_i^{sing}(X,\mathbb{Z}) \otimes \mathbb{Z}_l = H_i^{sing}(X,\mathbb{Z}_l),$$

$$\varprojlim_n H_{i-1}^{sing}(X,\mathbb{Z})[l^n] = 0$$

and hence the lemma follows. □

2.3 Specialization for Wreath Products

Our goal is to prove the following proposition which is inspired by [To, Lemma 8.1], a variant of which is stated in Lemma 2.3.4 below.

Proposition 2.3.1. *Let $p \neq l$ be prime numbers. Assume G is a finite abstract group satisfying the following conditions:*

(i) *The specialization map $A^*_{G_{\bar{\mathbb{Q}}_p}} \otimes \mathbb{Z}_l \to A^*_{G_{\bar{\mathbb{F}}_p}} \otimes \mathbb{Z}_l$ is an isomorphism.*

(ii) *$BG_{\bar{\mathbb{Q}}_p}$ and $BG_{\bar{\mathbb{F}}_p}$ can be approximated by schemes admitting a cell decomposition.*

(iii) *The cycle map $A^*_{G_{\bar{\mathbb{Q}}_p}} \otimes \mathbb{Z}_l \to H^*(BG_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l)$ is split injective.*

Then the same conditions hold for the wreath product $\mathbb{Z}/l \wr G$.

In order to prove this proposition we need to say something about the cyclic product of a quasi-projective scheme, since these are the spaces that approximate $B(\mathbb{Z}/l \wr G)$.

Cyclic Products. Let S be a quasi-projective scheme over an arbitrary field k. (We note that since we are interested in classifying spaces of finite groups, the assumption on S to be quasi-projective is no loss of generality by Lemma 1.3.4.) Let l be a prime. Consider the permutation action of $\mathbb{Z}/l$ on S^l. Since S is quasi-projective the geometric quotient $S^l/(\mathbb{Z}/l)$ exists. If we take out the diagonal of S^l, then the action of $\mathbb{Z}/l$ on $S^l - S$ is free and

$$\pi\colon S^l - S \to (S^l - S)/(\mathbb{Z}/l)$$

is a principal bundle quotient with structure group $\mathbb{Z}/l$. We will write $X = S^l - S$, $Y = (S^l - S)/(\mathbb{Z}/l)$ and $Z^l S = S^l/(\mathbb{Z}/l)$ and call $Z^l S$ the cyclic product of S. Note that π is finite, etale of degree l.

Assume $(l, \operatorname{char} k) = 1$ and k contains the l-th roots of unity. Fix a character $\mu_l \hookrightarrow \mathbb{G}_m$ and let $c_1 \in A^1 Y$ be its first Chern class. For $i > 0$ and $j \leq il - 1$ we consider the operations

$$\gamma_i \colon A_i(S) \to A_{li}(Y), \quad \alpha_i^j \colon A_i(S) \to A_j(Y)$$

constructed in [To, Section 7], where $\alpha_i^j = c_1^{li-j}\gamma_i$. We note that

$$\pi^*\gamma_i(a) = a^{\otimes l}|_X, \text{ hence } l\gamma_i(a) = \pi_*(a^{\otimes l}|_X),$$

and that α_i^j is a homomorphism of abelian groups.
For convenience we briefly recall the construction of γ_i. Let $C \in Z_*(S)$ be a cycle on S of dimension greater than 0. Then the support of C^l is not contained in the diagonal of S^l and we may consider the restriction $C^l - C$ of C^l to a cycle on $X = S^l - S$. The cycle $C^l - C$ is invariant under the action of $\mathbb{Z}/l$ and hence is the pull-back of a unique cycle $Z^l(C) - C$ on Y under the etale map $X \to Y$. This defines a map $Z_{\geq 1}(S) \to Z_{\geq l}(Y)$ which passes through rational equivalence (see loc. cit.) and induces the maps γ_i. Note that γ is not additive. More precisely, let $C = \sum_{i=1}^n m_i[V]$ be a cycle in S. Then

$$C^l - C = \sum_{\underline{i} \in \{1,\ldots,n\}^l} m_{i_1} \ldots m_{i_l}[V_{i_1} \times \ldots \times V_{i_l}],$$

where by abuse of notation we also write $[V_{i_1} \times \ldots \times V_{i_l}]$ for its restricition to $X = S^l - S$. The unique cycle on Y whose pull-back to X is $C^l - C$ is then given by

$$\gamma(C) = \sum_i m_i^l[(V_i^l - V_i)/(\mathbb{Z}/l)] \quad + \sum_{\underline{i} \in (\{1,\ldots,n\}^l - \{1,\ldots,n\})/(\mathbb{Z}/l)} m_{i_1} \ldots m_{i_l} \pi_*([V_{i_1} \times \ldots \times V_{i_l}]),$$

where $\{1, \ldots, n\}^l - \{1, \ldots, n\}$ denotes the complement of $\{1, \ldots, n\}$ when embedded diagonally in $\{1, \ldots, n\}^l$. For this note that $\pi^*\pi_* = \sum_{g \in \mathbb{Z}/l} g \colon Z_*(X) \to Z_*(X)$. This follows from [Fu, Proposition 1.7] applied to the fibersquare

$$\begin{array}{ccc} \mathbb{Z}/l \times X & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y, \end{array}$$

where m denotes the action map and p the projection to X, yielding $\pi^*\pi_* = m_*p^* = \sum_{g \in \mathbb{Z}/l} g$.

Remark 2.3.2. In fact, Totaro's operations map into the Chow group of $Z^l S$, so our operations are Totaro's composed with the pull-back to the open subset Y in $Z^l S$. However, in the end we will only be interested in the Chow group resp. homology of Y for dimension greater than $\dim S$ resp. $2 \dim S$ and in this case the Chow group resp. homology of $Z^l S$ and Y coincide.

Totaro then defines a functor F_l from graded abelian groups to graded abelian groups in the following way. Let A_* be a graded abelian group. Then $F_l(A_*)$ is the graded abelian group generated by the graded abelian group $A_*^{\otimes l}$ together with $A_i \otimes Z/l$ in degree j for $i+1 \leq j \leq li-1$ and elements $\gamma_i x_i$ in degree li for $x \in A_i$ and $i > 0$ subject to the relations

$$\begin{aligned} x_1 \otimes \ldots \otimes x_l &= x_2 \otimes \ldots \otimes x_l \otimes x_1 \\ l\gamma_i x &= x^{\otimes l} \\ \gamma_i(x+y) &= \gamma_i x + \sum_\alpha \alpha_1 \otimes \ldots \otimes \alpha_l + \gamma_i(y). \end{aligned}$$

Here α runs through the $\mathbb{Z}/l$-orbits in $\{x,y\}^l - \{(x,\ldots,x),(y,\ldots,y)\}$. If A_* is isomorphic to a finite direct sum $\bigoplus_{i=1}^n \mathbb{Z}/a_i \cdot e_i$ with a_i being 0 or a prime power, i.e. A_* is finitely generated, we can give a more precise description of $F_l(A_*)$ in the following way. Let R be the set of i such that $\dim e_i > 0$ and $a_i = 0$ or a power of l, then

$$\begin{aligned} F_l(A_*) = & \bigoplus_{\underline{i} \in (\{1,\ldots,n\}^l - R)/(\mathbb{Z}/l)} \mathbb{Z}/(a_1,\ldots,a_l) \cdot e_{i_1} \otimes \ldots \otimes e_{i_l} \oplus \bigoplus_{i \in R} \mathbb{Z}/(la_i) \cdot \gamma(e_i) \\ & \oplus \bigoplus_{\substack{i \in R \\ \dim e_i < j < l \dim e_i}} \mathbb{Z}/l \cdot \alpha^j(e_i) \end{aligned} \tag{2.3.1}$$

where again $\{1,\ldots,n\}^l - R$ denotes the complement of R when embedded diagonally in $\{1,\ldots,n\}^l$.
Using the operations

$$\begin{aligned} \otimes l \colon A_i(S) &\to A_{li}(X) \xrightarrow{\pi_*} A_{li}(Y) \\ \gamma_i \colon A_i(S) &\to A_{il}(Y) \\ \alpha_i^j \colon A_i(S) &\to A_j(Y) \end{aligned}$$

we obtain a homomorphism of graded abelian groups

$$\Psi_k \colon F_l(A_*(S)) \to A_*(Y).$$

We shall need one last piece of notation. For a scheme S and $r \in \mathbb{N}$ we write $F_l^{<r}(S)$ for the subgroup of elements of $F_l(A_*S)$ of degree $> l \dim S - r$. Clearly $F_l^{<r}(S) = F_l(A^{<r}S)$.

Lemma 2.3.3. *Let S be a smooth quasi-projective scheme over $\mathbb{Z}_p$. Then the diagram*

$$\begin{array}{ccc} F_l^{<r}(S_{\bar{\mathbb{Q}}_p}) & \xrightarrow{\Psi_{\bar{\mathbb{Q}}_p}} & A^{<r}(Y_{\bar{\mathbb{Q}}_p}) \\ \downarrow & & \downarrow \\ F_l^{<r}(S_{\bar{\mathbb{F}}_p}) & \xrightarrow{\Psi_{\bar{\mathbb{F}}_p}} & A^{<r}(Y_{\bar{\mathbb{F}}_p}) \end{array}$$

commutes. Here the vertical maps are given by specialization.

Proof. First we note that the exterior product map is compatible with specialization meaning that the diagram

$$\begin{array}{ccc} A_*(S_K \times_K S_K) & \xrightarrow{\sigma_R} & A_*(S_k \otimes_k S_k) \\ \times \uparrow & & \uparrow \times \\ A_*(S_K) \otimes A_*(S_K) & \xrightarrow[\sigma_R^{\otimes 2}]{} & A_*(S_k) \otimes A_*(S_k) \end{array}$$

commutes. This follows in the same way as Lemma 2.1.8 by using Remark 2.1.1. Moreover, push-forward and intersecting with Chern classes of line bundles are compatible with pull-back and refined Gysin homomorphisms. Hence from the definition of the maps $\Psi_{\bar{\mathbb{Q}}_p}$ and $\Psi_{\bar{\mathbb{F}}_p}$ we see that it suffices to show that the operation γ is compatible with specialization.

For this let $C = \sum_i m_i[V_i]$ be a cycle on $S_{\bar{\mathbb{Q}}_p}$ and consider a finite extension K of $\mathbb{Q}_p$ such that each subvariety V_i is defined over K. Let A be the integral closure of $\mathbb{Z}_p$ in K and k the residue field of A. Denote $i\colon \operatorname{Spec} k \hookrightarrow \operatorname{Spec} A$. Replace S by S_A and write as usual $X = S^l - S$, $Y = (S^l - S)/(\mathbb{Z}/l)$ and $\pi\colon X \to Y$ for the quotient map. Consider the diagram

$$\begin{array}{ccccc} A_*(S_k) & \xleftarrow{i^!} & A_*(S) & \longrightarrow & A_*(S_K) \\ \gamma_k \downarrow & & \gamma_A \downarrow & & \gamma_K \downarrow \\ A_*(Y_k) & \xleftarrow{i^!} & A_*(Y) & \longrightarrow & A_*(Y_K) \end{array}$$

Let $\bar{V}_i$ be the closure of V_i in S, then $\bar{C} = \sum_i m_i[\bar{V}_i]$ is a cycle that restricts to C. Now since the right side of the above diagram clearly commutes we see from the definition of the spezialization map that it suffices to prove the following assertion: Let $C = \sum_{i=1}^n m_i[V_i] \in Z_*(S)$ such that each subvariety $V_i \subset S$ maps dominantly to $\operatorname{Spec} A$ and such that $(V_i)_K$ is geometrically integer. Then

$$\gamma_k(i^! C) = i^!(\gamma_A C).$$

Since $(V_i)_k \hookrightarrow V_i$ is again a regular embedding of codimension 1 we have $i^!(C) = \sum_i m_i[(V_i)_k]$ (cf. Remark 2.1.1 (i)) and therefore

$$\gamma_k(i^! C) = \sum_i m_i^l[((V_i)_k^l - (V_i)_k)/(\mathbb{Z}/l)] \quad + \sum_{\underline{i} \in (\{1,\ldots,n\}^l - \{1,\ldots,n\})/(\mathbb{Z}/l)} m_{i_1} \ldots m_{i_n} (\pi_k)_*([(V_{i_1})_k \times \ldots \times (V_{i_l})_k]).$$

On the other hand we have

$$\gamma_A(C) = \sum_i m_i^l[(V_i^l - V_i)/(\mathbb{Z}/l)] \quad + \sum_{\underline{i} \in (\{1,\ldots,n\}^l - \{1,\ldots,n\})/(\mathbb{Z}/l)} m_{i_1} \ldots m_{i_n} \pi_*([V_{i_1} \times \ldots \times V_{i_l}]).$$

Now by assumption on the V_i we have that $(V_i^l - V_i)/(\mathbb{Z}/l) \subset Y$ is again a subvariety mapping dominantly to $\operatorname{Spec} A$ and therefore $i^![(V_i^l - V_i)/(\mathbb{Z}/l)] = [((V_i)_k^l - (V_i)_k)/(\mathbb{Z}/l)]$. Finally since

$$(\pi_k)_*([(V_{i_1})_k \times \ldots \times (V_{i_l})_k]) = (\pi_k)_*(i^!([V_{i_1} \times \ldots \times V_{i_l}])$$

the lemma follows from the compatibility of refined Gysin homomorphisms with proper push-forward ([Fu, Theorem 6.2 (a)]). □

Lemma 2.3.4. *Let S be a smooth quasi-projective scheme over $\bar{\mathbb{Q}}_p$ and $r \leq \dim S$. Assume that $A^{<r}(S) \otimes \mathbb{Z}_l \to H^{<2r}(S, \mathbb{Z}_l)$ is split injective. Then the composition*

$$F_l^{<r}(S) \otimes \mathbb{Z}_l \to A^{<r}(Y) \otimes \mathbb{Z}_l \to H^{<2r}(Y, \mathbb{Z}_l)$$

is split injective.

The above Lemma is a variant of [To, Lemma 8.1 (3)]:

Lemma 2.3.5. *(Totaro) Let S be a quasi-projective scheme over $\mathbb{C}$. If the cycle map $A_*(S) \to H_*^{BM}(S, \mathbb{Z})$ is split injective, then the composition $F_l(S) \to A_*(Z^l S) \to H_*^{BM}(Z^l S, \mathbb{Z})$ is split injective.*

Here $H_*^{BM}(S, \mathbb{Z})$ denotes Borel-Moore homology ([Fu, Section 19.1]). Let us explain how the two Lemmata above are related. Chose an isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$. By Theorem 2.2.13 and Lemma 2.2.14 we have identifications

$$\begin{aligned} H^{<2r}(Y, \mathbb{Z}_l) &= H_{sing}^{<2r}(Y, \mathbb{Z}) \otimes \mathbb{Z}_l \\ &= H^{\mathrm{BM}}_{>2l \dim S - 2r}(Y, \mathbb{Z}) \otimes \mathbb{Z}_l, \end{aligned}$$

Note that $2l \dim S - 2r \geq 2 \dim S$ and therefore

$$H^{\mathrm{BM}}_{>2l \dim S - 2r}(Y, \mathbb{Z}) = H^{\mathrm{BM}}_{>2l \dim S - 2r}(Z^l S, \mathbb{Z}).$$

Under this identifications Lemma 2.3.4 is the assertion of Totaro's Lemma, which is discussed in more detail below after we finished the proof of Proposition 2.3.1.

