Some operator algebraic techniques in Loop Quantum Gravity

Presented by

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Submitted to the University of Paderborn

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Faculty of Electrical Engineering, Computer Science and Mathematics
for the Degree of "doctor rerum naturalium"

Prepared at the University of Hamburg and the University of Paderborn

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Submitted: December 17, 2010
Date of Defense: April 4, 2011

Second Revised Version: August 19, 2011

A short overview about this work

The aim of the work is to give a mathematial description for a theory of Quantum Gravity. The following objects will be studied

- the two known algebras of Quantum Gravity in the Loop Quantum Gravity approach, which are the holonomy-flux *-algebra (given in [64]) and the Weyl C^*-algebra (given in [39]),
- modifications of these algebras and new algebras of Loop Quantum Gravity,
- states and representations of the algebras and
- the concept of quantum constraints and KMS-Theory in Loop Quantum Gravity.

In comparison to other theories of quantum physics it is obtained that, the two known algebras are not the only algebras in the Loop Quantum Gravity framework. Surprisingly, a huge amount of different algebras in Loop Quantum Gravity will be presented in this dissertation. The idea of the construction of these algebras is to establish a finite set of operators, which generates (in the sense of Woronowicz, Schmüdgen and Inoue) the different O^*- or C^*-algebras of quantum gravity. In the Loop Quantum Gravity approach usually the basic classical variables are connections and fluxes. Studying the three constraints appearing in the canonical quantisation of classical general relativity in the ADM-formalism some other variables like curvature appear. Consequently, the main difficulty of the quantisation of gravity is to find a suitable replacement of the set of elementary classical variables and constraints. The important aim of this project is to modify the holonomy-flux *-algebra and the Weyl C^*-algebra in such a way that the set of constraints of Quantum Gravity in the formulation of the Ashtekar variables is a subalgebra of the modified O^*- or C^*-algebra, which is generated by a set of the operators associated to holonomies, fluxes, diffeomorphisms and in some cases even the curvature. In this dissertation an exceptional algebra satisfying this property, will be proposed.

Ein kurzer Überblick über die Arbeit

Das Ziel dieser Dissertation ist eine mathematische Formulierung einer Theorie der Quantengravitation. Auf der Suche nach einer solchen physikalischen Theorie werden verschiedene Zugänge, wie zum Beispiel die Stringtheorie oder aber auch die Schleifenquantengravitation, favorisiert. Diese Arbeit greift den zuletzt genannten Ansatz auf und behandelt folgende Gesichtspunkte:

- die Untersuchung vorhandener Algebren für eine Theorie der Quantengravitation mit Hilfe von Schleifen und Wegen, wie zum Beispiel die Holonomie - Fluß *-Algebra (siehe [64]) und die Weyl C^*-Algebra (siehe [39]),
- die Modifikation bestehender und die Entwicklung neuer Algebren,
- das Studium von Zuständen und Darstellungen dieser Algebren und
- das Konzept von quantisierten Zwangsbedingungen und der KMS - Theorie im Zugang der Schleifenquantengravitation.

Im Vergleich zu anderen physikalischen Quantentheorien ist es möglich eine Mannigfaltigkeit an verschiedenen Algebren im Zugang der Schleifenquantengravitation herzuleiten. In dieser Arbeit wird hierfür ein besonderer Zugang benutzt, welcher unter Anderem durch Woronowicz, Schmüdgen und Inoue entwickelt wurde. Eine endliche Menge von beschränkten oder unbeschränkten Operatoren, welche durch eine Quantisierungsabbildung klassischer Größen definiert werden, erzeugt in einer geeigneten Weise eine *- oder C^*-Algebra. Die grundlegenden quantisierten Größen der Theorie der Schleifenquantengravitation sind Holonomien entlang von Wegen und quantisierte Flußoperatoren, welche jeweils zu einer Fläche und einem Weg assoziiert sind. Bei dem Studium der klassischen Zwangsbedingungen, welche bei der kanonischen Formulierung der Gravitationstheorie unter Berücksichtung von speziellen ADM-Variablen auftreten, fallen insbesondere zusätzliche Größen auf. Zum Beispiel beinhaltet die Hamiltonzwangsbedingung die Krümmung. Dies führt zu Schwierigkeiten bei der Quantisierung dieser Zwangsbedingung, denn eine quantisierte Krümmung ist bisher nicht definierbar. In dieser Dissertation werden die Voraussetzungen für eine physikalische Algebra der Quantentheorie untersucht. Eine solche Algebra ist dadurch ausgezeichnet, dass sie in einem geeigneten Sinne die Algebra der quantisierten Zwangsbedingungen enthält bzw. durch diese unter Anderem erzeugt wird. Die bisher definierten Algebren erfüllen die Anforderung an eine physikalische Algebra nicht. Daher ist das Ziel dieser Dissertation eine geeignete Algebra zu finden, welche durch die quantiserte Konfigurationsvariable Holonomie und der quantisierten kanonisch-konjugierten Flußvariable sowie andere Größen, wie z.B. ein quantisiertes Analogon zur Krümmmung, erzeugt wird.

Contents

Tao Te Ching – Lao Tzu

Chapter 14

Because the eye gazes but can catch no glimpse of it,
It is called elusive.
Because the ear listens but cannot hear it,
It is called the rarefied.
Because the hand feels for it but cannot find it,
It is called the infinitesimal.
These three because they cannot be further
scrutinized,
Blend into one.
Its rising brings no light;
Its sinking, no darkness.
Endless the series of things without name
On the way back to where there is nothing.
They are called shapeless shapes;
Forms without form;
Are called vague semblances.
Go towards them, and you can see no front;
Go after them and you see no rear.
Yet by seizing on the Way that was
You can ride the things that are now.
For to know what once there was, in the
Beginning,
This is called the essence of the Way.

Preface

During the last years two quantum algebras for a gravitational quantum theory in the concept of Loop Quantum Gravity have been developed. The holonomy-flux *-algebra has been introduced by the project group of Lewandowski, Okolow, Sahlmann and Thiemann [64] and the Weyl C^*-algebra has been presented by Fleischhack [39]. Both algebras are generated by the quantised canonical variables of gravity, which are given by the holonomies and the fluxes. The fundamental aspects of the holonomy-flux *-algebra and the Weyl C^*-algebra are given by the uniqueness of the representation of these algebras with respect to diffeomorphism invariance and the unitary (weakly) continuous representation of the fluxes on some Hilbert space. A general overview about the development of the theory, at the point of time of the preparation of this dissertation, can be found in Thiemann [104], Ashtekar and Lewandowski [11] and Rovelli [82].

The structure of this dissertation is the following. There are two parts. The first part is about the quantisation procedure of classical gravity in the framework of Loop Quantum Gravity. A quantisation of a classical theory provides a set of quantum operators, which generate for example *-, O^*-, C^*- or von Neumann algebras. In the first chapter motivations for a construction of different algebras in the context of the Loop Quantum Gravity are given. Therefore different simple examples for common physical systems are presented. The underlying mathematical theory is studied very briefly in subsection 1.4.2. Using this mathematical framework different algebras for Loop Quantum Gravity is considered and is introduced in section 1.5. The second chapter illustrates an outline of the implementation of quantum constraints in the framework of Loop Quantum Gravity and a motivation for the study of KMS-states. Furthermore the analysis indicates that, a study of different quantum algebras is necessary for the application of constraints. For readers who are familiar with the construction of O^*-, C^*- and von Neumann algebras and readers who are not interested in physical arguments can skip the chapters one and two of the first part. In the third chapter the classical and quantum variables of the theory of Loop Quantum Gravity are presented. These fundamental quantum objects allow by using the ideas formulated in chapter one to construct different algebras for a theory of quantum gravity. Consequently the second part of this dissertation decomposes into six units, which are given by

(i) a new formulation of the C^*-algebras of Loop Quantum Cosmology
(sections 4 and 7.4)

(ii) the smooth holonomy C^*-algebra
(section 5)

(iii) a new formulation of the Weyl C^*-algebra for Loop Quantum Gravity
a new algebra - the holonomy-flux von Neumann algebra
(chapter 6)

(iv) new algebras - the flux group, flux transformation group, holonomy and heat kernel holonomy C^*-algebra
a new algebra - the holonomy-flux cross-product C^*-algebra
(chapter 7)

(v) new analytic holonomy *-algebras,
a new algebra - the holonomy-flux cross-product *-algebra, which is a comparable with
the holonomy-flux *-algebra,
a new algebra - the localised holonomy-flux cross-product *-algebra
other new holonomy-flux *-algebras
(chapter 8)

(vi) a new algebra - the holonomy groupoid C^*-algebra for a gauge theory
(chapter 9)

Now a short suggestion for the reading of this work is given as follows. For a brief introduction to this dissertation in section 1.5 an overview about the different algebras for LQG and LQC is presented. The algebras discovered in the units (i), (ii), (iii), (v) and (vi) can be studied separately without the knowledge of the others. The unit (iv) is based on the concept of covariant representations and C^*-dynamical systems, which are used in (iii) for the new formulation of the Weyl C^*-algebra. In particular readers only interested in specific algebras of the second part do not need to read the whole definitions of the chapter 3. Consequently only particular sections of chapter 3 are necessary for the understanding of the construction of a specific algebra. In sections 1.5.1 and 10 a closer analysis of the different algebras in comparison to physical examples is given. A short overview about the structures of the different units presented above and the issue of quantum constraints is given in the following paragraphs.

The algebras of quantum operators in LQG and LQC

Different algebras in Loop Quantum Cosmology are presented in section 4 and 7.4. In the model of Loop Quantum Cosmology the C^*-algebra of almost periodic functions on the real line is used. An overview about the mathematical structure used in LQC has been presented by Ashtekar, Bojowald and Lewandowski [5]. In the particular framework of this dissertation two possibilities for a construction of the quantum algebra of LQC are presented in section 4.1. These C^*-algebras are isomorphic. There is no previous knowledge of other sections of this work necessary to follow the ideas. Furthermore in a general context the Weyl algebras over symplectic spaces are studied briefly. The investigations are directly used in the Loop Quantum Cosmology approach, which are shortly presented in section 4.2. The full development of the structures are given in section 7.4. The basic construction procedure is first analysed for a more general case in the section 7 and is transmitted to the LQC approach.

The smooth holonomy C^*-algebra has been introduced by Ashtekar and Isham [7]. In this dissertation some of the ideas mentioned by Ashtekar and Isham are resumed and are used to construct the modified smooth holonomy C^*-algebras in section 5. The fundametal objects of the construction of the algebras are smooth loops or paths and structures derived from these smooth paths. These geometric objects are introduced in the sections 3.1.1, 3.1.2 and 3.3.5.

The analytic holonomy C^*-algebra and the Weyl C^*-algebra are based on analytic paths,which are studied in section 3.1.3, and derived objects, which are investigated in section 3.3.4. The new variables constructed from analytic (or semi-analytic) paths has been used by Ashtekar and Lewandowski [8] to construct the analytic holonomy C^*-algebra. The Weyl C^*-algebra has been developed by Fleischhack in [39] in terms of semi-analytic paths. This algebra is constructed from the unital commutative analytic holonomy C^*-algebra and homeomorphisms on the configuration space. In this dissertation the Weyl C^*-algebra is associated to a set of surfaces, whereas Fleischhack have been used more general geometric objects for the definition of his Weyl C^*-algebra. Furthermore in this work finite path groupoids are generalised to finite graph systems, which are derived from semi-analytic paths and are presented in the section 3.1.3. The holonomy maps for the loop group has been presented by Barrett [16] and is generalised to finite path groupoids and finite graph systems in section 3.3.4. The analytic holonomy C^*-algebra is constructed from holonomy maps for finite graph systems. The quantum flux operators, which are defined in section 3.4, and the analytic holonomy C^*-algebra generate the Weyl C^*-algebra associated to surfaces, which is presented in section 6. The uniqueness result of a particular representation of a certain Weyl C^*-algebra associated to surfaces is reformulated in the new framework. The **holonomy-flux von Neumann algebra** is analysed in section 6.5. In particular it is shown that this von Neumann algebra is not of the standard form.

The holonomy-flux cross-product C^*-algebra is developed in section 7. In Loop Quantum Gravity approach this algebra has not been considered so far. The development can be comprehended after reading the sections 3.1.3, 3.3.4 and 3.4. The construction is independently from the Weyl C^*-algebra. But the different actions, which are defined in section 6.1 and which define different C^*-dynamical systems, are necessary for the development of this new algebra. For a short overview about the construction the definition of a particular action, which is presented in lemma 6.1.16, is sufficient. The concept of cross-product C^*-algebras in general allows to construct some other C^*-algebras. In particular different algebras like the **flux group, flux transformation group, holonomy or heat-kernel holonomy C^*-algebra** are generated by the flux operators or the holonomies along paths. In analogy of the previous constructions a cross-product C^*-algebra in the context of Loop Quantum Cosmology is presented in section 7.4.

Some analytic holonomy *-algebras are constructed similarly to the analytic holonomy C^*-algebra. The algebras are generated by matrix elements. In contrast to the analytic holonomy C^*-algebra, these algebras are constructed from a different multiplication operation and are derived in section 8.1. Readers, who are interested in the holonomy-flux *-algebra, which has been given in [64], can skip this section. **The holonomy-flux cross-product *-algebra** is comparable with the *-algebra presented by Lewandowski, Okołów, Sahlmann and Thiemann [64]. The holonomy-flux cross-product *-algebra is generated by the identity $\mathbb{1}$, the holonomies along paths and the quantum flux associated to surfaces and paths satisfying the canonical commutator relations. This *-algebra is studied in section 8. The construction of the holonomy-flux *-algebra can be comprehend after reading the sections 3.1.3, 3.3.4 and 3.4. There is a uniqueness result for a certain diffeomorphism-invariant state of the holonomy-flux cross-product *-algebra, which is comparable with the uniqueness result of the holonomy-flux *-algebra given in [64]. In comparison with the Heisenberg cross-product algebra presented by Klimyk and Schmüdgen [53], the Heisenberg holonomy-flux cross-product *-algebras are defined in section 8.2.2. The holonomy-flux cross-product *-algebra is modified by tensor products, which is given in section 8.3. Some more fundamental modifications are based on the concept of algebras generated by unbounded or bounded elements. This allows to define new algebras, which are not considered in the LQG approach until now. For example the **localised holonomy-flux cross-product *-algebra**, which is studied in section 8.4. Moreover there are other holonomy-flux *-algebras constructable. The new *-algebras are called the holonomy-flux Nelson transform *-algebra and are constructed in section 8.5.

The holonomy groupoid C^*-algebra for gauge and gravitational theories is based on a generalisation of Barrett's ideas about holonomy maps and connections, which has been presented in [16]. The further development of his concept is presented in sections 3.3.4 and 3.3.5 and is based on the mathematical theory of path connections, which has been introduced by Mackenzie [66]. In section 3.3.1 and 3.3.2 a reformulation of Barrett's ideas for gauge and gravitational theories is generalised in terms of the objects introduced by Mackenzie. The generalisation of Barrett's holonomy maps is used to define holonomy maps in the gauge groupoid. This groupoid depends on the fundamental structure of a chosen principal bundle $P(\Sigma, G, \pi)$. The new holonomy maps define parallel transports in this principal fibre bundle. Due to the generalised Ambrose-Singer theorem given by Mackenzie [66] in this framework even curvature is studied.

Barrett has been suggested that, the configuration space of a theory is given by all holonomy maps. This idea is transferred to the holonomy maps for a gauge theory in the more general framework, which is called the holonomy groupoid formulation of LQG. A new C^*-algebra of continuous functions on the space of holonomy maps is introduced in section 9. There exists a left (or right) action of the exponentiated Lie algebroid associated to $P(\Sigma, G)$ on this algebra. Hence there is an action related to infinitesimal connections and curvature, since both objects are encoded as elements of the Lie algebroid. This is studied in section 9.1. The full development of the C^*-algebra can be understood after reading the basic sections 3.1.1 and 3.1.2 about loops and paths, the sections 3.1.4 and 3.1.5 about Lie groupoids, the section 3.2 about the duality and section 3.3.1 about the generalised holonomy mappings for a gauge theory. There are some ideas about a cross-product C^*-algebra for holonomy maps and fluxes in this new holonomy groupoid formulation. The construction for general groupoids has beeen given by Masuda [68, 69] and is shortly presented in section 9.2. For a gravitational theory, which is defined on a certain frame bundle, the structures are more complicated. The ideas presented in section 3.3.2 are used to develop a new algebra for quantum gravity, which is not comparable with the Weyl C^*-algebra, the holonomy-flux *-algebra or any other algebra presented in this work.

The issue of quantum constraints and KMS-Theory

After the investigations about different algebras of quantum operators the question arises, which algebra is the physically distinguished algebra describing a theory of quantum gravity. This issue cannot be solved without a study of quantum constraints and complete quantum observables. The implementation of the constraints and complete observables and the relation of the underlying theory to KMS-Theory are analysed in chapter 2. A mathematical description for a treatment of classical constraints and complete observables has been presented for example by Thiemann [98, 94, 102], Thiemann and Dittrich [32] in the Loop Quantum Gravity approach. A treatment of quantum constraints and complete quantum observables in the operator algebraic framework of LQG is not available so far. In this dissertation in chapter 2 a mathematical concept is invented. Furthermore representations and states of the quantum algebras of LQG are important. It is analysed that, for the definition of quantum constraints and complete quantum observables KMS- states of the algebras are exceptional states. It is proven in section 6.5 that, the Weyl C^*-algebra does not admit KMS-states. Consequently different modifications of the analytic holonomy C^*-algebra and the extended algebras of holonomies and fluxes have to be analysed with respect to the existence KMS-states. In section 8.4 the new localised holonomy-flux cross-product *-algebra is presented. This algebra is constructed such that the problems occuring during the analysis of the algebras presented before, do not appear. There exists a state, which is invariant under certain diffeomorphism, on this new *-algebra. This state is a KMS-state for a suitable C^*-algebra, which is derived from a subalgebra of the localised holonomy-flux cross-product *-algebra. The algebra is related to a fixed surface set. Therefore the physical objects defining the algebra are localised on a finite number surfaces in the hypersurface Σ. The quantum Hamilton constraint is a generator of an automorphism group $\mathbb{R} \ni t \mapsto \alpha(t) \in \mathfrak{Aut}(\mathfrak{A})$ on a subalgebra $\mathfrak{A}$ of the localised holonomy-flux cross-product *-algebra. Furthermore there are α-invariant states on $\mathfrak{A}$, which are also invariant under some certain diffeomorphisms. The quantum algebra generated by holonomies, flux operators and all constraints of the theory of quantum gravity is not constructable so far. Nevertheless in this dissertation the author presents a notion of Dirac states and the set of complete quantum observables for a modified theory of quantum gravity with respect to a specific quantum Hamiltonian constraint and for certain diffeomorphisms.

Part I

Quantisation of gravity in the Loop Quantum Gravity approach

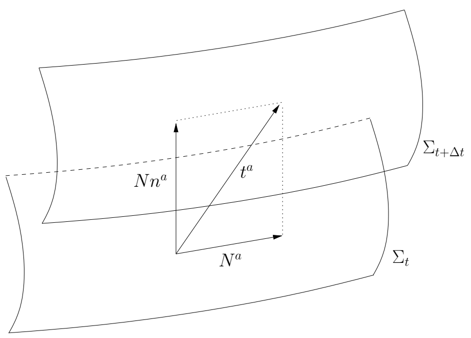
Chapter 1

An overview about General Quantum Physics and Quantum Gravity

A physical quantum theory is based on a choice of canonical variables, a set of constraints and a way of quantising these objects. In the LQG approach to quantum gravity the canonical variables and the constraints have been introduced by the Ashtekar in terms of certain variables of classical gravity. A quantum map relates these classical variables to quantum operators that may form an algebra and are defined on some Hilbert space. In comparison with other quantum theories different quantisation maps can be taken into account such that there are different quantised variables. The quantum objects derived from the Ashtekar variables form certain algebras describing a quantum theory of gravity. In this chapter ideas for a construction of algebras for quantum gravity are presented by studying some well-known physical examples. The main issue will be to understand how different algebras of quantum variables can be developed by using different mathematical concepts and choices of classical variables.

1.1 The Ashtekar formulation of Classical Gravity

The Ashtekar formulation of gravity starts from a global hyperbolic spacetime (M, g), which decomposes into $\Sigma \times \mathbb{R}$ such that there is a foliation by 3-dimensional Cauchy surfaces Σ_t parametrized by a global "time" function t. Then there exists the following decomposition into normal and tangential parts to Σ_t. The lapse function $N = -t^a n_a$ and shift vector $N_a = q_{ab}t^b$, where n^a is the unit normal vector field on Σ_t and t^a describes the "flow" of time. The spatial metric is the dynamical variable in general relativity, since, moving forward in time corresponds to a change of the spatial metric $q_{ab}(0)$ to $q_{ab}(t)$.



The extrinsic curvature K_{ab} represents the "time derivate" of the spatial metric on Σ_t embedded in the spacetime M. In fact, the failure of coincidation of the evolved normal vector n^a from p to q corresponds to the embedding of Σ_t in spacetime M. The inital data of the initial value formulation for canonical gravity introduced by Arnowitt-Deser-Misner (1962) is, therefore, given by the triple $(\Sigma_0, q_{ab}(0), K_{ab}(0))$ consisting of the 3-dimensional manifold, the Riemanian metric $q_{ab}(0)$ on Σ_0 and symmetric tensor field $K_{ab}(0)$ on Σ_0. For vacuum the solution of the initial-value problem unique and exists. The proof establishes the existence of a unique "maximal Cauchy development"

$(\Sigma_0, q_{ab}(t), K_{ab}(t))$ to an entire range of values of t in such a manner that a globally hyperbolic Lorentzian manifold (M, g) emerges, into which Σ_0 can be embedded as a Cauchy surface. Diffeomorphically equivalent initial conditions of such Cauchy developments lead to unique Lorentzian manifolds only modulo diffeomorphisms. This result can be understood as a result of Einstein's "hole" argument.

The Ashtekar connection form on a principal $SO(3)$-bundle of orthonormal frames $O^+(\Sigma_t, q)(\Sigma_t, SO(3), \pi)$ is a sum $A^i_a = \Gamma^i_a + \beta k^i_a$ where Γ^i_a is the Levi-Cevita spin connection and k^i_a is an one-form corresponding to the extrinsic curvature and the triad such that $k^i_a := K_{ab}(t)e^{bi}$. Therefore the important elementary variables of the theory are connection forms associated to the principal bundle with a spatial base manifold Σ_t and the embedding of Σ_t into the spacetime manifold M and vector densities of weight one called fluxes E^i. Hence the main objects are connection k-forms $\Omega^k(O^+(\Sigma_t, q), so(3))$ with values in $so(3)$ given by the sections of the bundle defined by $\wedge^D T^*\Sigma \otimes \underline{so(3)}$ where D is the dimension of Σ. Furthermore the Ashtekar connection A defines a curvature $R_A : T\Sigma \times T\Sigma \rightarrow \frac{O^+(\Sigma_t,q)\times so(3)}{SO(3)}$.

The extrinsic curvature is constructed from a tensor field $k : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \rightarrow C^\infty(\Sigma)$, a metric g on M, a normal vector n such that $g(n, n) = -1$ and a covariant derivative $\bigtriangledown^M$ on M. The relation is given by

$$\begin{aligned} &g(\bigtriangledown^M_X n, Y) = k(X, Y) \text{ for } X, Y \in \Gamma(T\Sigma) \\ &K(X, Y) = k(X, Y)n \end{aligned} \tag{1.1}$$

where $K : T\Sigma \times T\Sigma \rightarrow T\Sigma^\perp$.

One of the scalar Hamilton constraints have the following form

$$C := \frac{1}{\sqrt{|\det E)|}}(R_A - (1 + \beta^2)k^2)E^2 \tag{1.2}$$

An overview about the structure of the Ashtekar connection and the quantised objects refer to Rovelli in [81]. For a mathematical viewpoint the work of Levermann in [58] can be considered.

The classical algebra of a system describing gravity contains the Ashtekar connection A^i_a and the field E^b_j satisfying the canonical Poisson bracket

$$\{E^b_j(x), A^i_a(y)\} = \kappa\delta^b_a\delta^i_j\delta(x, y) \tag{1.3}$$

and the contraints: spatial diffeomorphisms C_a, Gauss constraint C_j and Hamilon constraint C satisfying Poisson brackets on their own. To quantise this classical Poisson algebra one has to find a quantisation map from the Poisson algebra to a suitable *- or C^*-algebra.

The main concept for the quantisation of the infinitesimal connections and the fields depending on points in spacetime is to replace these infinitesimal objects with holonomies along paths and fields smeared over a surface S.

1.2 Quantisation procedures of a classical system

In mathematical quantum physics the following quantisation procedures for classical systems play a fundamental role:

(i) the Canonical Quantisation Procedure based on Hilbert space methods introduced by Dirac (1948/49)

(ii) the Algebraic Quantisation of observables related to *-, C^*-, von Neumann or generally Jordan algebras and

(iii) Path Integral Methods.

In general a classical Hamiltonian dynamical system with n degrees of freedom is a $2n$-dimensional manifold P, which is called the phase space. Often the phase space is encoded in a cotangent bundle over a classical configuration space. The canonical variables are given by a set of $2n$ functions (x_i, p_i) on the phase space for $i = 1, ..., n$. The differential-geometric structure on P is given by Poisson bracket, which defines a suitable subspace $\mathcal{F} \subset C^\infty(P)$ of the space of real-valued, infinitely differentiable functions and the structure of a Lie algebra.

The Dirac or Canonical Quantisation Procedure is given by the assignment of functions in $\mathcal{F}$ with symmetric operators on a Hilbert space $\mathcal{H}$. In particular the derivations of the associative Poisson algebra are replaced by a symmetric operators on a dense domain and all quantum operators (for example the constraints) are defined on a common invariant dense subspace of $\mathcal{H}$. In some examples one can construct a quantisation map $\mathcal{Q}$ from a suitable subspace of the Poisson algebra to a suitable Lie algebra of symmetric operators on a Hilbert space on a common invariant dense domain. In the LQG approach to quantum gravity the classical algebra of position and momentum variables and constraints form a difficult algebra. The Algebraic Quantisation of classical canonical variables are encoded in a suitable algebra without referring to a Hilbert space.

In this dissertation the Algebraic Quantisation ansatz for a theory of Loop Quantum Gravity, whereas in literatur often the Canonical Quantisation approach, is used. The new algebraic formulation allows to study a huge amount of different algebras. The ideas for a construction of different algebras of quantum variables is presented for different simple physical examples in the next section.

1.3 Some algebras in Physics

1.3.1 Algebras in Quantum Mechanics

First of all quantum algebras are generated by bounded or unbounded operators. Quantised configuration and momentum variables, which are bounded operators, generate C^*-algebras, whereas quantum operators, which are unbounded, form O^*-algebras. O^*-algebras are certain *-algebras. Since unbounded operators are defined on Hilbert spaces only on dense domains, the *-operation is required to respect dense domains. This condition is encoded in the concept of O^*-algebras, which can be found in [51] or appendix 12.2.3. To understand the ideas the diversity of quantum algebras in Quantum Mechanics, Quantum Field Theory and Statistical Mechanics is studied in the next subsections very briefly.

1.3.1.1 O^*-algebras of Quantum Mechanics

In Quantum mechanics the classical variables of the theory are position and momentum variables. On the one hand the quantisation map replaces the position operator by a single projection-valued measure (PVM)[1] P_E on $\mathbb{R}^3$ with values in a separable Hilbert space $L^2(\mathbb{R}^3)$ where $E \subset \mathbb{R}^3$. Then the classical variable x is replaced by $P_E(x)$. For a measurable function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ the quantised operator of a classical function f is given by

$$\mathcal{Q}(f)\psi := \int_{\mathbb{R}^3} \mathrm{d}\, P_E(x) f(x)\psi$$

This operator acts as a multiplication operator on the Hilbert space and is self-adjoint on the domain

$$D(f) := \left\{ \psi \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \mathrm{d}\langle \psi, P_E(x)\psi\rangle |f(x)|^2 < \infty \right\}$$

The quantum position operator is given by

$$\mathcal{Q}(x^j)\psi := \int_{\mathbb{R}^3} \mathrm{d}\, P_E(x) x^j \psi$$

and is defined on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ of all infinitely differentiable rapidly decreasing functions on $\mathbb{R}^3$.

On the other hand the momentum operators p are replaced by a single strongly continuous unitary group representation V of the group $\mathbb{R}^3$ on the Hilbert space $L^2(\mathbb{R}^3)$, which implements translations on $\mathbb{R}^3$ by $V(y)\psi(x) = \psi(x-y)$. Then the quantised momentum operators are represented on the Hilbert space by the assignment

$$\mathcal{Q}(p_i)\psi := i\hbar \lim_{t_i \rightarrow 0} t_i^{-1} \left(V(t_i) - 1 \right) \psi,$$

[1] With other words there is a map $E \mapsto P_E$ from a Borel subset $E \subset \mathbb{R}^3$ to the projections on $L^2(\mathbb{R}^3)$ that satisfies $P_\varnothing = 0$, $P_{\mathbb{R}^3} = 1$, $P_E P_F = P_F P_E = P_{E\cap F}$ for all measurable $E, F \subset \mathbb{R}^3$, and $P_{\cup E_i} = \sum P_{E_i}$ for all countable collections of mutually disjoint $E_i \subset \mathbb{R}^3$

where $V(t_i)$ is defined at $x^i = t_i$ and $x^j = 0$ for $j \neq i$. The operators $\mathcal{Q}(p_i)$ are self-adjoint on the set of all ψ for which the limit exists (Stone theorem). This indicates that, the quantum momentum operators are given by infinitesimal representations $\mathrm{d}\, V(x_j) := \partial_{x^i}$ of the group $\mathbb{R}^3$ on the Hilbert space $L^2(\mathbb{R}^3)$.

The representation of the quantum operators $\mathcal{Q}(x^i) := x^i$ and $\mathcal{Q}(p_i) := i\hbar\partial_{x^i}$ on the Hilbert space $L^2(\mathbb{R}^3)$ is called the Schrödinger representation of quantum mechanics. The canonical commutation relations of the quantum operators is encoded in the relation $[\mathcal{Q}(p_i), \mathcal{Q}(x^j)] = -i\hbar\delta_i^j$. The bounded operators $\mathbb{1}$, x^i and the unbounded operators operators ∂_{x^j} for all $1 \leq i, j \leq 3$ satisfying these relations generate an associative Lie *-algebra or, in general, a closed operator O^*-algebra. The elements of this algebra are of the form

$$\sum_{1\leq k\leq m} \sum_{1\leq l\leq n} \lambda_{kl} x^k \Big(\frac{\mathrm{d}}{\mathrm{d}\, x}\Big)^l \text{ for } \lambda_{kl} \in \mathbb{C}$$

This O^*-algebra is called the Heisenberg algebra $\mathfrak{O}_{\text{Heis}}$ of Quantum Mechanics.

Furthermore there exists another *-algebra. The canonical commutator relations of the quantum operators $\mathcal{Q}(p_i) := i\hbar\partial_{x^i}$ and $\mathcal{Q}(f) := f$ for every function $f \in C^\infty(\mathbb{R}^3)$ are $[\mathcal{Q}(p_i), \mathcal{Q}(f)] = i\hbar\mathcal{Q}(Xf)$, where the canonical vector field X on $\mathbb{R}^3$, which is defined by

$$(Xf)(y) := \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0} f(\exp(-tx^i)y)$$

Then the algebra generated by $\mathcal{Q}(f)$ for every function $f \in C^\infty(\mathbb{R}^3)$ and $\mathcal{Q}(p_i)$ for $1 \leq i \leq 3$, which satisfy the canonical commutator relations, is an O^*-algebra, too. The elements are of the form

$$\sum_{1\leq k\leq m} \sum_{1\leq l\leq n} f_k(x) \Big(\frac{\mathrm{d}}{\mathrm{d}\, x}\Big)^l \text{ for every } f_k \in C^\infty(\mathbb{R}^3)$$

Hence to summarise two different O^*-algebras are illustrated by using only different functions of the configuration variables in Quantum Mechanics. In LQG approach the ideas is used to define different holonomy-flux *-algebras.

1.3.1.2 C^*-algebras in Quantum Mechanics

C^*-algebras are isomorphic to norm-closed *-subalgebras of the C^*-algebra of bounded operators on some Hilbert space. Consequently bounded operaors on a Hilbert space are used to define C^*-algebras.