Proof. (of Proposition 2.3.1) Fix $r_o \in \mathbb{N}$. Let $S \to \mathbb{Z}_p$ be a smooth approximation of $BG_{\mathbb{Z}_p}$ up to codimension r_o (cf. Remark 2.1.2). We then have $A^r(S_{\bar{\mathbb{Q}}_p}) = A^r_{G_{\bar{\mathbb{Q}}_p}}$ as well as $A^r_{\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}} = A^r((S^l_{\bar{\mathbb{Q}}_p} - S_{\bar{\mathbb{Q}}_p})/\mathbb{Z}/l)$ for all $r < r_o$ and similary over $\bar{\mathbb{F}}_p$. Consider the diagram

$$\begin{array}{ccc} F_l^{<r_o}(S_{\bar{\mathbb{Q}}_p}) & \xrightarrow{\Psi_{\bar{\mathbb{Q}}_p}} & A^{<r_o}_{\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}} \\ \cong \downarrow & & \downarrow \\ F_l^{<r_o}(S_{\bar{\mathbb{F}}_p}) & \xrightarrow{\Psi_{\bar{\mathbb{F}}_p}} & A^{<r_o}_{\mathbb{Z}/l \wr G_{\bar{\mathbb{F}}_p}} \end{array}$$

where the left vertical arrow is induced by the specialization map for G. It is an isomorphism by condition (i). The surjectivity of the horizontal maps follow from [To, Lemma 8.1 (2)], whose proof is valid not only over $\mathbb{C}$ but over an arbitrary field. We can apply [To, Lemma 8.1 (2)] since condition (ii) holds for G. The diagram commutes by Lemma 2.3.3. Adding the cycle maps to etale cohomology we obtain a commutative diagram

$$\begin{array}{ccccc} F_l^{<r_o}(S_{\bar{\mathbb{Q}}_p}) \otimes \mathbb{Z}_l & \twoheadrightarrow & A^{<r_o}_{\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}} \otimes \mathbb{Z}_l & \longrightarrow & H^{<2r_o}(B(\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}), \mathbb{Z}_l) \\ \cong \downarrow & & \downarrow & & \downarrow \cong \\ F_l^{<r_o}(S_{\bar{\mathbb{F}}_p}) \otimes \mathbb{Z}_l & \twoheadrightarrow & A^{<r_o}_{\mathbb{Z}/l \wr G_{\bar{\mathbb{F}}_p}} \otimes \mathbb{Z}_l & \longrightarrow & H^{<2r_o}(B(\mathbb{Z}/l \wr G_{\bar{\mathbb{F}}_p}), \mathbb{Z}_l). \end{array}$$

Here the right vertical map is an isomorphism by Corollary 2.2.12. Since condition (iii) holds for G, it follows from Lemma 2.3.4 that the composition

$$F_l^{<r_o}(S_{\bar{\mathbb{Q}}_p}) \otimes \mathbb{Z}_l \twoheadrightarrow A^{<r_o}_{\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}} \otimes \mathbb{Z}_l \longrightarrow H^{<2r_o}(B(\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}), \mathbb{Z}_l)$$

is split injective, using that by definition $H^{<2r_o}(B(\mathbb{Z}/l \wr G_{\bar{\mathbb{Q}}_p}), \mathbb{Z}_l) = H^{<2r_o}((S^l_{\bar{\mathbb{Q}}_p} - S_{\bar{\mathbb{Q}}_p})/\mathbb{Z}/l, \mathbb{Z}_l)$. This shows that condition (i) and (iii) hold again for $\mathbb{Z}/l \wr G$. The fact that condition (ii) also holds for $\mathbb{Z}/l \wr G$ is proven in [To, Lemma 8.1 (2)]. □

Let us now discuss the proof of Lemma 2.3.5 in more detail. The proof of loc. cit just remarks, that this follows from Nakaoka's description of a basis for the homology of $Z^l S$ in [Na] together with the equation (2.3.1) for $F_l(S)$. However, since Nakaoka only computes the cohomology of $Z^l S$ with $\mathbb{Z}/l$-coefficents for S a finite simplicial complex, the desired conclusion of Lemma 2.3.5 is not completely clear. It seems that [St] and [Yo] are more useful. In [St] Stein computes the integral homology of the 2-fold cyclic product of a finite simplicial complex and Yoshioka generalizes Steins method to the l-fold cyclic products of a finite simplicial complex for l an odd prime. Now in order to apply Totaros argument we still need to show the following.

(I) The results of Stein and Yoshioka also compute $H_*^{\mathrm{BM}}(Z^l S, \mathbb{Z})$ (at least in dimension $\geq 2(l-1)\dim S$ which is the case of interest for us).

(II) The basis elements of equation (2.3.1) for $F_l(S)$ are mapped bijectively to a subset of a basis for $H_*^{\mathrm{BM}}(Z^l S, \mathbb{Z})$ described in [St, Theorem 13.2 (f)] for $l = 2$ and in [Yo, Section 10 Proposition (j)] for $l \neq 2$.

We first explain (I). If $X \to \operatorname{Spec}\mathbb{C}$ is an arbitrary scheme such that the one-point-compactification $X^c = X(\mathbb{C}) \cup \{*\}$ of $X(\mathbb{C})$ (with its complex topology) is a CW-complex then

$$H_i^{\mathrm{BM}}(X, \mathbb{Z}) = H_i(X^c, \{*\})$$

where the right hand side denotes relative homology ([Fu, Example 19.1.1]). In order to apply the results of Stein and Yoshioka it thus suffices to prove the following lemma.

Lemma 2.3.6. *Let $S \to \operatorname{Spec} \mathbb{C}$ be a quasi-projective scheme. Then the one-point-compactification S^c of $S(\mathbb{C})$ (with its complex topology) has a finite triangulation.*

This will be a consequence of Hironoka's semi-algebraic triangulation theorem ([Hi]). We recall that a subset of $\mathbb{R}^n$ is called semi-algebraic if it belongs to the Boolean class of subsets of $\mathbb{R}^n$ generated by $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ with $f \in \mathbb{R}[X_1, \ldots, X_n]$. Moreover, a map $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is called semi-algebraic when its graph Γ_f is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$. In particular, the image of a semi-algebraic set under a semi-algebraic map is again semi-algebraic. This is Proposition II in [Hi]. Note that every polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ or more generally every regular rational map $\mathbb{R}^n \to \mathbb{R}^m$ is semi-algebraic.
We also note that in Hironaka's article simplices are open, i.e. an r-simplex Δ in $\mathbb{R}^n$ is defined as

$$\Delta = \{\sum_{i=0}^{r} a_i v_i \mid a_i > 0, \sum_{i=0}^{r} a_i = 1\},$$

for affinely independent vectors $v_0, \ldots, v_r$ in $\mathbb{R}^n$.

Theorem 2.3.7. *([Hi]) Let $\{X_\alpha\}_\alpha$ be a finite system of bounded semi-algebraic sets in $\mathbb{R}^n$. Then there exists a simplicial decomposition $\mathbb{R}^n = \cup_a \Delta_a$ and a semi-algebraic automorphism κ of $\mathbb{R}^n$ such that each X_α is a finite union of $\kappa(\Delta_a)$.*

Lemma 2.3.8. *Let K be a simplicial complex in some euclidean space and $A \subset K$ be a subcomplex. Then the quotient space K/A has a triangulation.*

Proof. Let $\mathcal{S}(K)$ resp. $\mathcal{S}(A)$ be the corresponding simplicial sets. That is, if Δ denotes the category of non-empty finite totally ordered sets with non-decreasing maps, then $\mathcal{S}(K)$ is the contravariant functor $\Delta \to (\text{Sets})$ that maps the set $\underline{n} = \{1, \ldots, n\}$ to the set of non-decreasing maps $f \colon \underline{n} \to \text{Vert}(K)$ such that the convex hull of the image of f is a simplex of K. Since A is a subcomplex of K we have $\mathcal{S}(A)(\underline{n}) \subset \mathcal{S}(K)(\underline{n})$ for all n. Consider the quotient simplicial set $\mathcal{S}(K)/\mathcal{S}(A)$, i.e. the functor $\underline{n} \mapsto \mathcal{S}(K)(\underline{n})/\mathcal{S}(A)(\underline{n})$. Forming the geometric realization of this simplicial set we obtain $|\mathcal{S}(K)/\mathcal{S}(A)| = |\mathcal{S}(K)|/|\mathcal{S}(A)| = K/A$. This shows that K/A is the geometric realization of a simplicial set. By [FP, Corollary 4.6.12] the geometric realization of any simplicial set is triangulable. □

Proof. (Lemma 2.3.6) We may assume that S is open in a closed subscheme X of $\mathbb{P}^n_{\mathbb{C}}$. Let T denote the complement of S in X. Let $\mathbb{S}^{2n+1}$ denote the real $2n+1$-sphere. We view $\mathbb{S}^{2n+1}$ as a subset of $\mathbb{C}^{n+1}$. We can then write $\mathbb{P}^n(\mathbb{C})$ as the quotient $\mathbb{S}^{2n+1}/U(1)$ with the usual action of the unitary group $U(1)$ on $\mathbb{S}^{2n+1}$. The map

$$\mathbb{S}^{2n+1} \to \mathbb{C}^N, \quad (z_0, \ldots, z_n) \mapsto ((z_i \bar{z}_i)_i, (\bar{z}_i z_j)_{i<j})$$

thus induces a closed embedding $\mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{C}^N$ of topological spaces such that each composition

$$\mathbb{R}^{2n} = \mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C}) \hookrightarrow \mathbb{C}^N = \mathbb{R}^{2N}$$

is a regular rational embedding. Using this embedding we may embed $T(\mathbb{C})$ and $X(\mathbb{C})$ as bounded semi-algebraic sets in $\mathbb{R}^{2N}$. Applying the semi-algebraic triangulation theorem to the system $\{X(\mathbb{C}), T(\mathbb{C})\}$ realizes $X(\mathbb{C})$ as a finite simplicial complex with subcomplex $T(\mathbb{C})$. Since $S^c = X(\mathbb{C})/T(\mathbb{C})$ the previous lemma yields the claim. □

We now explain point (II). Write $G = \mathbb{Z}/l$ and let g_0 denote a fixed generator of G. We keep the notation $X = S^l - S$, $Y = X/G$ and $Z^l S = S^l/G$ for S a quasi-projective scheme over $\mathbb{C}$. Recall that we denoted by $c_1 \in H^2(Y, \mathbb{Z})$ the first Chern class of the line bundle on Y corresponding to the character $G \hookrightarrow \mathbb{C}^*$ mapping g_0 to $\exp(2\pi i/l)$. The assertion will be clear after showing that the elements $c_1 \cap \alpha \in H^{\mathrm{BM}}_{i-2}(Y, \mathbb{Z})$ for $\alpha \in H^{\mathrm{BM}}_i(Y, \mathbb{Z})$ obtained by capping with c_1 correspond to the elements in [St] and [Yo] obtained by the operation of cascades ([St, Section 3]). Let us recall the operation of cascades.
Consider a chain complex K_* of free $\mathbb{Z}[G]$-modules. We have the following operations on K_*

$$\sigma = \sum_{g \in G} g \colon K_* \to K_*, \quad \tau = id - g_0 \colon K_* \to K_*.$$

In the following we will use the letter ρ to either mean τ or σ. If $\rho = \sigma$ and the context is fixed $\bar{\rho}$ will denote τ and vice versa. Following the notation in [St] and [Yo] we will denote by $K_*^{\rho^{-1}}$ resp. K_*^{ρ} the kernel resp. image of ρ. The i-th homology group of $K_*^{\rho^{-1}}$ resp. K_*^{ρ} will be denoted by $H_i^{\rho^{-1}}(K_*)$ resp. $H_i^{\rho}(K_*)$. By assumption on the action of G on K_* one easily verifies the following lemma.

Lemma 2.3.9. $K_*^{\rho^{-1}} = K_*^{\bar{\rho}}$

From the short exact sequence

$$0 \longrightarrow K_*^{\rho^{-1}} \longrightarrow K_* \longrightarrow K_*^{\rho} \longrightarrow 0$$

we obtain a boundary operator $H_*^{\rho}(K_*) \to H_{*-1}^{\rho^{-1}}(K_*)$ on the level of homology. In view of the above lemma we may compose the boundary operator of σ and τ to obtain a map

$$\Gamma_\rho \colon H_*^{\rho}(K_*) \to H_{*-2}^{\rho}(K_*)$$

that decreases the degree by 2. We note that the image of Γ_ρ is l-torsion ([Na, Theorem 1.7]). Stein calls the elements that are derived from an element $\alpha \in H_i^{\rho}(K_*)$ by repeated application of Γ_ρ the cascades of α.
In view of point (I) we may replace S by its one-point compactification and henceforth assume that S has the structure of a finite simplicial complex. Consider now

the relative simplicial chain complex $C_* = C_*^{\Delta}(S^l, S)$. We then have an action of G on C_* that makes C_* into a complex of free $\mathbb{Z}[G]$-modules. By [St, Section 4] we have an isomorphism $H_*^{\sigma}(C_*) \cong H_*^{\Delta}(Z^l S, S)$ and hence an operation

$$\Gamma_\sigma\colon H_*^{\Delta}(Z^l S, S) \to H_{*-2}^{\Delta}(Z^l S, S).$$

Since simplicial homology and singular homology coincide we may further identify $H_*^{\Delta}(Z^l S, S) = H_*(Z^l S, S) = H_*^{\mathrm{BM}}(Y)$. We then need to prove the following lemma.

Lemma 2.3.10. *Under the identification $H_*^{\Delta}(Z^l S, S) = H_*^{BM}(Y)$ the operation Γ_σ equals capping with c_1 up to multiplication with an element of $(\mathbb{Z}/l)^*$.*

Proof. In the proof we will use the notation and results of Appendix A. We shall need one more ingredient. Let

$$\delta\colon \operatorname{Hom}(G, \mathbb{C}^*) \to H^2(G, \mathbb{Z})$$

be the boundary operator derived from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow 0.$$

We will construct a natural map

$$H^*(G, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$$

such that for each character χ the element $\delta(\chi)$ is mapped to $c_1(L_\chi) \in H^2(Y, \mathbb{Z})$, where L_χ is the line bundle over Y induced by χ. For this let BG be a model of the classifying space of G and EG its universal covering space. By the universal property of BG there is a unique (up to homotopy) map $Y \to BG$ such that $X \to Y$ is the pull-back of $EG \to BG$ along $Y \to BG$. Since $H^*(BG, \mathbb{Z}) = H^*(G, \mathbb{Z})$ we obtain the desired map $H^*(G, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$. The fact that $\delta(\chi)$ is mapped to $c_1(L_\chi)$ follows easily from the long exact sequence in cohomology derived from the exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C^0(Y) & \xrightarrow{\exp} & C^0(Y)^* & \longrightarrow & 0
\end{array}$$

of sheaves on Y. Here $C^0(Y)$ resp. $C^0(Y)^*$ denotes the sheaf of $\mathbb{C}$-valued continuous resp. $\mathbb{C}$-valued continuous and non-vanishing functions on Y.
Now, since $C_* = C_*^{\Delta}(S^l, S)$ is a complex of free $\mathbb{Z}[G]$-modules we have

$$\mathbb{H}_*(G, C_*) = H_*(C_*/C_*^{\tau}) = H_*(C_*^{\sigma}) = H_*(Z^l S, S) = H_*^{\mathrm{BM}}(Y, \mathbb{Z}).$$

Moreover, we see from the construction of cap products in group (co)homology (see Appendix A) that the diagram

$$\begin{array}{ccc} H^i(G,\mathbb{Z}) \times \mathbb{H}_j(G,C_*) & \xrightarrow{\cap} & \mathbb{H}_{j-i}(G,C_*) \\ \downarrow & & \| \\ H^i(Y,\mathbb{Z}) \times H_j^{\mathrm{BM}}(Y,\mathbb{Z}) & \xrightarrow{\cap} & H_{j-i}^{\mathrm{BM}}(Y,\mathbb{Z}) \end{array}$$

commutes. We consider the element $\alpha \in H^2(G,\mathbb{Z}) \cong \mathbb{Z}/l$ corresponding to the extension

$$\alpha: \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\text{diag}} \mathbb{Z}[G] \xrightarrow{\tau} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Here $\mathbb{Z}[G] \to \mathbb{Z}$ is the map $\sum \lambda_g g \mapsto \sum \lambda_g$. Again since C_* is a complex of free $\mathbb{Z}[G]$-modules we see that the morphisms $\alpha \cap' \cdot \colon H_j^{\mathrm{BM}}(Y,\mathbb{Z}) \to H_{j-2}^{\mathrm{BM}}(Y,\mathbb{Z})$ and $\Gamma_\sigma \colon H_j^{\mathrm{BM}}(Y,\mathbb{Z}) \to H_{j-2}^{\mathrm{BM}}(Y,\mathbb{Z})$ coincide. Finally, since α is mapped to some multiple of c_1 under the map $H^2(G,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ depending on the choice of the generator $g_0 \in G$, the claim follows from Lemma A.4. $\square$

2.4 Specialization for the Classical Groups over finite Fields

In this section we investigate the specialization map $CH^*BG_{\mathbb{C}} \to CH^*BG_{\bar{\mathbb{F}}_p}$, where G belongs to the class of classical groups $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, $\mathrm{O}_n(\mathbb{F}_q)$ and $\mathrm{SO}_n(\mathbb{F}_q)$ over some finite field $\mathbb{F}_q$ of characteristic p. For this we need to know the structure of their l-Sylow subgroups. Of course, this is well-known so let us gather the results we shall need.

Lemma 2.4.1. *Let r be the order of q in $(\mathbb{Z}/l)^*$ and $b = v_l(q^r - 1)$.*

(i) *If $l \neq 2, p$ the l-Sylow subgroups of the above list of classical groups are isomorphic to a product of groups of the form $\mathbb{Z}/l^{l^i} \wr \mathbb{Z}/l^b$.*

(ii) *If $q \equiv 1$ mod 4 the same assertion holds for the 2-Sylow subgroups of the groups $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ and $\mathrm{O}_{2m+1}(\mathbb{F}_q)$.*

Proof. The first part is proven in [Wei] by A.J. Weir. We will give a slightly different version of Weir's proof using an argument of Quillen used in the proof of Lemma 13 in [Qu]. This way we will also obtain part (ii).

Let us write $n = dr + e$ for $0 \leq e < r$. If we denote $C = \mathbb{F}_q(\mu_l)^*$ then the natural action of C on $\mathbb{F}_q(\mu_l)$ gives a faithfull r-dimensional representation over $\mathbb{F}_q$. Now let S_d act on C^d by permuting the factors then $S_d \ltimes C^d$ has an dr-dimensional faithfull representation and after adding a trivial e-dimensional representation we may view $S_d \wr C = S_d \ltimes C^d$ as a subgroup of $GL_n(\mathbb{F}_q)$.

We claim that the index of $S_d \wr C$ in $\mathrm{GL}_n(\mathbb{F}_q)$ is prime to l if $l \neq 2$ or $l = 2$ and $q \equiv 1$ mod 4. Note that this index is given as

$$[\mathrm{GL}_n(\mathbb{F}_q) : S_d \wr C] = q^{n(n-1)/2} \prod_{\substack{i=1 \\ i \not\equiv 0(r)}}^{n} (q^i - 1) \prod_{j=1}^{d} \frac{q^{jr} - 1}{j(q^r - 1)},$$

so one only needs to check that the last factor is an l-adic unit. For this one can use the l-adic logarithm

$$log\colon 1 + l^u\mathbb{Z}_l \to l^u\mathbb{Z}_l,$$

which is an isomorphism preserving the valuation for all $u \geq 1$ if $l \neq 2$ and all $u \geq 2$ if $l = 2$. So if $l \neq 2$ or $v_l(q^r - 1) \geq 2$ we obtain

$$\begin{aligned} v_l(q^{jr} - 1) &= v_l(log(q^{jr})) \\ &= v_l(jlog(q^r)) \\ &= v_l(j) + v_l(log(q^r)) \\ &= v_l(j) + v_l(q^r - 1) \end{aligned}$$

Let us now assume $l = 2$ and $q \equiv 1 \bmod 4$. Then $r = 1$ and $d = n$. Moreover, $\frac{q^j-1}{q-1} \equiv j \bmod 4$ and therefore $v_l(\frac{q^j-1}{q-1}) = 1$ if and only if $v_l(j) = 1$. The claim follows.
If we write $d = b_0 + b_1 l + \ldots + b_s l^s$ with $0 \leq b_i < l$ we therefore obtain

$$v_l(\#GL_n(\mathbb{F}_q)) = v_l(\#(S_d \wr C)) = bd + \sum_{i=1}^{s} b_i\mu_i(l)$$

with $\mu_i(l) = \sum_{k=0}^{i-1} l^k$. Setting $N_0 = b$ and $N_i = bl^i + \mu_i(l)$ for $i > 0$ we can rewrite this as

$$v_l(\#GL_n(\mathbb{F}_q)) = \sum_{i=0}^{s} b_i N_i.$$

From this point we can use the argumentation of [Wei]: Let G_i be an l-Sylow subgroup of $GL_{rl^i}(\mathbb{F}_q)$. Then G_i has order N_i. Hence $\prod_{i=0}^{s} G_i^{b_i}$ is an l-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$. Since $C \subset \mathrm{GL}_r(\mathbb{F}_q)$ we see $G_0 = \mathbb{Z}/l^b$. Inductively we obtain $G_i = \mathbb{Z}/l^{l^i} \wr \mathbb{Z}/l^b$ for then G_i has the right order.
For $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ we recall $\mathrm{Sp}_{2m}(\mathbb{F}_q) = q^{m^2}(q^2 - 1)(q^4 - 1) \cdot \ldots \cdot (q^{2m} - 1)$. If r is even an l-Sylow subgroup of $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ is already an l-Sylow subgroup of $GL_{2m}(\mathbb{F}_q)$. We may thus assume that r is odd. In this case the factors that are divisible by l are $q^{2r} - 1, \ldots, q^{2rk} - 1$, where $2m = 2rk + a$ for $0 \leq a < 2r$. Writing $k = b_0 + b_1 l + \ldots + b_t l^t$ with $0 \leq b_i < l$ we thus obtain

$$v_l(\mathrm{Sp}_{2m}(\mathbb{F}_q)) = v_l(\#GL_{rk}(\mathbb{F}_{q^2})) = bk + \sum_{i=1}^{s} b_i\mu_i(l)$$

In the case of $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ we note that $\#\,\mathrm{O}_{2m+1}(\mathbb{F}_q) = \#\,\mathrm{Sp}_{2m}(\mathbb{F}_q)$. One can then deduce the assertion for $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ and $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ in the same way as for $\mathrm{GL}_n(\mathbb{F}_q)$. For the details we refer to [Wei]. □

Remark 2.4.2. If l is a prime that does not divide $q - 1$ then every l-Sylow subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$ is also an l-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$. This follows from $|\,\mathrm{GL}_n(\mathbb{F}_q)|/|\,\mathrm{SL}_n(\mathbb{F}_q)| = q - 1$.

Consider now an odd prime divisor l of $q-1$ and an l-Sylow subgroup P of the symmetric group S_n. Since l is odd we may view P as a subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$. It follows from the proof of the above lemma that an l-Sylow subgroup of $\mathrm{SL}_n(\mathbb{F}_q)$ is then of the form

$$P \ltimes \{diag(a_1, \ldots, a_n) \mid \prod_i a_i = 1\},$$

where P acts via permutations and the a_i belong to a fixed l-Sylow subgroup of $C = \mathbb{F}_q(\mu_l)^*$. This group is not a product of groups of the form $\mathbb{Z}/l^{\wr i} \wr \mathbb{Z}/l^b$.

Proposition 2.4.3. (i) *The specialization map $CH^*BG_{\mathbb{C}} \to CH^*BG_{\bar{\mathbb{F}}_p}$ for the classical groups $G = \mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2m}(\mathbb{F}_q)$, $\mathrm{O}_n(\mathbb{F}_q)$ and $\mathrm{SO}_n(\mathbb{F}_q)$ over some finite field $\mathbb{F}_q$ of characteristic p become injective after inverting $2p$.*

(ii) *If $q \equiv 1 \mod 4$ the specialization map for $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{Sp}_{2m}(\mathbb{F}_q)$ and $\mathrm{O}_{2m+1}(\mathbb{F}_q)$ become injective after inverting p.*

(iii) *If S denotes the product of p and all prime divisor of $q-1$ the specialization map for $\mathrm{SL}_n(\mathbb{F}_q)$ becomes injective after inverting S.*

Proof. We need to check that the specialization map of the respective l-Sylow subgroups are injective, where in (i) we consider a prime l not dividing $2p$, in (ii) we consider a prime l different from p and in (iii) we consider a prime l not dividing S.

In any case we know that these l-Sylow subgroups are products of groups of the form $\mathbb{Z}/l^{\wr i} \wr \mathbb{Z}/l^b$. Let us check that conditions (i)-(iii) of Proposition 2.3.1 hold for $G = \mathbb{Z}/l^b$. Choosing an l^b-th root of unity we may identify $\mathbb{Z}/l^b = \mu_{l^b}$. Note that reduction induces an isomorphism $\mu_{l^b}(\bar{\mathbb{Q}}_p) = \mu_{l^b}(\bar{\mathbb{F}}_p)$. Thus (i) follows from Example 1.2.12 and (iii) is an immediate consequence of Lemma 2.2.5. We know that the complement of the zero section in $\mathcal{O}_{\mathbb{P}^m}(l^b)$ approximates $B(\mu_{l^b})$ (cf. the proof of Lemma 2.2.5). This space can be cut open into spaces of the form $\mathbb{A}^1 - \{0\} \times \mathbb{A}^k$. Hence condition (ii) holds. Then using Proposition 2.3.1 we see that the specialization map for the group $\mathbb{Z}/l\mathbb{Z}^{\wr i} \wr \mathbb{Z}/l^b\mathbb{Z}$ is an isomorphism. The general case follows from the Künneth formula (Lemma 1.8.6). □

Remark 2.4.4. A p-Sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ is given by the upper triangular matrices with 1's on the diagonal. The specialization morphism is not injective in this case. For $n = 2$ this Sylow subgroup is just $(\mathbb{Z}/p\mathbb{Z})^a$ with $a = v_p(q)$ and its Chow ring in characteristic p is trivial while in characteristic 0 it is $\mathbb{Z}[t_1, \ldots, t_a]/(pt_1, \ldots, pt_a)$.

2.5 Specialization for arbitrary finite Groups

We have seen that the specialization map

$$\sigma_G \colon CH^*BG_{\bar{\mathbb{Q}}_p} \to CH^*BG_{\bar{\mathbb{F}}_p}$$

is an isomorphism after inverting p for the finite cyclic groups and wreath products. In the next Chapter we will see that it is also an isomorphism for some classical groups over finite fields. Morever, the Chow ring $CH^*BG_{\bar{\mathbb{F}}_p}$ is p-torsion free in these cases. One can ask whether this is true for every finite abstract group G.

Question 2.5.1. *Let G be a finite abstract group. Is the specialization map*

$$\sigma_G\colon CH^*BG_{\bar{\mathbb{Q}}_p} \to CH^*BG_{\bar{\mathbb{F}}_p}$$

injective (an isomorphism) after inverting p?

If $CH^*BG_{\bar{\mathbb{F}}_p}$ is generated by Chern classes of representations of $G_{\bar{\mathbb{F}}_p}$, we know that σ_G is surjective by the theory of Brauer lifts. In this section we conduct a brief discussion of the above Question. By the usual transfer argument it suffices to prove injectivity of $\sigma_G\colon CH^*BG_{\bar{\mathbb{Q}}_p} \to CH^*BG_{\bar{\mathbb{F}}_p}$ in the case of G being an l-group, where l is a prime not equal to p. The assertion for etale cohomology is true by Corollary 2.2.12. Hence by looking at the commutative square

$$\begin{array}{ccc} CH^*BG_{\bar{\mathbb{Q}}_p} \otimes \mathbb{Z}_l & \longrightarrow & H^*(BG_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l) \\ \downarrow & & \downarrow \cong \\ CH^*BG_{\bar{\mathbb{F}}_p} \otimes \mathbb{Z}_l & \longrightarrow & H^*(BG_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l) \end{array}$$

we see that it would be sufficient to prove that for each l-group G the cycle map

$$CH^*BG_{\bar{\mathbb{Q}}_p} \otimes \mathbb{Z}_l \to H^*(BG_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l)$$

is injective. If G is an l-group with exponent > 1 we can chose a normal subgroup H in G such that $G/H = \mathbb{Z}/l$. Using induction on the exponent of G we see that injectivity would follow from an affirmative answer to the following question.

Question 2.5.2. *Let X be a smooth scheme over some field k. Assume that for a prime $l \neq$ char k we have a free action of $G = \mathbb{Z}/l$ on X. If the cycle map $CH^*X \otimes \mathbb{Z}_l \to H^*(X, \mathbb{Z}_l)$ is injective does the same hold true for $CH^*(X/G) \otimes \mathbb{Z}_l \to H^*(X/G, \mathbb{Z}_l)$?*

We have seen that a similar assertion holds in the case of cyclic products Z^lS for S quasi-projective and smooth over $\mathbb{C}$ admitting a cell decomposition by using the results of Stein and Yoshioka (cf. Lemma 2.3.4). We highly expect that the same is true for any field of characteristic different from l that contains the l-th roots of unity. For this one could try to compute the etale cohomology groups $H^*(Z^lS, \mathbb{Z}_l)$ in terms of a basis for $H^*(S, \mathbb{Z}_l)$ similar to what Stein and Yoshioka did.
At last we want to mention another example in which the specialization map becomes an isomorphism after inverting p.

Proposition 2.5.3. *Let S be a scheme over $\mathbb{Z}_p$ which has a decomposition into affine cells. Then $\sigma\colon A_*(S_{\bar{\mathbb{Q}}_p}) \to A_*(S_{\bar{\mathbb{F}}_p})$ is an isomorphism after inverting p.*

Proof. Let $\mathbb{A}^n_{\mathbb{Z}_p}$ an affine open cell in $S_{\mathbb{Z}_p}$ and W its complement in S. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_*(W_{\bar{\mathbb{Q}}_p}) & \longrightarrow & A_*(S_{\bar{\mathbb{Q}}_p}) & \longrightarrow & A_*(\mathbb{A}^n_{\bar{\mathbb{Q}}_p}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & A_*(W_{\bar{\mathbb{F}}_p}) & \longrightarrow & A_*(S_{\bar{\mathbb{F}}_p}) & \longrightarrow & A_*(\mathbb{A}^n_{\bar{\mathbb{F}}_p}) & \longrightarrow & 0 \end{array}$$

We claim the upper row is exact and the lower row becomes exact after inverting p. This holds since the cycle map $A_*(X) \otimes \mathbb{Z}_l \to H_*(X_{k^{sep}}, \mathbb{Z}_l)$ is an isomorphism for schemes X with a decomposition into affine cells. Here X is defined over an arbitrary field k and l is a prime different from the characteristic of k. In particular, $H_i(X, \mathbb{Z}_l) = 0$ for odd i. The proposition thus follows from noetherian induction. □

3 The Chow Ring of the Classifying Space of some Chevalley Groups

We recall that Chevalley groups are the finite groups of the form $G(\mathbb{F}_q)$, where G is a connected split reductive group scheme over $\operatorname{Spec} \mathbb{Z}$. The goal of this chapter is to compute $CH^*BG(\mathbb{F}_q)_{\mathbb{C}}$ in some special cases.
In [Gu] Guillot computes the mod l Chow ring of $\mathrm{GL}_n(\mathbb{F}_q)$ considered as an algebraic group over $\mathbb{C}$ for prime numbers $l \neq 2, p$ by using a similar approach as Quillen in [Qu], where he computes the cohomology ring of $\mathrm{GL}_n(\mathbb{F}_q)$ with mod-l coefficients. Guillot's result is

$$A^*_{\mathrm{GL}_n(\mathbb{F}_q)}/l = \mathbb{Z}/l[c_r, c_{2r}, \dots, c_{mr}],$$

where r is the order of q in $(\mathbb{Z}/l)^*$ and $n = mr + e$ for $0 \leq e < r$. He also shows if $\mathbb{F}_q$ contains the l^b-th roots of unity for some integer b, then

$$A^*_{\mathrm{GL}_n(\mathbb{F}_q)}/l^b = \mathbb{Z}/l^b[c_1, \dots, c_n].$$

Our approach will be the following. In view of Proposition 2.1.3 and Proposition 2.4.3 we will first consider $G(\mathbb{F}_q)$ as an algebraic group over $\bar{\mathbb{F}}_p$, where $p = \operatorname{char} \mathbb{F}_q$, and then compute $CH^*BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$. This turns out to be a much simpler task. The reason is that by Corollary 2.1.7 we have a canonical isomorphism $BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p} \cong [G_{\bar{\mathbb{F}}_p}/G_{\bar{\mathbb{F}}_p}]$ of stacks, where the action of $G_{\bar{\mathbb{F}}_p}$ on $G_{\bar{\mathbb{F}}_p}$ is given by conjugation with the q-th power Frobenius. It is this presentation of $BG(\mathbb{F}_q)_{\bar{\mathbb{F}}_p}$ that enables us to compute its Chow ring.
Let us first take a look at the case $G = \mathrm{GL}_n$ for $n = 1, 2$. In the following examples σ will denote the q-th power Frobenius on $\mathbb{G}_m$ resp. GL_2 and T will denote the maximal torus of diagonal matrices.

Example 3.0.4. ($\mathbb{G}_m \curvearrowright \mathbb{A}^n - \{0\}$ via weights $a_1, \dots, a_n$.) This is an application of Lemma 1.6.3 and Proposition 1.2.8. Let us write $V = \mathbb{A}^n$ for the $\mathbb{G}_m$-representation with weights $a_1, \dots, a_n$. If we let $\mathbb{G}_m$ act on $\mathbb{P}(V)$ with the same weights then $V - \{0\} \to \mathbb{P}(V)$ is a $\mathbb{G}_m$-equivariant principal $\mathbb{G}_m$-bundle and the corresponding line bundle is given by $\mathcal{O}(1)$. Hence by Proposition 1.2.8 we have $A^*_{\mathbb{G}_m}(V - \{0\}) = A^*_{\mathbb{G}_m}(\mathbb{P}(V))/c_1(\mathcal{O}(1))$. Then using Lemma 1.6.3 we get

$$A^*_{\mathbb{G}_m}(V - \{0\}) = \mathbb{Z}[t]/c_n(V) = \mathbb{Z}[t]/(\prod_i a_i)t^n.$$

As a special case we may consider $\mathbb{G}_m$ acting on $\mathbb{G}_m$ via conjugation or σ-conjugation with σ the q-th power Frobenius. This means weight 0 resp. weight $q-1$ and therefore $A^*_{\mathbb{G}_m}(\mathbb{G}_m) = \mathbb{Z}[t]$ in the conjugation case and $A^*_{\mathbb{G}_m}(\mathbb{G}_m) = \mathbb{Z}[t]/(q-1)t$ in the σ-conjugation case.

Example 3.0.5. ($\mathrm{GL}_n \curvearrowleft \mathrm{GL}_n \supset T$ via conjugation, i.e. $X \cdot G = G^{-1}XG$.) The unit section of $\mathrm{GL}_n \to \operatorname{Spec} k$ is GL_n-equivariant. Therefore the pull-back $A^*_{\mathrm{GL}_n} \to A^*_{\mathrm{GL}_n}(\mathrm{GL}_n)$ is injective. Since GL_n is open in $\mathbb{A}^{n^2}$ this pull-back factors as $A^*_{\mathrm{GL}_n} \to A^*_{\mathrm{GL}_n}(\mathbb{A}^{n^2}) \to A^*_{\mathrm{GL}_n}(\mathrm{GL}_n)$. Here the first map is an isomorphism and the second is surjective. It follows that $A^*_{\mathrm{GL}_n} \to A^*_{\mathrm{GL}_n}(\mathrm{GL}_n)$ is an isomorphism. Thus we obtain $A^*_{\mathrm{GL}_n}(\mathrm{GL}_n) = \mathbb{Z}[c_1, \dots, c_n]$ and $A^*_T(\mathrm{GL}_n) = \mathbb{Z}[t_1, \dots, t_n]$.
Note that we cannot apply this argument in the σ-conjugation case since there is no fixpoint for this action on GL_n.