Now the quantisation map of the position operator $\mathcal{Q}(x)$ is given by the unitary operator $u_x := \exp(ix)$ and the quantum momentum $\mathcal{Q}(p)$ is equal to the unitary $v_p = \exp(-ip)$. Then the canonical commutation relations for the abelian locally compact group $\mathbb{R}^3$ change to

$$v_p u_x v_p^* = \exp(i\langle x, p\rangle) u_x \tag{1.4}$$

where $v_p^* = v_{-p}$ and $\langle ., .\rangle$ denotes the euclidean inner product on $\mathbb{R}^3$. The *-algebra generated by the Weyl elements $W(x, p) := \exp(i(x - p))$ and $W^*(x, p) := \exp(-i(x - p))$ satisfying the commutator relations can be completed in a C^*-norm. This completition is called the Weyl C^*-algebra of Quantum Mechanics.

The quantum algebra generated by the configuration variables only is given by the continuous functions $C_0(\mathbb{R}^3)$ vanishing at infinity with pointwise multiplication and supremum norm. Notice that, $C_0(\mathbb{R}^3)$ is isomorphic to the group algebra $C^*(\mathbb{R}^3)$. The group algebra can be related to the a certain Weyl C^*-algebra, which is analysed in section 4.2 and for the discretised group $\mathbb{R}^2_{\text{d}}$ in section 7.4.

There is a $\mathbb{R}^3$-covariant representation $(\mathcal{H}, V, \pi)$ of the C^*-algebra $C_0(\mathbb{R}^3)$ such that there exists an action $(L_p f)(y) := f(y-p)$ of $\mathbb{R}^3$ on $C_0(\mathbb{R}^3)$, a continuous unitary representation V of $\mathbb{R}^3$ on the Hilbert space $\mathcal{H}$, and a non-degenerate representation π of $C_0(\mathbb{R}^3)$ on $\mathcal{H}$ satisfying

$$V(p)\pi(f)V(p)^* = \pi(L_p f) \text{ for all } p \in \mathbb{R}^3, f \in C_0(\mathbb{R}^3) \tag{1.5}$$

whenever $V(p) = v_p$. Consequently another Weyl C^*-algebra is constructed by the Weyl elements v_p and a the C^*-algebra $C_0(\mathbb{R}^3)$, which satisfy the canonical commutator relation (1.5).

Until now the idea of the quantisation map defined at the beginning of subsection 1.3.1.1 is not considered. Consequently consider the quantum position operator $\mathcal{Q}(f) := f$ for a function f in $C_0(\mathbb{R}^3)$ depending on x, the quantum momentum operator $\mathcal{Q}(p)$ for $p \in \mathbb{R}^3$, which is given by the operator-valued measure $\mathcal{Q}(p) := \int_{\mathbb{R}^3} \mathrm{d}\,\mu(p) v_p$ and the quantum momentum $\mathcal{Q}(m)$ for functions m in $C(\mathbb{R}^3)$ depending on the momentum, which is defined by $\mathcal{Q}(m) := \int_{\mathbb{R}^3} \mathrm{d}\,\mu(p) m(p) v_p$. Then another C^*-algebra for Quantum Mechanics is available. This algebra is called the cross-product C^*-algebra. The reduced cross-product C^*-algebra is the completion of the Banach *-algebra $L^1(\mathbb{R}^3, C_0(\mathbb{R}^3))$ in the $L^2(\mathbb{R}^3)$-norm. The representation π_I of the Banach *-algebra $L^1(\mathbb{R}^3, C_0(\mathbb{R}^3))$ on the Hilbert space $L^2(\mathbb{R}^3)$ is given by

$$\pi_I(f)\psi = \int_{\mathbb{R}^3} \mathrm{d}\,\mu(p) f(p) v_p \psi$$

whenever $f \in L^1(\mathbb{R}^3, C_0(\mathbb{R}^3))$ and $\psi \in L^2(\mathbb{R}^3, \mathrm{d}\,\mu)$. The representation π_I is called the integrated representation. The cross-product C^*-algebra in general is completion of the Banach *-algebra $L^1(\mathbb{R}^3, C_0(\mathbb{R}^3))$ with respect to the universal norm.

The unbounded operator $\mathrm{d}\,V(x)$, which is defined as the infinitesimal representation $\mathrm{d}\,V$ of the group $\mathbb{R}^3$, is not contained in this C^*-algebra. These operators are affiliated with the cross-product C^*-algebra. In this context for example Woronowicz and Schmüdgen speak about the C^*-algebra, which is generated by the bounded quantum position operators $\mathcal{Q}(f)$ for every $f \in C_0(\mathbb{R}^3)$ and unbounded quantum momentum operators $\mathcal{Q}(p_i) := i\hbar\,\mathrm{d}\,V(x_i)$ for every $x_i \in \mathbb{R}$ and $i = 1, 2, 3$.

This particular cross-product C^*-algebra is a tranformation group C^*-algebra, and therefore is denoted by $C^*(\mathbb{R}^3, \mathbb{R}^3)$. Furthermore this C^*-algebra is Morita equivalent to the C^*-algebra $\mathcal{K}(L^2(\mathbb{R}))$ of compact operators. The representation theory of the C^*-algebra $\mathcal{K}(L^2(\mathbb{R}))$ is rather simple, since there is only one irreducible representation up to unitary equivalence. Morita equivalence of C^*-algebras implies that, the representation theories of both C^*-algebras are the same. Hence all irreducible representations of the cross-product C^*-algebra are unitarily equivalent to π_I. This result generalises the famous Stone - von Neumann theorem about the uniqueness of the irreducible Schrödinger representation of the Weyl C^*-algebra of Quantum Mechanics.

Furthermore in the same context the twisted transformation group C^*-algebra $C^*_\sigma(\mathbb{R}^3, \mathbb{R}^3)$, where σ is a bilinear form on $\mathbb{R}^3 \oplus \mathbb{R}^3$ is inherited by the exponentiated euclidean inner product $\exp(i\langle .,. \rangle)$ in $\mathbb{R}^3$, is defined similarly to the transformation group C^*-algebra $C^*(\mathbb{R}^3, \mathbb{R}^3)$ of Quantum Mechanics.

Summarising in the presentation above the following C^*-algebras are presented:

· two Weyl C^*-algebras,

· quantum C^*-algebra of position operators, which is isomorphic to the group C^*-algebra,

· the cross-product C^*-algebra, which is also called the transformation group C^*-algebra of Quantum Mechanics, and

· the twisted transformation group C^*-algebra.

In this dissertation the different algebras derived from the concepts presented above are used to define the Weyl C^*-algebra, the analytic holonomy C^*-algebra and the holonomy-flux cross-product C^*-algebra.

1.3.2 Algebras in Quantum Field Theory

A survey on the C^*-algebra approach to Quantum Field Theory has been explored briefly in Streater [80]. A more detailed study can be found in Haag [49].

In Quantum Field Theories the Weyl algebra of a free scalar field in Minkowski spacetime is constructed from the real vector space $C_0^\infty(\mathbb{R}^n)$ and a non-degenerate symplectic form $s : C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \to \mathbb{R}$. On the one hand the Weyl elements are given by a family $\{W(f) : f \in C_0^\infty(\mathbb{R}^n)\}$ of linear independent Weyl elements, which satisfy the canonical commutator relations

$$W(f)W(g) = \exp\Big(-\frac{i}{2}s(f,g)\Big)W(f+g), \quad W(f)^* = W(-f), \quad \forall f, g \in C_0^\infty(\mathbb{R}^n) \tag{1.6}$$

The linear hull of Weyl elements with support in a bounded, open region $\mathcal{O}$ in $\mathbb{R}^n$ forms a *-algebra $\mathcal{W}_0(\mathcal{O})$. The Weyl C^*-algebra $\mathcal{W}(\mathcal{O})$ is defined by the completion of the Weyl *-algebra $\mathcal{W}_0(\mathcal{O})$ with respect to the norm $\|\pi(W)\|_{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$. Moreover another C^*-algebra is constructed from the Weyl *-algebra $\mathcal{W}_0(\mathcal{O})$ completed in the norm $\|.\|$ such that

$$\|W(f)\| := \sup\{\|\pi(W(f))\|_{\mathcal{H}} : \pi \text{ representation of } \mathcal{W}_0(\mathcal{O}) \text{ on a Hilbert space } \mathcal{H}\}$$

whenever $W(f) \in \mathcal{W}_0(\mathcal{O})$. This C^*-algebra is called the universal Weyl C^*-algebra.

On the other hand consider operator-valued distributions $C_0^\infty(\mathbb{R}^n, \mathbb{R}) \ni F \mapsto \exp(i\Phi(F))$, where $\Phi(F)$ is the Segal operator, which act as multiplication operators on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$. Then the Weyl elements given by $W(F) = \exp(i\Phi(F))$ satisfying certain canonical commutator relations define a Weyl *-algebra, too. This *-algebra is completed to a C^*-algebra, which is isomorphic to the Weyl *-algebra $\mathcal{W}_0(\mathbb{R}^n)$. Moreover there exists a polar decomposition of the Segal operator $\Phi(F) := |\Phi(F)| J_{\Phi(F)}$ on the Hilbert space $\mathcal{H}$. The operators $|\Phi(F)|$ and $J_{\Phi(F)}$ are affiliated with a von Neumann algebra $\mathcal{R}(\mathbb{R}^n)$ and hence they generate the von Neumann algebra. The von Neumann algebra is isomorphic to $\pi''(\mathcal{W}(\mathbb{R}^n))$, where π is a suitable representation of the Weyl C^*-algebra on a Hilbert space.

Since for two functions f and k in $C_0^\infty(\mathbb{R}^n)$ with disjoint support in the open region $\mathcal{O}$ the Weyl elements $W(f)$ and $W(k)$ commute, there is a local structure of the Weyl C^*-algebra $\mathcal{W}(\mathcal{O})$. Together with the inductive structure of open sets in $\mathbb{R}^3$ it is possible to define inductive families of Weyl C^*-algebras. Let Q be an unbounded open set in $\mathbb{R}^n$. Then the C^*-algebra $\mathcal{W}(Q)$ is the inductive C^*-limit of an inductive system of families $\{(\mathcal{W}(\mathcal{O}), \beta_{\mathcal{O},\mathcal{O}'}) : \mathcal{O} \subset \mathcal{O}' \subset Q\}$ of C^*-algebras. The family of Weyl C^*-algebras satisfy the following conditions:

(i) Isotony: If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{W}(\mathcal{O}_1) \subset \mathcal{W}(\mathcal{O}_2)$ holds.

(ii) Locality: If $\mathcal{O}_1$ and $\mathcal{O}_2$ spacelike separated regions then for $A \in \mathcal{W}(\mathcal{O}_1)$ and $B \in \mathcal{W}(\mathcal{O}_2)$ it is true that $[A, B] = 0$ yields.

(iii) Translational covariance: The translation group $\mathbb{R}^n$ act as an automorphism on $\mathcal{W}(\mathbb{R}^n)$ where $Q = \mathbb{R}^n$:

$$\forall x \in \mathbb{R}^n : \exists \alpha_x \in \mathfrak{Aut}(\mathcal{W}(\mathbb{R}^n)) : \alpha_x(\mathcal{W}(\mathcal{O})) = \mathcal{W}(\mathcal{O} + x)$$

(iv) Lorentz covariance: The Poincaré group act as an automorphism on $\mathcal{W}(\mathbb{R}^n)$:

$$\beta_{(a,\Lambda)}(\mathcal{W}(\mathcal{O})) = \mathcal{W}(\mathcal{O}_{(a,\Lambda)})$$

where $\mathcal{O}_{(a,\Lambda)}$ denotes the transformed set, which is obtained if each element of $\mathcal{O}$ is translated by the four-vector a and rotated by a Lorentz transformation Λ.

The physically important representations of $\mathcal{W}(\mathbb{R}^n)$ are the non-degenerate particle representations. These representations are strongly continuous unitary representation of the translation group $x \mapsto U(x)$ such that the spectrum of $U(x)$ is contained in the forward light cone and $(\mathcal{H}, U, \pi)$ is a $\mathbb{R}^n$-covariant representation of $\mathcal{W}(\mathbb{R}^n)$. A particular particle representation is associated to a state ω, which is invariant under the action β of the Poincaré group P, i.o.w. $\omega(\beta_{(a,\Lambda)}(W(f))) = \omega(W(f))$ for all $(a, \Lambda) \in P$. Such a representation is for example given by the vacuum representation, which is defined by a GNS-triple associated to the state ω such that $\omega(W(f)) = 0$ for all $f \neq 0$ and $\omega(\mathbb{1}) = 1$.

The theory of covariant representations is related to C^*-dynamical systems and is intensively used in the mathematical description of scalar fields on Minkowski or curved spacetimes. An overview has been presented by Borchers [20, 19]. For example the triple $(\mathcal{W}(\mathcal{O}), \beta, P)$ forms a C^*-dynamical system. This C^*-dynamical system is not the only one, which can be considered. In the theory of QFT different physical symmetries are implemented by different automorphisms on the Weyl C^*-algebra $\mathcal{W}(\mathbb{R}^n)$ or on the local C^*-algebra $\mathcal{W}(\mathcal{O})$ and define C^*-dynamical systems. The implementation of symmetries on different local Weyl C^*-algebras leads for example to the Goldstone phenomenon [80, p.6]. Hence the study of different C^*-dynamical systems is used to analyse physical phenomena.

To summarise the last paragraphs notice that, the Weyl C^*-algebra associated to $\mathbb{R}^n$ or open subsets of $\mathbb{R}^n$, inductive limit Weyl C^*-algebras and von Neumann algebras associated to $\mathbb{R}^n$ or open subsets of $\mathbb{R}^n$ have been presented.

In the LQG approach to quantum gravity the inductive limit of an inductive family of graphs are used to construct the quantum C^*-algebra of configuration variables. The quantum configuration operator is given by the holonomies along a particular set of semi-analytic paths that form a graph. Functions of the quantum holonomies define the analytic holonomy C^*-algebra, which is presented in section 6. Furthermore in this dissertation the Weyl C^*-algebra for LQG is constructed as the inductive limit C^*-algebra of an inductive family of Weyl C^*-algebras. The construction is based on the inductive family of graphs. In this dissertation the author encodes the group-valued quantum flux operators by elements of the flux group associated to a finite set of surfaces and graphs. Precisely for each suitable set of surfaces there exists a certain flux group. In particular the interplay of covariant representations of each flux group, C^*-dynamical systems of each flux group and the analytic holonomy C^*-algebra are analysed. Hence a big bunch of C^*-dynamical systems are constructable for the quantum algebra of configuration variables. The Weyl C^*-algebra is constructed from all unitaries that define covariant representations of a flux group. Since the Weyl algebra depends on the surface set, the C^*-algebra is called the Weyl C^*-algebra for surfaces. Furthermore the author of this work studies C^*-dynamical systems of different groups of special diffeomorphisms and the analytic holonomy C^*-algebra restricted to graphs, and C^*-dynamical systems of different groups of special diffeomorphisms and the Weyl C^*-algebra associated to surfaces and graphs. In this framework it is necessary to restrict the group of diffeomorphisms to certain diffeomorphisms, which preserve the structure of the graphs and the surfaces. Since otherwise they wouldn't define automorphisms on the quantum algebras restricted to graphs. Finally a von Neumann algebra in the context of LQG is derieved, too.

1.3.3 Algebras for Lattice Gauge Theories and Statistical Mechanics

In the following a very short survey of a particular concept, which has been used in lattice gauge theories or statistical mechanics, is illustrated. There is a detailed mathematical theory for inductive limit C^*-algebras, which are derived from families of different physical quantum algebras, and which has been studied by Bratteli and Robinson [22, 21].

Consider a fixed lattice L consisting of a countable number of vertizes. Then the algebra $\mathfrak{A}_L$ for a lattice gauge theory is an infinite C^*-tensor product $\otimes_{v\in L}\mathfrak{A}_v$ of finite-dimensional full matrix algebras $\mathfrak{A}_v = M_n(\mathbb{C})$. For a non-empty finite subset Λ of L study C^*-subalgebras $\mathfrak{A}_\Lambda$ of $\mathfrak{A}_L$.

A classical interaction will be quantised and hence there is a corresponding quantum operator Θ_Λ, which is assumed to be contained in $\mathfrak{A}_\Lambda$. The quantum Hamiltonian H is an operator, which depends on the quantum interaction. Furthermore the quantum Hamiltonian H is defined as a sum over all subsystems Λ in L of the Hamiltonians restricted to subsystems. In general the operator H is mathematically ill-defined. The problem is solved by the concept of symmetric *-derivations on C^*-algebras. The quantum Hamiltonian H is replaced by a *-derivation δ, which is given by

$$\delta(A) := i\sum_{\Lambda\in L}[\Theta_\Lambda, A] \text{ for all } A\in D(\delta) \tag{1.7}$$

where $D(\delta)$ is the domain of the *-derivation δ on a C^*-algebra such that $D(\delta)\supseteq\cup_{\Lambda\in L}\mathfrak{A}_\Lambda$ is satisfied. Notice that, $\cup_{\Lambda\in L}\mathfrak{A}_\Lambda$ is a dense subalgebra of $\mathfrak{A}_L$. The sum converges in operator-norm since if $A\in\mathfrak{A}_{\Lambda'}$ and $\Lambda'\cap\Lambda=\varnothing$ is true, then $[\Theta_\Lambda, A]=0$ holds.

If the lattice is specified, for example if $L=\mathbb{Z}^n$ is considered, then the C^*-dynamical system $(\mathfrak{A}_{\mathbb{Z}^n},\beta,\mathbb{Z}^n)$ is obtained. Denote by C_L the commutative classical C^*-algebra of the quantum lattice L. Then assume that, C_L is a C^*-subalgebra of the C^*-algebra $\mathfrak{A}_L$. Furthermore the interaction is required to be classical, i.o.w. $\Theta_\Lambda\in C_L$ yields, and translation-invariant, i.o.w. $\Theta_\Lambda=\alpha_z(\Theta_\Lambda)=\Theta_{\Lambda+z}$ for every $z\in\mathbb{Z}^n$ is satisfied. Then there exists a self-adjoint operator $\bar{H}$, which is the generator of the translations of $\mathbb{R}$. This operator defines a time evolution of the infinite system, since it is true that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha_{\bar{H}}(t)(A) = \lim_{\breve{\Lambda}\rightarrow\mathbb{Z}^n} c\sum_{\Lambda\in\breve{\Lambda}}[\Theta_\Lambda, A]$$

where c is a constant and suitable $A\in\mathfrak{A}_L$, yields. Hence the dynamics of the system is defined by $\bar{H}$.

Summarising in this subsection an infinite C^*-tensor product of matrix algebras is used to define a quantum algebra for a simple gauge theory. This idea is used to define the localised holonomy-flux cross-product *-algebra. In particular the algebra of quantum configuration variables, which are located on surfaces and on a certain inductive limit graph, is defined to be an infinite C^*-tensor product of matrix algebras instead of an inductive limit C^*-algebra of the inductive family of analytic holonomy C^*-algebras restricted to graphs. Furthermore the Hamiltonian in LQG approach contains the quantum volume operator. This quantum volume operator is constructed from quantum flux operators. This operator is defined as a *-derivation similarly to the *-derivation given by (1.7). The C^*-dynamical system $(\mathfrak{A}_L, \beta, L)$ is replaced by the C^*-algebra of quantum variables, which are located on surfaces and on a certain inductive limit graph, and an action ζ of a particular group, which implements restricted graph-diffeomorphisms. These diffeomorphisms are chosen such that they preserve the inductive limit structure of graphs and the surfaces. Precisely they are invented in section 6.2. Loosely speaking these diffeomorphisms map independent paths to independent paths and surfaces to surfaces with the same orientation with respect to the independent paths. Fortunately they preserve the structure of the quantum flux operators and hence the quantum volume operator. Moreover the particular restricted graph-diffeomorphisms are necessary to define the quantum Hamiltonian constraint, which is given by equation (2.3) in section 2.1 and which is an infinite sum over graphs, as the generator of an automorphism of the algebra of quantum variables.

1.4 General mathematical concepts for the construction of C^*-algebras

In this section the mathematical concept introduced by the physical examples, is investigated. The facts, which will be presented, are well-known in mathematics. In the proceeding sections it is illustrated that, there are difficulties if a more general concept is analysed. These problems have not been studied in the simple physical examples presented before.

1.4.1 Pontryagin duality, quantum groups and cross-product algebras

C^*-algebras of quantum configuration variables

For every locally compact abelian group G the Pontryagin duality holds. This implies that, there is an identification of unitary representations of an abelian locally compact group G with representations of the algebra $C_0(\hat{G})$ of functions on the Pontryagin dual $\hat{G}$ vanishing at infinity. $\hat{G}$ is the set of all continuous group characters on G taking values in the unit circle. The group multiplication of $\hat{G}$ is the pointwise multiplication of characters. The topology of $\hat{G}$ is the compact-open topology, in which a net of elements in $\hat{G}$ converges to an element in $\hat{G}$ if the net converges uniformly to the element $\hat{g}$ on each compact subset of G. Clearly $\hat{G}$ equipped with this structure is a commutative locally compact group.

Let $C_r^*(G)$ be the reduced group C^*-algebra, which is generated by matrix elements of the fundamental representation of a abelian locally compact group G. Therefore via the Gel'fand-Naĭmark theorem there is a *-isomorphism (the Fourier transform) between $C_0(\hat{G})$ and $C_r^*(G)$. Define the following function $\hat{f} \in C_0(\hat{G})$ by

$$\hat{f}(\hat{g}) := N \int_G \mathrm{d}\,\mu_H(g) f(g) \langle g, \hat{g}\rangle \quad \forall f \in C_r^*(G) \tag{1.8}$$

where $U_{\hat{g}}(g) := \langle g, \hat{g}\rangle$ defines a finite-dimensional unitary continuous representation of G and N is a suitable constant. Then there is an isomorphism $\mathcal{F} : C_r^*(G) \to C_0(\hat{G})$, which is defined by $f \mapsto \mathcal{F}(f) = \hat{f}$. Note that, for $G = \mathbb{R}$ the Fourier transform reads

$$\hat{f}(x) := N \int_{\mathbb{R}} \mathrm{d}\,\mu(p) f(p) \exp(i\langle p, x\rangle) \quad \forall f \in C_r^*(\mathbb{R}), \hat{f} \in C_0(\mathbb{R})$$

The the pointwise product in $C_0(\hat{G})$ is given by

$$\hat{f}(\hat{g}) \cdot \hat{k}(\hat{g}) = M \int_G \mathrm{d}\,\mu_H(g) (f * k)(g) U_{\hat{g}}(g) \quad \forall f \in C_r^*(G)$$

For general non-abelian locally compact groups there is in general no isomorphism between the two C^*-algebras.

[1] Pontryagin duality states that there is a mapping $U : G \to \hat{\hat{G}}$ defined by $U(\hat{g})(g) = \hat{g}(g)$ for all $\hat{g} \in \hat{G}$ and $g \in G$, is a group isomorphism and a homeomorphism.

Consequently one has to argue which algebra is more fundamental or which algebra encodes the information about the group G. This problem has been argued by Woronowicz [110]. First assume that G is locally compact and abelian. Then Woronowicz have considered two quantum groups, which are defined by the pairs $(C_r^*(G), \hat{\triangle}_r)$ and $(C_0(\hat{G}), \triangle)$, where $\hat{\triangle}_r$ and $\hat{\triangle}$ are comultiplications. A generalisation of (1.8) is given by the so called generalised Fourier transform:

$$\pi_I(f) := \int_G \mathrm{d}\,\mu_H(g) f(g) U(g) \tag{1.9}$$

whenever U is a continuous unitary representation of G on a Hilbert space $\mathcal{H}$. The map π_I is a representation of $C_r(G)$ on the Hilbert space $\mathcal{H}$. Furthermore the map π_I is compatible with these quantum group structure and therefore $(C_r^*(G), \hat{\triangle}_r)$ and $(C_0(\hat{G}), \triangle)$ are isomorphic as quantum groups. For non-abelian locally compact groups the integrated representation of $C_r^*(G)$ is defined by (1.9), too. But in this case the C^*-algebra $C_r^*(G)$ is non-commutative and hence $(C_r^*(G), \hat{\triangle}_r)$ is not isomorphic (as quantum groups) to a quantum group $(C_0(H), \triangle)$, where H is a locally compact group. The appropriate dual group H of G is not constructable. Woronowicz [114] has argued that, the non-commutative reduced group algebra $C_r^*(G)$ encodes as a quantum group all information about G. For some further comments refer to appendix 12.2.2. Summarising for non-commutative locally compact groups either the group C^*-algebra $C^*(G)$ or the C^*-algebra $C_0(G)$ can be analysed. Unfortunately there need not exists an isomorphism between the two algebras.

Notice that, if G is a connected Lie group, then the basis $T_1, .., T_N$ of the Lie algebra $\mathfrak{g}$ of G are skewadjoint unbounded operators, which are affiliated with the group C^*-algebra $C^*(G)$. Moreover Woronowicz and Napiórkowski [115] have shown that, the group C^*-algebra $C^*(G)$ is generated even by unbounded elements. The arguments are generalised such that it can be used in the framework of the holonomy-flux cross-product C^*-algebra. In this dissertation the classical flux variables are encoded either as G-valued quantum flux operators or Lie algebra-valued quantum flux operators. The latter is defined mostly for a structure group G being a compact Lie group and $\mathfrak{g}$ denotes the associated Lie algebra. In this context the Lie-algebra-valued quantum flux operators are affiliated operators with the holonomy-flux cross-product C^*-algebra. This algebra is constructed from the quantum configuration and quantum momentum operators, which are given by the holonomies along paths and the G-valued quantum flux operators associated to surfaces.

The C^*-algebras of quantum configuration and quantum momentum variables

A general Weyl C^*-algebra is generated by Weyl elements satisfying some canonical commutator relations. In the following the focus lies on abelian locally compact groups. The Weyl algebra $\mathcal{W}$ is generated by Weyl elements, which are constructed by an unitary continuous representation π of G and an unitary continuous representation Π of $\hat{G}$ on a Hilbert space $\mathcal{H}$ such that the canonical commutator relations

$$\pi(g)\Pi(\hat{g})\pi(g)^* = \langle g, \hat{g}\rangle \Pi(\hat{g}) \tag{1.10}$$

are satisfied.

In the case of an arbitrary non-commutative locally compact group G the concept of a G-covariant representation $(\mathcal{H}, U, \pi)$ corresponding to a C^*-dynamical system is useful. The C^*-dynamical system is a triple $(G, C_0(X), \alpha)$, which is given by a point-norm continuous automorphic action α defined by $(\alpha_x f)(y) = f(x^{-1}y)$ of G on $C_0(X)$. The Weyl algebra is constructed from Weyl elements, which are defined by the unitary continuous representations of G on a Hilbert space $\mathcal{H}$. Each unitary representation U and a representation of the C^*-algebra $C_0(X)$ define a covariant pair (U, Φ_M) such that

$$U(g)\Phi_M(f)U^*(g) = \Phi_M(\alpha(g)(f))$$

is satisfied. Clearly if an abelian locally compact group is considered, then this Weyl algebra is equivalent to the Weyl algebra $\mathcal{W}$ defined in the previous paragraph.

For simplicity assume that the set X is equal to G. Then a new C^*-algebra is constructed from the C^*-dynamical system $(G, C_0(G), \alpha)$. This C^*-algebra is called the transformation group C^*-algebra $C^*(G, G)$. Furthermore the C^*-algebra $C^*(G, G)$ is isomorphic to the algebra $\mathcal{K}(L^2(G))$ of compact operators on $L^2(G)$. Rieffel has generalised this result. He has proven the generalised Stone - von Neumann theorem, which states that there is an equivalence of categories of representations of the C^*-algebras. This equivalence is called Morita equivalence. The category of integrated representations of the transformation group C^*-algebra is equivalent to the category of representations of $\mathbb{C}$. Consequently there is a theorem available, which guaranties a uniqueness result for irreducible representations of the transformation group C^*-algebra.

Assume that G is non-commutative locally compact group. Then in this work the Weyl algebra associated to surfaces and graphs is constructed in the context of Loop Quantum Gravity by using the following objects. The algebra is constructed from a set of holonomies along paths, which defines the configuration space $\bar{\mathcal{A}}_\Gamma$, the G-valued quantum flux operators associated to surfaces and graphs, which define different flux groups, and automorphic actions of each flux group on $C_0(\bar{\mathcal{A}}_\Gamma)$. Furthermore the C^*-dynamical system $(C_0(\bar{\mathcal{A}}_\Gamma), \alpha, \bar{G}_{\breve{S},\Gamma})$, which is defined for an action α of a particular flux group $\bar{G}_{\breve{S},\Gamma}$ associated to a surface set $\breve{S}$ and a graph Γ, defines a new algebra. This certain algebra is called the holonomy-flux cross-product C^*-algebra.

1.4.2 O^*- and C^*-algebras generated by unbounded and bounded operators

The well-known Gelfand-Naĭmark theorem states that, any unital commutative C^*-algebra is isomorphic to the algebra of continuous functions on a compact topological space. But even more is true, the category of commutative unital C^*-algebras is dual to the category of topological spaces. The idea of the non-commutative Gelfand-Naĭmark theorem has been presented by Woronowicz and Kruszyński [114]. They have obtained a correspondence between unital non-commutative C^*-algebras and non-commutative spaces. This duality is performed by the concept of a C^*-algebra, which is generated by a finite set of operators. In fact the examples of Quantum Mechanics show that a finite set of unbounded or bounded elements generate a O^*-algebra or even a C^*-algebra. In the context of C^*-algebras unbounded elements are not elements of the algebra, but they are affiliated. The concept of affiliated operators has been introduced by Woronowicz [113, 110]. In the context of O^*-algebras for example Inoue [51] and Schmüdgen [89] have presented a mathematical theory for algebras generated by unbounded elements.

Thus the concept allows to define a huge number of different algebras generated by the quantum configuration and quantum momentum operators of a theory. Ashtekar and Isham [7] have defined the holonomy C^*-algebra, which is generated by the Wilson functions. The Wilson function $\mathrm{tr}(\mathfrak{h}(\gamma))$ depends on a holonomy $\mathfrak{h}$ along a smooth loop γ at v in the loop group[2] $\mathrm{LG}(v)$ and some relations. Until now the spectrum of the holonomy C^*-algebra is not explicity known. The philosophy of Woronowicz has been to define the continuous function algebra of a space vanishing at infinity by using a finite number of unbounded operators. Notice that, for example for a compact Lie group the unitary continuous representations are finite-dimensional. If the holonomy along a path γ is identified with such a representation π on G, i.e. $\mathfrak{h}(\gamma) = \pi(g)$ for $g \in G$ holds, then the Wilson functions $\mathrm{tr}(\mathfrak{h}(\gamma))$ for every loop γ in the loop group $\mathrm{LG}(v)$ for a vertex $v \in \Sigma$ generate a *-algebra, which can be completed to a C*-algebra. Another possibility for a construction of an algebra is to construct the holonomy C^*-algebra by using the algebra of almost periodic functions on the topological loop group $\mathrm{LG}(v)$. The constructions are analysed in section 5.

1.5 A short summary about algebras in Loop Quantum Gravity and Cosmology

The examples of the last sections show that, there are many different degrees of freedom for a construction of algebras of quantum variables. Now the huge variety of algebras for LQG and LQC is studied. To summarise different algebras arise if

(i) the loops or paths are chosen to be smooth, analytic or semi-analytic,

(ii) only certain functions of holonomies along loops or paths are used.

(iii) Different multiplication operations and involutions define different quantum *-, O^*- or C^*-algebras.

(iv) The quantum algebras are completed with respect to different norms such that different Banach *-algebras or C^*-algebras arise.

[2]Notice that the loop group in this project is not the loop group, which is often used in mathematics. The loop group in this context is a larger group, which will be defined in section 3.1.1.

(v) The quantum flux operators are contained in the quantum algebra or are affiliated operators with the algebras of quantum variables.

(vi) If inductive families of graphs are considered, then quantum C^*-algebras are defined as inductive limit C^*-algebras of inductive families of quantum C^*-algebras restricted to graphs.

(vii) Other quantum operators, for example the quantum curvature or a generalised holonomy map, are contained in the quantum algebra or are affiliated operators with the algebras of quantum variables.

In the next section the algebras derived from the different choices are presented shortly with their main properties.