Example 3.0.6. $\mathrm{GL}_2 \curvearrowleft \mathrm{GL}_2 \supset T$ via σ-conjugation, i.e. $X \cdot G = G^{-1}X\sigma(G)$.) This case can be easily dealt with using a similar approach as in [EF]. We first compute $A^*_T(\mathrm{GL}_2)$. T operates on $V = Mat(2 \times 2, k)$ via

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^q \lambda_1^{-1} x_1 & \lambda_2^q \lambda_1^{-1} x_2 \\ \lambda_1^q \lambda_2^{-1} x_3 & \lambda_2^q \lambda_2^{-1} x_4 \end{pmatrix}$$

Thus the Chern roots of V in A^*_T are given by

$$\alpha_1 = (q-1)t_1, \quad \alpha_2 = qt_2 - t_1, \quad \alpha_3 = qt_1 - t_2, \quad \alpha_4 = (q-1)t_2.$$

Note that the underlying scheme of GL_2 is isomorphic to $\mathbb{A}^4 - V(x_1x_4 - x_2x_3)$. Applying Proposition 1.2.8 to the T-equivariant $\mathbb{G}_m$-bundle $\mathbb{A}^4 - V(x_1x_4 - x_2x_3) \to \mathbb{P}^3 - V(x_1x_4 - x_2x_3)$ shows

$$A^*_T(\mathrm{GL}_2) = A^*_T(\mathbb{P}^3 - \Delta)/c_1(\mathcal{O}(1))$$

with $\Delta = V(x_1x_4 - x_2x_3)$. Hence we have to compute

$$A^*_T(\mathbb{P}^3 - \Delta) = A^*_T(\mathbb{P}^3)/I,$$

where I is the ideal in $A_T^*(\mathbb{P}^3)$ given by the image of the push-forward $A_*^T(\Delta) \to A_*^T(\mathbb{P}^3)$. Note that we computed $A_T^*(\mathbb{P}^3)$ in Lemma 1.6.3. To compute I we note that the image of the Segre embedding

$$\pi \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$$

is equal to Δ. If we let T act on $\mathbb{P}^1 \times \mathbb{P}^1$ via $D \cdot (x, y) = (D^{-1}x, \sigma(D)y)$ then π becomes a T-equivariant map. Let ζ_i denote the pull-back of $c_1^T(\mathcal{O}_{\mathbb{P}^1}(1))$ via the projection $p_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ for $i = 1, 2$. Using Lemma 1.7.4 on $\mathbb{P}^1$ we compute

$$(1) \quad \zeta_1 = [0 \times \mathbb{P}^1]_T - t_1 \text{ and } \zeta_2 = [\mathbb{P}^1 \times 0]_T + q t_1.$$

Let as usual $\zeta = c_1^T(\mathcal{O}_{\mathbb{P}^3}(1))$. From $\pi^*\mathcal{O}_{\mathbb{P}^3}(1) = p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^3}(1)$ we obtain $\pi^*(\zeta) = \zeta_1 + \zeta_2$. Hence by the projection formula

$$(2) \quad \zeta \pi_*(1) = \pi_*(\zeta_1) + \pi_*(\zeta_2).$$

Now by Lemma 1.8.5 we have that $A_T^*(\mathbb{P}^1 \times \mathbb{P}^1) = A_T^*(\mathbb{P}^1) \otimes_{A_T^*} A_T^*(\mathbb{P}^1)$ is generated as a A_T^*-module by $1, \zeta_1, \zeta_2$ and $\zeta_1\zeta_2$. From equation (2) we see that the image of π_* is generated as a $A_T^*(\mathbb{P}^3)$-module by $\pi_*(1)$,$\pi_*(\zeta_1)$ and $\pi_*(\zeta_1\zeta_2)$. To compute $\pi_*(1)$ we can use Lemma 1.7.4 since $\pi_*(1) = [V(x_1x_4 - x_2x_3)]_T$. We obtain

$$\pi_*(1) = 2\zeta - (q-1)(t_1 + t_2).$$

From equation (1) we deduce

$$\begin{aligned} \pi_*(\zeta_1) = [V(x_1, x_2)]_T - t_1\pi_*(1) &= [V(x_1)]_T[V(x_2)]_T - t_1\pi_*(1) \\ &= (\zeta - \alpha_1)(\zeta - \alpha_2) - t_1\pi_*(1) \\ &= \zeta^2 - qc_1\zeta + (q^2 - 1)c_2, \end{aligned}$$

where $c_1 = t_1 + t_2$ and $c_2 = t_1 t_2$. Here $[V(x_i)] = \zeta - \alpha_i$ holds by Lemma 1.7.4. We are left to compute $\pi_*(\zeta_1\zeta_2)$. For this we deduce from equation (1)

$$\zeta_1\zeta_2 = [0 \times 0]_T - t_1(\zeta_2 - q\zeta_1) - qt_1^2$$

and thus

$$\begin{aligned} \pi_*(\zeta_1\zeta_2) &= [V(x_1, x_2, x_3)]_T - t_1(\pi_*(\zeta_2) - q\pi_*(\zeta_1)) - qt_1^2\pi_*(1) \\ &= (\zeta + \alpha_1)(\zeta + \alpha_2)(\zeta + \alpha_3) - t_1(\pi_*(\zeta_2) - q\pi_*(\zeta_1)) - qt_1^2\pi_*(1). \end{aligned}$$

Here we have used the equality $[V(x_1, x_2, x_3)]_T = [V(x_1)]_T[V(x_2)]_T[V(x_3)]_T$. We see that $\pi_*(\zeta_1\zeta_2)$ already lies in the $A_T^*(\mathbb{P}^3)$-module generated by $\pi_*(1)$ and $\pi_*(\zeta_1)$. Putting all this together we thus obtain

$$A_T^*(\mathrm{GL}_2) = \mathbb{Z}[t_1, t_2]/((q-1)c_1, (q^2-1)c_2, c_4(V)).$$

Recall $V = Mat(2 \times 2, k)$. Writing $c_4(V) = \alpha_1 \ldots \alpha_4$ as a polynomial in c_1 and c_2 we get $c_4(V) = (q-1)^2(q+1)^2c_2^2 - q(q-1)^2c_2c_1^2$. Hence we obtain

$$A_T^*(\mathrm{GL}_2) = \mathbb{Z}[t_1, t_2]/((q-1)c_1, (q^2-1)c_2)$$

$$A_{\mathrm{GL}_2}^*(\mathrm{GL}_2) = \mathbb{Z}[c_1, c_2]/((q-1)c_1, (q^2-1)c_2),$$

where the result for $A_{\mathrm{GL}_2}^*(\mathrm{GL}_2)$ follows from Corollary 1.9.14 since the image $\mathrm{Im}(A_*^T(V(x_1x_4 - x_2x_3)) \to A_*^T(V)) = ((q-1)c_1, (q^2-1)c_2)$ is generated by elements lying in $A_{\mathrm{GL}_2}^*(V) = A_{\mathrm{GL}_2}^*$.

Let us now treat the general case.

Proposition 3.0.7. *Let G be a connected split reductive group over $\mathbb{F}_q$ with split maximal torus T. We write $S = \mathrm{Sym}(\hat{T}) = A_T^*$ and $S_+ = A_T^{\geq 1}$. If σ denotes the q-th power Frobenius, we have a natural action of σ on S, that we will also denote by σ.*
Let $P \supset T$ be a parabolic subgroup with Levi component L and consider the action of L on G by σ-conjugation. If $W_G = W(G,T)$ and $W_L = W(L,T)$ denote the respective Weyl groups we have

$$A_L^*(G)_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/(S_+^{W_G}).$$

If G and L are both special we have

$$A_L^*(G) = S^{W_L}/(f - \sigma f \mid f \in S_+^{W_G}).$$

Before we proof this proposition we state two lemmata.

Lemma 3.0.8. *Let $A \to B$ be a faithfully flat ring homomorphism and I an ideal of A. Then $IB \cap A = I$.*

Proof. We have to see that $A/I \to B/IB$ is injective. But this map is the base change of $A \to B$ by $A \to A/I$, thus again faithfully flat and hence injective. □

Lemma 3.0.9. (i) *Let $R \subset S$ be an extension of rings. Assume there exists an R-linear surjective map $f\colon R^n \to S^n$ for some n. Then $R = S$.*
(ii) *If $R \subset S \subset T$ is an extension of rings such that T is a free module over S and R of the same finite rank, then $R = S$.*

Proof. (i) Taking the highest exterior power $\wedge_R^n(R^n)$ of R^n as an R-modul and the highest exterior power $\wedge_S^n(S^n)$ of S^n as an S-modul, the map f induces a surjective R-linear map $\wedge_R^n(R^n) \to \wedge_S^n(S^n)$. In other words $S = Rx$ for $x \in S^*$. In particular, we find $r \in R$ such that $x^2 = rx$. It follows $x = r \in R$ and hence $R = S$. Part (ii) follows from (i). □

Proof. (of Proposition 3.0.7.) Let us first consider the case that G and L are special. To compute $A^*_T(G)$ we consider the action of $T \times T$ on G given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Using the embedding $T \hookrightarrow T \times T, g \mapsto (g, \sigma(g))$ we get a morphism

$$[G/T] \to [G/(T \times T)]$$

which is a principal $(T \times T)/T$-bundle. But $(T \times T)/T \cong T$ via the map induced by $(g_1, g_2) \mapsto \sigma(g_1) g_2^{-1}$. Hence by Proposition 1.2.8 we have

$$A^*([G/T]) \cong A^*([G/(T \times T)])/\hat{T} A^*([G/(T \times T)]).$$

Thus we need to compute $A^*([G/(T \times T)])$. For this let B be a Borel subgroup of G such that $T \subset B \subset P$. By Lemma 1.5.7 we can then identify

$$A^*([G/(T \times T)]) = A^*([G/(B \times T)]) = A^*_T(G/B)$$

and since G is special we obtain from Proposition 1.6.6

$$A^*([G/(T \times T)]) = S \otimes_{S^{W_G}} S = (S \otimes_{\mathbb{Z}} S)/(1 \otimes f - f \otimes 1 \mid f \in S_+^{W_G})$$

Let $\chi \in \hat{T}$ be a character of T. Then χ acts on $A^*([G/(T \times T)]$ by multiplying with the element $\sigma\chi \otimes 1 - 1 \otimes \chi$ as follows from the definition of the isomorphism $(T \times T)/T \cong T$. Therefore

$$A^*_T(G) = S/(f - \sigma f \mid f \in S_+^{W_G}).$$

From this we deduce the result for $A^*_L(G)$ in the following way. Let us write I for the ideal $(f - \sigma f \mid f \in S_+^{W_G})$ in S^{W_G}. We remark that $L \cap B$ is a Borel subgroup of L containing T by [Bo, Proposition 14.12]. Consider the $L/(L \cap B)$-bundles $[\operatorname{Spec} k/(L \cap B)] \to [\operatorname{Spec} k/L]$ and $[G/(L \cap B)] \to [G/L]$. Since L is special we obtain from Proposition 1.9.8

$$A^*_L \otimes A^*(L/(L \cap B)) \cong A^*_{L \cap B} = A^*_T$$
$$A^*_L(G) \otimes A^*(L/(L \cap B)) \cong A^*_{L \cap B}(G) = A^*_T(G).$$

Since $A^*_L = S^{W_L}$ by Theorem 1.9.6 and since $A^*(L/(L \cap B))$ is a free abelian group of rank $|W_L|$, we deduce from the first equation that S is a free S^{W_L}-module of rank $|W_L|$. In particular, $S^{W_L} \hookrightarrow S$ is faithfully flat. It follows $IS \cap S^{W_L} = IS^{W_L}$ and that S/IS is a free S^{W_L}/IS^{W_L}-module of the same rank $|W_L|$. The second equation tells us that $A^*_T(G)$ is a free $A^*_L(G)$-module of rank $|W_L|$.
Therefore

$$S^{W_L}/IS^{W_L} \subset A^*_L(G) \subset A^*_T(G) = S/IS$$

and $A^*_T(G)$ is free over S^{W_L}/IS^{W_L} and over $A^*_L(G)$ of the same finite rank $|W_L|$. Hence $A^*_L(G) = S^{W_L}/IS^{W_L}$ by Lemma 3.0.9.
It remains to show $A^*_L(G)_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/(S_+^{W_G})$ in the non-special case. Using the same argumentation as in the special case we arrive at

$$A^*_T(G)_{\mathbb{Q}} = S_{\mathbb{Q}}/(f - \sigma f \mid f \in S_+^{W_G}).$$

It follows from [De, Theorem 3] that $S_{\mathbb{Q}}^{W_G}$ is generated as a $\mathbb{Q}$-algebra by $\dim T$ homogeneous algebraically independent elements. Hence $A_T^*(G)_{\mathbb{Q}} = S_{\mathbb{Q}}/(S_+^{W_G})$. Now by Theorem 1.9.6 we know $A_L^*(G)_{\mathbb{Q}} = A_T^*(G)_{\mathbb{Q}}^{W_L}$. Since $S_{\mathbb{Q}}^{W_L} \hookrightarrow S_{\mathbb{Q}}$ is finite free ([De, Theorem 2 (d)]) it is also faithfully flat. Hence by Lemma 3.0.8 we obtain $S_{\mathbb{Q}}^{W_L} \cap S_+^{W_G} S_{\mathbb{Q}} = S_+^{W_G} S_{\mathbb{Q}}^{W_L}$ and the assertion follows. □

Theorem 3.0.10. *Let k be a field containing $\mathbb{F}_q$. Then the following equations hold.*

$$CH^* B\,\mathrm{GL}_n(\mathbb{F}_q)_k = \mathbb{Z}[c_1, \ldots, c_n]/((q-1)c_1, \ldots, (q^n - 1)c_n)$$
$$CH^* B\,\mathrm{Sp}_{2m}(\mathbb{F}_q)_k = \mathbb{Z}[c_2, c_4, \ldots, c_{2m}]/((q^2-1)c_2, (q^4-1)c_4, \ldots, (q^{2m}-1)c_{2m}),$$
$$CH^* B\,\mathrm{SL}_n(\mathbb{F}_q)_k = \mathbb{Z}[c_2, c_3, \ldots, c_n]/((q^2-1)c_2, (q^3-1)c_3, \ldots, (q^n-1)c_n),$$

where the c_i are the i-th Chern classes of the canonical representation of the respective groups.

Proof. Let G be one of the groups GL_n, Sp_{2m} or SL_n. By Corollary 2.1.7 we have $BG(\mathbb{F}_q)_k = [G_k/(G_k]$, where the action is given by conjugation with the q-th power Frobenius. Moreover, G is special ([Se2, Section 4.4]) and therefore the theorem follows from the above proposition. (Recall that $S^{W_G} = A_G^*$ by Theorem 1.9.6 and A_G^* was computed in Example 1.2.12.) □

Theorem 3.0.11. *Let S be the product of p and the primes that divide $q-1$. Then the following equations hold.*

$$\begin{aligned} &CH^* B(\mathrm{GL}_n(\mathbb{F}_q)_{\mathbb{C}})_{2p} \\ &\qquad = \mathbb{Z}[(2p)^{-1}][c_1, \ldots, c_n]/((q-1)c_1, (q^2-1)c_2, \ldots, (q^n-1)c_n) \\ &CH^* B(\mathrm{Sp}_{2m}(\mathbb{F}_q)_{\mathbb{C}})_{2p} \\ &\qquad = \mathbb{Z}[(2p)^{-1}][c_2, c_4, \ldots, c_{2m}]/((q^2-1)c_2, (q^4-1)c_4, \ldots, (q^{2m}-1)c_{2m}), \\ &CH^* B(\mathrm{SL}_n(\mathbb{F}_q)_{\mathbb{C}})_S \\ &\qquad = \mathbb{Z}[S^{-1}][c_2, c_3, \ldots, c_n]/((q^2-1)c_2, (q^3-1)c_3, \ldots, (q^n-1)c_n), \end{aligned}$$

where c_i denotes the i-th Chern class of the Brauer lift of the canonical representation of the respective groups.
If $q \equiv 1 \bmod 4$ it suffices to invert p in the first and second equation.

Proof. If G is one of the groups GL_n, Sp_{2m} or SL_n the previous theorem shows that the Chow ring of $BG(\mathbb{F}_q)_k$ is generated by Chern classes of the canonical representations of $G(\mathbb{F}_q)_k$, where k is any field containing $\mathbb{F}_q$. The theorem thus follows from Proposition 2.4.3 and Proposition 2.1.3 (i). □

4 The Chow Ring of the Stack of level-n Barsotti-Tate Groups

4.1 The Stack of truncated Displays

Let R be an $\mathbb{F}_p$-algebra. We denote by $W_n(R)$ the ring of truncated Witt vectors of length n. Let $I_{n,R} \subset W_n(R)$ be the image of the Verschiebung $W_{n-1}(R) \to W_n(R)$ and $J_{n,R} \subset W_n(R)$ be the kernel of the projection $W_n(R) \to W_{n-1}(R)$. The Frobenius on R induces a ring homomorphism $\sigma\colon W_n(R) \to W_n(R)$ and the inverse of the Verschiebung induces a bijective σ-linear map $\sigma_1\colon I_{n+1,R} \to W_n(R)$. Note that $pR = 0$ implies $I_{n,R}J_{n,R} = 0$, hence we may view $I_{n+1,R}$ as a $W_n(R)$-module.
Truncated displays were introduced in [La]. Let us recall the necessary notations. For now we are only going to need the following description of truncated displays.

Definition 4.1.1. *A truncated display of level n over an $\mathbb{F}_p$-algebra R is a triple (L, T, Ψ) consisting of projective $W_n(R)$-modules L and T of finite rank and a σ-linear automorphism $\Psi\colon L \oplus T \to L \oplus T$.*

A morphism between truncated displays is defined as follows. First we can use Ψ to define σ-linear maps

$$F\colon L \oplus T \to L \oplus T, \quad l + t \mapsto p\Psi(l) + \Psi(t),$$

$$F_1\colon L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) \to L \oplus T, \quad l + (t \otimes \omega) \mapsto \Psi(l) + \sigma_1(\omega)\Psi(t).$$

Then a morphism between two truncated displays (L, T, Ψ) and (L', T', Ψ') of level n is given by a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \operatorname{Hom}(L, L')$, $B \in \operatorname{Hom}(T, L')$, $C \in \operatorname{Hom}(L, T' \otimes_{W_n(R)} I_{n+1,R})$ and $D \in \operatorname{Hom}(T, T')$ such that

$$\begin{array}{ccc} L \oplus T & \xrightarrow{F} & L \oplus T \\ \downarrow & & \downarrow \\ L' \oplus T' & \xrightarrow{F'} & L' \oplus T' \end{array} \qquad \begin{array}{ccc} L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F_1} & L \oplus T \\ \downarrow & & \downarrow \\ L' \oplus (T' \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F_1'} & L' \oplus T' \end{array}$$

commute.
The height of a truncated display is defined as the rank of $L \oplus T$ and the dimension as the rank of the projective R-module $T/I_{n,R}T$. Both are locally constant functions on $\operatorname{Spec} R$.
Let $\mathcal{D}\mathrm{isp}_n \to \operatorname{Spec} \mathbb{F}_p$ denote the stack of truncated displays of level n. That is for R an $\mathbb{F}_p$-algebra $\mathcal{D}\mathrm{isp}_n(\operatorname{Spec} R)$ is the groupoid of truncated displays of level n. It is proved in [La, Proposition 3.15] that $\mathcal{D}\mathrm{isp}_n$ is a smooth Artin algebraic stack of dimension zero over $\mathbb{F}_p$ with affine diagonal.
For $h \in \mathbb{N}$ and $0 \leq d \leq h$ we denote by $\mathcal{D}\mathrm{isp}_n^{h,d}$ the open and closed substack

of truncated displays of level n with constant height h and constant dimension d. Then

$$\mathcal{D}\mathrm{isp}_n = \coprod_{h,d} \mathcal{D}\mathrm{isp}_n^{h,d}.$$

By the lemma below it suffices to compute the Chow ring of $\mathcal{D}\mathrm{isp}_n^{h,d}$.

Lemma 4.1.2. *Let $\mathscr{X}_i, i \in I$ be a family of stacks. Then $A^*(\coprod_i \mathscr{X}_i) = \prod_i A^*(\mathscr{X}_i)$.*

Proof. We write $\mathscr{X} = \coprod_i \mathscr{X}_i$. Let T be a scheme and $T \to \mathscr{X}$ a morphism. This morphism defines a decomposition $\coprod_i T_i$ of T into open and closed subschemes. Note that only finitely many T_i are non-empty since T is of finite type. We know $A_*(T) = \bigoplus_i A_*(T_i)$. Thus we get a natural map

$$\prod_i A^*(\mathscr{X}_i) \to A^*(\mathscr{X}).$$

To see that this defines an isomorphism, it suffices to note that for $c \in A^*(\mathscr{X})$ the map $c(T \to \mathscr{X})$ sends $A_*(T_i)$ to $A_*(T_i)$. This follows since by definition of the operational Chow ring the maps $c(T \to \mathscr{X})$ and $c(T_i \to \mathscr{X})$ are compatible with the push-forward of $T_i \hookrightarrow T$. □

A Presentation of $\mathcal{D}isp_n^{h,d}$. We will adopt the notation of the proof of Proposition 3.15 in [La]. Let $X_n^{h,d}$ be the functor on affine $\mathbb{F}_p$-schemes with $X_n^{h,d}(R) = \mathrm{GL}_h(W_n(R))$. This is an affine open subscheme of $\mathbb{A}^{nh^2}$. Furthermore, let $G_n^{h,d}$ be the functor such that $G_n^{h,d}(R)$ is the group of invertible matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{GL}_{h-d}(W_n(R)), B \in \mathrm{Hom}(W_n(R)^d, W_n(R)^{h-d}), C \in \mathrm{Hom}(W_n(R)^{h-d}, I_{n+1,R}^d)$ and $T \in \mathrm{GL}_d(W_n(R))$. Then $G_n^{h,d}$ is a connected algebraic group of dimension nh^2.

Remark 4.1.3. Since $I_{2,R}$ is in bijection to R via σ_1 we may view $G_1^{h,d}(R)$ as the group of invertible matrices with entries in R with respect to the multiplication given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BD' \\ C\sigma(A') + \sigma(D)C' & DD' \end{pmatrix},$$

where in the four blocks we have the usual matrix multiplication.

Let $\pi_n^{h,d} \colon X_n^{h,d} \to \mathcal{D}\mathrm{isp}_{n,d}$ be the functor that assigns to an invertible matrix $\Psi \in \mathrm{GL}_h(W_n(R))$ the truncated display $(W_n(R)^{h-d}, W_n(R)^d, \Psi)$, where we view Ψ as a σ-linear map $W_n(R)^h \to W_n(R)^h$ via $x \mapsto \Psi \cdot \sigma x$. Now if we let $G_n^{h,d}$ act on $X_n^{h,d}$ via

$$\Psi \cdot G = G^{-1} \Psi \sigma_1(G)$$

where $\sigma_1(G) = \begin{pmatrix} \sigma(A) & p\sigma(B) \\ \sigma_1(C) & \sigma(D) \end{pmatrix}$, then every $G \in G_n^{h,d}$ defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(G \cdot \Psi)$ of truncated displays. On the contrary if G defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(\Psi')$ then necessarily $\Psi' = G^{-1}\Psi\sigma_1(G)$. We thus obtain

Theorem 4.1.4. *The functor $\pi_n^{h,d}$ induces an isomorphism of stacks*

$$[X_n^{h,d}/G_n^{h,d}] \cong \mathcal{D}isp_n^{h,d}.$$

There are the following two obvious vector bundles on $\mathcal{D}\mathrm{isp}_n^{h,d}$.

Definition 4.1.5. *Let* $\operatorname{Spec} R \to \mathcal{D}isp_n^{h,d}$ *be a map corresponding to a truncated display* $\mathcal{P} = (L, T, \Psi)$.

(i) *We denote by $\mathcal{L}ie$ the vector bundle of rank d over $\mathcal{D}isp_n^{h,d}$ that assigns to* $\operatorname{Spec} R \to \mathcal{D}isp_n^{h,d}$ *the vector bundle* $\mathrm{Lie}(\mathcal{P}) = T/I_{n,R}T$ *of rank d over R.*

(ii) *By ${}^t\mathcal{L}ie^\vee$ we denote the vector bundle of rank $h-d$ that assigns to* $\operatorname{Spec} R \to \mathcal{D}isp_n^{h,d}$ *the vector bundle $L/I_{n,R}L$ of rank $h-d$ over R.*

Remark 4.1.6. The notation ${}^t\mathcal{L}ie^\vee$ in the above definition stems from the fact that the dual of $L/I_{n,R}L$ gives the Lie algebra of the dual display $\mathcal{P}^t$. For the definition of the dual display see [Zi, Definition 19].

The Truncated Display Functor. As already mentioned in the introduction the strategy for computing the Chow ring of the stack of truncated Barsotti-Tate groups is to relate it to the stack of truncated displays. This happens via the truncated display functor

$$\phi_n \colon BT_n \to \mathcal{D}\mathrm{isp}_n$$

constructed in [La]. Let us briefly sketch the construction.
Let G be a p-divisible group over an $\mathbb{F}_p$-algebra R. The Witt ring $W(R)$ is p-adically complete and the ideal I_R in $W(R)$ carries natural divided powers compatible with the canonical divided powers of p. Let $\mathbb{D}(G)$ denote the covariant Dieudonne crystal of G. We can evaluate $\mathbb{D}(G)$ at $W(R) \to R$ and set $P = \mathbb{D}(G)_{W(R)\to R}$ and $Q = \mathrm{Ker}(P \to \mathrm{Lie}(G))$. Furthermore, let $F^\sharp \colon P^\sigma \to P$ and $V^\sharp \colon P \to P^\sigma$ be the maps induced by Frobenius and Verschiebung of G. One can now show that there are σ-linear maps $F \colon P \to P$ resp. $\dot{F} \colon Q \to P$ compatible with base change in R such that $(P, Q, F, \dot{F})$ is a display which induces the maps $F^\sharp$ and $V^\sharp$. See [La, Proposition 2.4] for the precise statement. This construction yields a 1-morphism

$$\phi \colon BT \to \mathcal{D}\mathrm{isp}$$

from the stack of Barsotti-Tate groups to the stack of displays. It is clear from the construction that the Lie algebra of G is equal to the Lie algebra of $\phi(G)$ defined by P/Q.
Moreover, one can prove that for all n there are maps $\phi_n \colon BT_n \to \mathcal{D}\mathrm{isp}_n$ compatible with the truncation maps on both sides such that ϕ is the projective limit of the system $(\phi_n)_{n\geq 1}$. The following theorem is the central result in [La].

Theorem 4.1.7. *ϕ_n is a smooth morphisms of smooth algebraic stacks over $\mathbb{F}_p$ which is an equivalence on geometric points.*

4.2 Grouptheoretic Properties of $G_n^{h,d}$

We denote by $K_{(n,m)}^{h,d}$ the kernel of the projection $G_n^{h,d} \to G_m^{h,d}$ for $m < n$ and by $\tilde{K}_n^{h,d}$ the kernel of the projection $G_n^{h,d} \to \mathrm{GL}_{h-d} \times \mathrm{GL}_d$. Note $G_n^{h,0} = \mathrm{GL}_h(W_n(\cdot))$. We recall the following well known facts about the Witt ring. For an $\mathbb{F}_p$-algebra R we denote by $[\cdot]\colon R \to W_n(R)$ the map $r \mapsto (r, 0, \ldots, 0)$ and ${}^V(\cdot)\colon W(R) \to W(R)$ is the Verschiebung.

Lemma 4.2.1. *Let R be an $\mathbb{F}_p$-algebra and $x, y \in R$. Then $[x+y]-[x]-[y]$ lies in ${}^V W(R)$. Furthermore, ${}^{V^r}W(R) \cdot {}^{V^s}W(R) \subset {}^{V^{r+s}}W(R)$ holds.*

Proof. The first part follows immediately from the fact that ${}^V W(R)$ is the kernel of the ring homomorphism $\mathbb{W}_0\colon W(R) \to R$ and the fact that $\mathbb{W}_0([x]) = x$ holds for all $x \in R$.
For the second part we may assume $r \geq s$. We then write ${}^{V^r}x {}^{V^s}y = {}^{V^r}(x^{F^r V^s}y) = p^s \cdot {}^{V^r}(x^{F^{r-s}}y)$. Since $pR = 0$ we have the equality $p(x_0, x_1, \ldots) = (0, x_0^p, x_1^p, \ldots)$ in $W(R)$ and the lemma follows. □

Lemma 4.2.2. (i) *$K_{(n,m)}^{h,d}$ is unipotent.*
(ii) *$\tilde{K}_n^{h,d}$ is unipotent.*

Proof. (i) First note that $K_{(n,n-1)}^{h,0} = ker(\mathrm{GL}_h(W_n(\cdot)) \to \mathrm{GL}_h(W_{n-1}(\cdot)))$ is unipotent. To see this we consider the Verschiebung ${}^V(\cdot)$ as a map $W_n(R) \to W_n(R)$. Then by the above lemma the map

$$\mathbb{G}_a^{h^2} \to K_{(n,n-1)}^{h,0}, \quad A \mapsto I_h + {}^{V^{n-1}}[A]$$

is an isomorphism of algebraic groups.
Next we show that $K_{h,d}^{(n,n-1)}$ is unipotent. This is the group of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in K_{(n,n-1)}^{h-d,0}$, $B \in J_n^{(h-d)\times d}$, $C \in J_{n+1}^{d\times(h-d)}$ and $D \in K_{(n,n-1)}^{d,0}$. The multiplication in this group is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BD' \\ CA' + DC' & DD' \end{pmatrix}$$

Starting with the normal subgroup $\begin{pmatrix} I_{h-d} & J_n^{(h-d)\times d} \\ J_{n+1}^{d\times(h-d)} & I_d \end{pmatrix}$, which is isomorphic to $\mathbb{G}_a^{2d(h-d)}$, and then using the fact that $K_{(n,n-1)}^{h-d,0}$ resp. $K_{(n,n-1)}^{d,0}$ are isomorphic to $\mathbb{G}_a^{(h-d)^2}$ resp. $\mathbb{G}_a^{d^2}$ one obtains a filtration of $K_{(n,n-1)}^{h,d}$ by normal subgroups, whose successive quotients are isomorphic to a product of copies of $\mathbb{G}_a$. Now we have an exact sequence

$$0 \longrightarrow K_{(n,n-1)}^{h,d} \longrightarrow K_{(n,m)}^{h,d} \longrightarrow K_{(n-1,m)}^{h,d} \longrightarrow 0$$

and by induction we may assume that $K^{h,d}_{(n-1,m)}$ is unipotent. It follows that $K^{h,d}_{(n,m)}$ is unipotent.
(ii) For $n = 1$ the assertion is obvious in view of Remark 4.1.3. For $n > 1$ we use the exact sequence

$$0 \longrightarrow K^{h,d}_{(n,n-1)} \longrightarrow \tilde{K}^{h,d}_{n} \longrightarrow \tilde{K}^{h,d}_{n-1} \longrightarrow 0.$$

By induction and part (i) it follows that $\tilde{K}^{h,d}_{n}$ is unipotent. □

Corollary 4.2.3. (i) *$G^{h,d}_n$ is special.*
(ii) *$\tilde{K}^{h,d}_n$ is the unipotent radical of $G^{h,d}_n$.*
(iii) *The projection $X^{h,d}_n \to X^{h,d}_1$ is a trivial $K^{h,0}_{(n,1)}$-torsor.*

Proof. We have the exact sequence

$$0 \longrightarrow \tilde{K}^{h,d}_{n} \longrightarrow G^{h,d}_{n} \longrightarrow \mathrm{GL}_{h-d} \times \mathrm{GL}_{d} \longrightarrow 0.$$

Now $\tilde{K}^{h,d}_n$ is unipotent, thus special. Since $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ is also special part (i) follows.
Clearly the projection $X^{h,d}_n \to X^{h,d}_1$ is a $K^{h,0}_{(n,1)}$-torsor by definition of $K^{h,0}_{(n,1)}$. It is trivial since $K^{h,0}_{(n,1)}$ is unipotent and $X^{h,d}_1$ is affine. □

4.3 The Chow Ring of $\mathcal{D}\mathrm{isp}_n$

We start with the following theorem which reduces the calculation of $A^*(\mathcal{D}\mathrm{isp}_n)$ to the case $n = 1$.

Theorem 4.3.1. *The pull-back*

$$\tau_n^* \colon A^*(\mathcal{D}isp^{h,d}_1) \to A^*(\mathcal{D}isp^{h,d}_n)$$

of the truncation $\tau_n \colon \mathcal{D}isp^{h,d}_n \to \mathcal{D}isp^{h,d}_1$ is an isomorphism.

Proof. Under the presentation $\mathcal{D}\mathrm{isp}^{h,d}_n = [X^{h,d}_n/G^{h,d}_n]$ the truncation τ_n is induced by the natural projections $X^{h,d}_n \to X^{h,d}_1$ and $G^{h,d}_n \to G^{h,d}_1$. Thus τ_n factors as

$$[X^{h,d}_n/G^{h,d}_n] \to [X^{h,d}_1/G^{h,d}_n] \to [X^{h,d}_1/G^{h,d}_1].$$

The pull-back of the second map is an isomorphism by Lemma 4.2.2 and Corollary 1.5.2. To show that the pull-back of the first map is also an isomorphism let us abbreviate $X = X^{h,d}_1$ and $G = G^{h,d}_n$. By part (iii) of Corollary 4.2.3 we know $X^{h,d}_n = X \times K$ with $K = K^{h,0}_{(n,1)}$ and the projection $X \times K \to X$ is G-equivariant. Moreover, K is an affine space by Lemma 4.2.2. After replacing $[X/G]$ by an appropiate mixed space (cf. Convention 1.2.1) we may assume that $[X/G]$ is a scheme. We claim that $(X \times K)/G \to X/G$ is a Zariksi locally-trivial affine bundle. Since G is special by part (i) of Corollary 4.2.3 the principal G-bundle $X \to X/G$ is locally trivial for the Zariski topology and after replacing X/G by an appropiate open subset we may assume $X = G \times (X/G)$. We then have an isomorphism $(G \times (X/G) \times K)/G \cong (X/G) \times K$ given by the assignment $(g, x, k) \mapsto (x, k')$, where k' is defined by $g^{-1}(g, x, k) = (1, x, k')$. This proves the claim and hence the pull-back of the first map is also an isomorphism. □

When $d = h$ or $d = 0$ the group $G_1^{h,d}$ is just GL_h and the action on $X_1^{h,d}$ is the usual σ-conjugation. If $0 < d < h$ we have that $G_1^{h,d}$ is a split extension of the group $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ by the unipotent group $\{\begin{pmatrix} E_{h-d} & * \\ * & E_d \end{pmatrix}\}$, where $*$ denotes an arbitrary matrix with entries in R (cf. Remark 4.1.3). The splitting is given by the canonical inclusion $\mathrm{GL}_{h-d} \times \mathrm{GL}_d \hookrightarrow G_1^{h,d}$.