1.5.1 Algebras in Loop Quantum Cosmology

In the model of Loop Quantum Cosmology the algebra of almost periodic functions on the real line has been often used in literature and are presented in section 4. This algebra is isomorphic to the C^*-algebra of continuous functions on the Bohr compactification of the real line. In comparison with the examples in the context of QM and QFT, Weyl algebras over symplectic spaces are constructed for the LQC approach, too. The investigations are shortly studied in section 4.2 and are analysed further in section 7.4. Consequently the concept of algebras generated by the quantum configuration variables leads to the algebra of almost periodic functions on the real line, whereas the Weyl algebras are generated by quantum configuration and quantum momentum variables. Hence in particular the degree of freedom (v) is taken into account.

1.5.2 Algebras in Loop Quantum Gravity

The quantum configuration and momentum operators of the theory

The classical configuration space of Loop Quantum Gravity is the space of smooth connections $\breve{\mathcal{A}}_s$ on an arbitrary principal fibre bundle $P(\Sigma, G)$. In this work the quantum operator $\mathcal{Q}(A)$ of the infinitesimal connection A is given by the G-valued holonomy $\mathfrak{h}$ along a path γ. The operator $\mathcal{Q}(A)$ is implemented as a multiplication operator on a certain Hilbert space $\mathcal{H} := L^2(\bar{\mathcal{A}}, \mu)$, where $\bar{\mathcal{A}}$ is the quantum configuration space and μ is a measure thereon. Let $\mathfrak{g}$ be the associated Lie algebra of a compact Lie group G. Then the quantum operator $\mathcal{Q}(E^i)$ of the classical fluxes E^i is given by the $\mathfrak{g}$-valued quantum flux operator $E_S(\gamma)$, which is associated to a surface S and a path γ. The canonical commutator relation is given by

$$[E_S(\gamma), \mathfrak{h}(\gamma)] = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \exp(tE_S(\gamma))\mathfrak{h}(\gamma) - \mathfrak{h}(\gamma)E_S(\gamma) \tag{1.11}$$

whenever $t \in \mathbb{R}$. Set

$$E_S(\gamma)\mathfrak{h}(\gamma)] := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \exp(tE_S(\gamma))\mathfrak{h}(\gamma)$$

Furthermore the *right-invariant flux vector field* $e^{\overrightarrow{L}}$ is defined by

$$[E_S(\gamma), f_\Gamma] := e^{\overrightarrow{L}}(f_\Gamma)$$

where

$$e^{\overrightarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\exp(tX_S)\mathfrak{h}_\Gamma(\gamma)) \text{ for } X_S \in \mathfrak{g}, \mathfrak{h}_\Gamma(\gamma) \in G, t \in \mathbb{R} \tag{1.12}$$

whenever $f_\Gamma \in C_0^\infty(\bar{\mathcal{A}}_\Gamma)$.

The differential operators $\frac{\mathrm{d}}{\mathrm{d}\,t}\exp(tE_S(\gamma))$ act on the Hilbert space $\mathcal{H}$. The flux $E_S(\gamma)$ is given by the value of a map $E_S : P\Sigma \to \mathfrak{g}$ evaluated for a path γ in the set $P\Sigma$ of paths in Σ. This definition does not coincide with the usual definition presented in LQG literature completely. In this dissertation the full intersection behavior is encoded in the $\mathfrak{g}$-valued quantum flux operator $E_S(\gamma)$ or the G-valued quantum flux operator $\exp(E_S(\gamma))$. These definitions are presented in detail in section 3.4.

Then the following algebras are constructed.

The smooth holonomy C^*-algebra

The set of smooth loops modulo a suitable equivalence relation forms a fundamental group at a vertex v in Σ and a set of paths modulo some equivalence relations forms a fundamental groupoid. Notice that, for different equivalence relations different loop groups exists. These groups are analysed in section 3.1.1 and 3.1.2. One exceptional fundamental group is given by the loop group $\mathrm{LG}(v)$. This group has been originally introduced by Barrett [16]. He have presented the concept of holonomy maps for this loop group, which is generalised to fundamental or path groupoids in section 3.3.5. Furthermore Barrett have equipped the loop group with a topology by using the holonomy map.

Since one distinguishes between the smooth and (semi-)analytic loops and paths in a smooth and (semi-)analytic manifold, there exists the smooth and the analytic holonomy C^*-algebra. This correponds to the degree of freedom (i). Both algebras are unital commutative C^*-algebras. Therefore both algebras are isomorphic to a continuous function algebra on compact quantum configuration spaces, which are denoted either by $\bar{\mathcal{A}}_s$ or $\bar{\mathcal{A}}$. Notice if only loops are considered, then the quantum configuration spaces are denoted by $\bar{\mathcal{A}}_s^L$ and $\bar{\mathcal{A}}^L$.

The idea of the definition of cylindrical function algebra for the loop group $\mathrm{LG}(v)$ has been introduced by Ashtekar and Isham [7] and is shortly presented in section 5. In this dissertation the author reformulates and extends the idea of Ashtekar and Isham. The *-algebra $AP(\mathrm{LG}(v))$ of almost periodic functions w.r.t. the topological loop group $\mathrm{LG}(v)$ at $v \in \Sigma$ is studied. It is assumed that, the structure group G is the associated compact group to $\mathrm{LG}(v)$ in the sense of Dixmier [33]. The completion of $AP(\mathrm{LG}(v))$ in the supremum norm is denoted by $\mathrm{Cyl}(\mathrm{LG}(v))$. The unital commutative C^*-algebra $\mathrm{Cyl}(\mathrm{LG}(v))$ is isomorphic to the C^*- algebra $C(\bar{\mathcal{A}}_s^L)$ of continuous function on the compact quantum configuration space $\bar{\mathcal{A}}_s^L$ restricted to loops.

A new C^*-algebra is given by the group C^*-algebra $C^*(\mathrm{LG}(v))$ generated by matrix elements of the fundamental representation of G and the loops in $LG(v)$. Apart from the difficulties of a precise definition of this algebra, there is no obvious reason for an isomorphism between the C^*-algebras $\mathrm{Cyl}(\mathrm{LG}(v))$ and $C^*(\mathrm{LG}(v))$. Recall that, in the context of quantum mechanics there is an isomorphism between the group algebra w.r.t. $\mathbb{R}$ and the algebra of continuous functions on $\mathbb{R}$ vanishing at infinity. The algebra constructed from the Wilson functions $\mathrm{tr}(\mathfrak{h}(\gamma))$ by Ashtekar and Isham [7] is called the Wilson C^*-algebra in this dissertation. This algebra does not coincide with the C^*-algebra $\mathrm{Cyl}(\mathrm{LG}(v))$ apriori. In this dissertation a modified Wilson C^*-algebra for the discretised loop group $\mathrm{LG}_\mathrm{d}(v)$ is presented in section 5.3. This algebra is generated by certain Wilson functions and a convolution product between the Wilson functions. This algebra is suggested to be isomorphic to the center of the group C^*-algebra $C^*(\mathrm{LG}_\mathrm{d}(v))$. The possibility of the definition of the C^*-algebra of almost periodic functions on $\mathrm{LG}(v)$, the group C^*-algebra on $\mathrm{LG}(v)$ or $\mathrm{LG}_\mathrm{d}(v)$, the Wilson C^*-algebra and the modified Wilson C^*-algebra is related to the degrees of freedom (ii) and (iii).

Furthermore there is a generalisation of the concept of almost periodic functions to path groupoids and systems of paths. Note that, a system of paths is a set of paths that are not necessarily independent. This is contrary to the definition of a graph, which is a set of independent paths, and the definition of graph systems, which are sets of graphs. The inductive limit system of paths of the inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of systems of paths is denoted by $\mathcal{P}_\infty\Sigma$. The commutative C^*-algebra $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$ of cylindrical functions w.r.t. the limit system $\mathcal{P}_\infty\Sigma$ is constructed. Therefore it is used that the limit is defined by an inductive family $\{(\mathrm{Cyl}(\mathcal{P}_\Gamma\Sigma), \beta_{\Gamma,\Gamma'}) : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}\}$ of C^*-algebras. The space $\bar{\mathcal{A}}_s$ is equivalent to the set $\mathrm{Hom}(\mathcal{P}_\Gamma\Sigma, G)$ of all holonomy maps and is called of the space of generalised connections.

The analytic holonomy and Weyl C^*-algebra

In the context of (semi-)analytic paths, geometric objects like finite path groupoids, graphs and finite graph systems are defined in section 3.1.3. The holonomy maps for such finite path groupoids and finite graph systems are investigated in section 3.3.4. Originally Ashtekar and Lewandowski [8] have introduced the analytic holonomy C^*-algebra only for holonomy maps along analytic loops. Later Fleischhack [39] has presented a construction for a Weyl C^*-algebra for semi-analytic paths. In this work instead of paths only finite graph systems are used for the development of different algebras of quantum operators related to semi-analytic paths and surfaces. The reason for this choice is related to the precise construction, which has been developed by Ashtekar and Lewandowski [8], and is presented in section 3.3.4. Moreover the author assumes that the analytic holonomy C^*-algebra is given by the inductive limit $C(\bar{\mathcal{A}})$ of the inductive family $\{C(\bar{\mathcal{A}}_\Gamma)\}$ of unital commutative analytic holonomy C^*-algebras associated to finite graph systems. Let G be a Lie group and $\mathfrak{g}$ the associated Lie algebra to G. Then the quantum analogue of a classical flux variable is defined by a $\mathfrak{g}$-valued quantum operator associated to a surface and a path. The exponentiated $\mathfrak{g}$-valued quantum operator associated to a surface and a path is called the G-valued quantum flux operator. If G is chosen to be a locally compact group, then the quantum flux operator is always G-valued. The detailed structure is given in section 3.4. In section 6 the Weyl C^*-algebra $\mathfrak{Weyl}(\mathbb{S})$ for surfaces is developed similar to the Weyl C^*-algebra of Fleischhack [39]. There exits an exceptional state, which is invariant under some particular actions of different groups, of a certain Weyl C^*-algebra. In particular invariance for some particular diffeomorphisms, which preserve the structure of the Weyl C^*-algebra associated to surfaces and graph systems, is required. For example the diffeomorphisms are assumed to preserve the graphs structure and hence they are called graph-diffeomorphisms. This issue is presented in section 6.4. The existence of this state is comparable with the uniqueness result for the Weyl C^*-algebra of Fleischhack [39]. It is used that pure states correspond to irreducible representations via GNS-construction. The uniqueness depends on the identification of the quantum configuration space $\text{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|})$ of holonomy maps along independent paths of a graph Γ and the product group $G^{|\Gamma|}$. Moreover the uniqness theorem is not given for the full Weyl C^*-algebra for surfaces, since the certain graph-diffeomorhisms do not define automorphisms on the full Weyl C^*-algebra for surfaces in general. The action of a (surface or surface-orientation preserving) graph-diffeomorphism and the action of a flux group associated to surfaces and graphs on the analytic holonomy C^*-algebra restricted to a graph system do not commute in general. Summarising, the degrees of freedom (i), (v) and (vi) are used for the development of the Weyl C^*-algebra for surfaces.

Finally the same objects, which define the Weyl algebras, also generate the holonomy-flux von Neumann algebra. This algebra is presented in section 6.5. This feature is related to the degree of freedom (iii).

The holonomy-flux cross-product C^*-algebra

In section 7 the flux group, the flux transformation group, the non-commutative holonomy, the heat kernel holonomy, the holonomy-flux cross-product and the holonomy-flux-graph-diffeomorphism cross-product C^*-algebra are presented for a certain surface set, a graph system and an inductive limit of an inductive family of graph systems. This new C^*-algebras are defined by using the degrees of freedom (i), (iv), (v) and (vi). Furthermore, graph-diffeomorphism invariant states of a holonomy-flux cross-product C^*-algebra are studied. It turns out that, there is a state ω^Γ_E, which is not graph-diffeomorphism invariant, on the holonomy-flux cross-product C^*-algebra associated to a surface set and a graph system. For particular surfaces and graph systems the holonomy-flux cross-product C^*-algebra is identified with the transformation group C^*-algebra $C(G^{|\Gamma|}, G^{|\Gamma|})$, where $G^{|\Gamma|}$ is the product group of a locally compact group G. Hence the generalised Stone - von Neumann theorem implies the uniqueness of the representation associated to ω^Γ_E on the Hilbert space $L^2(G^{|\Gamma|}, \mu_{|\Gamma|})$. Notice that, $\mu_{|\Gamma|}$ is the Haar measure on the product group $G^{|\Gamma|}$. Finally the multiplier algebra of a certain holonomy-flux cross-product C^*-algebra is defined and it is shown that every other holonomy-flux cross-product C^*-algebra is contained in this multiplier algebra.

The concept of cross-product C^*-algebras is also introduced in the context of Loop Quantum Cosmology and is presented in section 7.4.

Some analytic holonomy *-algebras

The analytic holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ is generated by matrix elements $T_{\gamma,\pi_s,m,n} := \pi_s(\mathfrak{h}(\gamma))^m_n$ of the holonomy map $\mathfrak{h}$ of a path groupoid $\mathcal{P} \rightrightarrows \Sigma$ along a path γ, where γ runs over all paths in $\mathcal{P}$, π_s runs over all (equivalence classes of) irreducible representations of a compact group G, and m and n runs over all the corresponding matrix indices. In section 8.1 another *-algebras constructed from matrix elements $T_{\gamma,\pi_s,m,n}$ are presented. This is related to the degrees of freedom (ii) and (iii).

The holonomy-flux cross-product *-algebras

Guided by the ideas of the construction of O^*-algebras in the example 1.3.1.1 new *- and O^*-algebras can be constructed for LQG. Let G be a compact Lie group and let $\mathfrak{g}$ be the associated Lie algebra of G. Then the algebras are generated by the holonomies along paths and $\mathfrak{g}$-valued quantum fluxes associated to surfaces. In comparison to the *-algebra presented by Lewandowski, Okołów, Sahlmann and Thiemann[64], the new holonomy-flux cross-product *-algebra is generated by the identity $\mathbb{1}$, the holonomies $\mathfrak{h}(\gamma)$ and the quantum flux $E_S(\gamma)$ satisfying the canonical commutator relations. The construction of both *-algebras only distinguishes in the definition of a particular multiplication operation of the vector space. This is studied in section 8. There is a uniqueness result for a particular diffeomorphism invariant state of a certain holonomy-flux cross-product *-algebra associated to a surface set. Furthermore an argument for the difficulty of a construction of other states, which are not necessarily invariant under a huge particular diffeomorphsim group, is presented.

A simple modification of the holonomy-flux cross-product *-algebra is given in section 8.3. A new *-algebra construction, which is influenced by the ideas of the example 1.3.3 in Lattice Gauge Theory and Statistical Mechanics, is presented in section 8.4. The new *-algebra is called the the localised holonomy-flux cross-product *-algebra. This *-algebra will be analysed further in the context of quantum constraints and quantum complete observables. Based on the ideas of Woronowicz in [115] and of Buchholz and Grundling in [26], the holonomy-flux Nelson transform *-algebra is constructed in section 8.5. This algebra is generated in the sense of Woronowicz by a finite set of operators. The flux operators associated to one surface S are replaced by an object, called the flux Nelson transform, which depends on three surfaces in Σ. The idea is to construct an operator, which can be related to the quantum volume in LQG. This is not finished until now and will be a further project for the author.

Summarising the degrees of freedom (i), (ii), (iii), (iv), (v) and (vi) are used for the construction of the different *-algebras.

The holonomy groupoid C^*-algebras for gauge and gravitational theories

However none of these *- or C^*-algebras of the previous paragraphs contain a quantum analogue of the classical variable curvature. This is the degree of freedom (vii). Another reason for a reformulation of the algebras of quantum variables is related to the fact that, the classical Hamilton constraint, which is given by (2.2), contains curvature. This Hamilton constraint cannot be quantised without a modification of this operator by replacing the curvature. The aim of this dissertation is to find a suitable algebra of quantum variables of the theory. This algebra is specified by the fact that, the quantum Hamilton constraint is an element of (or is affiliated with) this new algebra. Moreover the algebra is assumed to be generated by certain holonomies along paths, quantum fluxes and the quantum analogueue of curvature.

Quantisation of a gravitational theory in the context of LQG uses substantially the duality between infinitesimal objects and holonomies along paths in a path groupoid. Barrett [16] has presented a roadmap for the construction of the configuration space of two physical examples: Yang-Mills and gravitational theories. He have suggested to consider all holonomy maps along loops in a certain loop group. In section 3.1.1 and 3.1.2 different groups of loops and different groupoids of paths are presented. The holonomy maps of Barrett are further generalised for the example of a gauge theory. The construction is based on the concept of path connections in a Lie groupoid, which has been introduced by Mackenzie [66]. In section 3.1.4 several examples for Lie groupoids and some of their properties are collected. The simpliest Lie groupoid is given by a Lie group G over $\{e_G\}$. The new framework of Mackenzie allows to study instead of the holonomy maps, which have been presented by Barrett [16], generalised holonomy maps associated to path connections in a groupoid. In general the holonomy map in a Lie groupoid is a groupoid morphism from a path groupoid to a Lie groupoid, which satisfy some new conditions. In section 3.3.4.2 some new path groupoids, which are called path groupoids along germs, are studied. For these certain path groupoids a general holonomy map in the groupoid G over $\{e_G\}$ is defined. This new holonomy map corresponds one-to-one to a smooth path connection. There exists another example of a Lie groupoid, which is given by the gauge groupoid associated to a principal fibre bundle. This groupoid is introduced in section 3.1.4. For gauge theories the holonomy maps in the gauge groupoid are defined as a groupoid morphisms from a path groupoid to the gauge groupoid. These groupoid morphisms correspond uniquely to path connections, too. In this situation a path connection is an integrated infinitesimal smooth connection over a lifted path in the gauge groupoid w.r.t. a principal fibre bundle $P(\Sigma, G, \pi)$. Then the holonomy map is related to a parallel transport in the principal fibre bundle. Furthermore a holonomy map for a gauge theory defines a holonomy groupoid for a gauge theory. Notice that the ordinary holonomy map in the framework of LQG has been defined by a groupoid morphism form the path groupoid to the simple Lie groupoid G over $\{e_G\}$. Since in this case the holonomy mapping maps paths to the elements of the structure group G, the holonomy map does not define a parallel transport in $P(\Sigma, G)$. Finally the generalisation of Barrett's idea is to consider all general holonomy maps in a Lie groupoid as the configuration space of the theory. For example for gauge theories the set of holonomy maps in the gauge groupoid is defined in section 3.3.3.1. The duality between infinitesimal connections and these new holonomy maps are reviewed very briefly in section 3.2. A detailed analysis why the theory of Mackenzie really generalises the examples of Barrett is presented in section 3.3.1 and 3.3.2.

The next step is to find a replacement of the curvature. Moreover the particular decomposition of the Ashtekar connection and curvature associated to the Ashtekar connection and the way how the foliation is embedded into the spacetime M is required to be encoded somehow in the quantum algebra of gravity. Clearly this is very hard to be obtained in a background independent manner. The problems of implementing infinitesimal structures like infinitesimal diffeomorphisms and curvature arise from the special choice of the analytic holonomy C^*-algebra defined in section 6. Furthermore it turns out that the modification of the conditions for the holonomy mappings presented in section 3.3.4 are not sufficient for the implementation of quantum curvature. This is the reason why the author suggests to replace the configuration space of the theory. Hence all holonomy maps in a gauge groupoid replace the original quantum configuration variables. Each holonomy map defines a holonomy groupoid such that a family of C^*-algebra depending on a holonomy groupoid associated to a path connection is constructable. The C^*-algebras are called holonomy groupoid C^*-algebras of a gauge theory and are presented in chapter 9. In this framework it is used that, instead of a measure on the configuration space a family of measures are defined on a Lie groupoid. This is more general than the original approach of a measure on the quantum configuration space in LQG. There the quantum configuration space of generalised connections restricted to a graph is identified with the product group $G^{|\Gamma|}$ of the structure group G. Finally for gauge theories the quantum curvature is implemented by the following theorem. The generalised Ambrose-Singer theorem, which has been given by Mackenzie [66], states that the Lie algebroid of the holonomy groupoid of a path connection is the smallest Lie algebroid, which is generated from infinitesimal connections and curvature. Furthermore there is an action related to infinitesimal connections and curvature, since both objects are encoded as elements of a Lie algebroid, on the holonomy groupoid C^*-algebra. This is studied in section 9.1. Summarising the algebra of quantum variables for a gauge theory is generated by the holonomy groupoid C^*-algebra of a gauge theory, the quantum flux operators and the Lie algebroid of the holonomy groupoid of a path connection. The construction of the algebra generated by the holonomy groupoid C^*-algebra of a gauge theory and the quantum flux operators is similarly to the holonomy-flux cross-product C^*-algebra, which is defined in chapter 7. The development for groupoids has been presented by Masuda [68, 69]. In this dissertation the ideas are illustrated in section 9.2. For the construction a left (or right) action of the holonomy groupoid on the C^*-algebra $C_0(G)$ of continuous functions on the Lie group G is necessary to define a C^*-groupoid dynamical system. This object replaces the C^*-dynamical systems of C^*-algebras, groups and actions of these groups on these algebras. The new cross-product algebra contains holonomies and in some appropriate sense this algebra is generated by curvature and fluxes. A detailed description of this construction is a further project.

Finally two important remarks are presented as follows. First recognize that, there exists a set of holonomy groupoids each associated to a path connection, which is contained in a set of path connections. This implies that, there is a set of holonomy groupoid C^*-algebras each associated to a path connection, too. Furthermore there are morphisms between these algebras. Hence there is a natural structure of a category available. Secondly the whole construction of the algebras basically depends on the chosen principal fibre-bundle $P(\Sigma, G, \pi)$ and consequently in particular on the base manifold Σ. Therefore the background independence is not fulfilled. Now the idea is to use the covariance principle, which has been developed by Brunetti, Fredenhagen and Verch [24] in the framework of algebraic quantum field theory. Some first analysis is presented in section 9.3 and will be studied in a further work.

The next step is to take into account that, the Hamiltonian framework of gravity is given w.r.t. the orthonormal framebundle on Σ. Therefore the gauge groupoid has to be replaced by the orthornormal frame groupoid $\mathcal{F}\left(\frac{O^+(\Sigma,q)\times so(3)}{SO(3)}\right)$ over Σ. An analysis of this object has been presented by Mackenzie [66] and the implementation into the new holonomy groupoid formulation is a future project.

To summarise, the new ideas presented above will be used to solve the following problems:

- the assignement of the quantum analogue to the classical Hamilton constraint, which contains curvature;
- there is an enlargement of the original quantum configuration space;
- background independence of the holonomy groupoid formulation of LQG and
- a description for quantised gravitational theories and gauge theories.

1.6 Comparison of QM, QFT and LQG algebras

The Weyl C^*-algebras in Quantum Mechanics for particles moving in $\mathbb{R}^3$ and in Quantum Field Theory for scalar fields in Minkowski spacetime are based on different symplectic forms on different real vector spaces. The canonical commutation relations (the CCR's) describe the significant incompatibility of quantum position and quantum momentum in the quantum theory, whereas classically these variables commute. Furthermore the Weyl elements that generate the Weyl C^*-algebra of QM are encoded as (projective) unitary (weakly) continuous representation of $\mathbb{R}^n$ on a Hilbert space.

In Loop Quantum Gravity (LQG) the non-commutativity is manifest in both in the classical and quantum regime. In the classical theory the Ashtekar connection and the field satisfy the commutator relation 1.3. Furthermore from the non-commutativity of the the group G, it follows that the G-valued holonomy along a path and the G-valued the quantum flux operator associated to a surface and a path do not commute. The CCR's of holonomies and fluxes have a more complicated form, since moreover they depend on surfaces and path orientations. Furthermore the Weyl algebra for LQG is not constructed by symplectic forms on vector spaces such that canonical commutation relations similarly to 1.4 or 1.6 exist. Consequently the ideas for a construction of quantum algebras, which has been invented for QM and QFT in the last section, have to be carefully transferred. In particular there exists not necessarily isomorphisms between different algebras of quantum configuration operators in the Loop Quantum Gravity approach.

In general quantum physics, especially quantum mechanics, has been also studied in the language of von Neumann algebras. A Neumann algebra is a C^*-algebra of a special kind. The nature of quantum physics is given by the probabilistic behaviour of physical events. In mathematics, the classical Borel integration and measure theory describing probability is generalised to von Neumann algebras and weights. Hence it is natural to consider von Neumann algebras in quantum physics. A von Neumann algebra of holonomies and quantum fluxes is constructed in the Loop Quantum Gravity approach in this dissertation. In the context of von Neumann algebras the theory of Tomita-Takesaki is very useful. But in section 6.5 it is easily shown that, the von Neumann algebra is not of the standard form. Consequently the basis for the KMS-theory is not given. Mathematically KMS-theory has been also derived for C^*-algebras. But it is proven that, the Weyl C^*-algebra for surfaces does not admit any KMS-states. Nevertheless the theory of KMS-states is available for other C^*-algebras constructed from holonomies and is used for the implementation of quantum constraints in section 8.4.

Chapter 2

Quantum constraints, KMS-Theory and dynamics

In this section a first short overview about the following issues is presented in the framework of Loop Quantum Gravity:

· the classical and quantum system of constraints,

· the Dirac states,

· complete observables and

· KMS-states.

2.1 The implementation of quantum constraints on algebras of Loop Quantum Gravity

In the Hamilton formalism of General Relativity a set of constraints appears. The constraints form a classical constraint algebra on the hypersurface Σ. The classical variables are replaced by the quantisation map with Hilbert space operators or operator algebra elements. The structure of the constraint algebra is very difficult, since there is an infinite number of constraints. Thiemann [98, 94] has analysed the quantum constraint algebra derived from holonomy and flux operators on a certain kinematical Hilbert space. In particular a formula for the quantum Hamilton constraint has been presented. There is another idea to deal with a set of infinite classical constraints due to Thiemann. This is the Master constraint project, which has been developed by Thiemann and Dittrich [32, 102]. The concept of the Master constraint is reformulated in terms of quantum algebras and states. Originally in the LQG framework the quantum constraints have been usually given by Hilbert space operators. The implementation of the constraints has been done for example by using rigging maps on Hilbert spaces. These ideas have been presented by Ashtekar, Lewandowski, Marolf, Mourão and Thiemann [12], Giulini [46] and Giulini and Marolf [47, 48]. In the operator algebra viewpoint used in this dissertation the constraints define particular states, which are contained in the state space of the algebra of quantum variables. It is required that, this algebra contains the set of constraints (or at least the constraints are affiliated to the algebra). A state that implements in this sense the constraint is called a *Dirac state*. In particular these states are invariant under automorphisms of the algebra, which are derived from the constraints. Furthermore the time avarage of an operator is defined as a suitable state on the algebra. In LQG a contrary viewpoint has been often used, there the time avarage is given by an operator T on a Hilbert space $\mathcal{H}$.

The implementation of constraints in a classical theory of gravity has been studied by many authors. In this dissertation the work of Dittrich [30, 31] is focused. Dittrich has used the Dirac formalism to perform a set of classical constraints that defines a constraint function algebra on the phase space. This algebra is equipped with certain Poisson brackets on the phase space. Physical or Dirac observables are suitable phase space functions implemented by some particular Poisson brackets. In LQG approach to quantum gravity the classical Hamiltonian, which implements the dynamics of the system, is a constraint of the classical system, too. Therefore the physical observables do not evolve with respect to this Hamiltonian. The evolution of a physical observable has to be related to a physical freedom of the system. This corresponds to a choice of a so called clock variable. In this dissertation these concepts are introduced in the next sections.

2.1.1 The classical hypersurface deformation constraint algebra

Thiemann [103, 101, 102] has presented for the classical gravitational theory and the manifold $M = \Sigma \times \mathbb{R}$ with hypersurface metric $q = (q_{ab})$ the following constraints:

- $C_b(x)$ ... the spatial diffeomorphism constraint,
- $C(x)$ ... the Hamilton constraint and
- $\mathcal{G}(x)$ ... the Gauss constraint.

for every $x \in \Sigma$. The set of smeared constraints contains in particular all linear combinations of the smeared constraints:

- $\vec{C}(\vec{N}) = \int_\Sigma d^3x N^a(x) C_a(x)$ ... the smeared spatial diffeomorphism constraint
- $C(N) = \int_\Sigma d^3x N(x) C(x)$ ... the smeared Hamilton constraint

which satisfy the following relations

$$\begin{aligned}
\{\vec{C}(\vec{N}), \vec{C}(\vec{N}^{'})\} &= \kappa \vec{C}(\mathcal{L}_{\vec{N}} \vec{N}^{'}) \\
\{\vec{C}(\vec{N}), C(N')\} &= \kappa \vec{C}(\mathcal{L}_{\vec{N}} N') \\
\{C(N), C(N')\} &= \kappa \int_\Sigma d^3x (N_{,a}N' - NN'_{,a})(x) q^{ab}(x) C_b(x)
\end{aligned} \tag{2.1}$$

whenever κ is a constant, $\mathcal{L}_{\vec{N}}\overrightarrow{N}'$ is the Lie derivative for a vector valued function $\overrightarrow{N}'$ on Σ and $\mathcal{L}_{\vec{N}}N$ is the Lie derivative for a scalar valued function N on Σ. This set forms the *classical hypersurface deformation constraint algebra*.

Summarising the *algebra of hypersurface quantum constraints* contains the quantum analogues of the classical spatial diffeomorphism constraints, the classical Hamilton constraints and the classical gauge constraints such that these quantum operators satisfy some certain commutator relations.

In the LQG framework the quantum constraint algebra is generated by the quantum gauge constraints, the quantum diffeomorphism constraints and the quantum Hamilton constraint. In this dissertation the quantum gauge constraints are replaced by elements of the *local flux group*. The elements of the *fixed point algebra associated to the action of the local flux group* presented in section 6.2, are invariant under the action of the local flux group. In the next subsection the quantum diffeomorphism constraints are analysed.

2.1.2 The quantum spatial diffeomorphism constraints

The quantum spatial diffeomorphism constraints are assumed to be bisections of a path groupoid, a finite path groupoid or a finite graph system. The group $\text{Diff}(\Sigma)$ of classical diffeomorphisms in the spatial manifold Σ is replaced by the group $\mathfrak{B}(\mathcal{P})$ of bisections in a path groupoid $\mathcal{P}$ over Σ. In the next paragraph this issue is treated in detail.

There exists a parameter group, which is given by $\text{Diff}(\Sigma) \ni \varphi \mapsto \zeta_\varphi \in \mathfrak{Aut}(\mathfrak{A})$, of automorphisms on the commutative Weyl C^*-algebra $\mathfrak{A}$ for surfaces of Loop Quantum Gravity. Indeed a diffeomorphism φ in $\text{Diff}(\Sigma)$ is given by a bisection σ in $\mathfrak{B}(\mathcal{P})$ through the map $t \circ \sigma$. Consequently this parameter group of automorphisms is reformulated by the parameter group $\mathfrak{B}(\mathcal{P}) \ni \sigma \mapsto \zeta_\sigma \in \mathfrak{Aut}(\mathfrak{A})$ of automorphisms on $\mathfrak{A}$. The remarkable property of this parameter group is that, the automorphism ζ of the C^*-algebra $\mathfrak{A}$ is not point-norm continuous. This is verified by the following argument: For a sequence of paths that converges to the constant path at a vertex, the sequence of holonomy maps along the paths does not converge. Hence finite classical diffeomorphisms and bisections of a finite path groupoid have to be considered. The parameter group of automorphisms defined by morphisms from the group of bisections on a finite path groupoid to the automorphism group on the C^*-algebra $\mathfrak{A}$, is point-norm continuous. Note that, since the Weyl algebra for surfaces is constructed from an inductive limit of an inductive family of C^*-algebras restricted to finite graph systems, the group of bisections are related to finite graph systems instead of finite path groupoids.

2.1.3 The quantum Hamilton constraint

In the last subsection the classical Hamilton constraint $C(x)$ of LQG has been shortly introduced. In the next paragraphs the quantisation of this constraint, which has been proposed by Thiemann [98], is reviewed briefly.

Thiemann has presented the following classical expression for the classical Hamilton constraint

$$C(x) = \frac{1}{\sqrt{det(q)}} tr((F_{ab} - [K_a, K_b])\,[E^a, E^b]) \tag{2.2}$$

where F_{ab} is the curvature of the connection A, $\beta K_a = A_a - \Gamma_a$ is the extrinsic curvature and β is the Immirzi parameter.