The case $h = 2$. Here the basic calculation of Example 3.0.6 already yields the following result.

Theorem 4.3.2.

$$A^*(\mathcal{D}isp_1^{2,1}) = \mathbb{Z}[t_1, t_2]/((p-1)c_1, (p^2-1)c_2)$$
$$A^*(\mathcal{D}isp_1^{2,0}) = A^*(\mathcal{D}isp_1^{2,2}) = \mathbb{Z}[c_1, c_2]/((p-1)c_1, (p^2-1)c_2),$$

where $c_1 = t_1 + t_2$ and $c_2 = t_1 t_2$.

Proof. By Lemma 1.2.10 and Proposition 1.2.6 we have

$$A^*(\mathcal{D}\mathrm{isp}_1^{2,1}) = A^*_{G_1^{2,1}}(X_1^{2,1}) = A^*_T(X_1^{2,1}),$$

where T is the torus of diagonal matrices in GL_2. Now the last ring is just the equivariant Chow ring of T acting on GL_2 via σ-conjugation and this case was done in Example 3.0.6. □

Corollary 4.3.3.

$$\mathrm{Pic}(\mathcal{D}isp_1^{2,1}) = \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$$
$$\mathrm{Pic}(\mathcal{D}isp_1^{2,0}) = \mathrm{Pic}(\mathcal{D}isp_1^{2,2}) = \mathbb{Z}/(p-1)\mathbb{Z}$$

Proof. Recall $\mathrm{Pic}\,\mathcal{D}\mathrm{isp}_n^{h,d} = A^1\mathcal{D}\mathrm{isp}_n^{h,d}$ by Proposition 1.2.6. □

Remark 4.3.4. There is also a more direct approach to compute the above Picard groups. By using a theorem of Rosenlicht, namely that for irreducible varieties X and Y the natural map $\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$ is surjective, it is not difficult to establish the following exact sequence

$$\mathcal{O}(X)^*/k^* \longrightarrow \hat{G} \longrightarrow \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}(X)$$

for G connected and X an irreducible G-scheme. The first map assigns to a non-vanishing regular function on X its eigenvalue. In our case we have $G = T = \mathbb{G}_m^2$ and $X = \mathrm{GL}_2$. Then $\mathcal{O}(\mathrm{GL}_2)^*/k^* = \mathbb{Z}$ with generator given by the determinant and eigenvalue given by the character $(p-1)(t_1 + t_2) \in \hat{T}$. Since $\mathrm{Pic}(\mathrm{GL}_2) = 0$ we again obtain $\mathrm{Pic}^T(\mathrm{GL}_2) = \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$.

Let us look for generators of the free resp. torsion part of the Picard group. Using the representation of $\mathcal{D}\mathrm{isp}_1^{2,1}$ as a quotient stack it is easy to see that $c_1({}^t\mathcal{L}ie^\vee) = t_1$ and $c_1(\mathcal{L}ie) = t_2$ hold in $A^*(\mathcal{D}\mathrm{isp}_1^{2,1})$, where $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$ are the vector bundles of Definition 4.1.5. See also Theorem 4.3.5 below. Thus we see that $\mathcal{L}ie$ is a generator for the free part and $\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee$ for the torsion part. In particular, we obtain that $(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)^{p-1}$ is trivial. This can also be seen directly as follows: $(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)^{p-1}$ being trivial means that $\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee$ is fixed under the pull-back of the Frobenius map $Frob\colon \mathcal{D}\mathrm{isp}_1^{2,1} \to \mathcal{D}\mathrm{isp}_1^{2,1}$ assigning to a display $\mathcal{P}$ over an $\mathbb{F}_p$-algebra R the display $\mathcal{P}^\sigma$ obtained by base change via the Frobenius $\sigma\colon R \to R$. But by definition of a truncated display we have an isomorphism $\Psi\colon L \oplus T \cong L^\sigma \oplus T^\sigma$ of R-modules. Let us write Ψ as a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The mapping $l \otimes t \mapsto A(l) \otimes D(t) - C(l) \otimes B(t)$ then yields the desired isomorphism $L \otimes T \cong L^\sigma \otimes T^\sigma$.

Let us put this result into context by relating it to the corresponding result for elliptic curves. Let $\mathcal{M}_{1,1} \to \operatorname{Spec} k$ denote the moduli stack of elliptic curves. A morphism $\operatorname{Spec} R \to \mathcal{M}_{1,1}$ corresponds to a pair $(C \to \operatorname{Spec} R, \sigma)$ where $C \to \operatorname{Spec} R$ is a smooth projective curve of genus 1 and $\sigma\colon \operatorname{Spec} R \to C$ is a smooth section. We now have the following diagram

$$\begin{array}{ccccc} \mathcal{M}_{1,1} & \longrightarrow & BT^{h=2,d=1} & \xrightarrow{\phi} & \mathcal{D}\mathrm{isp}^{h=2,d=1} \\ & & \downarrow & & \downarrow \\ & & BT^{h=2,d=1}_{n=1} & \xrightarrow[\phi_1]{} & \mathcal{D}\mathrm{isp}^{h=2,d=1}_{n=1} \end{array}$$

Let us consider the pull-back map $A^*(\mathcal{D}\mathrm{isp}_1^{2,1}) \to A^*(\mathcal{M}_{1,1})$. In characteristic p different from 2 and 3 Edidin and Graham computed $A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$, where t is given by the first Chern class of the Hodge bundle on $\mathcal{M}_{1,1}$ ([EG, Proposition 21]).
By construction of the truncated display functor the pull-back of $\mathcal{L}ie$ to $\mathcal{M}_{1,1}$ is the dual of the Hodge bundle on $\mathcal{M}_{1,1}$. Since the dual of an elliptic curve is the elliptic curve it follows from Remark 4.1.6 that the pull-back of ${}^t\mathcal{L}ie^\vee$ is given by the Hodge bundle. Hence $A^*(\mathcal{D}\mathrm{isp}_1^{2,1}) \to A^*(\mathcal{M}_{1,1})$ is the map

$$\mathbb{Z}[t_1, t_2]/((p-1)c_1, (p^2-1)c_2) \to \mathbb{Z}[t]/(12t)$$

that sends t_1 to $-t$ and t_2 to t. Note that $p^2 - 1$ is divisible by 12 (in fact even by 24) since we assume that p is different from 2 and 3. In particular, there can be no such map for $p = 2, 3$ so that the description $A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$ does not hold in characteristic 2 and 3.

The general case. As in the case $h = 2$ we are reduced to consider the action of $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ on GL_h by σ-conjugation. In the following we will write c_i for the i-th elementary symmetric polynomial in the variables $t_1, \dots, t_h$ and $c_i^{(j,k)}$ will denote the i-th elementary symmetric polynomial in the variables $t_j, \dots, t_k$, where $1 \leq j < k \leq h$ and $1 \leq i \leq k - j + 1$. We recall that $\mathbb{Z}[t_1, \dots, t_n]^{S_{h-d} \times S_d} = \mathbb{Z}[c_1^{(1,h-d)}, \dots, c_{h-d}^{(1,h-d)}, c_1^{(h-d+1,h)}, \dots, c_d^{(h-d+1,h)}]$.

Theorem 4.3.5.

$$\begin{aligned} A^*(\mathcal{D}isp_1^{h,d}) &= A^*_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}(\mathrm{GL}_h) \\ &= \mathbb{Z}[t_1, \dots, t_n]^{S_{h-d} \times S_d}/((p-1)c_1, \dots, (p^h - 1)c_h), \end{aligned}$$

where the $c_i^{(1,h-d)}$ *resp.* $c_i^{(h-d+1,h)}$ *are the Chern classes of* ${}^t\mathcal{L}ie^\vee$ *resp.* $\mathcal{L}ie$.

Proof. By Lemma 1.2.10 we know $A^*(\mathcal{D}\mathrm{isp}_1^{h,d}) = A^*_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}(\mathrm{GL}_h)$, where the action of $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ on GL_h is given by σ-conjugation. Since $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ is special with Weyl group $S_{h-d} \times S_d$ we obtain from Proposition 3.0.7

$$A^*_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}(\mathrm{GL}_h) = \mathbb{Z}[t_1, \dots, t_n]^{S_{h-d} \times S_d}/((p-1)c_1, \dots, (p^h - 1)c_h).$$

The assertion that the $c_i^{(1,h-d)}$ resp. $c_i^{(h-d+1,h)}$ are the Chern classes of $\mathcal{L}ie$ resp. ${}^t\mathcal{L}ie^\vee$ follows from the following simple fact. Let us write $\mathcal{E}_d$ resp. $\mathcal{E}_{h-d}$ for the vector bundle over $[*/\,\mathrm{GL}_d]$ resp. $[*/\,\mathrm{GL}_{h-d}]$ that corresponds to the canonical representation of GL_d resp. GL_{h-d}. Then $\mathcal{L}ie$ is the pull-back of $\mathcal{E}_d$ under the natural map

$$\mathcal{D}\mathrm{isp}_1^{h,d} = [GL_h/G_1^{h,d}] \longrightarrow [*/(\mathrm{GL}_d \times GL_{h-d})] \longrightarrow [*/GL_d]$$

and similary for ${}^t\mathcal{L}ie^\vee$. □

Corollary 4.3.6.

$$\mathrm{Pic}(\mathcal{D}isp_1^{h,d}) = \begin{cases} \mathbb{Z}/(p-1)\mathbb{Z} & \textit{if } d = 0, h \\ \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} & \textit{else.} \end{cases}$$

A generator for the free resp. torsion part is $\det(\mathcal{L}ie)$ *resp.* $\det(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)$.

4.4 The Chow Ring of the Stack of G-Zips

Let us first consider the case of F-zips introduced in [MW]. We denote by F-zip the stack of F-zips over a field k of characteristic $p > 0$ that is for S a k-scheme F-zip(S) is the groupoid of F-zips over S. If $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ is a function with finite support we denote by F-zip$^\tau$ the open and closed substack of F-zips of type τ. Note that

$$\text{F-zip} = \coprod_\tau \text{F-zip}^\tau\,.$$

The stacks F-zip$^\tau$ are smooth Artin algebraic stacks over k which follows for example from the following representation as a quotient stack. Let X_τ denote the k-scheme whose S-valued points are given by

$$X_\tau(S) = \{\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \mid \underline{M} \text{ F-zip of type } \tau, M = \mathcal{O}_S^h\}.$$

This is a smooth scheme of dimension h^2. Here $h = \sum_{i\in\mathbb{Z}} \tau(i)$ is also called the height of $\underline{M}$. The group GL_h acts on X_τ by

$$G \cdot \underline{M} = (\mathcal{O}_S^h, G(C^\bullet), G(D_\bullet), G\varphi_\bullet(G^{-1})^\sigma).$$

It is easy to see that two F-zips over S of the above form are isomorphic if and only if they lie in the same $\mathrm{GL}_h(S)$-orbit. Thus

$$\text{F-zip}^\tau = [X_\tau / \mathrm{GL}_h].$$

An F-zip $\underline{M}$ over an $\mathbb{F}_p$-algebra R of type τ with support lying in $\{0, 1\}$ is just a tuple

$$\underline{M} = (M, C, D, \varphi_0, \varphi_1),$$

where M is a projective R-module with submodules C and D, which are direct summands of M and isomorphisms

$$\varphi_0 \colon C^\sigma \to M/D, \quad \varphi_1 \colon (M/C)^\sigma \to D.$$

Lemma 4.4.1. *Let R be an $\mathbb{F}_p$-algebra. Then we have an equivalence of categories*

$$\mathcal{D}isp_1(R) \to \coprod_{\tau, \mathrm{Supp}(\tau)\in\{0,1\}} \text{F-zip}^\tau(R)$$

given in the following way

$$(L, T, \Psi) \mapsto (L \oplus T, T, \Psi^\sigma(L^\sigma), \Psi^\sigma\mid_{T^\sigma}, \Psi^\sigma\mid_{L^\sigma}).$$

The above assignment commutes with pulling back. In particular, we get an isomorphism of stacks

$$\text{F-zip}^\tau \cong \mathcal{D}isp_1^{\tau(0)+\tau(1),\tau(1)}$$

for every type τ with support lying in $\{0, 1\}$.

Proof. An inverse functor is given by the assignment

$$(M, C, D, \varphi_0, \varphi_1) \mapsto (C, M/C, \varphi_0 \oplus \varphi_1).$$

□

There is more generally the stack of G-zips introduced in [PWZ]. Here G refers to an arbitrary reductive group. It is defined as follows. Let $\mathcal{Z}$ be an algebraic zip datum that is a 4-tupel (G, P, Q, φ) consisting of a reductive group G, parabolic subgroups P and Q and an isogeny $\varphi\colon P/R_u(P) \to Q/R_u(Q)$. To $\mathcal{Z}$ one associates the group

$$E_{\mathcal{Z}} = \{(p, q) \in P \times Q \mid \varphi(\pi_P(p)) = \pi_Q(q)\}.$$

Now $E_{\mathcal{Z}}$ acts on G by the rule

$$((p, q), g) \mapsto pgq^{-1}$$

and the quotient stack $[G/E_{\mathcal{Z}}]$ is called the stack of G-zips. If G is connected $\mathcal{Z}$ is called a connected zip datum ([PWZ, Definition 3.1]).
Let us recall how the stack of F-zips is just a special case of this construction. For this let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support, say $i_1 \leq \ldots \leq i_r$. If we denote $n_k = \tau(i_k)$, then $(n_1, \ldots, n_r)$ defines a partition of $h = \sum_k n_k$. We denote the standard parabolic of type $(n_1, \ldots, n_r)$ in GL_h by P_τ.

Lemma 4.4.2. *Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support and $\mathcal{Z} = (\mathrm{GL}_h, P_\tau, P_\tau^-, \phi)$ be the algebraic zip datum with P_τ^- the opposite parabolic of P_τ and ϕ the Frobenius isogeny. Then there is an isomorphism of stacks*

$$[\mathrm{GL}_h / E_{\mathcal{Z}}] \xrightarrow{\sim} \text{F-zip}^\tau .$$

Proof. Let S be an k-scheme. We denote by $C_\tau^\bullet$ the descending filtration

$$C_\tau^\bullet = \mathcal{O}_S^h \supset \mathcal{O}_S^{n_1+\ldots+n_{r-1}} \supset \ldots \supset \mathcal{O}_S^{n_1} \supset 0$$

in $\mathcal{O}_S^h$ given by the standard flag of type $(n_1, \ldots, n_r)$ and by $D_\bullet^{\tau^-}$ the ascending filtration

$$D_\bullet^{\tau^-} = 0 \subset \mathcal{O}_S^{n_r} \subset \ldots \subset \mathcal{O}_S^{n_r+\ldots+n_2} \subset \mathcal{O}_S^h.$$

given by the flag of type opposite to $(n_1, \ldots, n_r)$. To $g \in \mathrm{GL}_h(S)$ we assign the F-zip

$$\underline{M}_g = (\mathcal{O}_S^h, C_\tau^\bullet, g(D_\bullet^{\tau^-}), \varphi_\bullet),$$

where φ is given by the restriction of g to the succesive quotients of $C_\tau^\bullet$. Note that we can consider g as a σ-linear map.
If (p, q) is an element of $E_{\mathcal{Z}}$ we get an isomorphim $M_g \to M_{pgq^{-1}}$ of F-zips induced by p. The fact that p commutes with the φ_i is exactly the condition $\phi(\pi(p)) = \pi(q)$.
On the other hand if an isomorphism $p\colon M_g \to M_{g'}$ of F-zips is given, we see that $g'^{-1}pg$ preserves the flag of type opposite to $(n_1, \ldots, n_r)$. Thus $q = g'^{-1}pg \in P_\tau^-$ and again the compatibility of p with the φ_i implies the condition $\phi(\pi(p)) = \pi(q)$. □

Theorem 4.4.3. *Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support $i_1 \leq \ldots \leq i_r$ and $n_k = \tau(i_k)$. Let $h = \sum_i n_i$ be its height. Then*

$$A^* \operatorname{F-zip}^\tau \cong A^*_{\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}}(\mathrm{GL}_h),$$

where $\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$ acts on GL_h by σ-conjugation. Therefore

$$A^* \operatorname{F-zip}^\tau = \mathbb{Z}[t_1, \ldots, t_h]^{S_{n_1} \times \ldots \times S_{n_r}} / ((p-1)c_1, \ldots, (p^h - 1)c_h)$$

with c_i the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$.

Proof. By the previous lemma we have

$$A^* \operatorname{F-zip}^\tau = A^*_{E_{\mathcal{Z}}}(\mathrm{GL}_h),$$

where $\mathcal{Z}$ is the algebraic zip datum $(\mathrm{GL}_h, P_\tau, P_\tau^-, \phi)$. By definition P_τ and P_τ^- have the same Levi component $L = \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_r}$. Therefore we have a split exact sequence

$$0 \longrightarrow R_u(P) \times R_u(P^-) \longrightarrow E_{\mathcal{Z}} \longrightarrow L \longrightarrow 0,$$

where the splitting is given by $L \hookrightarrow E_{\mathcal{Z}}$, $l \mapsto (l, \phi(l))$. Since the restriction of the action of $E_{\mathcal{Z}}$ to L is the usual Frobenius conjugation the theorem again follows from Lemma 1.2.10 and Proposition 3.0.7. □

Corollary 4.4.4.

$$\operatorname{Pic}(\operatorname{F-zip}^\tau) = \mathbb{Z}^{r-1} \times \mathbb{Z}/(p-1)\mathbb{Z}$$

We can also use Proposition 3.0.7 to say something about the Chow ring of the stack of G-zips for an arbitrary connceted Frobenius zip datum.

Definition 4.4.5. *We call a connected algebraic zip datum $\mathcal{Z} = (G, P, Q, \varphi)$ a Frobenius zip datum if it is defined over $\mathbb{F}_q$ with $q = p^a$, Q is the opposite parabolic of P and φ is the q-th power Frobenius σ^a.*
We call a Frobenius zip datum $\mathcal{Z} = (G, P, P^-, \sigma^a)$ special, if G is special and if P admits a special Levi component.

Theorem 4.4.6. *Let G be a connected split reductive group over $\mathbb{F}_q$ and $\mathcal{Z} = (G, P, P^-, \sigma^a)$ be a Frobenius zip datum. Let $W_G = W(G, T)$ be the Weyl group of G and $W_L = W(L, T)$ be the Weyl group of a Levi component L of P with respect to a split maximal torus $T \subset L$ of G. Writing $S = \operatorname{Sym}(\hat{T})$ one has*

$$A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L} / (S_+^{W_G}).$$

If $\mathcal{Z}$ is special we have

$$A^*([G/E_{\mathcal{Z}}]) = S^{W_L} / (f - \sigma^a f \mid f \in S_+^{W_G}).$$

Proof. Arguing as in the proof of Theorem 4.4.3 but using Lemma 1.5.7 we obtain

$$A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = A_L^*(G)_{\mathbb{Q}},$$

where the action of L on G is given by σ^a-conjugation. If G is special the above equality holds over $\mathbb{Z}$. We conclude by Proposition 3.0.7. □

Example 4.4.7. We consider the case $\mathcal{Z} = (\mathrm{Sp}(2n), P, P^-, \sigma^a)$. Recall that $\mathrm{Sp}(2n)$ is special and the Weyl group of $\mathrm{Sp}(2n)$ is the wreath product $S_n \wr (\mathbb{Z}/2\mathbb{Z}) = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. It acts on $\mathrm{Sym}(\hat{T}) = \mathbb{Z}[t_1, \dots, t_n]$ in the following way. S_n acts by permuting the variables $t_1, \dots, t_n$ and after identifying $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ an element $(\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}/2\mathbb{Z}^n$ acts by $(\varepsilon_1, \dots, \varepsilon_n) \cdot t_i = \varepsilon_i t_i$.
If P is the Borel we obtain from the above theorem

$$A^*([\mathrm{Sp}(2n)/E_{\mathcal{Z}}])) = \mathbb{Z}[t_1, \dots, t_n]/((q^2 - 1)c_1(\underline{t}^2), \dots, (q^{2n} - 1)c_n(\underline{t}^2)).$$

If P is the maximal parabolic subgroup fixing a maximal isotropic subspace then $L = \mathrm{GL}_n$ and $W_L = S_n$ and therefore

$$A^*([\mathrm{Sp}(2n)/E_{\mathcal{Z}}]) = \mathbb{Z}[c_1, \dots, c_n]/((q^2 - 1)c_1(\underline{t}^2), \dots, (q^{2n} - 1)c_n(\underline{t}^2)).$$

From the above description we can easily deduce the dimension of $A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}}$ as a $\mathbb{Q}$-vectorspace for a connected Frobenius zip datum $\mathcal{Z}$.

Corollary 4.4.8. *Let $\mathcal{Z} = (G, P, P^-, \sigma^a)$ be a connected Frobenius zip datum. Then $\dim_{\mathbb{Q}} A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = |W_G/W_L|$, where as usual $W_G = W(G,T)$ is the Weyl group of G and $W_L = W(L,T)$ is the Weyl group of a Levi component L of P.*

Proof. $S_{\mathbb{Q}}$ is free over $S_{\mathbb{Q}}^{W_G}$ resp. $S_{\mathbb{Q}}^{W_L}$ of rank $|W_G|$ resp. $|W_L|$ by [De, Theorem 2 (d)]. Since $S_{\mathbb{Q}}^{W_G} \hookrightarrow S_{\mathbb{Q}}^{W_L}$ is a finite map of polynomial rings it is also flat, hence $S_{\mathbb{Q}}^{W_L}$ is locally free over $S_{\mathbb{Q}}^{W_G}$ necessarily of rank $|W_G/W_L|$. In fact it follows from the famous theorem of Quillen-Suslin ([Qu3]) that $S_{\mathbb{Q}}^{W_L}$ is free over $S_{\mathbb{Q}}^{W_G}$ of rank $|W_G/W_L|$, but we will not need this. Since $A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L} \otimes_{S_{\mathbb{Q}}^{W_G}} (S_{\mathbb{Q}}^{W_G}/S_+^{W_G})$ the corollary follows. □

In the case of F-zips the above corollary reads as follows.

Corollary 4.4.9. *Let $\tau\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with support $i_1 < \ldots < i_r$ and set $n_k = \tau(i_k)$. Let $h = n_1 + \ldots + n_r$. Then*

$$\dim_{\mathbb{Q}} A^*(\text{F-zip}^\tau)_{\mathbb{Q}} = \frac{h!}{n_1! \cdot \ldots \cdot n_r!}.$$

It turns out that a $\mathbb{Q}$-basis of the Chow ring of the stack of G-zips is given by the closures of the orbits of the action of $E_{\mathcal{Z}}$ on G. To prove this let us introduce the naive Chow group of a quotient stack.

Definition 4.4.10. *Let G be an algebraic group and X be a G-scheme. Let $Z_*([X/G])$ be the free abelian group generated by the set of G-invariant subvarieties of X graded by dimension. Let $W_i([X/G])$ be the group $\bigoplus_Y k(Y)^G$, where the sum goes over all G-invariant subvarieties of X of dimension $i+1$. There is the usual divisor map $div\colon W_i([X/G]) \to Z_i([X/G])$ and we define the i-th naive Chow group of $[X/G]$ to be*

$$A_i^o[X/G] = Z_i([X/G])/div(W_i([X/G])).$$

Remark 4.4.11. There is more generally a definition of naive Chow groups for arbitrary algebraic stacks ([Kr, Definition 2.1.4]) which in the case of a quotient stack agrees with the one given above. Thus the above definition is independent of the presentation as a quotient stack.

Remark 4.4.12. There is a natural map $A_*^o[X/G] \to A_*[X/G]$. When X is Deligne-Mumford, i.e. the stabilizer of every point is finite and geometrically reduced, the induced map $A_*^o[X/G]_{\mathbb{Q}} \to A_*[X/G]_{\mathbb{Q}}$ is an isomorphism of groups and an isomorphism of rings if $[X/G]$ is smooth ([Kr, Theorem 2.1.12 (ii)]).

The stack of G-zips is not Deligne-Mumford. However, we still have the following proposition.

Proposition 4.4.13. *Let G be a connected algebraic group and X be an admissible G-scheme (cf. Definition 1.3.5) with finitely many orbits such that the stabilizer of every point is an extension of a finite group by a unipotent group. Then $A_*^o[X/G]_{\mathbb{Q}} \to A_*[X/G]_{\mathbb{Q}}$ is an isomorphism.*

Proof. We prove this by induction on the number of orbits. Let U denote the open G-orbit and W its complement. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_*^o[W/G]_{\mathbb{Q}} & \longrightarrow & A_*^o[X/G]_{\mathbb{Q}} & \longrightarrow & A_*^o[U/G]_{\mathbb{Q}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_*[W/G]_{\mathbb{Q}} & \longrightarrow & A_*[X/G]_{\mathbb{Q}} & \longrightarrow & A_*[U/G]_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

and we claim that the rows of this diagram are exact. Since there are only finitely many orbits every G-invariant subvariety Y of X is the closure of a G-orbit. Since Y admits a dense G-invariant subset every G-invariant rational function on Y is constant. It follows $A_*^o[X/G] = \bigoplus_Z \mathbb{Z}[\bar{Z}]$ where the sum goes over all G-orbits Z of X. From this we obtain the exactness of the top row. For the exactness of the lower row we need to see that the pull-back map $A_*([X/G],1)_{\mathbb{Q}} \to A_*([U/G],1)_{\mathbb{Q}}$ is surjective. But $[U/G]$ is isomorphic to the classifying space of the stablizer group scheme of U. By assumption and Corollary 1.5.4 we get that $A_*([U/G],m)_{\mathbb{Q}} \to A_*(B\{0\},m)_{\mathbb{Q}}$ is an isomorphism. Equivalently the pull-back of the structure morphism $[U/G] \to \operatorname{Spec} k$ is an isomorphism for the higher Chow groups with rational coefficents and hence the claim follows.

Now the right vertical arrow is an isomorphism since both groups are isomorphic to $\mathbb{Q}$. By induction we may assume that the first vertical arrow is also an isomorphism. $\square$

Recall that an algebraic zip datum $\mathcal{Z}$ is called orbitally finite if G has finitely many $E_{\mathcal{Z}}$-orbits ([PWZ, Definition 7.2]). By [PWZ, Remark 7.4] every Frobenius zip datum is orbitally finite.

Theorem 4.4.14. *Let $\mathcal{Z}$ be an orbitally finite connected algebraic zip datum and $[G/E_{\mathcal{Z}}]$ be the corresponding stack of G-Zips. Then the following assertions hold.*

(i) $A_*^o[G/E_{\mathcal{Z}}]_{\mathbb{Q}} \to A_*[G/E_{\mathcal{Z}}]_{\mathbb{Q}}$ *is an isomorphism.*

(ii) $A_*^o[G/E_{\mathcal{Z}}] = \bigoplus_Z \mathbb{Z}[\bar{Z}]$ *where the sum goes over all orbits Z.*

In particular, the dimension of $A_*[G/E_{\mathcal{Z}}]_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space is equal to the number of orbits.

Proof. The assumption of the previous proposition on the stabilizer group schemes hold by [PWZ, Theorem 8.1]. $\square$

Remark 4.4.15. If $\mathcal{Z} = (G, P, P^-, \sigma^a)$ is a connected Frobenius zip datum, then it follows from the above theorem and Corollary 4.4.8 that the number of $E_{\mathcal{Z}}$-orbits in G is given by the order of the coset space W_G/W_L. The same holds more generally for an arbitrary connected orbitally finite algebraic zip datum by [PWZ, Theorem 7.5]. Moreover, a description of the closure relations between the orbits is given in [PWZ, Theorem 6.2].

4.5 The Chow Ring of BT_n

The goal of this section is to prove the following theorem.

Theorem 4.5.1. *The pull-back $\phi_n^*\colon A^*(\mathcal{D}isp_n) \to A^*(BT_n)$ is injective and an isomorphism after inverting p.*

We know that $\mathcal{D}\mathrm{isp}_n = \coprod_{d\leq h} \mathcal{D}\mathrm{isp}_n^{h,d}$ is a decomposition into open and closed substacks. The same holds for BT_n and the morphism ϕ_n maps $BT_n^{h,d}$ to $\mathcal{D}\mathrm{isp}_n^{h,d}$. By Lemma 4.1.2 it suffices to prove the theorem for the restriction of ϕ_n to $BT_n^{h,d}$. The following proposition is the crucial point in the proof of Theorem 4.5.1.

Proposition 4.5.2. *Let L be a field extension of k and $\operatorname{Spec} L \to \mathcal{D}isp_n$ be a morphism. Then there is a finite field extension L' of L of p power degree and an infinitesimal commutative group scheme A over L' such that the fiber $\phi_n^{-1}(\operatorname{Spec} L')$ is the classifying space of A.*

Proof. The diagonal $\Delta\colon BT_n \to BT_n \times_{\mathcal{D}\mathrm{isp}_n} BT_n$ is flat and surjective by [La, Theorem 4.7]. This means that two Barsotti-Tate groups of level n having the same associated display become isomorphic when pulled back to a suitable fppf-covering. It follows that the fiber $(BT_n)_L$ of a display P over some field L is a gerbe over L. If L is perfect there is a truncated Barsotti-Tate group G over L with $\phi_n(G) = P$, i.e. $(BT_n)_L$ is a neutral gerbe. In this case $(BT_n)_L = B\underline{\mathrm{Aut}}^o(G)$ where $\underline{\mathrm{Aut}}^o(G) = \mathrm{Ker}(\underline{Aut}G \to \underline{Aut}P)$ is commutative and infinitesimal again by [La, Theorem 4.7]. If L is not perfect we may consider the perfect hull $L^{p^{-\infty}}$ in an algebraic closure of L. Then $L \subset L^{p^{-\infty}}$ is purely inseparable and $(BT_n)_L(L^{p^{-\infty}})$ is non-empty. Since $(BT_n)_L(L^{p^{-\infty}}) = \varinjlim_{L'}(BT_n)_L(L')$, where the limit goes over all finite subextensions $L \subset L' \subset L^{p^{-\infty}}$, we find some L' such that $(BT_n)_{L'}$ has a section corresponding to a truncated Barsotti-Tate group G over L'. Thus $A = \underline{\mathrm{Aut}}^o(G)$ and L' have the desired properties. □

Remark 4.5.3. Over the open and closed substack of BT_n consisting of level-n BT-groups of constant dimension d and codimension c the degree of $\mathrm{Aut}^o(G^{univ})$ is p^{ncd}. See Remark 4.8 in [La].

Note that $\mathcal{D}\mathrm{isp}_n^{h,d}$ and $BT_n^{h,d}$ both admit admissible presentations in the sense of Definition 1.3.5. In the case of $\mathcal{D}\mathrm{isp}_n^{h,d}$ this follows from Theorem 4.1.4 and Lemma 1.3.4. To obtain the assertion for $BT_n^{h,d}$ we use [We, Proposition 1.8] which yields a presentation $BT_n^h = [Y_n^h / \mathrm{GL}_{p^{nh}}]$ with Y_n^h quasi-affine and of finite type over k. Now BT_n^h is smooth over $\mathrm{Spec}\, k$ ([La]). Hence Y_n^h is also smooth and in particular normal and equidimensional.
We now consider the flat pull-back map $\phi_n^* \colon A_*(\mathcal{D}\mathrm{isp}_n^{h,d}, m) \to A_*(BT_n^{h,d}, m)$ from Lemma 1.3.11.

Proposition 4.5.4. $\phi_n^* \colon A_*(\mathcal{D}isp_n^{h,d}, m) \to A_*(BT_n^{h,d}, m)$ *is an isomorphism after inverting p.*

Proof. Let us write $\mathscr{X} = BT_n^{h,d}$ and $\mathscr{Y} = \mathcal{D}\mathrm{isp}_n^{h,d}$. We fix some $i_o \in \mathbb{Z}$ and show that $\phi_n \colon A_{i_o}(\mathcal{D}\mathrm{isp}_n^{h,d}, m)_p \to A_{i_o}(BT_n^{h,d}, m)_p$ is an isomorphism.
Consider an approximation of $\mathscr{Y}$ (cf. Convention 1.2.1) by a quasi-projective scheme $Y \to \mathscr{Y}$ so that $A_{i_o}(\mathscr{Y}, m) = A_{i_o}(Y, m)$ and similary an approximation $X \to \mathscr{X}$ of $\mathscr{X}$. Let r denote the relative dimension of $X \to \mathscr{X}$. Let Z be the fibre product $X \times_{\mathscr{Y}} Y$. The morphism $Z \to Y$ is then smooth of relative dimension r and we need to see that the pull-back $A_{i_o}(Y, m)_p \to A_{i_o+r}(Z, m)_p$ is an isomorphism. Note that Z is again quasi-projective since it is open in a vector bundle over the quasi-projective scheme X (cf. Remark 1.3.6). We have the following cartesian diagram

$$\begin{array}{ccccc} Z_y & \longrightarrow & \mathscr{X}_{k(y)} & \longrightarrow & \mathrm{Spec}\, k(y) \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & \mathscr{X}_Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \mathscr{X} & \longrightarrow & \mathscr{Y} \end{array}$$

By Lemma 1.4.5 it suffices to see that $A_i(\mathrm{Spec}\, k(y), m)_p \to A_{i+r}(Z_y, m)_p$ with $i = i_o - \dim \overline{\{y\}}$ is an isomorphism. According to the previous proposition there

is a finite field extension K of $k(y)$ of p-power degree such that $\mathscr{X}_K = BA$ holds for an infinitesimal group scheme A over K.
Since Z_K is open in a vector bundle over $\mathscr{X}_K$ of rank r we have $Z_K = U/A$, where U is open in a representation V of A. Note that V is of dimension r. Hence by chosing $\operatorname{codim} X^c$ to be big enough, we may assume $A_i(\operatorname{Spec} K, m) \to A_{i+r}(U, m)$ is an isomorphism. Since A is of p-power degree it follows that the map $A_i(\operatorname{Spec} K, m)_p \to A_{i+r}(Z_K, m)_p$ is an isomorphism . Now since the field extension $K \supset k(y)$ is of p-power degree it follows from Corollary 1.4.3 that $A_i(\operatorname{Spec} k(y), m)_p \to A_{i+r}(Z_y, m)_p$ is also an isomorphism. We are done. □

Proof. (of Theorem 4.5.1) Since BT_n and $\mathcal{D}\mathrm{isp}_n$ are smooth $(\phi_n)_p^*\colon A^*(\mathcal{D}\mathrm{isp}_n)_p \to A^*(BT_n)_p$ is an isomorphism by Lemma 1.3.11 and the proposition above. We already know $A^*(\mathcal{D}\mathrm{isp}_n)$ is p-torsion free by Theorem 4.3.5 and Theorem 4.3.1. Thus ϕ_n^* is injective. □

Gathering the results of Chapter 4 we obtain

Theorem 4.5.5. (i) *We have*

$$A^*(BT_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1, \dots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \dots, (p^h-1)c_h),$$

where c_i denotes the i-th elementary symmetric polynomial in the variables $t_1, \dots, t_h$ and $t_1, \dots, t_d$ resp. $t_{d+1}, \dots, t_h$ are the Chern roots of $\mathcal{L}ie$ resp. ${}^t\mathcal{L}ie^\vee$.