Then Thiemann has used the classical identity:

$$\frac{[E^a, E^b]}{\sqrt{det(q)}} = \epsilon^{abc}\{A_c, V\} \text{ with } V = \int \mathrm{d}^3\, x\sqrt{det(q)}$$

The volume V is encoded in the quantum operator $\mathcal{Q}(V)$, which is mainly given by a product of fluxes.

Then the quantum map of the Poisson bracket $\{A, V\}$ is given by

$$\mathcal{Q}(\{A, V\}) = \mathfrak{h}_A(e_\triangle)[\mathfrak{h}_A(e_\triangle)^{-1}, \mathcal{Q}(V)]$$

where $\mathfrak{h}_A(e_\triangle)$ denotes the holonomy of a connection A along a path $e_\triangle$, which is given by the triangulation T of Σ. Moreover the quantum operator associated to the classical variable F is presented by

$$\mathcal{Q}(2\epsilon^2 F(x)) = \mathfrak{h}_A(l_\triangle) - \mathfrak{h}_A(l_\triangle)^{-1}$$

where $l_\triangle$ is a loop at some base point x and which is given by the triangulation T of Σ.

The quantum Hamilton constraint for a suitable triangulation T of Σ is given by

$$\mathcal{Q}(C(N))_T = \sum_{\triangle\in T} Tr(\big(\mathfrak{h}_A(l_\triangle) - \mathfrak{h}_A(l_\triangle)^{-1}\big)\,\mathfrak{h}_A(e_\triangle)[\mathfrak{h}_A(e_\triangle)^{-1}, \mathcal{Q}(V)]) \tag{2.3}$$

The triangulation T of a manifold Σ and the choice of loops $l_\triangle$ and edges $e_\triangle$ with respect to the triangulation define a certain graph Γ. Finally the quantum Hamilton constraint operator with respect to the whole hypersurface Σ is presented by a (norm-)limit of operators

$$\mathcal{Q}(C(N)) := \lim_{T\rightarrow\Sigma} \mathcal{Q}(C(N))_T \tag{2.4}$$

The quantum Hamilton constraint operator and some other Hamiltonians have been analysed by Thiemann in many articles. For example the Hamilton constraint has been implemented in [98, 96, 94, 93, 95, 97, 100] by using a certain regularisation of this constraint operator and particular triangulations of the manifold. Furthermore Aastrup, Grimstrup and Nest [3, 2, 4, 1] have pointed out that the holonomies along paths are differentiable continuous function on $\bar{\mathcal{A}}$ with values in a finite matrix algebra $M_n(\mathbb{C})$ and hence the quantum Hamilton constraint is a (pseudo) differential operator on the algebra of differentiable continuous function on $\bar{\mathcal{A}}$ with values in a finite matrix algebra $M_n(\mathbb{C})$. In the simpliest case the quantum Hamilton constraint operator contains the the holonomies along paths and the quantum flux operators, which are operators represented as multiplication or (pseudo) differential operators on the Ashtekar-Lewandowski Hilbert space. If other Hilbert space representations of the operators are considered, then the existence of a well-defined quantum Hamilton constraint, which is given by a limit of the sum over all triangulations, is not clear. Similarly the one-parameter group of automorphisms derived from the quantum Hamilton constraint on the analytic holonomy C^*-algebra need not to be point-norm continuous. The difficulties that can arise, are investigated in the context of the localised holonomy-flux cross-product *-algebra in section 8.4.

2.1.4 The classical Thiemann Master constraint, Dirac and complete observables

The issue of quantum constraints

In the following considerations the ideas of Thiemann [103, 101, 102, 104] are reviewed and reinterpreted in the language of operator algebras. In Loop Quantum Gravity usually the configuration and momentum variables and the diffeomorphism, Gauss and Hamilton constraints have been implemented as symmetric closed operators on a Hilbert space. In LQG [11, 103, 104] it has been possible to construct a Hilbert space such that the quantum configuration and momentum variables and the Gauss constraint are Hilbert space operators. The Hamilton constraint has only been imposed partly. In general the idea is to impose the quantum analogues of these constraints as one Master constraint, a set of constraint operators or an algebra of quantum constraints. This is studied in the next paragraphs.

The Master constraint approach has been invented by Thiemann [102]. He has proposed to consider only one self-adjoint positive operator $\mathbf{M}$ instead of a set of constraints or an algebra.

The basic idea of Thiemann has been to replace a system of infinitely many constraints $\mathfrak{C}(x) = (-q^{ab}C_aC_b)(x) + C(x)^2 = 0$ for every $x \in \Sigma$ by a single Master constraint $\mathbf{M}$ formally given by

$$\mathbf{M} = \frac{1}{2}\int_\Sigma \mathrm{d}^3\, x \frac{\mathfrak{C}(x)}{\sqrt{det(q)(x)}} \tag{2.5}$$

where q_{ab} is the spatial metric.

In the following paragraphs a more general Master constraint is studied. Let J be a discrete finite index set. Consider the Hilbert space $\mathcal{H}^I_{\mathcal{M}} = L^2(X^I_{\mathcal{M}}, \mu^I_{\mathcal{M}})$, where $I \in J$ and $X^I_{\mathcal{M}}$ is a Borel subset of a metrizable phase space $\mathcal{M}$ and $\mu_{\mathcal{M}}$ a Borel measure thereon. Let Σ be a (metrizable) space and X_Σ a Borel subset of Σ with Borel measure ν_Σ. Then for every $x \in \Sigma$ the constraint $C_I(x)$ depends also on the momentum space $\mathcal{M}$, hence consider $\mathcal{M} \ni m \mapsto C_I(x)[m] \in \mathbb{C}$ such that $C_I(x) \in C^\infty(\mathcal{M})$. Then it is assumed that, $C_I(x)$ is a multiplication operator acting on $\mathcal{H}^I_{\mathcal{M}}$. Therefore the quantum Master constraint is a symmetric operator acting on the Hilbert space $\mathcal{H}_{\mathcal{M}}$. The operator is also denoted by $\mathbf{M}$ and is defined by

$$\mathbf{M}(m) := \int_{X_\Sigma} \mathrm{d}\,\nu_\Sigma(x) \sum_I C_I(x)[m]^* C_I(x)[m] \qquad \text{such that}$$

$$\langle\psi, \mathbf{M}\phi\rangle_{\mathcal{H}_{\mathcal{M}}} = \int_{X_\Sigma} \mathrm{d}\,\nu_\Sigma(x) \sum_I \langle C_I(x)[m]\psi(m), C_I(x)[m]\phi(m)\rangle_{\mathcal{H}^I_{\mathcal{M}}}$$

holds on a suitable dense domain. Assume that $\mathbf{M}$ is positive, unbounded and essentially self-adjoint or positive, bounded and symmetric on $\mathcal{H}_{\mathcal{M}}$. Then the constraint condition on $\mathcal{M}$, which is given by

$$C_I(x)[m] = 0\ \forall x \in X^I_\Sigma,$$

is reformulated by the relation $\langle\psi, \mathbf{M}\psi\rangle = 0$ for all $\psi \in \mathcal{H}_{\text{phys}}$. This relation is called the *Master constraint relation* in analogy to Thiemann. Therefore there is a condition for the Hilbert space to be a *physical Hilbert space*, which is defined by

$$\mathcal{H}_{\text{phys}} := \{\phi \in \mathcal{H}_{\mathcal{M}} : \mathbf{M}\phi = 0\}$$

for the Hilbert space operator $\mathbf{M}$ for which $0 \in \sigma_{\mathrm{d}}(\mathbf{M})$ (discrete spectrum).

In this dissertation the ideas are reformulated in the language of operator algebras. Let $\mathfrak{A}$ be an appropriate C^*-algebra of quantum variables and assume that $\mathfrak{A}$ is a non-degenerate C^*-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{M}})$. Furthermore let ω be a state on $\mathfrak{A}$. In the following paragraphs different constraints are studied.

Assume that, the single exponentiated unitary Master constraint $\exp(i\mathbf{M})$ is contained in the multiplier algebra of the algebra $\mathfrak{A}$. Then the Master constraint condition is replaced by the condition that the state ω corresponding to the GNS-representation of $\mathbf{M}$ on the Hilbert space $\mathcal{H}_{\mathcal{M}}$ is a Dirac state. The Dirac state space is defined by

$$\begin{aligned}\mathcal{S}_D &:= \{\omega \in \mathcal{S}(\mathfrak{A}) : \pi_\omega(\exp(it\mathbf{M}))\Omega = \Omega\ \forall t \in \mathbb{R}\}\\ &= \{\omega \in \mathcal{S}(\mathfrak{A}) : \omega(\exp(it\mathbf{M})O) = \omega(O) = \omega(O\exp(it\mathbf{M})) \quad \forall O \in \mathfrak{A} \text{ and } \forall t \in \mathbb{R}\}\end{aligned}$$

whenever $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ is a GNS-representation associated to ω. If the Master constraint $\mathbf{M}$ is contained in $\mathfrak{A}$, then ω is a Dirac state if $\omega(\mathbf{M}) = 0$ holds.

In the previous paragraphs $\mathbf{M}$ has been assumed to be a Hilbert space operator. This corresponds to a point-norm continuous one-parameter group $t \mapsto \alpha_t(\mathbf{M})$ of automorphisms on $\mathfrak{A}$ that satisfies

$$i[\mathbf{M}, O] = \lim_{t\to 0} \frac{\alpha_t(\mathbf{M})(O) - O}{t}$$

Then a state is called $\mathbb{R}$-*invariant* with respect to the automorphism group $t \mapsto \alpha_t(\mathbf{M})$ on $\mathfrak{A}$, if

$$\omega \circ \alpha_t(\mathbf{M}) = \omega$$

holds for all $t \in \mathbb{R}$. The set of $\mathbb{R}$-invariant states is denoted by $\mathcal{S}^\alpha$.

In analogy to Thiemann a bounded Hilbert space operator O, which satisfy

$$\langle \psi, [[O, \mathbf{M}], O]\, \psi\rangle = 0 \text{ for all } \psi \in \mathcal{H}_{\text{phys}} \tag{2.6}$$

is called a *weak Dirac observable*. If

$$\langle \psi, [O, \mathbf{M}]\psi\rangle = 0 \text{ for all } \psi \in \mathcal{H}_{\text{phys}} \tag{2.7}$$

holds, then O is called a *strong Dirac observable*. In the following paragraph these objects are replaced.

The commutativity condition (2.7) is substituted by the condition for the state to be $\mathbb{R}$-invariant with respect to the automorphism group $t \mapsto \alpha_t(\mathbf{M})$ on $\mathfrak{A}$. In particular Dirac states of the algebra $\mathfrak{A}$ of quantum observables are $\mathbb{R}$-invariant states with respect to this automorphism group. Since it is assumed that, $\mathbf{M}$ is an element of the multiplier algebra $M(\mathfrak{A})$, the automorphisms $\alpha_t(\mathbf{M})$ are inner. In general a covariant representation of $(\mathfrak{A}, \mathbb{R}, \alpha(\mathbf{M}))$ in $\mathcal{L}(\mathcal{H})$ is given by a pair $(\Phi_M, U_t(\mathbf{M}))$ such that $U_t(\mathbf{M}) := \exp(it\mathbf{M})$. Then the sets $\mathcal{S}^\alpha$ and $\mathcal{S}_D$ coincide. Consequently all elements of the algebra $\mathfrak{A}$ of quantum variables satisfy

$$\omega(i[O, \mathbf{M}]) = \lim_{t\to 0} \frac{\omega(\exp(it\mathbf{M})O) - \omega(O\exp(it\mathbf{M}))}{t} = 0 \tag{2.8}$$

for every Dirac state ω of $\mathfrak{A}$. A state independent formulation is the following. The set $\mathfrak{A}^\alpha$ of observables in $\mathfrak{A}$ such that $\alpha_t(\mathbf{M})(A) = A$ for all $t \in \mathbb{R}$ is called the *algebra of generalised strong Dirac observables* in this dissertation. Then the condition $\omega([O, \mathbf{M}]) = 0$ is fulfilled for every element O in $\mathfrak{A}^\alpha$ and every state ω in the state space of $\mathfrak{A}$ (and where the convention $0/0 = 0$ is assumed).

For generality assume that, $t \mapsto \alpha_t(\mathbf{M})$ is a one-parameter group of automorphisms on $\mathfrak{A}$. Thiemann [103] has proposed an ergodic-mean operator on a Hilbert space. This is generalised to arbitrary operators. Since in this dissertation the states are focused, the following object is used. Let O be an element of the C^*-algebra $\mathfrak{A}$ of quantum variables and let ω be a certain state on $\mathfrak{A}$ with GNS-representation $(\mathcal{H}_\Sigma, \pi_\Sigma, \Omega_\Sigma)$. Then a new state on $\mathfrak{A}$ is defined

$$\begin{aligned}\omega_{\mathbf{M}}(O) &:= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\,t\ \langle \Omega_\Sigma, \pi_\Sigma(\alpha_t(\mathbf{M})(O))\Omega_\Sigma\rangle \text{ for } (\pi_\Sigma, \Omega_\Sigma, \mathcal{H}_\Sigma) \text{ GNS-triple assoc. to } \omega \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\,t\ \omega(\alpha_t(\mathbf{M})(O))\end{aligned} \tag{2.9}$$

such that $\omega_{\mathbf{M}} \circ \alpha_t(\mathbf{M}) = \omega_{\mathbf{M}}$ for all $t \in \mathbb{R}$. Consequently $\omega_{\mathbf{M}}$ is a Dirac state and O is a strong Dirac observable. Note that, the right side of equation (2.9) is not necessarily well-defined and finite for all states on the algebra. In particular if ω is a density matrix state and a KMS-state for $\alpha_t(\mathbf{M})$, then the state $\omega_{\mathbf{M}}$ defined by (2.9) exists. Furthermore this state is invariant under the automorphism α and it is a density matrix state, too. In general if the state $\omega_{\mathbf{M}}$ exists, then the state is called *time average*. The remarkable properties of the time avarage, which is constructed from the KMS-state $\omega_{\mathbf{M}}$, is one reason for the study of KMS-states in this dissertation. But often the state ω is not suitable. In these cases the following operator is defined

$$E(O) := = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\,t\ \alpha_t(\mathbf{M})(O)$$

in the operator norm-limit and is called the *time avarage operator*. Note that, $\mathrm{d}\,t\ \alpha_t(\mathbf{M})(O)$ is a positive operator-valued measure. This implies that, a weight[1] $\tilde{\omega}_{\mathbf{M}}$ on $\mathfrak{A}$ is defined by

$$\tilde{\omega}_{\mathbf{M}}(O) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\,t\ \omega(\alpha_t(\mathbf{M})(O)) \in [0,\infty]$$

for an arbitrary state ω on $\mathfrak{A}$.

Summarising for a Master constraint operator, which is contained in the algebra $\mathfrak{A}$ of quantum observables in LQG, the time avarage construced from a KMS-state of $\mathfrak{A}$ is an α-invariant state and $\mathbf{M}$ is imposed as a constraint on this state.

The situation is more difficult if a set $\breve{C} := \{C^*_J C_J\}$ of unbounded closed constraint operators C_J, which are not contained in a general C^*-algebra $\mathfrak{A}$ of quantum observables, replaces the Master constraint. Assume that every element Z_C-transform defined by $C^*_J C_J$ is contained in the multiplier algebra of $\mathfrak{A}$, and that each operator $C^*_J C_J$ is essentially self-adjoint. Then the Dirac state space is given by

$$\begin{aligned}\mathcal{S}_D &:= \{\omega \in \mathcal{S}(\mathfrak{A}) : \pi_\omega(C^*_J C_J)\Omega_\omega = 0 \quad \forall C^*_J C_J \in \breve{C}\}\\ &= \{\omega \in \mathcal{S}(\mathfrak{A}) : \omega(C^*_J C_J) = 0 \quad \forall C^*_J C_J \in \breve{C}\}\end{aligned}$$

The *set of strong Dirac observables* is given by the weak relative commutant

$$\mathcal{O}^s_D := \{O \in \mathfrak{A} : \omega([O, C^*_J C_J]) = 0 \quad \forall C^*_J C_J \in \breve{C} \text{ and } \forall \omega \in \mathcal{S}_D\}$$

and the *set of weak Dirac observables* is given by

$$\mathcal{O}^w_D := \{O \in \mathfrak{A} : \omega([O, [O, C^*_J C_J]]) = 0 \quad \forall C^*_J C_J \in \breve{C} \text{ and } \forall \omega \in \mathcal{S}_D\}$$

The bracket $[O, C^*_J C_J]$ for a fixed $C^*_J C_J \in \breve{C}$ defines a *-derivation $\delta_{C_J}(O) := [O, C^*_J C_J]$ on $\mathfrak{A}$. Consequently redefine the set $\mathcal{O}^s_D$ by

$$\mathcal{O}^s_D = \{O \in \mathfrak{A} : \omega(\delta_{C_J}(O)) = 0 \quad \forall C_J \in \breve{C} \text{ and } \forall \omega \in \mathcal{S}_D\}$$

Now a more general concept is introduced. This replaces the notion of weak and strong Dirac observables. If products of constraints are used, then the Dirac states have to be analysed once more. Hence for generality assume that, the set $\breve{C}$ forms an algebra of quantum constraints and $\breve{C}$ is contained in the multiplier algebra $M(\mathfrak{A})$. Set

$$N_\omega := \{A \in \mathfrak{A} : \omega(A^* A) = 0\}$$

Then define the set $\mathcal{N}$ to be the closed left and right ideal generated by $\breve{C}$. Then for example AC, A^*C^*, CA and C^*A^* are elements of $\mathcal{N}$. Then denote the closure of all linear combinations of elements of $\mathcal{N}$ and $\breve{C}$ by $\mathcal{D}$. For a constraint C in $\breve{C}$ it is in particular true that, $C \in \mathcal{D}$, $C^*A^* \in \mathcal{D}$, $[A, C] \in \mathcal{D}$ and $[[A, C], A] \in \mathcal{D}$.

Redefine the *Dirac state space* by

$$\mathcal{S}_D := \{\omega \in \mathcal{S}(\mathfrak{A}) : \pi_\omega(D)\Omega_\omega = 0 \quad \forall D \in \mathcal{D}\}$$

Then $\mathcal{D} \subset N_\omega$, whenever $\omega \in \mathcal{S}_D$.

Finally in this dissertation the *set of Dirac observables* is given by

$$\mathcal{O}_D := \mathfrak{A}/\mathcal{D}$$

The set $\mathcal{O}_D$ forms in particular a *-algebra, which can be hopefully completed to a C^*-algebra.

Consider for every J the one-parameter group $\mathbb{R} \ni t \mapsto \alpha_t(C^*_J C_J) \in \mathfrak{Aut}(\mathfrak{A})$ of automorphism such that $C^*_J C_J$ defines the (infinitesimal) generator of this group. Then the set $\bigcap_J \mathcal{S}^{\alpha_J}$ of all states of $\mathfrak{A}$, which are invariant under all automorphism groups $\mathbb{R} \ni t \mapsto \alpha_t(C^*_J C_J) \in \mathfrak{Aut}(\mathfrak{A})$ for all constraints in $\breve{C}$, is not contained in the set $\mathcal{S}_D$ of Dirac states. It is only true that a Dirac state is invariant under every automorphism $\alpha_t(C^*_J C_J)$. Consequently

[1]A weight is a positive linear functional on the algebra, which is not necessarily normalizable.

the state $\omega_{\mathbf{M}}$, where $\mathbf{M}$ is replaced by a constraint in $\breve{C}$, defined by (2.9) need not be a Dirac state. Hence a Dirac state is a more general concept than states, which are invariant under automorphisms given by the constraints.

There is a problem, if the constraints or the exponentiated constraints are not contained in the algebra $\mathfrak{A}$ or the multiplier algebra of $\mathfrak{A}$. Then in some cases the quantum constraint is affiliated to a larger algebra of quantum configuration and momentum variables. Then similar investigations can be made with respect to this larger algebra. Note that, in this situation the algebra of quantum constraints is replaced by the algebra generated by all Z-transformations of the constraints C_J. The Z-transform of an operator C_J is given in [88] by $C_J(\mathbb{1}+C_J^*C_J)^{-1/2}$. These objects are in particular elements of the multiplier algebra of $\mathfrak{A}$, if the constraints C_J are affiliated operators with $\mathfrak{A}$.

A more general implementation of quantum constraints in terms of multipliers is the following. If the non-unital C^*-algebra $\breve{\mathcal{C}}$ of constraints is a concrete C^*-algebra of bounded operators on a separable Hilbert space $\mathcal{H}$, then the multiplier algebra $M(\breve{\mathcal{C}})$ of the C^*-algebra $\breve{\mathcal{C}}$ defines the C^*-algebra of quantum observables. For example in Loop Quantum Cosmology this characterisation can be used.

In general for the Loop Quantum Gravity approach the algebra, which is generated by the quantum constraint operators, has not been understood completely. The dissertation will change this. The algebras of quantum variables have to be chosen such that the operators derived from the quantum constraints are

- elements of (or affilliated with) the algebra of quantum variables, or
- elements of the multiplier algebra of this algebra.

Partial and complete quantum observables

Apart from the issue of the implementation of quantum constraints, the issue of partial and complete quantum observables has to be analysed. Assume that the Master constraint operator is contained in the multiplier algebra of $\mathfrak{A}$ and there are no other constraints. Let $\mathbf{M}$ be the Master constraint, which implements the dynamics of the gravitational system . Furthermore this operator is related to an one-parameter group of automorphisms $\tau \mapsto \alpha_{\mathbf{M}}(\tau)$, which is defined by

$$\alpha_{\mathbf{M}}(\tau)(A) := \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\tau^n}{n!}\delta^n_{\mathbf{M}}(A) \text{ for } \delta^n_{\mathbf{M}}(A) := [A,\mathbf{M}]_n = [...[[A,\mathbf{M}],\mathbf{M}],...,\mathbf{M}] \tag{2.10}$$

for all $A \in \mathfrak{A}$ such that the limit exists in norm-topology.

Finally let the state $\omega_{\mathbf{M}}$ of the algebra $\mathfrak{A}$ be a Dirac state. Then in particular this state is invariant under the one-parameter group of automorphisms, which is given by $s \mapsto \alpha_{\mathbf{M}}(s)$. Therefore the physical or Dirac observables do not evolve with respect to this Hamiltonian.

The *set of complete quantum observables* is defined by $A_\tau = \alpha_{\mathbf{M}}(\tau)(A)$ running over all $A \in \mathcal{O}_D$ such that the limit (2.10) exists in norm-topology. In particular the set

$$\mathcal{O}_D^{\alpha_{\mathbf{M}}} := \{A \in \mathcal{O}_D : \alpha_{\mathbf{M}}(t)(A) = A \quad \forall t \in \mathbb{R}\}$$

forms an algebra. This algebra is completed to a C^*-algebra, which is called the *C^*-algebra of complete quantum observables* in this dissertation.

The evolution of a physical observable has to be related to a clock variable T of a physical freedom of the system. Let T be an element of the C^*-algebra $\mathcal{O}_D$ or the multiplier algebra of $\mathcal{O}_D$. Furthermore let H_T be the Hamiltonian of the clock observable T and assume that $t \mapsto \alpha_{H_T}(t)$ is a one-parameter group of automorphisms on $\mathcal{O}_D$. Then the *time-of-occurence-of-an-event operator* is defined similarly to the operator, which has been introduced by Fredenhagen and Brunetti [23], and is given by the operator

$$E(A) := \int_\tau \mathrm{d}\, t\ \tilde{T}^{-1/2}\alpha_{H_T}(t)(A^*A)\tilde{T}^{-1/2}$$

Note that, $\tilde{T}$ is derived from the clock operator T and have to be chosen suitably. Furthermore the expectation value of an observable A^*A contained in $\mathcal{O}_D^{\alpha_{\mathbf{M}}}$ if the clock T measures a time endurance τ is given by

$$W_T(A^*A) := \int_\tau \mathrm{d}\, t\ \omega_{\mathbf{M}}(\tilde{T}^{-1/2}\alpha_{H_T}(t)(A^*A)\tilde{T}^{-1/2}) \tag{2.11}$$

Then W_T is a linear functional, but it is not necessarily a state on the C^*-algebra of complete quantum observables. In general it is a weight on this C^*-algebra. If it is a state, then it is called the *expectation of the time of occurence of an event.*

In the context of the thermal time hypothesis presented by Rovelli and Connes [29] the one-parameter group of automorphism $t \mapsto \alpha_{H_T}(t)$ is the modular group and $\omega_{\mathbf{M}}$ is the thermal equilibrium with respect to the thermal time t, which is given by the Hamiltonian of the clock. The elements of $\mathfrak{A}$ are called *partial quantum observables* in this dissertation.

Summarising the complete or partial quantum observables are certain elements of the C^*-algebra of quantum variables. In particular the complete quantum observables are assumed to define a certain C^*-subalgebra of the C^*-algebra of quantum variables. The constraints are imposed on a Dirac state and the complete quantum observables are assumed to be Dirac observables. Moreover a Dirac state, a clock operator and a clock Hamiltonian define a state or a weight on the C^*-algebra of complete quantum observables.

There exists several one-parameter groups of automorphisms on the C^*-algebra of complete observables. The modular automorphism group, which is related to a physical time evolution, is defined by the quantum clock Hamiltonian. The ideas are presented in more detail in subsection 10. Clearly the concept has to be further generalised if a set of clocks is used.

In LQG the algebra of quantum variables that contains the quantum constraint and quantum clock operators is very complicated and has yet not been developed completely. Moreover the algebra of complete quantum observables has not been derived from the full algebra of quantum variables so far.

2.2 KMS-Theory in Generally Covariant Theories

In the Hamiltonian formulation of Gravity the dynamical Hamiltonian is a constraint. A preferred time flow such that the physical observables evolve with respect to this single time parameter is related to the concept of clocks. The idea of Modular Theory is to encode the time flow in a one-parameter group of automorphisms, which depends on the thermal state of the system. Hence one can speak about thermal time. This concept has been analysed by Connes and Rovelli [29] in the context of general covariant quantum theories and Modular Theory for von Neumann algebras.

Mathematically for a von Neumann algebra $\mathfrak{M}$ there exists the modular automorphism group $t \mapsto \alpha_t^\omega \in \mathfrak{Aut}(\mathfrak{M})$ associated to a faithful and normal state ω. This automorphism group is unique up to inner automorphism and is independent of the choice of the state. Moreover the modular automorphism group is used for the study of type III factors of von Neumann algebras.

More precisely the KMS-theory contains the following objects. In the GNS representation $(\mathcal{H}, \pi, \Omega)$ associated to the state ω there exists a unitary one-parameter group $t \mapsto \Delta_\omega^{it} \in \mathcal{L}(\mathcal{H})$ such that $\pi(\alpha_t^\omega(M)) = \Delta_\omega^{it}\pi(M)\Delta_\omega^{it}$ for all $M \in \mathfrak{M}$. The operator Δ_ω is called the modular operator. There exists the modular generator $K_\omega := \log \Delta_\omega$, which is the generator of the automorphism group $t \mapsto \alpha_t^\omega$. Furthermore there exists a anti-linear isometry J in $\mathcal{H}$ and an isomorphism $\gamma : \pi(\mathfrak{M}) \to \pi(\mathfrak{M})'$ such that $\gamma(\pi(M)) = J\pi(M)J$ for all $M \in \mathfrak{M}$. The operator J is called the modular conjugation. The modular automorphism is the only one-parameter automorphism group satisfying the KMS-condition w.r.t. the state ω at inverse temperature $\beta = 1$. The KMS-condition states that, $\omega \circ \alpha_t^\omega = \omega$ and for all $M, N \in \mathfrak{M}$ there exists a map $F_{M,N} : \mathbb{R} \times [0, \beta] \to]0, \beta[$ such that $F_{M,N}$ is holomorphic on $\mathbb{R} \times [0, \beta]$, $F_{M,N}$ is bounded continuous on $\mathbb{R} \times [0, \beta]$, $F_{M,N}(t) = \omega(\alpha_t^\omega(B)A)$ and $F_{M,N}(i\beta + t) = \omega(A\alpha_t^\omega(B))$ yields for all $t \in \mathbb{R}$.

Consequently the dynamics defined by the modular operator Δ_ω, and a KMS-state ω are intrinsic objects of the von Neumann algebra. Physically the equilibrium thermal state ω and the modular automorphism group contains all information about the dynamics of the system. In particular the information about the Hamiltonian, which is the generator of the automorphism group. An overview abot these structures has been presented by Bratteli and Robinson [22] and for a detailed lecture refer to Takesaki [91, 92].

In the fundamental article [35] of Emch the role of KMS-Structures and a quantisation of a classical Poisson system has been analysed. He has considered the von Neumann algebra $\mathcal{M}$ generated by the unitary Weyl elements $W(x, p)$, which satisfy the canonical commutator relations and which are based on the phase space $\mathbb{R}^{2n}$. Then he has showed that, for every faithful normal state ω on $\mathcal{M}$ there is a cyclic and separating vector Φ in a Hilbert space $\mathcal{H}$ such that

$\omega(W) = \langle \Phi, W\Phi \rangle_{\mathcal{H}}$ for all $W \in \mathcal{M}$. Furthermore every faithful normal state ω on $\mathcal{M}$ satisfies the KMS-condition and hence the modular objects are constructable.

In Quantum Field Theories the Tomita-Takesaki theory leads to a surprising duality between geometric objects on Minkowski spacetime and the modular automorphism group on the algebra of local observables. For example Brunetti, Guido and Longo [25] have shown that certain one-parameter subgroups of the Poincaré group are related to certain modular groups constructed via Tomita-Takesaki theory. In particluar the Bisognano-Wichmann theorem relates the Lorentz boosts on restricted wegde regions in Minkowski spacetime to unitaries, which are defined by the modular generator. Consequently the boosts implement the dynamical evolution of free fields in Minkowski spacetime. The modular involution J implements the spacetime reflection about the edge of the wedge along with a charge conjugation. Note that, the algebra of observables is restricted to a certain subalgebra associated to wedges and the full Poincaré invariance of the representation is broken. Then Brunetti, Guido and Longo have assumed that, the subgroup of Lorentz boosts is implemented as a covariant representation on the C^*-dynamical system consisting of the algebra of local observables restricted to wedges, the boosts and the automorphisms, which implement the boosts. This representation is also called the thermal representation $(\pi_\beta, \mathcal{H}_\beta, \Omega_\beta)$. The self-adjoint thermal Hamiltonian H_β is the generator of the unitary group $U_\beta(t) := \exp(-i\beta H_\beta)$ such that $\langle \Omega_\beta, A\exp(i\beta H_\beta)B\Omega_\beta \rangle = \langle \Omega_\beta, BA\Omega_\beta \rangle$ holds for elements A, B contained in a suitable dense subset of the quantum algebra. Indeed the vacuum representation is not related to a KMS-state at a finite temperature. A consequence of the Connes cocycle theorem is that, there is only one thermal state and automorphism group (up to inner automorphisms) of the von Neumann algebra. This implies that, there is also only one preferred time evolution of the physical system. In general modular groups on von Neumann algebras in the QFT framework have been considered by Borchers in [18]. A short summary over Tomita-Takesaki Theory in QFT has been presented by Summers [90].

There is also a modular theory on C^*-algebras such that KMS-states and modular objects are defined. This leads physically to the concept that, the C^*-algebra of quantum operators and the modular automorphism are important objects for the definition of a theory of quantum gravity. Clearly for this viewpoint it is assumed that, the algebra of constraints is a subset of (or are affiliated with) the algebra of quantum variables.

In the framework of Loop Quantum Gravity KMS-sates and modular objects have not been studied until now. The questions that arise are the following:

· Which automorphism group of the algebra of quantum variables is a candidate for the modular automorphism group?

· Which Hamiltonian is required to be the generator of the modular automorphism group?

The answers will depend on the choice of the algebra of LQG. In this dissertation it is shown that already for some simple automorphisms of the known C^*-algebras of quantum variables, there exists no KMS-states. Furthermore the von Neumann algebra generated by holonomies and fluxes is not suitable. Consequently the study of new algebras is necessary, if one would like to explore KMS-theory in LQG.

2.3 A summary of physical algebras of quantum operators in LQG

In Loop Quantum Gravity approach the quantum constraints, which have been presented in section 2, are usually not contained in the algebra of quantum observables. This is the main problem for the implementation of quantum constraints in the LQG framework. Moroever there are no natural physical clocks contained in the quantum algebra. Usually matter fields are used as clocks. Since matter fields are localised objects, an idea is to define a localised algebra of holonomies and fluxes. The theory, which is described by this algebra, is not diffeomorphism invariant, but this invariance can be weakend. Hence only diffeomorphisms, which preserve the localised areas, in which the matter fields are situated, are taken into account. The full quantum algebra can be for example a tensor product of a matter field algebra and the algebra of holonomies and quantum fluxes. A first approach in this direction is presented in section 8.4. Summarising the main aim of this dissertation is to find a suitable physical algebra of quantum configuration and quantum momentum variables of the theory. The physical algebra satisfies the following conditions:

(i) the quantum constraint operators are affiliated or contained in the physical algebra and

(ii) the elements of the physical algebra are complete observables.

In the framework of Loop Quantum Cosmology a treatment of quantum constraints has been discovered by Lewandowski et al. [13, 60, 61, 62, 63, 65]. The Weyl C^*-algebra in LQC literature has been represented on the polymer Hilbert space $\mathcal{H}$. There is only one Hamilton constraint, which is implemented on the non-separable Hilbert space $\mathcal{H}$. It is possible to consider another Weyl C^*-algebra, which is presented in section 7.4. Then the Hamilton constraint is contained in this algebra and the objects defined before can be investigated.

Chapter 3

The configuration and momentum space of loop quantum gravity

3.1 Path, gauge and Lie groupoids

The basic variables of LQG theory are derived from paths, graphs or groupoids on a smooth or analytic manifold Σ. One distinguishes between a smooth, analytic or semianalytic category to refer to smooth (analytic/semianalytic) paths on a smooth (analytic/semianalytic) manifold in the LQG literature. The aim of the study is to analyse various classical and hence quantum variables.

The investigations start with the important work of Barrett [16], who has introduced the concepts of holonomy map and thin holonomy group. For a construction of different classical configuration variables it is worth to understand his ideas and how these concepts can be generalised. Indeed a more abstract theory has been developed by Mackenzie [66] independly from Barrett. In this dissertation it is studied why the theory of Mackenzie replaces the concepts of Barrett. A short overview is given in the next paragraphs.

The smooth paths and loops defined in section 3.1.1 are the fundamental objects, which define holonomy groups. Furthermore a certain holonomy group called the fundamental group at a specific point in the manifold for smooth paths is illustrated. The group structure is generalised to groupoids. Therefore the fundamental groupoid is introduced. The difference is the following. The fundamental group consists of homotopy classes of loops at a chosen point, whereas the fundamental groupoid is defined by the homotopy classes of paths between arbitrary points. In particular the fundamental groupoid is a Lie groupoid. For Lie groupoids new mathematical concepts are available, which have been introduced by Mackenzie. For example transformations in Lie groupoids are presented in section 3.1.5. Another example for a Lie groupoid is the gauge groupoid, which is associated to a principal bundle and is studied in section 3.1.4 in detail. The LQG theory is basically a gauge theory, since the fundamental object is a principal fibre bundle. Therefore the gauge groupoid is used for a generalisation of the quantum variables given by the holonomy mappings. These generalised holonomy maps are introduced in section 3.3. The idea for these new objects is based on the duality of infinitesimal and integrated objects, which has been reformulated by Mackenzie and is presented very briefly in section 3.2.

In the context of analytic paths the main geometric objects are transferred. The notion of finite path groupoids and finite graph systems are presented in section 3.1.3. The inductive limit of an inductive family of finite path groupoids is used in LQG before. In this dissertation the framework is extended to inductive limit of an inductive family of finite graph systems. The relation of diffeomorphisms in the manifold Σ and diffeomorphisms that preserve the groupoid structure, which has been introduced by Fleischhack in [37] first, is studied in a new context in section 3.3.4.4. To understand the precise defintion of group actions associated to path- (or graph-) diffeomorphisms on finite path groupoids (or graph systems) the ideas of transformations in Lie groupoids are transfered. These group actions are used later in the context of the analytic holonomy C^*-algebra and graph-diffeomorphism invariance.

The objects defined in the sections 3.1.1 and 3.1.3 are partly borrowed from Thiemann [104] and the objects in section 3.1.2, 3.1.4 and 3.1.5 from Barrett [16], Caetano and Picken [28] and Mackenzie [66].

3.1.1 Loop spaces, loop and holonomy group

In this section a curve in a smooth manifold Σ is a (piecewise) smooth map $\gamma : I \to \Sigma$ where $I = [0,1]$. In further sections continuous, (piecewise) C^k-curves in Σ are used, too. The basic objects are studied to derive the loop and holonomy group at a base point in the manifold Σ.

First the composition of parametrized curves $\gamma_i : [0,1] \to \Sigma$ for $i = 1,2$ given by

$$\gamma_1 \circ \gamma_2 = (\gamma_1\gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } t \in [0, 1/2] \\ \gamma_2(2t-1) & \text{for } t \in [1/2, 1] \end{cases} \tag{3.1}$$

This relation is not an associative operation. Therefore consider the following equivalence relation. Two curves γ and γ' are **reparamtrisation equivalent** if there is an orientation preserving diffeomorphism $\phi : I \to I$ such that $\gamma' = \gamma \circ \phi$.

Definition 3.1.1. *A* ***loop*** *is a smooth continuous mapping of the unit interval $I = [0,1]$ into a (topological) space Σ such that $\gamma(0) = \gamma(1) = v$. The collection of loops in Σ with base point v is called* ***loop space*** *$L\Sigma$.*

The quotient of the loop space at v and reparamtrisation equivalence does not form a group. If additionally to reparametrisation equivalence the algebraic relation $\gamma \circ \gamma^{-1} \simeq \mathbb{1}_v$ is required, then the loop space at v modulo these equivalence relations form a group.

There are several homotopy equivalences, which implements such branch lines, available in literature. A homotopy relation on loop space $L\Sigma$ has been defined for example by Barrett[16]. Two loops γ, γ' are said to be thinly homotopic if there exist a homotopy of the composed path ${\gamma'}^{-1} \circ \gamma$ to the trivial loop at v. The idea that the loop ${\gamma'}^{-1} \circ \gamma$ shrinks to the trivial loop, which encloses no area. In other words, the definition of thin homotopy is given as follows.

Definition 3.1.2. *A loop γ is* ***thin****, if there exists a smooth homotopy of γ to the trivial loop with the image of the homotopy lying entirely within the image of γ, i.e. the homotopy $\varrho : I \times I \to \Sigma$ for $s,t \in I$ satisfy $\varrho(1,t) = \gamma(t)$, $\varrho(0,t) = \mathbb{1}_v$ and $\varrho(s,0) = \rho(s,1) = \mathbb{1}_v$, where $\mathbb{1}_v$ is the trivial loop at v, for all $t,s \in I$ and $\mathrm{Im}(\varrho) = \mathrm{Im}(\gamma)$.*

Then Barrett call two loops α, β thinly equivalent if $\alpha \circ \beta^{-1}$ is thin. This relation turns out to be not transitive. Consequently, the authors Caetano and Picken [28] improved this definition.

Definition 3.1.3. *[28, p.837] Two loops γ, γ' are said to be* ***thinly homotopic*** *if there exists a finite sequence $\gamma_1, ..., \gamma_n$ of loops such that $\gamma_1 = \gamma$ and $\gamma_n = \gamma'$ and $\gamma_{i+1}^{-1} \circ \gamma_i$ is a thin loop for $i = 1, .., n-1$.*

Notice that, two loops only differing by a reparametrisation are thinly homotopic.

Lemma 3.1.4. *The thin homotopy relation is an equivalence relation.*

Denote the equivalence class by $\{\gamma\}$ and the relation by $\sim_{\text{thin}}$.

Proof. Reflexivity and symmetry are obvious. Transitivity is proofed as follows. Let $\alpha \sim_{\text{thin}} \beta$ and $\beta \sim_{\text{thin}} \theta$ then there exists appropriate sequences $\{\beta_1^\alpha, ..., \beta_n^\alpha\}$ and $\{\beta_1^\theta, ..., \beta_n^\theta\}$ such that

$$\beta_1^\alpha = \alpha, \beta_n^\alpha = \beta, \beta_1^\theta = \beta, \beta_n^\theta = \theta \tag{3.2}$$

Consequently, the finite sequence $\{\beta_1^\alpha, ..., \beta_n^\alpha, \beta_2^\theta, ..., \beta_n^\theta\}$ satisfies the requires properties. Hence, $\alpha \sim_{\text{thin}} \theta$. □

Observe that for two thinly homotopic loops α and β there exists a sequence
$\alpha, \beta_2, \beta_3, ..., \beta_{n-1}, \beta$ such that

$$\beta_2^{-1} \circ \alpha \circ \beta_3^{-1} \circ \beta_2 ... \circ \beta_{n-2} \circ \beta^{-1} \circ \beta_{n-1} \sim_{\text{thin}} \mathbb{1}_v \tag{3.3}$$

Moroever there is another modification of homotopy equivalence given in Mackenzie's book [66, p.218]. The author call a restriction of a curve γ on a subinterval of $I = [0,1]$ a **revision** of γ. Two curves γ, γ' are called **equally good** if $\gamma(1) = \gamma'(1)$ and either γ is a revision of γ' or, conversely, γ' is a revision of γ. A **lasso** (for a cover $\{U_i\}$ of Σ) is a loop of the form $\gamma^{-1} \circ \beta \circ \gamma$ where the loop β lies entirely in one neighborhood U_i.

Definition 3.1.5. *A loop α is said to be* ***approximated at*** *v by finite product γ' of lassos which is equally good as α, if each loop β_i of the lasso $\gamma_i^{-1} \circ \beta_i \circ \gamma_i$ is contained in a neighborhood U_i of v and the product of lassos are homotopic to $\mathbb{1}_v$.*

If α is (thinly) homotopic to a point, then from the smooth homotopy map $\varrho : I \times I \rightarrow \Sigma$ for $s, t \in I$ which satisfy $\varrho(1,t) = \gamma(t)$, $\varrho(0,t) = \mathbb{1}_v$ and $\varrho(s,0) = \varrho(s,1) = \mathbb{1}_v$ for all $t, s \in I$ and the finite sequence of appropriate loops, the product of lassos can be constructed easily. Hence, a loop, which is homotopic to a point v, is approximated at v by a suitable product of lassos, which is equally good as the loop.

For the one-to-one correspondence between holonomy maps and horizontal lifts a further developed definition is useful. The following homotopy was first introduced by Ceatano and Picken in [28].

Definition 3.1.6. *[28, section 4] Two smooth loops $\gamma, \gamma' : I \rightarrow \Sigma$ are said to be* ***intimate homotopic****, if there exists a map $\varrho : I \times I \rightarrow \Sigma$ such that*

(i) ϱ is smooth,

(ii) for every $s \in I$ it is true that $\varrho(s,t) \in L\Sigma$,

(iii) for $0 \leq \epsilon \leq 1/2$ the map ϱ satisfies

$$\begin{aligned} 0 \leq s \leq \epsilon, &\qquad \varrho(s,t) = \gamma(t) \\ 1-\epsilon \leq s \leq 1, &\qquad \varrho(s,t) = \gamma'(t) \\ 0 \leq t \leq \epsilon, &\qquad \varrho(s,t) = \gamma(0) \\ 1-\epsilon \leq t \leq 1, &\qquad \varrho(s,t) = \gamma(1) \end{aligned} \tag{3.4}$$

and

(iv) the rank of the differential $\mathrm{d}\,\varrho(s,t)$ is smaller or equal 1 for all $(s,t) \in [0,1]^2$.

Ceatano and Picken have been shown that this relation is indeed an equivalence relation. The equivalence relation is denoted by $\sim_{\mathrm{in}}$. Moreover, this relation is a weakening of the thin homotopy relation. Moreover, the intimate relation is stronger than the same-holonomy relation for all smooth connections introduced by Ashtekar and Isham [7], which is introduced in a later section.

Now, for the different equivalence relations the quotient spaces can be considered. For example the quotient of the loop space $L\Sigma$ and intimate homotopic equivalence is a group, which is called the **intimate fundamental group** $\pi_1^{\mathrm{in}}(\Sigma, v)$ **at base point** v. If the rank condition of the differential is omitted, then the quotient of $L\Sigma$ and homotopy is the **fundamental group** $\pi_1(\Sigma, v)$. Finally the quotient of the loop space $L\Sigma$ and thinly homotopic equivalence is a group, too. This group is called the **loop group** $\mathrm{LG}(v)$ **at** v or **thin fundamental group** $\pi_1^1(\Sigma)$ **at** v. Now follow the ideas of Barrett presented in [16]. He has required that the configuration space is given by the set of certain mappings from the loop space at a base point v to the structure group G of a principal bundle such that these mappings arise as holonomy mappings. There is a set of conditions introduced by Barrett, which are called the Barrett axioms in this dissertation.

The Barrett Axioms

Barrett Axiom 1. ***(Group homomorphism)*** *the map $\mathfrak{h}_A : \mathrm{LG}(v) \rightarrow G$ is a group homomorphism*

Barrett Axiom 2. ***(Reparametrisation invariance)*** *for every orientation preserving diffeomorphism $\phi : I \rightarrow I$ the map $\mathfrak{h}_A$ satisfy*

$\mathfrak{h}_A(\gamma) = \mathfrak{h}_A(\gamma \circ \phi)$ *for all* $\gamma \in \mathrm{LG}(v)$

Barrett Axiom 3. ***(Same-holonomy relation)*** *for two thinly homotopic loops α and β the maps are equal, i.e.*

$\mathfrak{h}_A(\alpha) = \mathfrak{h}_A(\beta) \Leftrightarrow \alpha \sim_{\mathrm{thin}} \beta$

Barrett Axiom 4. ***(Smothness)*** *for a smooth family $\{\gamma : U \rightarrow L\Sigma, U$ open subset of $\mathbb{R}^3\}$ of loops all compositions $\mathfrak{h}_A \circ \gamma : U \rightarrow G$ of the map $\mathfrak{h}_\Gamma$ with elements of this family is smooth.*

The axioms (BAxiom1) and (BAxiom2) implement the algebraic structure. It follows that $\mathfrak{h}_A(\gamma^{-1}) = \mathfrak{h}_A^{-1}(\gamma)$ and the value of two loops differing only by reparametrization are equivalent. The last axiom give rise to a topological structure on G. The smooth family of loops are maps γ such that $\gamma : U \times [0,1] \rightarrow LG(v)$ with $\gamma(v,t) = \check{\gamma}(t)$ are smooth. Moreover, every map $\mathfrak{h}_A$ which obeys (BAxiom4) is continuous. Therefore, a topology on $LG(v)$ inherited from (BAxiom4) is defined, which is called the Barrett topology. Summarising the last condition guarantees the differentiability of the bundle and lifting. A map $\mathfrak{h}_A : \mathrm{LG}(v) \rightarrow G$ is called **holonomy map** if the axioms: (BAxiom1), (BAxiom2), (BAxiom3) and (BAxiom4) are fulfilled. For a fixed holonomy map $\mathfrak{h}_A : LG \rightarrow G$ let the axioms: (BAxiom1), (BAxiom2), (BAxiom3) and (BAxiom4) be satisfied, then the **holonomy group** $HG(v)$ **at** v is defined by the set $\{\mathfrak{h}_A(\gamma) : \gamma \in \mathrm{LG}(v)\}$.

Now the reconstruction theorem of Barrett [16] explains the term holonomy map.

Theorem 3.1.7. *[16, **Reconstruction theorem**]*
For a given connected manifold Σ with base point v and a holonomy map $\mathfrak{h}_A : L\Sigma \rightarrow G$, then there exists a principal bundle $P(\Sigma, G)$, a point in the fibre $u \in \pi^{-1}(v)$, and a connection A on P such that $\mathfrak{h}_A$ is the holonomy map of the bundle.

In particular Barrett has given the mathematical background for treating the holonomy maps as the primary, and connections and curvature as derived objects of the theory.

Theorem 3.1.8. *[16, **Representation theorem**]*
For a given connected, Hausdorff manifold Σ there is a bijective correspondence between

(i) a triple (P, A, u) consisting of a principal G-bundle P, a connection A on P and a base point $u \in P$ and

(ii) a holonomy map $\mathfrak{h}_A : L\Sigma(v) \rightarrow G$.

Therefore, it is straight forward to consider for a point u in a principal bundle P, a connection A on P the set

$$\Phi_u := \{g \in G : \mathfrak{h}_A(\alpha) = ug \quad \forall \alpha \in \mathrm{LG}(v)\} \tag{3.5}$$

whenever $\mathfrak{h}_A$ is a holonomy map. This set is a Lie subgroup of G and it is equal to the holonomy group $HG(v)$, where $\pi(u) = v$. The constant loop $\mathbb{1}_v : [0,1] \rightarrow v$ at V defines the identity map $\mathfrak{h}_A(\mathbb{1}_v) = u$.

Caetano and Picken [28] have modified the reconstruction and representation theorems 3.1.7 and 3.1.8 by using the intimate homotopy equivalence relation 3.1.6. They have improved the representation theorem by showing that from $\alpha \sim_{\text{in}} \beta$, it follows that $\mathfrak{h}_A(\alpha) = \mathfrak{h}_A(\beta)$ and $\mathfrak{h}_A$ is a group homomorphism arising from a holonomy map defined as a horizontal lift associated to a smooth connection A. The complicated step was to show that intimate loops have the same holonomy. Furthermore, let $c : \pi_1^{\text{in}}(\Sigma, v) \rightarrow \pi_1(\Sigma, v)$ be the canonical morphism and consider the group homomorphism $h : \pi_1(\Sigma, v) \rightarrow G$, which is connected to $\mathfrak{h}_A : \pi_1^{\text{in}}(\Sigma, v) \rightarrow G$ by $\mathfrak{h}_A := h \circ c$. Then by the Ambrose-Singer theorem the class of holonomies, which are given by group homomorphisms h, are associated to flat connections.

In general a purely algebraic equivalence relation is given as follows.

Definition 3.1.9. *The quotient of the loop space $L\Sigma$ at v by the **algebraic homotopy equivalence** refering to the relations*

$$\begin{aligned} \gamma &\simeq \gamma \circ \mathbb{1}_v \\ \gamma \circ \gamma^{-1} &\simeq \mathbb{1}_v \\ (\gamma \circ \gamma') \circ \gamma'' &\simeq \gamma \circ (\gamma' \circ \gamma'') \end{aligned} \tag{3.6}$$

*is called the **(algebraic) loop group** $\mathcal{L}(\Sigma, v)$, i.e. where $\mathbb{1}_v$ is the trivial loop in $L\Sigma$ at v.*

The most general holonomy mapping maps the (algebraic) loop group $\mathcal{L}(\Sigma, v)$ to G. Furthermore Ashtekar and Isham [7], Ashtekar and Lewandowski [8] and Lewandowski [59] construct a group, which they call a **hoop group**. This group is the quotient of the loop space at v modulo thin equivalence, reparametrisation equivalence and the same-holonomy relation for thinly homotopic loops and all holonomy mappings.

3.1.2 Fundamental groupoids of path spaces

Now the loops in the smooth category are generalised to paths. Consider a collection of (piecewise) smooth curves starting at a source point v on a path connected manifold Σ. The collection of curves starting at v is called the path space $P\Sigma^v$ at v. The set of all collections for all $v \in \Sigma$ is called the **path space** $P\Sigma$. The elements of the path space are called paths. A fibre of the path space $P\Sigma$ is the loop space $L\Sigma(v)$ at v. Let $s_{P\Sigma} : P\Sigma \to \Sigma$ and $t_{P\Sigma} : P\Sigma \to \Sigma$ be two surjective maps.

There exists a generalisation of thinly homotopic and intimate homotopic equivalence on the path space, which lead to the definition of the thin and intimate fundamental groupoid.

Definition 3.1.10. *Two paths $\gamma, \gamma' : I \to P\Sigma$ such that $s(\gamma) = s(\gamma')$, respectively, $t(\gamma) = t(\gamma')$ are said to be **thin path-homotopic** iff*

- *(Thin Path-Homotopic 1) there exists a finite sequence $\gamma_1, ..., \gamma_n$ of paths such that $\gamma_1 = \gamma$ and $\gamma_n = \gamma'$ and $\gamma_{i+1}^{-1} \circ \gamma_i$ is a thin loop for $i = 1, .., n-1$.*

Denote with $[\gamma]$ the equivalence class of thin path-homotopy.

*Two paths $\gamma, \gamma' : I \to P\Sigma$ such that $s(\gamma) = s(\gamma')$, respectively, $t(\gamma) = t(\gamma')$ are said to be **intimate path-homotopic** iff there is a map $\varrho : I \times I \to \Sigma$ such that*

- *(Path-Homotopic 1) ϱ is (piecewise) smooth*
- *(Path-Homotopic 2) for $0 \leq \epsilon \leq 1/2$ the map ϱ satisfies*

$$\begin{aligned} 0 \leq s \leq \epsilon, &\qquad \varrho(s,t) = \gamma(t) \\ 1-\epsilon \leq s \leq 1, &\qquad \varrho(s,t) = \gamma'(t) \\ 0 \leq t \leq \epsilon, &\qquad \varrho(s,t) = \gamma(0) \\ 1-\epsilon \leq t \leq 1, &\qquad \varrho(s,t) = \gamma(1) \end{aligned}$$

- *(Intimate Path-Homotopic 3) the rank of the differential $\mathrm{d}\,\varrho(s,t)$ is smaller or equal 1 for all $(s,t) \in [0,1]^2$.*

*A map $\varrho : [0,1] \times [0,1] \to \Sigma$ satisfying (Path-Homotopic 1), (Path-Homotopic 2) and (Path-Homotopic 3) is called a **smooth rank-one homotopy**.*

The last condition (Path-Homotopic 3) of intimate path-homotopy guarantees that the homotopy sweeps out a surface of vanishing area. The (smooth) **path-homotopy** equivalence (relativ to endpoints) is defined by (Path-Homotopic 1) and (Path-Homotopic 2). Hence two paths, which differ only by a reparamtrization, are not path-homotopic equivalent. Recognize that intimate path-homotopy is stronger than (smooth) path-homotopy. Denote the intimate path-homotopic equivalence relation by $\sim_{\text{intimate path-hom.}}$.

Moreover for two paths $[\gamma], [\gamma']$ a composition operation $\cdot$ is defined if $\gamma(1) = \gamma'(0)$.

Definition 3.1.11. *The **thin fundamental groupoid** $\Pi_1^1(\Sigma, v)$ **at** v over Σ is the quotient of the path space $P\Sigma^v$ at v and thin path-homotopy equivalence. The source and target maps are $s_{\mathcal{P}\Sigma}([\gamma]) = s(\gamma) = v$ and $t_{\mathcal{P}\Sigma}([\gamma]) = t(\gamma) = \gamma(1)$, the constant path $\mathbb{1}_v$ at v give rise to the inclusion $v \mapsto \mathbb{1}_v$, the multiplication is given by the concatenation*

$$\begin{aligned} [\gamma \cdot \gamma'](t) &= \gamma(2t) &&\text{for } 0 \leq t \leq 1/2 \\ [\gamma \cdot \gamma'](t) &= \gamma'(2t-1) &&\text{for } 1/2 \leq t \leq 1 \end{aligned} \tag{3.7}$$

and the inverse element of a path is given by the reverse of the path $[\gamma]^{-1} = \gamma(1-t)$.

In the following omit the brackets $[\gamma]$ for elements of $\Pi_1^1(\Sigma, v)$. Notice that, the vertex or loop group $\Pi_1^1\Sigma_v^v$ of $\Pi_1^1\Sigma$ at a base point v is the thin fundamental group $\pi_1^1(\Sigma, v)$.

The quotient of the path space $P\Sigma$ and the (piecewise) intimate path-homotopic equivalence given in definition 3.1.10 is called the **intimate fundamental groupoid** $\Pi_1^{\text{in}}\Sigma$. The vertex group $\Pi_1^{\text{in}}\Sigma_v^v$, which is given by all loops at v, of the groupoid $\Pi_1^{\text{in}}\Sigma$ at a base point v is the intimate fundamental group $\pi_1^{\text{in}}(\Sigma, v)$.

If $P\Sigma$ and the usual path-homotopy (relativ to the endpoints) equivalence is used, then the appropriate quotient is called the fundamental groupoid $\Pi_1\Sigma$ and the vertex group is the fundamental group $\pi_1(\Sigma, v)$. If additionally Σ is connected, then the fibres $\Pi_1\Sigma^v$ are the universal covering spaces of Σ. Remark that the fundamental group does not coincide with the loop group defined by thin homotopy in general. But the thin fundamental group is a quotient of the fundamental group, since, thin homotopy is a restricted notion of the usual homotopy equivalence on loop spaces.

Remark 3.1.12. *The* ***fundamental groupoid*** *over* Σ *is presented by the set*

$$\Pi_1(\Sigma) := \{(v, [\gamma], w) : v, w \in \Sigma, [\gamma] \text{ path-homotopy class of paths in } \mathcal{P}\Sigma \text{ s.t. } \gamma(0) = v, \gamma(1) = w\}$$

equipped with quotient topology of the compact open topology on $\mathcal{P}\Sigma$.

The map $s_{\Pi_1\Sigma} \times t_{\Pi_1\Sigma} : \Pi_1\Sigma \to \Sigma \times \Sigma$ *is the covering map.*

In general consider the path space modulo reparametrisation equivalence. Denote the this quotient also by $\mathcal{P}\Sigma$. Then the following groupoid can be constructed.

Definition 3.1.13. *A (algebraic)* ***path groupoid*** $\mathcal{P}\Sigma$ *over* Σ *is a pair* $(\mathcal{P}\Sigma, \Sigma)$ *equipped with the following structures:*

(i) two surjective maps $s_{\mathcal{P}\Sigma}, t_{\mathcal{P}\Sigma} : \mathcal{P}\Sigma \to \Sigma$ *called the source and target map,*

(ii) the set $\mathcal{P}\Sigma^2 := \{(\gamma_i, \gamma_j) \in \mathcal{P}\Sigma \times \mathcal{P}\Sigma : t(\gamma_i) = s(\gamma_j)\}$ *of composable pairs of paths,*

(iii) the composition $\circ : \mathcal{P}\Sigma^2 \to \mathcal{P}\Sigma$, *where* $(\gamma_i, \gamma_j) \mapsto \gamma_i \circ \gamma_j$,

(iv) the inversion $\gamma_i \mapsto \gamma_i^{-1}$ *of a path* γ_i

(v) object inclusion map $\iota : \Sigma \to \mathcal{P}\Sigma$ *and*

(vi) $\mathcal{P}\Sigma$ *modulo the algebraic equivalence relations generated by*

$$\gamma_i^{-1} \circ \gamma_i \simeq \mathbb{1}_{s(\gamma_i)} \text{ and } \gamma_i \circ \gamma_i^{-1} \simeq \mathbb{1}_{t(\gamma_i)}$$

Shortly, write $\mathcal{P}\Sigma \overset{s_{\mathcal{P}\Sigma}}{\underset{t_{\mathcal{P}\Sigma}}{\rightrightarrows}} \Sigma$.

3.1.3 Finite path groupoids and graph systems

The aim of this section is to introduce a discretised version of an algebraic path groupoid in the (semi-)analytic category.

Let $c : [0, 1] \to \Sigma$ be continuous curve in the domain $[0, 1]$, which is (piecewise) C^k-differentiable ($1 \leq k \leq \infty$), analytic ($k = \omega$) or semi-analytic ($k = s\omega$) in $[0, 1]$ and oriented such that the source vertex is $c(0) = s(c)$ and the target vertex is $c(1) = t(c)$. Moreover assume that, the range of each subinterval of the curve c is a submanifold, which can be embedded in Σ. An **edge** is given by a reparametrisation invariant curve of class (piecewise) C^k. The maps $s_\Sigma, t_\Sigma : P\Sigma \to \Sigma$, where $P\Sigma$ is the path space, are surjective maps and are called the source or target map.

A set of edges $\{e_i\}_{i=1,...,N}$ is called **independent**, if the only intersections points of the edges are source $s_\Sigma(e_i)$ or $t_\Sigma(e_i)$ target points. Composed edges are called **paths**. An **initial segment** of a path γ is a path γ_1 such that there exists another path γ_2 and $\gamma = \gamma_1 \circ \gamma_2$. The second element γ_2 is also called a **final segment** of the path γ.

Definition 3.1.14. *A* ***graph*** Γ *is a union of finitely many independent edges* $\{e_i\}_{i=1,...,N}$ *for* $N \in \mathbb{N}$. *The set* $\{e_1, ..., e_N\}$ *is called the* ***generating set for*** Γ. *The number of edges of a graph is denoted by* $|\Gamma|$. *The elements of the set* $V_\Gamma := \{s_\Sigma(e_k), t_\Sigma(e_k)\}_{k=1,...,N}$ *of source and target points are called* ***vertices****.*

A graph generate a finite path groupoid in the sense that the set $\mathcal{P}_\Gamma\Sigma$ contain all independent edges, their inverses and all possible compositions of edges. All the elements of $\mathcal{P}_\Gamma\Sigma$ are called paths associated to a graph. Furthermore, the surjective source and target maps s_Σ and t_Σ are restricted to the maps $s, t : \mathcal{P}_\Gamma\Sigma \to V_\Gamma$, which are required to be surjective.

Definition 3.1.15. *Let Γ be a graph. Then a* ***finite path groupoid*** $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *is a pair* $(\mathcal{P}_\Gamma\Sigma, V_\Gamma)$ *of finite sets equipped with the following structures:*

(i) two surjective maps $s,t:\mathcal{P}_\Gamma\Sigma\rightarrow V_\Gamma$*, which are called the source and target map,*

(ii) the set $\mathcal{P}_\Gamma\Sigma^2 := \{(\gamma_i,\gamma_j)\in\mathcal{P}_\Gamma\Sigma\times\mathcal{P}_\Gamma\Sigma : t(\gamma_i)=s(\gamma_j)\}$ *of finitely many composable pairs of paths,*

(iii) the composition $\circ : \mathcal{P}^2_\Gamma\Sigma\rightarrow\mathcal{P}_\Gamma\Sigma,$ *where* $(\gamma_i,\gamma_j)\mapsto\gamma_i\circ\gamma_j$*,*

(iv) the inversion map $\gamma_i\mapsto\gamma_i^{-1}$ *of a path,*

(v) the object inclusion map $\iota : V_\Gamma\rightarrow\mathcal{P}_\Gamma\Sigma$ *and*

(vi) $\mathcal{P}_\Gamma\Sigma$ *is defined by the set* $\mathcal{P}_\Gamma\Sigma$ *modulo the algebraic equivalence relations generated by*

$$\gamma_i^{-1}\circ\gamma_i\simeq\mathbb{1}_{s(\gamma_i)} \text{ and } \gamma_i\circ\gamma_i^{-1}\simeq\mathbb{1}_{t(\gamma_i)} \tag{3.8}$$

Shortly, $\mathcal{P}_\Gamma\Sigma \underset{t}{\overset{s}{\rightrightarrows}} V_\Gamma$.

Clearly, a graph Γ generate freely the paths in $\mathcal{P}_\Gamma\Sigma$. Moreover, the map $s\times t : \mathcal{P}_\Gamma\Sigma\rightarrow V_\Gamma\times V_\Gamma$ defined by $(s\times t)(\gamma) = (s(\gamma),t(\gamma))$ for all $\gamma\in\mathcal{P}_\Gamma\Sigma$ is assumed to be surjective ($\mathcal{P}_\Gamma\Sigma$ over V_Γ is a transitive groupoid), too.

A general groupoid $\mathcal{G}$ over $\mathcal{G}^0$ defines a small category where the set of morphisms is denoted in general by $\mathcal{G}$ and the set of objects is denoted by $\mathcal{G}^0$. Hence in particular the path groupoid can be viewed as a category, since

· the set of morphisms is identified with $\mathcal{P}_\Gamma\Sigma$,

· the set of objects is given by V_Γ (the units)

From the condition (3.8) it follows that the path groupoid satisfies additionally

(i) $s(\gamma_i\circ\gamma_j)=s(\gamma_i)$ and $t(\gamma_i\circ\gamma_j)=t(\gamma_j)$ for every $(\gamma_i,\gamma_j)\in\mathcal{P}^2_\Gamma\Sigma$

(ii) $s(v)=v=t(v)$ for every $v\in V_\Gamma$

(iii) $\gamma\circ\mathbb{1}_{s(\gamma)}=\gamma=\mathbb{1}_{t(\gamma)}\circ\gamma$ for every $\gamma\in\mathcal{P}_\Gamma\Sigma$ and

(iv) $\gamma\circ(\gamma_i\circ\gamma_j)=(\gamma\circ\gamma_i)\circ\gamma_j$

(v) $\gamma\circ(\gamma^{-1}\circ\gamma_1)=\gamma_1=(\gamma_1\circ\gamma)\circ\gamma^{-1}$

The condition (iii) imply that the vertices are units of the groupoid.