(ii) $\dim_{\mathbb{Q}} A^*(BT_n^{h,d})_{\mathbb{Q}} = \binom{h}{d}$ *and a basis is given by the cyclces of the closures of the EO-Strata.*

(iii)

$$(\operatorname{Pic} BT_n^{h,d})_p = \begin{cases} \mathbb{Z}[p^{-1}]/(p-1) & \text{if } d = 0, h \\ \mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}]/(p-1) & \text{else,} \end{cases}$$

where the generator for the free resp. torsion part is $\det(\mathcal{L}ie)$ *resp.* $\det(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)$.

Proof. By Theorem 4.5.1 we know $A^*(\mathcal{D}\mathrm{isp}_n^{h,d})_p \cong A^*(BT_n^{h,d})_p$. Morever, we have $A^*(\mathcal{D}\mathrm{isp}_n^{h,d}) \cong A^*(\mathcal{D}\mathrm{isp}_1^{h,d})$ by Theorem 4.3.1 and $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})$ was computed in Theorem 4.3.5. This proves part (i). By Lemma 4.4.1 and Lemma 4.4.2 we know that $\mathcal{D}\mathrm{isp}_1^{h,d}$ is isomorphic to the stack $[\mathrm{GL}_h / E_{\mathcal{Z}}]$ corresponding to the Frobenius zip datum $\mathcal{Z} = (\mathrm{GL}_h, P, P^-, \sigma)$, where P is the standard parabolic of type (d, h), P^- is the opposite parabolic and σ is the Frobenius isogeny. Now the dimension of $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})_{\mathbb{Q}}$ as a $\mathbb{Q}$-vectorspace follows from Corollary 4.4.9 and a basis is given by Theorem 4.4.14. This proves (ii). Finally (iii) follows from (i) together with the fact that $\operatorname{Pic} BT_n^{h,d} = A^1(BT_n^{h,d})$. □

A Group (Co)homology

Let (L_*, d_*^L) and (K_*, d_K^*) be complexes of modules over some ring A. We assume that $K_i = 0 = L_i$ for $i < 0$.
The tensor chain complex $(K_* \otimes_A L_*, d)$ is defined by

$$(K_* \otimes_A L_*)_i = \bigotimes_{p+q=i} K_p \otimes_A L_q$$

and

$$d(x \otimes y) = d_K(x) \otimes y + (-1)^p x \otimes d^L(y).$$

For any group G the term G-module will refer to a left $\mathbb{Z}[G]$-module. If M and N are two G-modules the tensor product $M \otimes_{\mathbb{Z}[G]} N$ is formed by considering M as a right G-module via $m \cdot g = g^{-1} \cdot m$. Thus the equation $gm \otimes gn = m \otimes n$ holds in $M \otimes_{\mathbb{Z}[G]} N$.

Definition A.1. *Let G be an abelian group.*
(i) *Let K^* be a cochain complex of G-modules. We define $\mathbb{H}^i(G, K^*)$ to be the i-th hyperext group $\mathbb{E}\mathrm{xt}^i_{\mathbb{Z}[G]}(\mathbb{Z}, K^*)$.*
(ii) *Let L_* be a chain complex of G-modules. We define $\mathbb{H}_i(G, L_*)$ to be the i-th hypertor group $\mathbb{T}\mathrm{or}_i^{\mathbb{Z}[G]}(\mathbb{Z}, L_*)$.*

When K^* resp. L_* is concentrated in degree 0 we obtain the usual group cohomology resp. homology.

Cap products in group (co)homology. Next we want to define a cap product map

$$\cap\colon H^i(G, K) \times \mathbb{H}_j(G, L_*) \to \mathbb{H}_{j-i}(G, K \otimes_{\mathbb{Z}} L_*).$$

We use the standard resolution

$$P_* = [\ldots \to \mathbb{Z}[G^{i+1}] \xrightarrow{d_i} \mathbb{Z}[G^i] \to \ldots \to \mathbb{Z}[G] \to \mathbb{Z}]$$

of $\mathbb{Z}$ ([Se3, Chapter 7 §3]). Here G acts on $\mathbb{Z}[G^{i+1}]$ via

$$g \cdot (g_0, \ldots, g_i) = (gg_0, \ldots, gg_i)$$

and d_i is defined by

$$d_i(g_0, \ldots, g_i) = \sum_{k=1}^{i} (-1)^k (g_0, \ldots, \hat{g}_k, \ldots, g_i).$$

Note that the $\mathbb{Z}[G^i]$ are free $\mathbb{Z}[G]$-modules and that P_* is exact. Hence we have $H^i(G, K) = H^i(\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, K))$ and $\mathbb{H}_j(G, L_*) = H_j(P_* \otimes_{\mathbb{Z}[G]} L_*)$. There is a G-map

$$\mathbb{Z}[G^{i+1}] = P_i \to P_j \otimes_{\mathbb{Z}} P_{i-j}, \quad (g_0, \ldots, g_i) \mapsto (g_0, \ldots, g_j) \otimes (g_j, \ldots, g_i)$$

and for each G-module K and L and every $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, K)$ a morphism

$$(P_i \otimes_{\mathbb{Z}} P_{j-i}) \otimes_{\mathbb{Z}[G]} L \to P_{j-i} \otimes_{\mathbb{Z}[G]} (K \otimes_{\mathbb{Z}} L), \quad (p \otimes p') \otimes x \mapsto p' \otimes (\phi(p) \otimes x).$$

We thus obtain morphims

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}[G]}(P_k, K) \times (P_{k'} \otimes_{\mathbb{Z}[G]} L_l) &\to \mathrm{Hom}_{\mathbb{Z}[G]}(P_k, K) \times ((P_k \otimes_{\mathbb{Z}} P_{k'-k}) \otimes_{\mathbb{Z}[G]} L_l) \\ &\to P_{k'-k} \otimes_{\mathbb{Z}[G]} (K \otimes_{\mathbb{Z}} L_l)\end{aligned}$$

which in turn induce a map

$$\cap \colon \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, K)^i \times (P_* \otimes_{\mathbb{Z}[G]} L_*)_j \to (P_* \otimes_{\mathbb{Z}[G]} (K \otimes_{\mathbb{Z}} L_*))_{j-i}. \quad \text{(A.1)}$$

Lemma A.2. *The above map passes to (co)homology yielding a cap product map*

$$\cap \colon H^i(G, K) \times \mathbb{H}_j(G, L_*) \to \mathbb{H}_{j-i}(G, K \otimes_{\mathbb{Z}} L_*).$$

Proof. Let us denote the differential operators of $\mathrm{Hom}_{\mathbb{Z}[G]}(P_*, K)$, $P_* \otimes_{\mathbb{Z}[G]} L_*$ and $P_* \otimes_{\mathbb{Z}[G]} (K \otimes_{\mathbb{Z}} L_*)$ by ∂, δ and d. It then suffices to prove the following equation

$$(-1)^i d(\phi \cap b) = \phi \cap \delta b - \partial\phi \cap b \quad \text{(A.2)}$$

for all $\phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, K)^i$ and $b \in P_* \otimes_{\mathbb{Z}[G]} L_*$.
Let us first assume $L_* = L$ in degree 0. Consider $b = (g_0, \ldots, g_j) \otimes x \in (P_j \otimes_{\mathbb{Z}[G]} L)$. We compute

$$\begin{aligned} d(\phi \cap ((g_0, \ldots, g_j) \otimes x) &= d((g_i, \ldots, g_j) \otimes \phi(g_0, \ldots, g_i) \otimes x) \\ &= \sum_{k=0}^{j-i} (-1)^k (g_i, \ldots, \hat{g}_{i+k}, \ldots, g_j) \otimes \phi(g_0, \ldots, g_i) \otimes x, \end{aligned}$$

$$\partial\phi \cap ((g_0, \ldots, g_j) \otimes b) = \sum_{k=0}^{i+1} (-1)^k (g_{i+1}, \ldots, g_j) \otimes \phi(g_0, \ldots, \hat{g}_k, \ldots, g_{i+1}) \otimes x$$

and

$$\begin{aligned} \phi \cap \delta(b) =& \phi \cap (\sum_{k=0}^{j} (g_0, \ldots, \hat{g}_k, \ldots, g_j) \otimes x) \\ =& \sum_{k=0}^{i} (-1)^k (g_{i+1}, \ldots, g_j) \otimes \phi(g_0, \ldots, \hat{g}_k, \ldots, g_{i+1}) \otimes x \\ &+ \sum_{k=i+1}^{j} (-1)^k (g_i, \ldots, \hat{g}_k, \ldots, g_j) \otimes \phi(g_0, \ldots, g_i) \otimes x \\ =& \sum_{k=0}^{i} (-1)^k (g_{i+1}, \ldots, g_j) \otimes \phi(g_0, \ldots, \hat{g}_k, \ldots, g_{i+1}) \otimes x \\ &+ \sum_{k=1}^{j-i} (-1)^{k+i} (g_i, \ldots, \hat{g}_{i+k}, \ldots, g_j) \otimes \phi(g_0, \ldots, g_i) \otimes x. \end{aligned}$$

It follows

$$
\begin{aligned}
\phi \cap \delta b - \partial\phi \cap b =& (-1)^{i+2}(g_{i+1},\dots,g_j)\otimes\phi(g_0,\dots,g_i)\otimes x\\
&+\sum_{k=1}^{j-i}(-1)^{k+i}(g_i,\dots,\hat{g}_{i+k},\dots,g_j)\otimes\phi(g_0,\dots,g_i)\otimes x\\
=&\sum_{k=0}^{j-i}(-1)^{k+i}(g_i,\dots,\hat{g}_{i+k},\dots,g_j)\otimes\phi(g_0,\dots,g_i)\otimes x\\
=&(-1)^i d(\phi\cap b)
\end{aligned}
$$

This proves the assertion in the first case.
For general L_* we write again $b = (g_0,\dots,g_j)\otimes x \in P_j \otimes_{\mathbb{Z}[G]} L_l$. We then have

$$
\begin{aligned}
d(\phi\cap b) &= (d(g_i,\dots,g_j)\otimes(\phi(g_0,\dots,g_i)\otimes x), (-1)^{j-i}(g_i,\dots,g_j)\otimes(\phi(g_0,\dots,g_i)\otimes dx))\\
&= (d(g_i,\dots,g_j)\otimes(\phi(g_0,\dots,g_i)\otimes x), (-1)^{j-i}\phi\cap((g_0,\dots,g_j)\otimes dx))
\end{aligned}
$$

and

$$
\phi\cap\delta((g_0,\dots,g_j)\otimes x) = (\phi\cap(d(g_0,\dots,g_j)\otimes x), (-1)^j\phi\cap((g_0,\dots,g_j)\otimes dx))
$$

Hence the general case follows from the computation in the case $L_* = L$ in degree 0. ☐

Remark A.3. (i) Certainly one can get a more general cap product map

$$
\cap\colon \mathbb{H}^i(G,K^*)\times\mathbb{H}_j(G,L_*)\to\mathbb{H}_{j-i}(G,K^*\otimes_{\mathbb{Z}}L_*)
$$

using the construction above. For this one has to find an appropiate sign convention for the differential operator of the Hom cochain complex $\mathrm{Hom}_{\mathbb{Z}[G]}(P_*,K^*)$. However, we will not need this.
(ii) We note that this definition of cap products in group (co)homology is analogous to the definition of cap products in simplicial (co)homology, where one uses the map

$$
C_i^{\Delta}(X)\to C_j^{\Delta}(X)\otimes_{\mathbb{Z}} C_{j-i}^{\Delta}(X),\quad \langle e_0,\dots e_i\rangle\mapsto\langle e_0,\dots e_i\rangle\otimes\langle e_i,\dots e_j\rangle.
$$

Here X is a simplicial complex and $C_k^{\Delta}(X)$ denotes the free abelian group generated by all k-simplices of X.

If $K=\mathbb{Z}$ there is another cap product

$$
\cap'\colon H^i(G,\mathbb{Z})\times\mathbb{H}_j(G,L_*)\to\mathbb{H}_{j-i}(G,L_*).
$$

We will see below that $\cap$ and $\cap'$ only differ by $(-1)^{\frac{i(i+1)}{2}}$. Let us first explain $\cap'$ in the case $i=1$. Let $\alpha\in H^1(G,\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z})$. We may view α as an equivalence class of extensions

$$
\alpha:\quad 0\longrightarrow\mathbb{Z}\longrightarrow P\longrightarrow\mathbb{Z}\longrightarrow 0
$$

of $\mathbb{Z}[G]$-modules. Applying the hypertor functor $\mathbb{T}\mathrm{or}_*^{\mathbb{Z}[G]}(\cdot, L_*)$ to this sequence we obtain a boundary operator

$$\alpha \cap' \cdot \colon \mathbb{H}_j(G, L_*) \to \mathbb{H}_{j-1}(G, L_*)$$

More generally, if $\alpha \in H^i(G, \mathbb{Z})$ is an extension

$$\alpha : \quad 0 \longrightarrow \mathbb{Z} \longrightarrow P_1 \longrightarrow \ldots \longrightarrow P_i \longrightarrow \mathbb{Z} \longrightarrow 0$$

of length i we can split up this extension into i short exact sequences. Each short exact sequence yields a boundary operator on the hypertor groups as above and by composing these boundary operators we obtain a cap product map

$$H^i(G, \mathbb{Z}) \times \mathbb{H}_j(G, L_*) \to \mathbb{H}_{j-i}(G, L_*)$$

which we will denote by $\cap'$.

Lemma A.4. *Let $\alpha \in H^i(G, \mathbb{Z})$. Then $\alpha \cap \cdot = (-1)^{\frac{i(i+1)}{2}} \alpha \cap' \cdot$.*

Proof. Assume $i = 1$ and let α be an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow P \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}[G]$-modules. Let P_* be the standard resolution of $\mathbb{Z}$ as above and consider $\beta \in \mathbb{H}_j(G, L_*)$ with representative $b \in (P_* \otimes_{\mathbb{Z}[G]} L_*)_j$. We have a commutative diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, \mathbb{Z}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, P) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_* \otimes_{\mathbb{Z}[G]} L_* & \longrightarrow & P_* \otimes_{\mathbb{Z}[G]} (L_* \otimes_{\mathbb{Z}} P) & \longrightarrow & P_* \otimes_{\mathbb{Z}[G]} L_* & \longrightarrow & 0 \end{array}$$

with exact rows. Here the vertical maps are the maps from (A.1) (that induce the cap product map $\cap$) applied to b. We note that the morphism

$$\cdot \cap b \colon \mathrm{Hom}_{\mathbb{Z}[G]}(P_*, P) \to P_* \otimes_{\mathbb{Z}[G]} (L_* \otimes_{\mathbb{Z}} P)$$

is only a morphism of complexes up to a factor ± 1. More precisely, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, P) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}[G]}(P_{i+1}, P) \\ \downarrow & & \downarrow \\ P_* \otimes_{\mathbb{Z}[G]} (L_* \otimes_{\mathbb{Z}} P)_{j-i} & \longrightarrow & P_* \otimes_{\mathbb{Z}[G]} (L_* \otimes_{\mathbb{Z}} P)_{j-i-1} \end{array}$$

commutes up to the factor $(-1)^{i+1}$ as follows from equation (A.2) since $\delta(b) = 0$. This holds for every G-module P. Thus taking (co)homology we obtain a diagram

$$\begin{array}{ccc} H^0(G, \mathbb{Z}) & \longrightarrow & H^1(G, \mathbb{Z}) \\ {\scriptstyle \cdot\cap\beta}\downarrow & & \downarrow{\scriptstyle \cdot\cap\beta} \\ \mathbb{H}_j(G, L_*) & \xrightarrow[\alpha\cap'\cdot]{} & \mathbb{H}_{j-1}(G, L_*). \end{array}$$

that commutes up to $(-1)^{0+1}$. Since the unit element in $H^0(G,\mathbb{Z})$ is mapped to α under the boundary operator $H^0(G,\mathbb{Z}) \to H^1(G,\mathbb{Z})$ we obtain the assertion in the case $\alpha \in H^1(G,\mathbb{Z})$. The assertion for $\alpha \in H^i(G,\mathbb{Z})$ with arbitrary i is proven by splitting up the extension α of length i into i short exact sequences. Repeating the above argument for each exact sequence we see that $\alpha \cap \beta$ and $\alpha \cap' \beta$ differ by the factor $(-1)^{\sum_{k=1}^{i} k} = (-1)^{\frac{i(i+1)}{2}}$. □

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