Usually in this dissertation surfaces play a fundamental role in the definition of quantum variables. In particular, there is a notion of discretised surfaces.

Definition 3.1.16. *Let* $\breve{S}$ *be a finite set of surfaces in* Σ*. A* ***discretised surface*** $S_{\rm d}$ *(associated to* $\breve{S}$*) is a set of points such that* $S_{\rm d}\subset S$ *for a surface* S *in* $\breve{S}$*. Denote a set of discretised surfaces (associated to* $\breve{S}$*) by* $\breve{S}_{\rm d}$*.*

The paths in a manifold start or end at certain points. There exists many different sets of such paths.

Definition 3.1.17. *Let* V_Γ *be the vertex set of a fixed graph* Γ *and* $\breve{S}_{\rm d}$ *a discretised surface set. The set of vertices in* V_Γ *that are not contained in* $\breve{S}_{\rm d}$ *are denoted by* $V_{\bar{\Gamma}}$*.*

Denote the set of all finitely generated paths by

$$\mathcal{P}_\Gamma\Sigma^{(n)} := \{(\gamma_1,...,\gamma_n)\in\mathcal{P}_\Gamma\times...\mathcal{P}_\Gamma : (\gamma_i,\gamma_{i+1})\in\mathcal{P}^{(2)}, 1\leq i\leq n-1\}$$

The set of paths with source point $v\in V_\Gamma$ *is given by*

$$\mathcal{P}_\Gamma\Sigma^v := s^{-1}(\{v\})$$

The set of paths with source points contained in each discretised surface of the set $\breve{S}_{\rm d}$ are denoted by $\mathcal{P}_\Gamma^{\breve{S}_{\rm d}}\Sigma$.
The set of paths with target point $v \in V_\Gamma$ is defined by

$$\mathcal{P}_\Gamma\Sigma_v := t^{-1}(\{v\})$$

The set of paths with target points contained each discretised surface of the set $\breve{S}_{\rm d}$ are denoted by $\mathcal{P}^\Gamma_{\breve{S}_{\rm d}}\Sigma$.
The set of paths with source point $v \in V_\Gamma$ and target point $u \in V_\Gamma$ is

$$\mathcal{P}_\Gamma\Sigma_u^v := \mathcal{P}_\Gamma\Sigma^v \cap \mathcal{P}_\Gamma\Sigma_u$$

The set of paths with source points and target points in $V_{\bar\Gamma}$ is denoted by $\mathcal{P}_{\bar\Gamma}\Sigma$

Definition 3.1.18. *The **isotropy group** $\mathcal{P}_\Gamma\Sigma_v^v$ **at a vertex** $v \in V_\Gamma$ is defined by the set $\{\gamma \in \mathcal{P}_\Gamma\Sigma : s(\gamma) = t(\gamma) = v\}$.*

*The **isotropy group bundle** $\mathcal{P}_\Gamma^\prime\Sigma$ is defined by the set $\{\gamma \in \mathcal{P}_\Gamma\Sigma : s(\gamma) = t(\gamma)\}$.*

Definition 3.1.19. *Let Γ be a graph. A **subgraph** Γ' **of** Γ is a given by a finite set of independent paths in $\mathcal{P}_\Gamma\Sigma$.*

For example, let $\Gamma := \{\gamma_1, ..., \gamma_N\}$ then for example $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3^{-1}, \gamma_4\}$ where $\gamma_1 \circ \gamma_2, \gamma_3^{-1}, \gamma_4 \in \mathcal{P}_\Gamma\Sigma$ is a subgraph of Γ, whereas the set $\{\gamma_1, \gamma_1 \circ \gamma_2\}$ is not a subgraph of Γ. Notice if additionally, $(\gamma_2, \gamma_4) \in \mathcal{P}_\Gamma^{(2)}$ then $\{\gamma_1, \gamma_3^{-1}, \gamma_2 \circ \gamma_4\}$ is a subgraph of Γ, too. Moreover, for $\Gamma := \{\gamma\}$ the graph $\Gamma^{-1} := \{\gamma^{-1}\}$ is a subgraph of Γ. As well, the graph Γ is a subgraph of Γ^{-1}. A subgraph of Γ that is generated by compositions of some paths, which are not reversed in their orientation, of the set $\{\gamma_1, ..., \gamma_N\}$ is called an **orientation preserved subgraph of a graph**. For example for $\Gamma := \{\gamma_1, ..., \gamma_N\}$ orientation preserved subgraphs are given by $\{\gamma_1 \circ \gamma_2\}$, $\{\gamma_1, \gamma_2, \gamma_N\}$ or $\{\gamma_{N-2} \circ \gamma_{N-1}\}$ if $(\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma\Sigma^{(2)}$ and $(\gamma_{N-2}, \gamma_{N-1}) \in \mathcal{P}_\Gamma\Sigma^{(2)}$.

Definition 3.1.20. *A **finite graph system** $\mathcal{P}_\Gamma$ **for** Γ is a finite set of subgraphs of a graph Γ. A finite graph system $\mathcal{P}_{\Gamma'}$ for Γ' is a **finite graph subsystem** of $\mathcal{P}_\Gamma$ for Γ if the set $\mathcal{P}_{\Gamma'}$ is a subset of $\mathcal{P}_\Gamma$ and Γ' is a subgraph of Γ. Shortly, one writes $\mathcal{P}_{\Gamma'} \leq \mathcal{P}_\Gamma$.*

*A **finite orientation preserved graph system** $\mathcal{P}_\Gamma^{\mathfrak{o}}$ for Γ is a finite set of orientation preserved subgraphs of a graph Γ.*

Recall that a finite path groupoid is constructed from a graph Γ, but a set of elements of the path groupoid need not be a graph again. For example, let $\Gamma := \{\gamma_1 \circ \gamma_2\}$ and $\Gamma' = \{\gamma_1 \circ \gamma_3\}$, then $\Gamma'' = \Gamma \cup \Gamma'$ is not a graph, since, this set is not independent. Hence, only appropriate unions of paths, which are elements of a fixed finite path groupoid will define graphs. The idea is to define a suitable action on elements of the path groupoid, which corresponds to an action of diffeomorphisms on the manifold Σ. The action has to be transfered to graph systems. But the action of bisection is defined by the use of the groupoid multiplication cannot easily generalised for graph systems.

Problem 3.1.1: Let $\breve{\Gamma} := \{\Gamma_i\}_{i=1,..,N}$ be a finite set such that each Γ_i is a set of not necessarily independent paths such that

(i) the set contain no loops and

(ii) each pair of paths satisfies one of the following conditions

- the paths intersect each other only in one vertex,
- the paths do not intersect each other or
- one path of the pair is a segment of the other path.

Then there is a map $\circ : \breve{\Gamma} \times \breve{\Gamma} \to \breve{\Gamma}$ of two elements Γ_1 and Γ_2 defined by

$$\{\gamma_1, ..., \gamma_M\} \circ \{\tilde\gamma_1, ..., \tilde\gamma_M\} := \Big\{\gamma_i \circ \tilde\gamma_j : t(\gamma_i) = s(\tilde\gamma_j)\Big\}_{1\leq i,j \leq M}$$

for $\Gamma_1 := \{\gamma_1, ..., \gamma_M\}, \Gamma_2 := \{\tilde\gamma_1, ..., \tilde\gamma_M\}$. Moreover, define a map $^{-1} : \breve{\Gamma} \to \breve{\Gamma}$ by

$$\{\gamma_1, ..., \gamma_M\}^{-1} := \{\gamma_1^{-1}, ..., \gamma_M^{-1}\}$$

Then the following is derived

$$\begin{aligned}\{\gamma_1,...,\gamma_M\}\circ\{\gamma_1^{-1},...,\gamma_M^{-1}\} &= \Big\{\gamma_i\circ\gamma_j^{-1} : t(\gamma_i)=t(\gamma_j)\Big\}_{1\leq i,j\leq M}\\ &= \Big\{\gamma_i\circ\gamma_j^{-1} : t(\gamma_i)=t(\gamma_j) \text{ and } i\neq j\Big\}_{1\leq i,j\leq M}\\ &\quad \cup\{\mathbb{1}_{s_{\gamma_j}}\}_{1\leq j\leq M}\\ &\neq \quad \cup\{\mathbb{1}_{s_{\gamma_j}}\}_{1\leq j\leq M}\end{aligned}$$

The equality would be true if the set $\breve{\Gamma}$ containes only graphs such that all paths are mutually non-composable. Consequently, this does not define a well-defined multiplication map. Notice that the same can be discovered if a similar map and inversion operation are defined for a finite graph system $\mathcal{P}_\Gamma$.

■

Consequently, the property of paths being independent need not be dropped for the definition of a suitable multiplication and inversion operation. In fact, the independence property is a necessary condition for the construction of the holonomy algebra for analytic paths. Only under this circumstance, each analytic path can be decomposed into a finite product of independent piecewise analytic paths again.

To conclude, it is more convincing to consider special graphs. A graph Γ is said to be **disconnected** if it contains only mutually pairs (γ_i,γ_j) of non-composable independent paths γ_i and γ_j for $i\neq j$ and $i,j=1,...,N$. In other words for all $1\leq i,l\leq N$ it is true that $s(\gamma_i)\neq t(\gamma_l)$ and $t(\gamma_i)\neq s(\gamma_l)$ where $i\neq l$ and $\gamma_i,\gamma_l\in\Gamma$.

Let Γ be a disconnected graph, which is equivalent to the set $\{\gamma_1,\gamma_2,\gamma_3\}$ such that $t(\gamma_1)\neq s(\gamma_2)$, $s(\gamma_1)\neq t(\gamma_2)$, $t(\gamma_1)\neq s(\gamma_3)$, $s(\gamma_1)\neq t(\gamma_3)$, $t(\gamma_2)\neq s(\gamma_3)$ and $s(\gamma_2)\neq t(\gamma_3)$. A finite graph system $\mathcal{P}_\Gamma^{\mathrm{I}}$ is said to be **totally disconnected**, if it contains only disconnected subgraphs of a disconnected graph Γ. If $\{\gamma_1,\gamma_2,\gamma_3\}\in\mathcal{P}_\Gamma^{\mathrm{I}}$, then $\{\gamma_1^{-1},\gamma_2^{-1},\gamma_3^{-1}\}\in\mathcal{P}_\Gamma^{\mathrm{I}}$.

Definition 3.1.21. *A finite path groupoid $\mathcal{P}_{\Gamma'}\Sigma$ over $V_{\Gamma'}$ is a **finite path subgroupoid** of $\mathcal{P}_\Gamma\Sigma$ over V_Γ if the set $V_{\Gamma'}$ is contained in V_Γ and the set $\mathcal{P}_{\Gamma'}\Sigma$ is a subset of $\mathcal{P}_\Gamma\Sigma$. Shortly, one writes $\mathcal{P}_{\Gamma'}\Sigma\leq\mathcal{P}_\Gamma\Sigma$.*

Clearly, for a subgraph Γ_1 of a graph Γ_2, the associated path groupoid $\mathcal{P}_{\Gamma_1}\Sigma$ over V_{Γ_1} is a subgroupoid of $\mathcal{P}_{\Gamma_2}\Sigma$ over V_{Γ_2}. This is a consequence of the fact that each path in $\mathcal{P}_{\Gamma_1}\Sigma$ is a composition of paths or their inverses in $\mathcal{P}_{\Gamma_2}\Sigma$.

Definition 3.1.22. *A **family of finite path groupoids** $\{\mathcal{P}_{\Gamma_i}\Sigma\}_{i=1,...,\infty}$, which is a set of finite path groupoids $\mathcal{P}_{\Gamma_i}\Sigma$ over V_{Γ_i}, is said to be **inductive** if for any $\mathcal{P}_{\Gamma_1}\Sigma,\mathcal{P}_{\Gamma_2}\Sigma$ exists a $\mathcal{P}_{\Gamma_3}\Sigma$ such that $\mathcal{P}_{\Gamma_1}\Sigma,\mathcal{P}_{\Gamma_2}\Sigma\leq\mathcal{P}_{\Gamma_3}\Sigma$.*

*A **family of graph systems** $\{\mathcal{P}_{\Gamma_i}\}_{i=1,...,\infty}$, which is a set of finite path systems $\mathcal{P}_{\Gamma_i}$ for Γ_i, is said to be **inductive** if for any $\mathcal{P}_{\Gamma_1},\mathcal{P}_{\Gamma_2}$ exists a $\mathcal{P}_{\Gamma_3}$ such that $\mathcal{P}_{\Gamma_1},\mathcal{P}_{\Gamma_2}\leq\mathcal{P}_{\Gamma_3}$.*

Definition 3.1.23. *Let $\{\mathcal{P}_{\Gamma_i}\Sigma\}_{i=1,...,\infty}$ be an inductive family of path groupoids and $\{\mathcal{P}_{\Gamma_i}\}_{i=1,...,\infty}$ be an inductive family of graph systems.*

*The **inductive limit path groupoid** $\mathcal{P}$ **over** Σ of an inductive family of finite path groupoids such that $\mathcal{P}:=\varinjlim_{i\to\infty}\mathcal{P}_{\Gamma_i}\Sigma$ is called the **(algebraic) path groupoid** $\mathcal{P}\rightrightarrows\Sigma$.*

*Moreover, there exists an **inductive limit graph** Γ_∞ of an inductive family of graphs such that $\Gamma_\infty:=\varinjlim_{i\to\infty}\Gamma_i$.*

*The **inductive limit graph system** $\mathcal{P}_{\Gamma_\infty}$ of an inductive family of graph systems such that $\mathcal{P}_{\Gamma_\infty}:=\varinjlim_{i\to\infty}\mathcal{P}_{\Gamma_i}$*

In this dissertation it is further assumed that, the inductive limit Γ_∞ of a inductive family of graphs is a graph, which consists of an infinite countable number of independent paths. The inductive limit $\mathcal{P}_{\Gamma_\infty}$ of a inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems contains an infinite countable number of subgraphs of Γ_∞ and each subgraph is a finite set of arbitrary independent paths in Σ.

If the curves are assumed to be smooth, then there exists a homotopy equivalence of paths such that a Lie groupoid, called the fundamental groupoid $\Pi_1(\Sigma)$ over Σ, is given.

3.1.4 General Lie and gauge groupoids

Definition 3.1.24. *A **Lie (or smooth) groupoid** $\mathcal{G}$ **over** $\mathcal{G}^0$ is a groupoid where $\mathcal{G}$ and $\mathcal{G}^0$ are smooth manifolds (additionally, $\mathcal{G}^0$ and $\mathcal{G}^v$ for all $v \in \mathcal{G}^0$ are Hausdorff), $s_{\mathcal{G}}$ and $t_{\mathcal{G}}$ are smooth surjective submersions such that $\mathcal{G}^{(2)}$ is a smooth submanifold of the product manifold $\mathcal{G} \times \mathcal{G}$, the inclusion $i : \mathcal{G}^0 \to \mathcal{G}$, the multiplication $\cdot$ and the inversion are smooth maps.*

*A groupoid $\mathcal{G}$ is **transitive** if for each pair $(v,w) \in \mathcal{G}^0 \times \mathcal{G}^0$ there is a morphism $\gamma \in \mathcal{G}$ such that $s_{\mathcal{G}}(\gamma) = v$ and $t_{\mathcal{G}}(\gamma) = w$. A Lie groupoid $\mathcal{G}$ is **locally trivial** if the map $s_{\mathcal{G}} \times t_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^0 \times \mathcal{G}^0$, called anchor of $\mathcal{G}$, is a surjective submersion.*

In particular each locally trivial Lie groupoid is transitive. A vector bundle $E \xrightarrow{q} \Sigma$, which admits a group bundle structure, since, $s_E = t_E = q$ and the composition is fibrewise addition $E_x \times E_x \to E_x$, is a Lie groupoid E over Σ such that this groupoid is not locally trivial.

Definition 3.1.25. *Let $\mathcal{G}$ be a Lie groupoid on Σ.*

*A **Lie subgroupoid** of $\mathcal{G}$ is a Lie groupoid $\mathcal{G}'$ on Σ' together with injective immersions $\iota : \mathcal{G}' \to \mathcal{G}$ and $\iota_0 : \Sigma' \to \Sigma$ such that (ι, ι_0) is a morphism of Lie groupoids.*

In this dissertation it is assumed that, the fibres $\mathcal{G}^u$ and $\mathcal{G}_u$ are connected for all $u \in \mathcal{G}^0$.

Definition 3.1.26. *A **Lie groupoid morphism** between two Lie groupoids $\mathcal{F}$ and $\mathcal{G}$ consists of two smooth maps $\mathfrak{h} : \mathcal{F} \to \mathcal{G}$ and $h : \mathcal{F}^0 \to \mathcal{G}^0$ such that*

$$(G1) \qquad \mathfrak{h}(\gamma \circ \gamma') = \mathfrak{h}(\gamma)\mathfrak{h}(\gamma') \text{ for all } (\gamma, \gamma') \in \mathcal{F}^{(2)}$$

$$(G2) \qquad s_{\mathcal{G}}(\mathfrak{h}(\gamma)) = h(s_{\mathcal{F}}(\gamma)), \quad t_{\mathcal{G}}(\mathfrak{h}(\gamma)) = h(t_{\mathcal{F}}(\gamma))$$

Definition 3.1.27. *A morphism $\mathfrak{h} : \mathcal{F} \to \mathcal{G}$ and $h : \mathcal{F}^0 \to \mathcal{G}^0$ is an **isomorphism of Lie groupoids** if $\mathfrak{h}$ and h are diffeomorphisms.*

The simpliest example for a Lie groupoid is given by the gauge groupoid.

Definition 3.1.28. *Let G act on the right of the product $P \times P$ of a principal bundle $P := P(\Sigma, G)$ by*

$$g(u,p) = (ug, pg)$$

*and denote the orbit of (u,p) by $\langle u,p \rangle$ and the set of orbits $\frac{P\times P}{G}$. Then $\mathcal{G}$ denote the groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$, called the **gauge groupoid associated to** $P(\Sigma, G)$ **with base** Σ, with*

$$\begin{aligned}
&s_P(\langle u,p\rangle) := (\pi \circ \mathrm{pr}_1)(\langle u,p\rangle) = \pi(u), \quad t_P(\langle u,p\rangle) := (\pi \circ \mathrm{pr}_2)(\langle u,p\rangle) = \pi(p)\\
&i(v) = \mathbb{1}_v := \langle u,u\rangle \text{ where } u \ni \pi^{-1}(v)\\
&\langle u_1,p_1\rangle \cdot \langle u_2,p_2\rangle := \langle u_1\delta(p_1,u_2),p_2\rangle \text{ for all } \langle u_i,p_i\rangle =: e_i, i = 1,2\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{such that } t_P(e_1) = s_P(e_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } \delta : P \times P \to G, \delta(u,ug) = g
\end{aligned}$$

such that the groupoid multiplication is smooth, the source map $s_P : \mathcal{G} \to \Sigma$ is a surjective submersion.

The inverse is given by

$$\langle u,p\rangle^{-1} := \langle p,u\rangle$$

Set $\delta : G \times G \to G$ to be the difference map which is defined by $\delta(g,h) := g^{-1}h$. Then compute

$$\langle u,p\rangle\langle u,p\rangle^{-1} = \langle u,p\rangle\langle p,u\rangle = \langle u\delta(p,p),u\rangle = \mathbb{1}_v \tag{3.9}$$

and

$$\langle u,p\rangle\langle u,p\rangle = \langle u\delta(p,u),p\rangle \tag{3.10}$$

Set $p = ux$ and $u' = py$ and derive

$$\begin{aligned}
\langle u,p\rangle\langle u,p\rangle\langle p,u'\rangle\langle p,u'\rangle &= \langle u\delta(p,u),p\rangle\langle p\delta(u',p),u'\rangle = \langle ux,p\rangle\langle py,u'\rangle = \langle p\delta(p,py),u'\rangle\\
&= \langle py,u'\rangle = \mathbb{1}_v
\end{aligned} \tag{3.11}$$

Lemma 3.1.29. *The map*

$$H : P \times P \to \mathcal{G}, h : P \to \Sigma$$

is a Lie groupoid homomorphism from the pair groupoid $P \times P$ over P.

Corollary 3.1.30. *The gauge groupoid $\mathcal{G}$ of a principal bundle $P(\Sigma, G, \pi)$ is a locally trivial Lie groupoid.*

Another Lie groupoid is the fundamental groupoid $\Pi_1\Sigma$ over Σ, which has been introduced in remark 3.1.12. There is a correspondence between this fundamental groupoid and the gauge groupoid.

Corollary 3.1.31. *The fundamental groupoid $\Pi_1\Sigma$ is the gauge groupoid of the principal bundle $\tilde{\Sigma}(\Sigma, \pi(\Sigma))$ where $\tilde{\Sigma}$ is the universal cover of Σ.*

Lemma 3.1.32. *Let Σ be a connected manifold.*

Then the Lie groupoid $\Pi_1\Sigma$ is connected.

For the generalisation of the concept of Barrett's duality betwen smooth connections and holonomy maps the following objects are necessary. The full detailed mathematics can be found in the book of Mackenzie [66].

Proposition 3.1.33. *Let $\mathcal{G}$ be a locally trivial Lie groupoid. Then*

(i) the isotropy groups $\mathcal{G}^u_u$ of $\mathrm{Iso}(\mathcal{G}) = \{\mathcal{G}^u_u\}_{u\in\mathcal{G}^0}$ are isomorphic to Lie groups.

(ii) For each $u \in \mathcal{G}^0$ the fibre $\mathcal{G}^u := s_{\mathcal{G}}^{-1}(u)$ is a differentiable principal bundle over $\mathcal{G}^0$ with the surjection $t_{\mathcal{G}}$, a smooth and free left action $L : \mathcal{G}^u_u \times \mathcal{G}^u \to \mathcal{G}^u$ and the isotropy group $\mathcal{G}^u_u$ as structure group. $\mathcal{G}^u(\mathcal{G}^0, \mathcal{G}^u_u, t_{\mathcal{G}})$ is called ***vertex bundle at*** *u.*

(iii) for $u, v \in \mathcal{G}^0$ and an element $\theta \in \mathcal{G}^u_v$ there is an isomorphism of principal bundles over $\mathcal{G}^0$

$$L_\theta(\mathrm{id}_{\mathcal{G}^0}, I_\theta) : \mathcal{G}^v(\mathcal{G}^0, \mathcal{G}^v_v, t_{\mathcal{G}}) \to \mathcal{G}^u(\mathcal{G}^0, \mathcal{G}^u_u, t_{\mathcal{G}})$$
$$\text{where } I_\theta : \mathcal{G}^v_v \to \mathcal{G}^u_u, \quad \gamma \mapsto \theta \circ \gamma \circ \theta^{-1}$$

Corollary 3.1.34. *Let $\mathcal{G}$ be a locally trivial Lie groupoid over $\mathcal{G}^0$.*

Then the map

$$\frac{\mathcal{G}^u \times \mathcal{G}^u}{\mathcal{G}^u_u} \to \mathcal{G}, \quad \langle \tilde{\gamma}', \tilde{\gamma} \rangle \mapsto \tilde{\gamma}' \circ \tilde{\gamma}^{-1} \tag{3.12}$$

from the gauge groupoid of the vertex bundle at u to the Lie groupoid is an isomorphism of Lie groupoids over $\mathcal{G}^0$.

The map

$$\frac{\mathcal{G}^u \times \mathcal{G}^u_u}{\mathcal{G}^u_u} \to \mathrm{Iso}(\mathcal{G}), \quad < \tilde{\gamma}, \alpha > \mapsto \tilde{\gamma} \circ \alpha \circ \tilde{\gamma}^{-1} \tag{3.13}$$

from the associated fibre bundle to the vertex bundle at u w.r.t. the action of inner-translation on $\mathcal{G}^u_u$ to the isotropy Lie group bundle is an isomorphism of Lie groupoids over $\mathcal{G}^0$.

Let $\mathcal{G}$ be a locally trivial Lie groupoid with connected fibres $\mathcal{G}^v$. Denote $\mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ be the set of continuous and piecewise-smooth paths $\tilde{\gamma} : I \to \mathcal{G}$ (where $I = [0, 1]$) for which $s_{\mathcal{G}} \circ \tilde{\gamma} : [0, t] \to \mathcal{G}^0$ is constant for all elements of $\tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ and $t \in I$.

Definition 3.1.35. *Two paths γ and γ' in $\mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ are called $s_{\mathcal{G}}$**-homotopic** $\gamma \overset{s_\mathcal{G}}{\approx} \gamma'$, if $s_{\mathcal{G}}(\gamma) = s_{\mathcal{G}}(\gamma')$, respectively, $t_{\mathcal{G}}(\gamma) = t_{\mathcal{G}}(\gamma')$ and there is a continuous and piecewise-smooth map $\varrho : I \times I \to \mathcal{G}$ such that (Path-Homotopic 2) property:*

$$\begin{aligned} 0 \leq s \leq \epsilon, &\qquad \varrho(s,t) = \gamma'(t) \\ 1-\epsilon \leq s \leq 1, &\qquad \varrho(s,t) = \gamma(t) \\ 0 \leq t \leq \epsilon, &\qquad \varrho(s,t) = \gamma(0) \\ 1-\epsilon \leq t \leq 1, &\qquad \varrho(s,t) = \gamma(1) \end{aligned}$$

is satisfied and for each $s \in I$ the map $\varrho(s,t)$ is an element of $\mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$. The equivalence class is denoted by $[\tilde{\gamma}]$ for all $\tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$.

Note that this definition is related to the ordinary homotopic equivalence relation.

Proposition 3.1.36. *[66, Prop.6.1.8] Let $\mathcal{G}$ be a locally trivial Lie groupoid on a connected base Σ and let $\theta : \Sigma \times \Sigma \rightarrow \mathcal{G}$ be a local morphism.*

Then there is a unique morphism of Lie groupoids $\mathfrak{h} : \Pi\Sigma \rightarrow \mathcal{G}$ such that $\mathfrak{h} \circ \tau = \theta$ where τ is the left inverse of $s_{\Pi\Sigma} \times t_{\Pi\Sigma}$.

Definition 3.1.37. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Lie groupoids and $\varphi : \mathcal{G}' \rightarrow \mathcal{G}$ be a morphisms of Lie groupoids over a smooth map $f : \Sigma' \rightarrow \Sigma$.*

Then $f^!\mathcal{G} = \mathcal{G} * \Sigma'$ denotes the pullback manifold

$$f^!\mathcal{G} = \{(\gamma, v') \in \mathcal{G} \times \Sigma' : s_{\mathcal{G}}(\gamma) = f(v')\}$$

and $\varphi^!$ denotes the map

$$\varphi^! : \mathcal{G}' \rightarrow f^!\mathcal{G}, \quad \gamma' \mapsto (\varphi(\gamma'), s_{\mathcal{G}'}(\gamma'))$$

Remark that, $f^!\mathcal{G}$ do not define a groupoid and the map $\varphi^!$ is not a morphism.

Definition 3.1.38. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Lie groupoids. Moreover, let $F : \mathcal{G}' \rightarrow \mathcal{G}$ and $f : \Sigma' \rightarrow \Sigma$ be a morphisms of Lie groupoids.*

Then (F, f) is called ***fibration*** *if $f : \Sigma' \rightarrow \Sigma$ and $F^! : \mathcal{G} \rightarrow f^!\mathcal{G}'$ are surjective submersions.*

Finally in general for every Lie groupoid $\mathcal{G}$ there exists an associated Lie algebroid $A\mathcal{G}$.

Definition 3.1.39. *A* ***Lie algebroid*** *$A\mathcal{G}$* ***associated to a transitive Lie groupoid*** *$\mathcal{G}$ is a vector bundle over $\mathcal{G}^0$, which is equipped with a vector bundle map $a : A\mathcal{G} \rightarrow T\mathcal{G}^0$, which is called anchor, a Lie bracket $[.,.]_{A\mathcal{G}}$ on the space $\Gamma(A\mathcal{G})$ of smooth sections of $A\mathcal{G}$, satisfying the following compatibility conditions*

(i) *$\mathbb{R}$-bilinear*

(ii) *alternating and Jacobi identity*

(iii) $[X, fY] = f[X, Y] + a(X)(f)Y$ *for all* $X, Y \in \Gamma(A\mathcal{G})$ *and* $f \in C^\infty(\Sigma)$

(iv) $a([X, Y]) = [a(X), a(Y)]$ *for all* $X, Y \in \Gamma(A\mathcal{G})$ *and* $f \in C^\infty(\Sigma)$.

In particular the vector bundle $T_v(\mathcal{G})$ over $\mathcal{G}^0$ is a Lie algebroid over $\mathcal{G}^0$.

3.1.5 Transformations in a Lie groupoid

After the definitions of the basic objects some transformation operations are introduced. The definitions are borrowed from Mackenzie [66].

Definition 3.1.40. *Let $\mathcal{G}$ be a Lie groupoid on the base $\mathcal{G}^0$, for $\tilde{\gamma} \in \mathcal{G}$ with $s_{\mathcal{G}}(\tilde{\gamma}) = v$ and $t_{\mathcal{G}}(\tilde{\gamma}) = w$ the* ***left-translation corresponding to*** *$\mathcal{G}$ is defined by*

$$L_{\tilde{\gamma}} : \mathcal{G}^w \rightarrow \mathcal{G}^v, \quad \tilde{\vartheta} \mapsto \tilde{\gamma} \circ \tilde{\vartheta}$$

and the ***right-translation corresponding to*** *$\mathcal{G}$*

$$R_{\tilde{\gamma}} : \mathcal{G}_v \rightarrow \mathcal{G}_w, \quad \tilde{\vartheta} \mapsto \tilde{\vartheta} \circ \tilde{\gamma}$$

Definition 3.1.41. ***A left-translation in a Lie groupoid*** *$\mathcal{G}$ over Σ is a pair of diffeomorphisms*

$$\Phi : \mathcal{G} \rightarrow \mathcal{G} \text{ and } \varphi : \Sigma \rightarrow \Sigma$$

such that

$$s_{\mathcal{G}}(\Phi(\gamma)) = \varphi(s_{\mathcal{G}}(\gamma)), \quad t_{\mathcal{G}}(\Phi(\gamma)) = \varphi(t_{\mathcal{G}}(\gamma)) \text{ for all } \gamma \in \mathcal{G}$$

and, moreover,

$$\Phi^v : \mathcal{G}^v \to \mathcal{G}^{\varphi(v)}, \quad \gamma \mapsto L_\theta(\gamma) \text{ for some } \theta \in \mathcal{G}_v^{\varphi(v)} \text{ and all } \gamma \in \mathcal{G}^v$$

A ***global bisection of*** $\mathcal{G}$ *is a smooth map* $\sigma : \Sigma \to \mathcal{G}$ *which is right-inverse to* $s_{\mathcal{G}} : \mathcal{G} \to \Sigma$ *(w.o.w.* $s_{\mathcal{G}} \circ \sigma = \mathrm{id}_\Sigma$*) and such that* $t_{\mathcal{G}} \circ \sigma : \Sigma \to \Sigma$ *is a diffeomorphism. The set of bisections on* $\mathcal{G}$ *is denoted* $\mathfrak{B}(\mathcal{G})$*. Denote the image of the bisection by* $L := \{\sigma(v) : v \in \Sigma\}$ *which is a closed embedded submanifold of* $\mathcal{G}$*.*

The set of global bisections $\mathfrak{B}(\mathcal{G})$ form a group, where the multiplication is given by

$$(\sigma * \sigma')(v) = \sigma'(v) \circ \sigma(t_{\mathcal{G}}(\sigma'(v))) \text{ for } v \in \Sigma, \tag{3.14}$$

the object inclusion $v \mapsto \mathbb{1}_v$ of $\mathcal{G}$, where id_v is the unit morphism at v in $\mathcal{G}_v^v$, and the inversion is given by

$$\sigma^{-1}(v) = \sigma((t_{\mathcal{G}} \circ \sigma)^{-1}(v))^{-1} \text{ for } v \in \Sigma \tag{3.15}$$

Define for a given bisection σ, the right-translation in $\mathcal{G}$ by

$$R_\sigma : \mathcal{G} \to \mathcal{G}, \quad \gamma \mapsto \gamma \circ \sigma(t_{\mathcal{G}}(\gamma)) \tag{3.16}$$

The map $\sigma \mapsto R_\sigma$ is a group isomorphism, i.e. $R_{\sigma * \sigma'} = R_\sigma \circ R_{\sigma'}$. Whereas $\sigma \mapsto t_{\mathcal{G}} \circ \sigma$ is a group morphism from $\mathfrak{B}(\mathcal{G})$ to the group of diffeomorphisms $\mathrm{Diff}(\Sigma)$.

Notice that local bisection are defined to be maps $\sigma : U \to \mathcal{G}$ where U is an open subset in Σ.

Definition 3.1.42. *Let* $\mathcal{G}$ *be a Lie groupoid on* Σ *, and fix a bisection* $\sigma \in \mathfrak{B}(\mathcal{G})$*.*

Then define the ***left-translation***

$$L_\sigma : \mathcal{G} \to \mathcal{G}, \quad \gamma \mapsto \sigma((t_{\mathcal{G}} \circ \sigma)^{-1}(s_{\mathcal{G}}(\gamma))) \circ \gamma$$

and the ***inner-translation*** *is given by*

$$I_\sigma : \mathcal{G} \to \mathcal{G}, \quad \gamma \mapsto \sigma(s_{\mathcal{G}}(\gamma))^{-1} \circ \gamma \circ \sigma(t_{\mathcal{G}}(\gamma))$$

which is an isomorphism of Lie groupoids over $t_{\mathcal{G}} \circ \sigma : \Sigma \to \Sigma$*.*

Clearly $L_{\sigma * \sigma'} = L_\sigma \circ L_{\sigma'}$ and $I_{\sigma * \sigma'} = I_\sigma \circ I_{\sigma'}$. $L_{\sigma^{-1}}(\gamma) = \sigma(s_{\mathcal{G}}(\gamma))^{-1} \circ \gamma$ and $I_\sigma = R_\sigma \circ L_{\sigma^{-1}} = L_{\sigma^{-1}} \circ R_\sigma$ yield.

Transformations in the gauge groupoid associated to a principal fibre bundle $P(\Sigma, G, \pi)$:

Lemma 3.1.43. *Consider the automorphisms* $\varphi : P \to P$ *on the principal bundle* $P(\Sigma, G, \pi)$*, a diffeomorphism* φ_0 *on* Σ *and the identity map* id *on the structure group* G*. Assume* $\pi \circ \varphi = \varphi_0 \circ \pi$*,* $\varphi(ug) = \varphi(u)g$ *for all* $u \in P$ *and* $g \in G$*.*

For a fixed element $u \in P$ *such that* $\pi^{-1}(u) = v$ *yields, set*

$$\sigma(v) := \langle u, \varphi(u) \rangle$$

Then σ *is smooth, since* π *is an surjective submersion and* σ *is a bisection of the gauge groupoid* $\frac{P \times P}{G} \rightrightarrows \Sigma$*.*

Moreover, there exists an action $I_\sigma : \frac{P \times P}{G} \longrightarrow \frac{P \times P}{G}$ *given by*

$$I_\sigma(\langle u, p \rangle) := \langle \varphi(u), \varphi(p) \rangle$$

The automorphism $\varphi(\varphi_0, \mathrm{id})$ *is called* ***gauge and diffeomorphism transformation*** *in* $P(\Sigma, G, \pi)$*.*

3.2 Duality of connections and holonomies

A generalisation of Barrett's duality [16] is given by the duality of infinitesimal connections of a principal bundle $P(\Sigma,\pi,G)$ and path connections in a corresponding Lie groupoid. In this particular case the Lie groupoid is given by the gauge groupoid. The theory of this duality in a more general framework (of transitive Lie groupoids) has been invented by Mackenzie [66]. In this section a very short overview about the basic structures is given. The infinitesimal geometric objects are presented for the special case of a gauge theory. Moreover these definitions coincide with the objects, which are usually given in books about differential geometry (refer to appendix). The duality of infinitesimal connections and holonomies is analysed in the more general context of Mackenzie. The idea is to use the more enhanced framework for a definition of quantum variables for a gauge theory and a gravitational theory and hence for new algebras.

3.2.1 Infinitesimal geometric objects for a gauge theory

In this section the basic geometric objects and their relations in the framework of Mackenzie are collected. For a study refer to the book [66] of Mackenzie.

Let $P(\Sigma,G,\pi)$ be a principal bundle where P is a smooth (or later analytic) manifold. The object $T\Sigma$ over Σ defines a Lie algebroid. The vector bundle morphism $\frac{TP}{G}\stackrel{\pi_*}{\to}T\Sigma$ constructed from the principal G- bundle $P\stackrel{\pi}{\to}\Sigma$ defines a Lie algebroid structure on $\frac{TP}{G}$. This follows from the fact that, there is a commutator map $[.,.]$ defined on the sections $\Gamma(T\Sigma)$ which is transfered to a bracket $[.,.]_P$ on $\frac{TP}{G}$. Hence $\frac{TP}{G}$ over Σ is a Lie algebroid. The object $\mathcal{G}:=\frac{P\times P}{G}\rightrightarrows\Sigma$ is a transitive Lie groupoid, called the **gauge groupoid** of a principal bundle $P(\Sigma,G,\pi)$.

On the one hand there exists an exact sequence of Lie groupoids

$$\frac{P\times G}{G}\stackrel{\iota}{\rightarrowtail}\frac{P\times P}{G}\stackrel{\tilde{\pi}}{\twoheadrightarrow}\Sigma\times\Sigma \tag{3.17}$$

where $\frac{P\times G}{G}$ is a Lie group bundle and $\Sigma\times\Sigma\rightrightarrows\Sigma$ is the **pair groupoid**. The maps $\iota,\tilde{\pi}$ are groupoid morphisms, ι is an embedding, $\tilde{\pi}$ is a surjective submersion and $\mathrm{Im}(\iota)=\ker(\tilde{\pi})$. Notice that, $\tilde{\pi}$ is given by the map $s_P\times t_P$, where s_P,t_P are the source and target map of the Lie groupoid $\frac{P\times P}{G}\rightrightarrows\Sigma$.

On the other hand there is an exact sequence of the Lie algebroids $\frac{TP}{G}$ and $T\Sigma$ over Σ, called the **Atiyah sequence**, which is given by

$$\frac{P\times\mathfrak{g}}{G}\stackrel{j}{\rightarrowtail}\frac{TP}{G}\stackrel{\pi_*}{\twoheadrightarrow}T\Sigma \tag{3.18}$$

such that j,π_* are vector bundle morphisms. The Lie algebroid bundle $\frac{P\times\mathfrak{g}}{G}=\ker(\pi_*)$ is called the **adjoint bundle**.

A **Lie algebroid connection** γ_A is a right splitting of the exact sequence (3.18), which is a map $\gamma_A:T\Sigma\to\frac{TP}{G}$ such that $\pi_*\circ\gamma_A=\mathrm{id}_{T\Sigma}$. A Lie algebroid connection γ_A is also called an **infinitesimal connection** and is shortly denoted by A (refer to appendix 12.1.1). The formulation as a right splitting of an exact sequence has the advantage that this definition simply generalises to transitive Lie groupoids and transitive Lie algebroids. This is derived in the book [66, Section 3.5, Section 5.2] of Mackenzie. In general an **infinitesimal connection in a transitive Lie groupoid** $\mathcal{G}$ over Σ is a morphism of vector bundles $\gamma:T\Sigma\to A\mathcal{G}$ over Σ such that $a\circ\gamma=\mathrm{id}_{T\Sigma}$ and a is the anchor of the Lie algebroid $A\mathcal{G}$ associated to a transitive Lie groupoid $\mathcal{G}$. Hence, in the context of a gauge theory the infinitesimal connections are also called the infinitesimal connections in the gauge groupoid. The adjoint connection $\nabla^{\mathrm{ad}}:=\mathrm{ad}\circ\gamma_A$ is defined as the commutator $j(\nabla^{\mathrm{ad}}_X(V))=[\gamma_A X,j(V)]$ for $V\in\Gamma\big(\frac{P\times\mathfrak{g}}{G}\big)$ and X is a smooth vector field $\mathfrak{X}(\Sigma)$.

Notice the following structure. There exists a map $l:T\Sigma\longrightarrow\frac{P\times\mathfrak{g}}{G}$ such that the Lie algebroid connection $\gamma_{A'}$ decomposes into a sum $\gamma_{A'}:=\gamma_A+j\circ l$, where γ_A is another Lie algebroid connection.

A **connection reform** $\omega:TP\to P\times\mathfrak{g}$ which is G-equivariant and horizontal, w.o.w. $\omega^{/G}\in\Omega^1_{\mathrm{basic}}(P,\mathfrak{g})^G$, corresponds to a morphism $\omega^{/G}:\frac{TP}{G}\longrightarrow\frac{P\times\mathfrak{g}}{G}$ of vector bundles over Σ such that $\omega^{/G}\circ j=\mathrm{id}_L$ where $L=\frac{P\times\mathfrak{g}}{G}$. There is a bijective correspondence betwen the connection γ_A and a connection reform ω, such that $j\circ\omega^{/G}+\gamma_A\circ\pi_*=\mathrm{id}_{A\mathcal{G}}$ where $A\mathcal{G}:=\frac{TP}{G}$.

The **curvature** (for a gauge theory) is a skew-symmetric vector bundle map $\bar{R}_A:T\Sigma\oplus T\Sigma\to\frac{P\times\mathfrak{g}}{G}$ such that $j(\bar{R}_A(X,Y))=\gamma_A([X,Y])-[\gamma_A(X),\gamma_A(Y)]$ for $X,Y\in\mathfrak{X}(\Sigma)$ being smooth sections in $T\Sigma$.

3.2.2 Integrated infinitesimals, path connections, holonomy groupoids and holonomy maps in groupoids

The concept of integrated infinitesimal connections over a lifted path in a Lie groupoid $\mathcal{G}$ over Σ, which is called a path connection Λ on $\mathcal{G}$, has been developed by Mackenzie [66]. In this section this more general framework of a path connection in a Lie groupoid is used. The main object derived from this path connection is the following: a Lie algebroid connection γ_A in a transitive Lie algebroid $A\mathcal{G}$ of a locally trivial Lie groupoid $\mathcal{G}$ over a connected smooth base manifold Σ. The Lie algebroid connection give rise to a notion of lifting paths in Σ to paths in a Lie groupoid $\mathcal{G}$ over $\mathcal{G}^0$. Moreover associated to a path connection there exists the holonomy groupoid and the holonomy map for a given Lie groupoid. Note that in the next section the fundamental definitions and theorems are collected and generalised to groupoids. For a detailed study refer to Mackenzie [66].

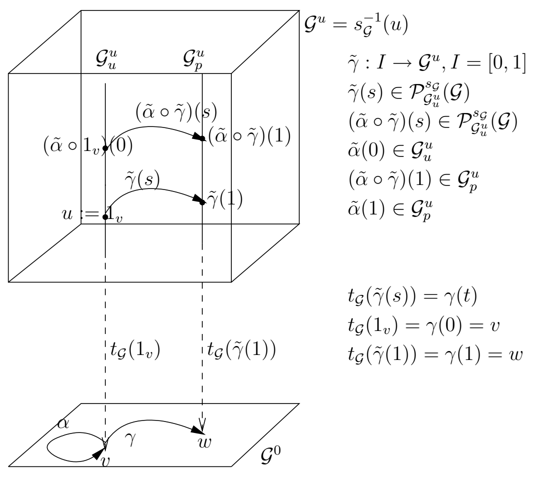
Path connection and holonomy maps in groupoids

Let $\mathcal{G}$ be a locally trivial Lie groupoid with connected fibres $\mathcal{G}^v$. Then recall the set $\mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ of continuous and piecewise-smooth paths $\tilde{\gamma} : I \rightarrow \mathcal{G}$ (where $I = [0,1]$) for which $s_\mathcal{G} \circ \tilde{\gamma} : [0,t] \rightarrow \mathcal{G}^0$ is constant for all elements of $\tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ and $t \in I$.

The paths in $\mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$, which commence at an identity of $\mathcal{G}$, are labeled by $\mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^u_u\}}(\mathcal{G})$. Every element of $\tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ is of the form $R_{\tilde{\gamma}(0)} \cdot \tilde{\vartheta} = \tilde{\gamma}$, where $\tilde{\vartheta} \in \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^u_u\}}(\mathcal{G})$ for $u = \tilde{\gamma}(0)$ and $R_{\tilde{\gamma}(0)} = \tilde{\alpha}$ for a loop $\tilde{\alpha}$ in $\mathcal{G}^u_u$. This corresponds to a right-translation R_w corresponding to $\mathcal{G}$ for $w \in \mathcal{G}^0$. In comparison to definition 3.1.40 one can also rewrite R_u by $R_{\tilde{\alpha}}$.

Consider a lift $\tilde{\gamma} : [0,1] \rightarrow \mathcal{G}^u$ such that $\tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^u_u\}}(\mathcal{G})$. Let $\tilde{\alpha} \in \mathcal{G}^u_u$ be a lifted loop, then it is true that $\tilde{\alpha} \circ \tilde{\gamma} \in \mathcal{P}^{s_\mathcal{G}}_\Sigma(\mathcal{G})$, where $\tilde{\alpha}(0) \in \mathcal{G}^u_u$ such that $(\tilde{\alpha} \circ \tilde{\gamma})(0) \in \mathcal{G}^u_u$, $\tilde{\gamma}_t(1) \in \mathcal{G}^u_p$, $(\tilde{\alpha} \circ \tilde{\gamma})(1) \in \mathcal{G}^u_p$. Moreover the surjections satisfy $t_{\mathcal{P}\mathcal{G}}(\tilde{\gamma}(s)) = \gamma(t)$, $t_\mathcal{G}(\mathbb{1}_v) = \gamma(0) = u$ and $t_\mathcal{G}(\tilde{\gamma}(1)) = \gamma(1) = w$.

Now consider the following picture, which illustrate the definition of a path connection



principal bundle $\mathcal{G}^u$ over $\mathcal{G}^0$ with fibre $\mathcal{G}^u_p$ and structure group $\mathcal{G}^u_u$

Definition 3.2.1. *Let $\mathcal{G}$ be a locally trivial Lie groupoid over a connected base Σ.*

*A **path connection** in a Lie groupoid $\mathcal{G}$ on a base space Σ is a map*

$$\Lambda : P\Sigma \rightarrow \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G}), \quad \gamma \mapsto \Lambda(\gamma) =: \tilde{\gamma}$$

where $I \ni s \mapsto \Lambda(\gamma, s) =: \tilde{\gamma}(s) \in \mathcal{G}^v$ and $I \ni t \mapsto \Lambda(\gamma)(t) =: \tilde{\gamma}_t \in \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^u_u\}}(\mathcal{G})$ such that the following conditions are satisfied:

(i) ***Start and target condition***: *The Λ-lift $\tilde{\gamma}$ of a path γ into the bundle $\{\mathcal{G}^v\}$ start at $\mathcal{G}^v_v$ and it is constant, i.o.w.*

$$s_{\mathcal{P}\mathcal{G}^v}(\Lambda(\gamma)) = \mathbb{1}_{\gamma(0)} \in P\Sigma^v_v \tag{3.19}$$

where $s_{\mathcal{G}}(\Lambda(\gamma)) = \gamma(0) =: v$. The projection $t_{\mathcal{P}\mathcal{G}^v} : \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G}^v) \rightarrow P\Sigma$ of a path $\Lambda(\gamma)$ starting at $\mathcal{G}^v_v$ in the Lie groupoid $\mathcal{G}^v$ onto the base space Σ is γ,

$$t_{\mathcal{P}\mathcal{G}^v}(\Lambda(\gamma)) = \tilde{\gamma}(1) =: \gamma(t) \in P\Sigma^v_w \tag{3.20}$$

and $t_{\mathcal{G}}(\Lambda(\gamma)) = \gamma(1) =: w$. Hence, $s \mapsto \Lambda(\gamma, s)$ is a path in $\mathcal{G}^{\gamma(0)}_{\gamma(t)}$.

(ii) ***Reparametrisation***: *for every diffeomorphism $\phi : I \rightarrow [a, b] \subset I$ there is a right-translation $R_v : \mathcal{P}^{s_\mathcal{G}}(\mathcal{G}) \rightarrow \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G})$ for every $v \in \Sigma$ such that for every path $\gamma : [0, 1] \rightarrow \Sigma$*

$$\Lambda(\gamma \circ \phi) = R_{\Lambda^{-1}(\tilde{\gamma})(\phi(0))} \cdot (\Lambda(\gamma) \circ \phi) \tag{3.21}$$

where $\Lambda^{-1} : \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G}) \rightarrow P\Sigma$ and $I \ni t \mapsto \Lambda^{-1}(\tilde{\gamma})(t) \in \Sigma$.

(iii) ***Smoothness***: *if $\gamma \in \mathcal{P}\Sigma$ is smooth at $t = t_0 \in I$, then $\tilde{\gamma}_t$ is also smooth at $t = t_0$*

(iv) ***Tangency***: *$\gamma, \gamma' \in \mathcal{P}\Sigma$ if the tangent vectors coincide at some $t_0 \in I$,*

$$\frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t}\Big|_{t=t_0} = \frac{\mathrm{d}\,\gamma'(t)}{\mathrm{d}\,t}\Big|_{t=t_0} \text{ then } \frac{\mathrm{d}\,\tilde{\gamma}_t(s)}{\mathrm{d}\,t}\Big|_{t=t_0} = \frac{\mathrm{d}\,\tilde{\gamma}'_t(s)}{\mathrm{d}\,t}\Big|_{t=t_0}$$

for $\tilde{\gamma}_t(s) = \Lambda(\gamma, s)(t)$ and $\tilde{\gamma}'_t(s) = \Lambda(\gamma', s)(t)$ and every $s \in I$

(v) ***Additivity***: *$\gamma, \gamma', \gamma'' \in \mathcal{P}\Sigma$ if the tangent vectors satisfy at some t_0,*

$$\frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t}\Big|_{t=t_0} + \frac{\mathrm{d}\,\gamma'(t)}{\mathrm{d}\,t}\Big|_{t=t_0} = \frac{\mathrm{d}\,\gamma''(t)}{\mathrm{d}\,t}\Big|_{t=t_0} \text{ then } \frac{\mathrm{d}\,\tilde{\gamma}_t(s)}{\mathrm{d}\,t}\Big|_{t=t_0} + \frac{\mathrm{d}\,\tilde{\gamma}'_t(s)}{\mathrm{d}\,t}\Big|_{t=t_0} = \frac{\mathrm{d}\,\tilde{\gamma}''_t(s)}{\mathrm{d}\,t}\Big|_{t=t_0}$$

for $\tilde{\gamma}_t(s) := \Lambda(\gamma, s)(t)$, $\tilde{\gamma}'_t(s) := \Lambda(\gamma', s)(t)$ and $\tilde{\gamma}''_t(s) := \Lambda(\gamma'', s)(t)$ and every $s \in I$.

In this dissertation it is assumed that the path connection in a Lie groupoid generalises to a **path connection in a groupoid $\mathcal{G}$ over $\mathcal{G}^0$** such that the conditions (i) till (v) are satisfied.

Proposition 3.2.2. *[66, Prop. 6.3.3] Let Λ be a path connection in a Lie groupoid $\mathcal{G}$ over $\mathcal{G}^0$.*

Then

(i) *Unit preserving: $\Lambda(\tilde{\mathbb{1}}_v) = \mathbb{1}_v$ where $\tilde{\mathbb{1}}_v$ is the constant path at v in $\mathcal{P}\Sigma$ and $\mathbb{1}_v$ the constant path in $\mathcal{P}\mathcal{G}$ where $\pi(\mathbb{1}_v) = v$.*

(ii) *Inverse preserving: $\Lambda(\gamma^{-1}) = (R_{\Lambda^{-1}(\tilde{\gamma})(1)} \cdot \Lambda)(\gamma)^{\leftarrow}$ where $\gamma(t)^{-1} = \gamma(1 - t)$, $\Lambda(\gamma)^{\leftarrow}(s) = \Lambda(\gamma, s - 1)$ are the reversal paths and $R_v : \mathcal{P}^{s_\mathcal{G}}(\mathcal{G}) \rightarrow \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G})$*

(iii) *Concatenation of paths: $\Lambda(\gamma \circ \gamma') = (R_{\tilde{\gamma}'(1)} \cdot \Lambda(\gamma)) \circ \Lambda(\gamma')$ for every $s \in I$ and where $R_{\gamma'} : \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}(\mathcal{G}) \longrightarrow \mathcal{P}^{s_\mathcal{G}}(\mathcal{G})$ such that $s_{\mathcal{P}\mathcal{G}^v}((R_{\tilde{\gamma}'(1)} \cdot \tilde{\gamma}) \circ \tilde{\gamma}') = \mathbb{1}_{(\gamma \circ \gamma')(0)}$ and $t_{\mathcal{P}\mathcal{G}^v}((R_{\tilde{\gamma}'(1)} \cdot \tilde{\gamma}) \circ \tilde{\gamma}') = \gamma \circ \gamma'$.*

The properties carry over for a path connection in a groupoid $\mathcal{G}$ over $\mathcal{G}^0$.

Definition 3.2.3. *For $\gamma \in P\Sigma$ the element $\mathfrak{h}_\Lambda(\gamma) := \Lambda(\gamma, 1) \in \mathcal{G}^v_w$ where $v = \gamma(0)$ and $w = \gamma(1)$ is the **holonomy map of the path γ in a groupoid $\mathcal{G}$ over $\mathcal{G}^0$**.*

Proposition 3.2.4. *Let Λ be a path connection in a groupoid $\mathcal{G}$ over Σ.*

Then

(i) *Unit: $\mathfrak{h}_\Lambda(\mathbb{1}_v) = e_{\mathcal{G}}(v)$ for $\mathbb{1}_v$ the constant function in $P\Sigma$ and $e_{\mathcal{G}}(v)$ the constant function in $\mathcal{G}$ whenever $v \in \Sigma$,*

(ii) *Inverse:* $\mathfrak{h}_\Lambda(\gamma^{-1}) = \mathfrak{h}_\Lambda^{-1}(\gamma)$ *for* $\gamma \in P\Sigma$ *and*

(iii) *Concatenation of paths:* $\mathfrak{h}_\Lambda(\gamma \circ \gamma') = \mathfrak{h}_\Lambda(\gamma)\mathfrak{h}_\Lambda(\gamma')$ *if* $t(\gamma) = s(\gamma')$ *and* $\gamma, \gamma' \in P\Sigma$.

Definition 3.2.5. *The set* $\mathrm{Hol}_\Lambda(\Sigma) = \{\mathfrak{h}_\Lambda(\gamma) : \gamma \in P\Sigma\}$ *is the* ***holonomy groupoid of*** Λ ***associated to a groupoid*** $\mathcal{G}$ ***over*** $\mathcal{G}^0$. *The vertex group* $\mathrm{Hol}_\Lambda(\Sigma, v)$ *at* $v \in \Sigma$ *is the* ***holonomy group of*** Λ ***at*** v. *The vertex bundle* $\{\mathrm{Hol}_\Lambda(\Sigma, v)\}$ *is the* ***holonomy group bundle of*** Λ

For a Lie groupoid $\mathcal{G}$ over $\mathcal{G}^0$ the holonomy groupoid is a transitive subgroupoid of $\mathcal{G}$.

Theorem 3.2.6. *[66, Theorem 6.3.19] Let* Λ *be a path connection in a locally trivial Lie groupoid* $\mathcal{G}$.

Then $\mathrm{Hol}_\Lambda(\Sigma)$ *is the holonomy Lie subgroupoid of* $\mathcal{G}$*, the vertex group* $\mathrm{Hol}_\Lambda(\Sigma, v)$ *is the holonomy Lie group and the vertex bundle* $\{\mathrm{Hol}_\Lambda(\Sigma, v)\}$ *is the holonomy Lie group bundle of* Λ.

Denote $H_\Lambda(\Sigma) := {}^{\mathrm{Hol}_\Lambda(\Sigma)}/_{\ker \mathfrak{h}_\Lambda}$ and $H_\Lambda^{\breve{\Lambda}}(\Sigma) := {}^{\mathrm{Hol}_\Lambda(\Sigma)}/_{\bigcap_{\Lambda\in\breve{\Lambda}} \ker \mathfrak{h}_\Lambda}$.

Definition 3.2.7. *Two paths* $\gamma, \gamma' \in \mathcal{P}\Sigma_w^v$ *are said to have the* ***same-holonomy w.r.t. a fixed path connection*** Λ *iff*

$$h_\Lambda(\gamma \circ {\gamma'}^{-1}) = \mathbb{1}_v \textit{ where } s_{\mathcal{P}\Sigma}(\gamma \circ {\gamma'}^{-1}) = v, \pi(u) = v \tag{3.22}$$

Two paths $\gamma, \gamma' \in P\Sigma_w^v$ *are said to have the* ***same-holonomy w.r.t. all path connections*** *iff* (3.22) *is true for all path connections* $\Lambda \in \breve{\Lambda}$.

Denote this relation by $\sim_{s.hol.\ \breve{\Lambda}}$.

This is obviously an equivalence relation. The consider the holonomy groupoid $\mathrm{Hoop}_{\breve{\Lambda}}(\Sigma) = \{\mathfrak{h}_\Lambda(\gamma) : \gamma \in \mathcal{P}\Sigma/\sim_{\text{s.hol. }\breve{\Lambda}}\}$. If the groupoid $\mathcal{G}$ over $\mathcal{G}^0$ is equal to a connected Lie group G over $\{e_G\}$, then the set $H_\Lambda^{\breve{\Lambda}}(\Sigma)$ and $\mathrm{Hoop}_{\breve{\Lambda}}(\Sigma)$ coincide.

Definition 3.2.8. *Let* $\mathcal{P}\Sigma$ *be the path groupoid modulo same holonomy w.r.t. all path connections in* $\breve{\Lambda}$. *Moreover let* $\mathcal{G}$ *be a groupoid over* $\mathcal{G}^0$.

Then the groupoid morphism $\mathfrak{h}_\Lambda : \mathcal{P}\Sigma \longrightarrow \mathcal{G}$ *such that* $\mathfrak{h}_\Lambda(\gamma) = \Lambda(\gamma, 1)$ *for all* $\gamma \in \mathcal{P}\Sigma$ *associated to a path connection* Λ *is called a* ***holonomy map for a groupoid*** $\mathcal{G}$ ***over*** $\mathcal{G}^0$.

Consider the example of the holonomy map constructed from the fundamental group $\pi_1(\Sigma, v)$, which is a group homomorphism

$$\mathfrak{h}_A : \pi_1(\Sigma, v) \to G$$

Obviously, two paths $\gamma, \gamma' \in \Pi_1(\Sigma)_w^v$ have the same-holonomy w.r.t. a smooth connection A in the set $\breve{\mathcal{A}}_s$ of smooth connections iff

$$\mathfrak{h}_A(\gamma \circ {\gamma'}^{-1}) = e_G \text{ and where } \mathcal{G}_v^v \simeq G \text{ and } e_G \text{ is the unit of the group } G.$$

In other words, the loop $\gamma \circ {\gamma'}^{-1}$ is contractible to the constant loop at v.

Observe that for $\mathcal{G}$ one can choose for example $\Pi_1(\Sigma)$ and $\Sigma \times \Sigma$. In fact, the fundamental groupoid $\Pi_1(\Sigma)$ is the biggest Lie groupoid and the pair groupoid the smallest among all Lie groupoid such that the corresponding Lie algebroid $\mathcal{AG}$ is equivalent to $T\Sigma$. However, in the case of a gauge theory it is more interesting to consider the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. Before the gauge groupoid is analysed in detail, some further properties are collected.

Duality and the generalised Ambrose-Singer theorem

In the context of the gauge groupoid, the duality of infinitesimal objects and path connections and holonomies is based on the following theorem.

Theorem 3.2.9. *[66, theorem 6.3.5])*
There is a bijective correspondence between path connections Λ *in the gauge groupoid* $\frac{P\times P}{G} \rightrightarrows \Sigma$ *and infinitesimal Lie algebroid connections* $\gamma_A : T\Sigma \to \frac{TP}{G}$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \tilde{\gamma}_t = T(R_{\Lambda(\gamma)(t_0)})\left(\gamma_A\Big(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\gamma(t)\Big)\right)$$

For $X \in T_v\Sigma$ and a path $\gamma \in P\Sigma$ with $\gamma(t_0) = v$ and $\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\gamma(t) = X$ for some $t_0 \in I$ define

$$\gamma_A(X) := T(R_{\Lambda^{-1}(\tilde{\gamma})(t_0)})\Big(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\tilde{\gamma}_t\Big) \tag{3.23}$$

Note that, the theorem has benn originally stated by Mackenzie [66] in the general context of transitive Lie groupoids and Lie algebroids. In this dissertation this theorem is formulated in the context of gauge theories for a comparison with Barrett [16]. The generalisation is deduced by replacing the Lie algeboid $\frac{TP}{G}$ by the transitive Lie algebroid $A\mathcal{G}$ associated to a transitive Lie groupoid $\mathcal{G}$.

There exists a **generalised exponential map** $\mathrm{Exp} : \Gamma A\mathcal{G} \longrightarrow \Gamma\mathcal{G}$ such that $\tilde{X} \mapsto \mathrm{Exp}(t\tilde{X}(v))$ for all $t \in \mathbb{R}$ and $v \in \mathcal{G}^0$. A local one-parameter group of (local) diffeomorphisms is given by the diffeomorphic maps $\phi_t : U \times [-\epsilon, \epsilon] \longrightarrow \Sigma$. Then a **local one-parameter group of (local) diffeomorphisms** $\tilde{\varphi}_t$ **on the Lie groupoid** $\frac{P\times P}{G} \rightrightarrows \Sigma$ with respect to ϕ_t is given by the gauge and diffeomorphism transformation $\tilde{\varphi}_t(\phi_t, \mathrm{id}_G)$ in $\frac{P\times P}{G} \rightrightarrows \Sigma$. Note that id_G is the identity map on G. Then the map $\mathbb{R} \ni t \mapsto t_P(\exp(t\tilde{X}(v)))$ defines a local one-parameter group, too. The map $(t, v) \mapsto \mathrm{Exp}(t\tilde{X}(v))$ is a **family of local bisections**. Note that, it is true that $\tilde{X}(v) = (\gamma_A(X))(v)$ for $\gamma_A : T\Sigma \longrightarrow A\mathcal{G}$, where $A\mathcal{G} := \frac{TP}{G}$, holds.

Corollary 3.2.10. *Let* γ_A *be an infinitesimal connection in* $\mathcal{G}$ *and let* Λ *be the corresponding path connection. Let* $\{\varphi_t(v)\}$ *be a local flow for a vector field* X *near* v*. Set* $\varphi(t) := \varphi_t(v)$*.*

Then

$$\mathrm{Exp}(t(\gamma_A(X))(v)) = \Lambda(\varphi, v)(t)$$

where $\Lambda(\varphi, v) : \mathbb{R} \longrightarrow \mathcal{G}^v$ *is the lift of the path* $t \mapsto \varphi_t(v)$*.*

Furthermore the next theorem give a correspondence between the an object constructed from curvature forms and infinitesimal connections and the holonomy groupoids associated to path connections in the context of a gauge theory.

Theorem 3.2.11. ***(Generalised Ambrose-Singer theorem** [66, theorem 6.4.20])*
Let $P(\Sigma, G)$ *be a principal bundle and* A *an infinitesimal connection in the gauge groupoid* $\frac{P\times P}{G} \rightrightarrows \Sigma$ *associated to a path connection* Λ*.*

Then there exists a least Lie subalgebroid of the Lie algebroid constructed from $P(\Sigma, G)$ *which contains the values of the right splitting* γ_A *associated to* A *and the values of its curvature, and this Lie subalgebroid is the Lie algebroid corresponding to the holonomy groupoid* $\mathrm{Hol}_\Lambda(\Sigma)$ *of* Λ*.*

Explicitly, in the example of the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$ this theorem has the following form. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\{\Omega(X, Y) \in \Omega^2(P, \mathfrak{g}) : X, Y \in TP\}$. Then $\mathfrak{k}$ is the least Lie subalgebroid of the Lie algebroid constructed from $P(\Sigma, G)$. The Lie algebroid $\{\Omega(X, Y) \in \Omega^2(P, \mathfrak{g}) : X, Y \in TP\}$ is related to a Lie algebroid subbundle of $\frac{P\times\mathfrak{g}}{G}$.

The theorem 3.2.11 has been originally stated by Mackenzie [66] in the general context of transitive Lie groupoids and Lie algebroids. The generalisation can be obtained by replacing the Lie algeboid $\frac{TP}{G}$ over Σ by the transitive Lie algebroid $A\mathcal{G}$ over Σ associated to a transitive Lie groupoid $\mathcal{G}$.

3.3 Holonomy maps and transformations in groupoids and graph systems

After the short summary about the important objects introduced by Mackenzie the relation to Barrett's theory of holonomy maps is presented in this section. Barrett has explained his ideas in the special case of a Yang Mills theory and for gravitational theories. Consequently the more general objects defined by Mackenzie are used for a reformulation of the examples given by Barrett [16] in the first two sections. It is shown that, the theory of Mackenzie really generalises the work of Barrett for Yang Mills theories (section 3.3.1) and gravitational theories (section 3.3.2).

The simpliest holonomy maps are group homomorphisms from the (thin/intimate) fundamental group to a chosen Lie group. The concept is enlarged by holonomy maps, which are groupoid morphisms. Therefore the concepts of holonomy maps, which have been presented by Barrett [16], is generalised to groupoid morphisms depending on different physical theories (refer to 3.3.1, 3.3.2, 3.3.3 and 3.3.5). Especially holonomy maps for a general gauge theory are presented in section 3.3.3. This formulation extends the work of Barrett for Yang Mills theories and is an application of Mackenzie's theory for Lie groupoids.

In section 3.3.4 the ideas are generalised in the context of (semi-)analytic paths. The concept of holonomy maps (in the sense of Barrett) for finite path groupoids is reformulated by the author in section 3.3.4.1 and is further generalised to the case of finite graph systems in 3.3.4.3. Moroever the generalisation of Barrett's objects to the framework of Mackenzie forces to generalise the holonomy mappings for a finite path groupoid once more. This is presented in section 3.3.4.2.

3.3.1 Holonomy maps for Yang Mills theories

In this section the ideas of Barrett in [16] for Yang Mills theories are rewritten in the language of Mackenzie's path connection (refer to [66]). The idea is to understand Barrett's ideas in a more general setting of path connections.

There are two main theorems in the work of Barrett. One of them is the *representation theorem for holonomies*, which states the following. For a given a principal G-bundle $P(\Sigma, G, \pi)$ equipped with a smooth connection A, a continuous homomorphism $\mathfrak{h} : \pi_1^1(\Sigma, v) \rightarrow G$ is constructed uniquely up to equivalence such that $\mathfrak{h}$ defines the holonomy map $\mathfrak{h}_\Lambda$ of this bundle associated to a path connection Λ.

Otherwise consider a group homomorphism $\mathfrak{h}$ satisfying the Barrett axioms (BAxiom1), (BAxiom2), (BAxiom3) and (BAxiom4), then by following the ideas of Barrett [16, p.1185 ff] a principal bundle $E(\Sigma, G, \pi)$ is constructed such that $\mathfrak{h}$ presents the holonomy map of that bundle. This is the content of the second theorem of Barrett, which is called the *reconstruction theorem for holonomies.* Note that, the next considerations are also valid, if the intimate homotopy is used. This construction of a principal bundle and a holonomy map of this bundle from a holonomy map satisfying Barrett axioms is studied in the following paragraphs.

Let $\mathcal{P}\Sigma$ be the path space over Σ. An element of $\mathcal{P}\Sigma^v \times G$ is given by a tuple $(\gamma, u(v))$ where $s(\gamma) = v$ and there is a map $u : \Sigma \rightarrow G$ such that $v \mapsto u(v) := u_v$ for each vertex $v \in \Sigma$.

There is a right action $R_\mathfrak{h} : \pi_1^1(\Sigma, v) \times G \rightarrow \mathcal{P}\Sigma^v \times G$, which is given by $R_\mathfrak{h}(\gamma, u_v) = (\gamma, u_v \mathfrak{h}(\gamma))$ for the map $\mathfrak{h} : \pi_1^1(\Sigma, v) \rightarrow G$ satisfying the Barrett axioms. Equipp $\mathcal{P}\Sigma^v$ with the thin path-homotopy stated in definition 3.1.10.

Definition 3.3.1. *Let $\Pi_1^1(\Sigma)^v$ be the thin-fundamental groupoid, which contain only paths that start at v and $\mathfrak{h}$ a holonomy map satisfying (BAxiom1), (BAxiom2), (BAxiom3) and (BAxiom4).*

*Two elements (γ, u_v) and (γ', u'_v) of $\Pi_1^1(\Sigma)^v \times G$ are said to be R_Σ-**equivalent**, iff for $u_v \in \pi^{-1}(v)$, $u'_v \in \pi^{-1}(v)$ and $\gamma, \gamma' \in \Pi_1^1(\Sigma)^v$ such that $s(\gamma) = s(\gamma') = v; t(\gamma) = t(\gamma')$ the elements satisfy*

$$u'_v = u_v \mathfrak{h}(\gamma^{-1} \circ \gamma') \quad \text{for } \gamma^{-1} \circ \gamma' \in \pi_1^1(\Sigma, v) \tag{3.24}$$

Write $(\gamma, u_v) \overset{R_\Sigma}{\sim} (\gamma', u'_v)$.

Lemma 3.3.2. *The $\overset{R_\Sigma}{\sim}$-relation is an equivalence relation.*

Denote the equivalence class of the relation R_Σ by $\langle ., . \rangle$.

For further constructions note that, in general the elements

$$\langle\gamma, u_v\rangle \in \hat{\mathcal{K}}^v_w = \frac{\mathcal{P}\Sigma^v_w \times G}{R_\Sigma}$$

where $s(\gamma) = v$, $t(\gamma) = w$ satisfy

$$\langle\gamma, u_v\rangle = \langle\gamma', u_v\mathfrak{h}(\gamma^{-1} \circ \gamma')\rangle \text{ for } \gamma, \gamma' \in \mathcal{P}\Sigma^v_w$$

whenever $\mathfrak{h} : \mathcal{P}\Sigma^v_v \to G$ is a general groupoid morphism.

Suppose that, γ and γ' are thin equivalent paths. Consequently, a holonomy map $\mathfrak{h}$ satisfies $\mathfrak{h}(\gamma^{-1} \circ \gamma') = e_G$, since $\mathfrak{h}$ is a groupoid morphism. Hence, if additionally (γ, u_v) and (γ', u'_v) are R_Σ-equivalent, then it follows that, $u_v = u'_v$ holds.

Let $e_G : \Sigma \to G$ be a map, which maps every point in Σ to the unit e_G of the group G. Remark that $(\alpha, e_G(v)) \overset{R_\Sigma}{\sim} (\mathbb{1}_v, \mathfrak{h}(\alpha^{-1}))$ holds for $\alpha \in \pi^1_1(\Sigma, v)$.

In general there is another action R_α of a loop α in $\pi^1_1(\Sigma, v)$ on $\mathcal{K}^v_w$ given by

$$R_\alpha\langle\gamma, u_v\rangle := \langle\gamma \circ \alpha, u_v\rangle = \langle\gamma, u_v\mathfrak{h}(\alpha^{-1})\rangle$$

Moreover there is a natural free left action L_g of an element g of G illustrated by

$$\begin{aligned} L_g\langle\gamma, u_v\rangle &:= \langle\gamma, gu_v\rangle = \langle\gamma', gu_v\mathfrak{h}(\gamma^{-1} \circ \gamma')\rangle \\ &= \langle\gamma', \breve{u}_v\mathfrak{h}(\gamma^{-1} \circ \gamma')\rangle \end{aligned}$$

if $\gamma, \gamma' \in \mathcal{P}\Sigma^v_w$ and $\breve{u}_v := gu_v$ is satisfied.

A gauge transformation $\mathcal{G}_\phi$ of a group homomorphism $\phi : G \to G$ acting on $\mathcal{K}^v_w$ is given by

$$\mathcal{G}_\phi\langle\gamma, u_v\rangle := \langle\gamma, \phi(u_v)\rangle$$

If $\phi(u_v) = u_v g$ for $g \in \mathcal{Z}(G)$ and $u_v \in G$, then notice that,

$$\begin{aligned} R_\alpha\mathcal{G}_\phi\langle\gamma, u_v\rangle &= R_\alpha\langle\gamma, u_v g\rangle = \langle\gamma, u_v g\mathfrak{h}(\alpha^{-1})\rangle \\ &= \langle\gamma, u_v\mathfrak{h}(\alpha^{-1})g\rangle = \mathcal{G}_\phi R_\alpha\langle\gamma, u_v\rangle \end{aligned} \tag{3.25}$$

yields.

Recall the surjection $t : \mathcal{P}\Sigma^v \to \Sigma$ such that $t(\gamma) = \gamma(1)$.

Proposition 3.3.3. *Let Σ be a smooth manifold and $\mathfrak{h}_A$ be the holonomy map satisfying the Barrett conditions (BAxiom1), (BAxiom2) and (BAxiom3).*

Then the bundle $E(\Sigma, G)$ characterised by total space $E = \frac{\Pi^1_1(\Sigma)^v \times G}{R_\Sigma}$, the fibre $\mathcal{K}^v_w = \frac{\Pi^1_1(\Sigma)^v_w \times G}{R_\Sigma}$, the base Σ and the surjection $\tilde{\pi}(\langle\gamma, u_v\rangle) = t(\gamma)$, is a principal fibre bundle.

Local trivialisations of $E(\Sigma, G, \tilde{\pi})$ can be constructed from the following maps. Let $\gamma : U \to \mathcal{P}\Sigma$ be a smooth family of paths such that $\gamma_t(v) = \breve{\gamma}(v, t)$, where $\breve{\gamma} : U \times I \to \Sigma$, is continuous and piecewise smooth, whenever $\gamma_t(v)|_{t=0} = v$ and $\gamma_t(v)|_{t=1} = \gamma(1)$. Then $\langle\gamma_t(v), u_v\rangle \in \mathcal{K}^v_w$ gives a local trivalisation.

Definition 3.3.4. *The lift Λ of each curve γ starting at v in Σ to a curve in E is a map*

$$\Lambda : \mathcal{P}\Sigma^v \times G \to \mathcal{P}E, (\gamma(s), e_G(v)) \mapsto \langle\tilde{\gamma}(s), e_G(v)\rangle =: \tilde{\gamma}_{e_G}(s) \in \mathcal{K}^v_{t(\gamma(s))}$$

where $\mathcal{P}E$ is the path space over E. Then the lifting map is determined by

$$\tilde{\gamma}_{e_G}(s) = \langle\gamma \circ \varrho_s, e_G(v)\rangle \textit{ where } \varrho_s : [0,1] \to [0,s], \quad t \mapsto \varrho_s(t) := ts \tag{3.26}$$

The holonomy is given by

$$\mathfrak{h}_\Lambda(\gamma) = \Lambda|_{s=1}(\gamma, e_G(v)) = \tilde{\gamma}_{e_G}(1)$$

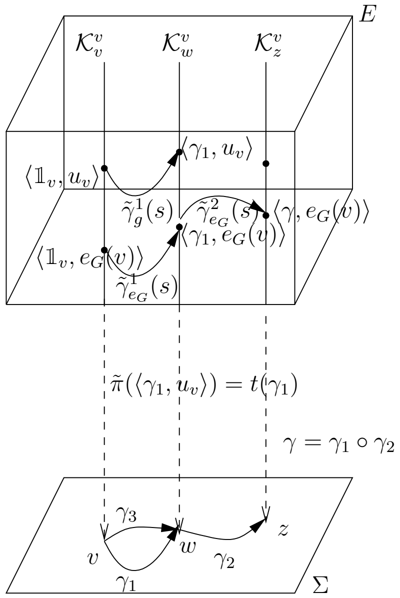


Figure 3.1: trivial bundle $E(\Sigma, \pi)$

Note that the map $\mathfrak{h} : \pi_1^1(\Sigma, v) \to G$ of Barrett determines the holonomy map $\mathfrak{h}_\Lambda$ on the bundle $E(\Sigma, G, \pi)$ via R_Σ-equivalence for $\alpha \in \pi_1^1(\Sigma, v)$ and

$$\mathfrak{h}_\Lambda(\alpha^{-1}) := \langle \alpha^{-1}, e_G(v)\rangle = \langle \mathbb{1}_v, e_G(v)\mathfrak{h}(\mathbb{1}_v \circ \alpha)\rangle = \langle \mathbb{1}_v, \mathfrak{h}(\alpha)\rangle \tag{3.27}$$

Set $\tilde{\gamma}_i := \tilde{\gamma}_{e_G}^i$ for $i = 1, 3$ and $\tilde{\gamma} := \tilde{\gamma}_3 \circ \tilde{\gamma}_1^{-1}$. Moreover, $\gamma = \gamma_3 \circ \gamma_1^{-1}$ is an element of LG(v) (or the thin fundamental group $\pi_1^1(\Sigma, v)$) such that $s(\gamma) = v$. For an element $\mathfrak{h}(\gamma) \in HG$ observe that, the equivalence relation R_Σ imply that

$$\langle \mathbb{1}_v, e_G(v)\mathfrak{h}(\gamma)\rangle = \langle \mathbb{1}_v, \mathfrak{h}(\gamma_3 \circ \gamma_1^{-1})\rangle = \langle (\tilde{\gamma}_1 \circ \tilde{\gamma}_3^{-1})(1), e_G(v)\rangle \in \mathcal{K}_v^v \tag{3.28}$$

holds. Therefore, one can either a change of the base point by an element in HG, or consider a loop at v.

Consequently, for the example presented in the picture 3.1 it is true that,

$$\begin{aligned}\langle (\tilde{\gamma}_1 \circ \tilde{\gamma}_2)(1), e_G(v)\rangle &= \langle (\tilde{\gamma}_3 \circ \tilde{\gamma}_2)(1), e_G(v)\mathfrak{h}(\gamma_3 \circ \gamma_2 \circ \gamma_2^{-1} \circ \gamma_1^{-1})\rangle \\ &= \langle (\tilde{\gamma}_3 \circ \tilde{\gamma}_2)(1), e_G(v)\mathfrak{h}(\gamma_3 \circ \gamma_1^{-1})\rangle\end{aligned} \tag{3.29}$$

holds.

Barrett has argued that, an infinitesimal connection A_c associated to the holonomy map h_Λ is a map $A_c : T_{\tilde{\pi}(c)}\Sigma \to T_cE$ for a point $c \in E$ such that this map A_c is given by

$$A_c\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\gamma(t)\right) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}(\Lambda_s(\gamma, e_G)|_{s=1} = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\tilde{\gamma}_{e_G}(s)|_{s=1} \tag{3.30}$$

whenever $\tilde{\gamma}_{e_G}(1) = c$ and $t_0 = \gamma(1)$. Consequently the map A_c maps a vector field $X(t_0) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\gamma(t)$ to a vector field $\tilde{X}(t_0) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0}\tilde{\gamma}_{e_G}(s)|_{s=1}$.

The image of A is equal to the horizontal distribution space of TE, and for each $c \in E$ it defines a horizontal subspace of T_cE. The vector field $\tilde{X}(t)$ defined by (3.30), projects to a tangent vector field in Σ and each vector $\tilde{X}$ depends only on the tangent vector of the path at the point. The extensive proof of this fact has been given by Barrett [16, p.1187 ff] in a slightly different framework. Denote $\tilde{\pi}_* : T_cE \to T_{\tilde{\pi}(c)}\Sigma$. Then Barrett has been shown that, $\tilde{\pi}_* A_c = \mathrm{id}$ yields and A_c is smooth. With no doubt

$$R_h A_c = A_{R_h(c)} \tag{3.31}$$

Let $c_0 := \langle \mathbb{1}_v, \mathfrak{h}(\alpha)\rangle \in E$. The infinitesimal connection is a map $A_{c_0} : T_{c_0}E \to T_v\Sigma$. Then set $A_{c_0} \circ \mathrm{pr}_2 =: A$, where

$$\mathrm{pr}_2 \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \mathbb{1}_v, \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \mathfrak{h}(\alpha) \right) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \mathfrak{h}(\alpha) \tag{3.32}$$

Summarising, the infinitesimal smooth connection A corresponds one-to-one to a holonomy map $\mathfrak{h}$ satisfying the Barrett axioms, if the intimate homotopy instead of thin homotopy is used (due to some arguments of Caetano and Picken).

In the next section, the results are reformulated for a gravitational and later for even more general theories.

3.3.2 Holonomy maps for gravitational theories

In this section the reconstruction theorem for holonomy maps on a frame bundle on a manifold for on gravitational theories given by Barrett [16] is reformulated. This new concept is based on path connections, which have been presented by Mackenzie [66]. During the considerations the intimate homotopy instead of the thin homotopy is used, since the results of Caetano and Picken [28] indicate that, the correspondence between horizontal lifts associated to connections and holomomy group homomorphisms require a rank-one homotopy. In fact, the consequence is that, there is an additional structure associated to the horizontal lifts, which is needed for the proof that, the Barrett holonomy map $\mathfrak{h}$ is derived from a smooth connection A.

Now a short overview about the certain structure of gravitational theories is presented. The gravitational field is described by a globally hyperbolic, connected Hausdorff manifold X, which is isomorphic to $\Sigma \times \mathbb{R}$ with a metric, and a connection on the bundle $O(X)$ of orthonormal frames. An orthonormal frame is an orthonormal ordered basis of T_vX for $v \in X$. The disjoint union of all orthonormal frames is given by the bundle $O(X)$.

In LQG the Ashtekar connection is constructed on a spatial, connected and orientable 3-dimensional (Riemmannian) manifold Σ. Therefore the following objects are often considered: the principal $SO(3)$-bundle $O^+(\Sigma, h)$ over a orientable 3-manifold Σ and the associated adjoint bundle $\frac{O^+(\Sigma,h)\times\mathfrak{so}(3)}{SO(3)}$ over the base Σ and w.r.t. Ad-action of $SO(3)$ on $\mathfrak{so}(3)$. The associated adjoint bundle is isomorphic to the tangent bundle $T\Sigma \xrightarrow{q} \Sigma$. Note that, there exists a spin-structure $(S(\Sigma), \tilde{\pi}, \Sigma, SU(2))$, which is indeed a principal $SU(2)$-bundle over Σ, and which is related to the principal $SO(3)$-bundle $O^+(\Sigma, h)$ over Σ.

The starting point of the construction of Barrett [16] is the tangent bundle. Assume that $T\Sigma$ is a trivial tangent bundle[1], and consequently $T\Sigma$ is isomorphic to $\Sigma \times \mathbb{R}^n$, where n is the dimension of the manifold Σ. Then each tangent space $T_v\Sigma$ is identified with $\mathbb{R}^n$ by using an orthonormal frame F_v at $v \in \Sigma$. Consequently, for each path $\gamma(t)$ starting at v in Σ there is a development w.r.t. to a parameter s of a horizontally lifted curve $\tilde{\gamma}_t(s)$ such that $\tilde{\gamma}_t \in \mathcal{P}\mathbb{R}^n$ and $\tilde{\gamma}_t(0) = 0$ is a curve in $\mathbb{R}^n$. Then $\tilde{\gamma}_t(1) = \tilde{\gamma}(t)$ is a curve in $\mathbb{R}^n$ and a frame F_{τ_v} is defined by

$$F_{\tau_{v,\gamma}} \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \tilde{\gamma}_t(1) \right) := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \gamma(t), \qquad F_{\tau_{v,\gamma}} : \mathbb{R}^n \to T_{\gamma(t_0)}\Sigma$$

whenever $\tau_{v,\gamma(t)} : T_v\Sigma \to T_{\gamma(t)}\Sigma$ denotes the parallel transport along a path γ in Σ, which is a linear isometry.

That means that, the geometry of Σ given by angles and proper distances is transferred to the path space $\mathcal{P}\mathbb{R}^n$. On the other hand, for a horizontal lift $\tilde{\vartheta}_t$ of the curve γ in $O^+(\Sigma)$ the interface to the horizontal lift in $\mathbb{R}^n$ is given by the integral

$$\tilde{\gamma}_s(t) := \int_{\tilde{\vartheta}_0(s)}^{\tilde{\vartheta}_t(s)} e^a$$

along the path $\tilde{\vartheta}_t(s)$ for a fixed $t \in I$ and the canonical $\mathbb{R}^n$-valued 1-form e^a on $O^+(\Sigma)$.

In analogy to the Yang-Mills case Barrett has developed in [16] a construction of an object $\frac{\mathcal{P}M^v\times G}{R_M}$, where he have chosen $\dim X = 4$ and M to be the Minkowski space. For Minkowski spacetimes there exists suitable group actions on $\mathbb{R}^n$ given by isometries. The Minkowski space itself is a homogenous space for the Poincaré group P, which is the semi-direct product of the translation group $\mathbb{R}^{1,3}$ and the Lorentz group $G = O(1,3)$, and which act

[1] This implies that Σ is assumed to be parallelisable.

as translations and rotations on M. Infact, the Lorentz group is a subgroup of P and the stabilizer of a point in M. Note that $\mathcal{P}M^v$ is the set of all paths in M starting at v.

In general it has been assumed that, the spatial manifold Σ has dimension 3 and that, the tangent space is identified with $\Sigma \times \mathbb{R}^3$. Then a frame, which is an element of $O^+(\Sigma)$, is identified with a subset of $\mathcal{P}\mathbb{R}^3 \times H$, where H is equal to the rotations $SO(3)$ and $\mathcal{P}\mathbb{R}^3$ is the path space above $\mathbb{R}^3$ without assuming reparametrisation invariance. Now the bundle $\frac{(\mathcal{P}\mathbb{R}^3)^v \times SO(3)}{R_{\mathbb{R}^3}}$ is constructed by using terms of Mackenzie as follows.

There exists an action R of translations T in $\mathbb{R}^3$ and rotations $SO(3)$, which is presented by the map $R_{(x,g)}$: $\mathcal{P}\mathbb{R}^3 \rightarrow \mathcal{P}\mathbb{R}^3$ for $(x,g) \in \mathbb{R}^3 \times SO(3)$ with

$$(R_{(x,g)}(\tilde{\gamma}))(t) := (\tilde{\gamma} \circ T_x)(t) \cdot g \tag{3.33}$$

such that the path $\tilde{\gamma}$ is translated by the map T_x in $\mathbb{R}^3$ by x and rotated by g.

Set $M := \mathbb{R}^3$. Denote the path space, which consists of all paths in M starting at x, by $\mathcal{P}M^x$ and $\mathcal{P}M^x_y$ denotes the set of paths that contain all paths in M starting at x and ending at y. Set G be equal to $\mathbb{R}^3 \times SO(3)$.

Holonomy Axiom 1. ***(Holonomy Group condition)***
[16, p. 1204 f. Axioms G1 and G2]

Let $\hat{\mathcal{K}}^x_y$ be a subset of $\mathcal{P}M^x_y \times G$ defined by the map $\mathfrak{h}_M : \mathcal{P}^x_{My} \rightarrow G$, where $\mathcal{P}^x_{My}$ is a subset of $\mathcal{P}M^x_y$, which contain all paths δ starting at x and ending at y such that the map $\mathfrak{h}_M$ maps δ to $\mathfrak{h}_M(\delta)$. Hence, each element of $\hat{\mathcal{K}}^x_y$ is of the form $(\delta, \mathfrak{h}_M(\delta)) \in \mathcal{P}^x_{My} \times G$.

Moroever, for a composable pair $(\tilde{\gamma}, \tilde{\gamma}') \in \mathcal{P}M^{(2)}$ the map $\mathfrak{h}_M$ satisfies $\mathfrak{h}_M(\tilde{\gamma})\mathfrak{h}_M(\tilde{\gamma}') = \mathfrak{h}_M(\tilde{\gamma} \circ \tilde{\gamma}')$ and for an inverse $\tilde{\gamma}^{-1}$ the map fulfill $\mathfrak{h}_M(\tilde{\gamma})^{-1} = \mathfrak{h}_M(\tilde{\gamma}^{-1})$.

The set $\hat{\mathcal{K}}^x_M := \{\hat{\mathcal{K}}^x_y\}_{y \in M}$ is a subset of $\mathcal{P}M^x \times G$. Set $\hat{\mathcal{K}}_M = \{\hat{\mathcal{K}}^x_M\}_{x \in M}$. Rewrite the elements $(\tilde{\gamma}, \mathfrak{h}_M(\tilde{\gamma})) \in \hat{\mathcal{K}}^x_y$ by $\mathfrak{H}_M(\tilde{\gamma})$.

Now restrict the map $\mathfrak{h}_M$ to the elements of the intimate fundamental groupoid $\Pi^{\text{in}}_1 M$ over M. Assume that, the subset of $\Pi^{\text{in}}_1 M^x_y$, which contains all paths δ starting at x and ending at y such that the map $\mathfrak{h}_M$ maps δ to $\mathfrak{h}_M(\delta)$, is equivalent to $\Pi^{\text{in}}_1 M^x_y$. Therefore the map $\mathfrak{h}_M : \Pi^{\text{in}}_1(M)^v \rightarrow G$ satisfies a similiar condition to (BAxiom1) for the intimate homotopy.

Moreover, for $\tilde{\gamma}$ and $\tilde{\gamma}'$ intimate equivalent paths in $\mathcal{P}M^v$, the maps coincide, i.e. $\mathfrak{h}_M(\tilde{\gamma}) = \mathfrak{h}_M(\tilde{\gamma}')$. The quotient $\Pi^{\text{in}}_1(M)$ of $\mathcal{P}M$ and the intimate path-homotopy is a groupoid. The quotient $\mathcal{K}_M$ of $\hat{\mathcal{K}}_M$ and intimate path-homotopy is a subset of $\Pi^{\text{in}}_1(M) \times G$.

Holonomy Axiom 2. ***(Intimate homotopy equivalence)***
[16, p. 1205 f. Axioms G3]

The intimate path-homotopy equivalent paths on $\mathcal{P}\Sigma$ are assumed to be intimate equivalent on $\mathcal{P}M$ and $\hat{\mathcal{K}}_M$ is required to contain the complete equivalence classes.

Let (K,k) be a groupoid morphism between the intimate fundamental $\Pi^{\text{in}}_1\Sigma$ over Σ and the intimate fundamental groupoid $\Pi^{\text{in}}_1 M$ over M, i.e. the maps satiyfy

$$\begin{aligned} K(\gamma \circ \gamma') &= K(\gamma)K(\gamma') \\ s_{\mathcal{P}M}(K(\gamma)) &= K(s_{\mathcal{P}\Sigma}(\gamma)), t_{\mathcal{P}M}(K(\gamma)) = k(t_{\mathcal{P}\Sigma}(\gamma)) \end{aligned}$$

whenever $\gamma, \gamma' \in \Pi^{\text{in}}_1\Sigma$. Note that, $s_{\mathcal{P}M} : \Pi^{\text{in}}_1 M \rightarrow M$ is the source map for the intimate fundamental groupoid over M. The target map for the intimate fundamental groupoid over M, respectively, the source and target maps for the intimate fundamental groupoid over Σ are denoted by $t_{\mathcal{P}M}$ or, respectively, $s_{\mathcal{P}\Sigma}$ and $t_{\mathcal{P}\Sigma}$.

Denote by $\mathcal{P}^v_{\Sigma w}$ a subset of $\mathcal{P}\Sigma^v_w$, which map all paths δ starting at v and ending at w to $\mathfrak{h}(\delta)$. Set $\mathcal{P}_\Sigma$ to be equal to $\{\mathcal{P}^v_{\Sigma w}\}_{v,w \in \Sigma}$. Then assume that, the groupoid morphism between $\mathcal{P}\Sigma$ over Σ and the intimate fundamental groupoid $\Pi^{\text{in}}_1 M$ over M is given by a pair (K,k) such that $K : \mathcal{P}^v_{\Sigma w} \rightarrow \Pi^{\text{in}}_1 M^x_y$, $k(v) = x$ and $k(w) = y$.

Definition 3.3.5. *Let $(\mathfrak{h}_M, h_M)$ be a pair of maps such that, $\mathfrak{h}_M$ is the map defined in Haxiom 1 and $h_M : M \rightarrow \{e_G\}$. Moreover, let (K, k) be a groupoid morphism between the path groupoid $\mathcal{P}\Sigma$ over Σ and the intimate fundamental groupoid $\Pi_1^{\text{in}} M$ over M such that Haxiom 2 is satisfied.*

Then the pair $(\mathfrak{h}, h)$, which contains a map $\mathfrak{h} : \mathcal{P}\Sigma \rightarrow G$ such that $\mathfrak{h} = \mathfrak{h}_M \circ K$ and $h = h_M \circ k$, is called the ***holonomy map for a gravitational field****.*

Holonomy Axiom 3. ***(Reparametrisation independence)***

There is an equivalence relation R_M on the space $\mathcal{P}M^x_y \times G$:
$(\tilde{\gamma}_1, g_1) \overset{R_M}{\sim} (\tilde{\gamma}_2, g_2)$ *iff*

$$\tilde{\gamma}_1 \overset{\text{intimate path-hom.}}{\sim} R_{(t_{\mathcal{P}M}(\tilde{\gamma}_3), g_3)}\tilde{\gamma}_2 = \tilde{\gamma}' \text{ and } \mathfrak{h}(\tilde{\gamma}') = g_2 = g_1.$$

The elements $\langle \tilde{\gamma}, g \rangle =: [\mathfrak{H}_M(\tilde{\gamma})]$ of the equivalence class are unique up to some reparametrization in M.

Recall the holonomy map $\mathfrak{h}_M : \mathcal{P} \rightarrow G$, where $\mathcal{P} \subset \mathcal{P}M$, and which satisfy the axiom (Haxiom1). Note $\mathcal{P} := \{\mathcal{P}^x_y\}_{x,y \in M}$. For two representatives of $[\mathfrak{H}_M(\tilde{\gamma})]$, which are given by $(\tilde{\gamma}_1, \mathfrak{h}_M(\tilde{\gamma}_1))$ where $\tilde{\gamma}_1 \in \mathcal{P}^x_y$ and $(\tilde{\gamma}_3, \mathfrak{h}_M(\tilde{\gamma}_3))$ where $\tilde{\gamma}_3 \in \mathcal{P}^x_y$ such that

$$\tilde{\gamma}_1 \overset{\text{intimate path-hom.}}{\sim} \tilde{\gamma}_3 \quad \text{and } g_3 = \mathfrak{h}_M(\tilde{\gamma}_3) = \mathfrak{h}_M(\tilde{\gamma}_1) = g_1$$

it is true that, $\langle \tilde{\gamma}_1, g_1 \rangle = \langle \tilde{\gamma}_3, g_1 \rangle$ yields.

Let $\mathcal{K}^x_M := \{\mathcal{K}^x_y\}_{y \in M}$, where $\mathcal{K}^x_y$ is a subset of the quotient $\hat{\mathcal{K}}^x_y$ modulo the equivalence R_M.

The space $\mathcal{K}^x_M$ implement the bundle of orthonormal frames $O^+(\Sigma)$. This is verified in the next steps. For each path $\tilde{\gamma} \in \mathcal{P}M$ the set $\{\langle \tilde{\gamma}, g \rangle : g \in G\}$ is equivalent to a fibre in $O^+(\Sigma)$.

Holonomy Axiom 4. ***(Frame bundle construction)***

The associated frame bundle is given by the total space $E_M = \frac{\Pi_1^{\text{in}} M^x \times G}{R_M}$ over the base Σ, where the element $\langle \tilde{\gamma}, g \rangle$ in $\frac{\Pi_1^{\text{in}} M^x_y \times G}{R_M}$ is projected by the surjection $\tilde{\pi} : E_M \rightarrow \Sigma$ through

$$\tilde{\pi}(\langle \tilde{\gamma}, g \rangle) := (\pi \circ t_{\mathcal{P}M} \circ \text{pr}_1)(\langle \tilde{\gamma}, g \rangle) = \pi(t_{\mathcal{P}M}(\tilde{\gamma})) = w = \pi(y)$$

whenever $\pi : M \rightarrow \Sigma$, $t_{\mathcal{P}M} : \Pi_1^{\text{in}} M^x \rightarrow M$ and $(\pi \circ s_{\mathcal{P}M})(\tilde{\gamma}) = v = \pi(x)$.

Note that, $\mathcal{P}M$ is the path space over M and the map $t_{\mathcal{P}M}(\tilde{\gamma})$ give the target point of the path $\tilde{\gamma}$ in M. Denote by $\mathcal{P}E_M$ the path space over E_M with target map $t_{\mathcal{P}E} : \mathcal{P}E_M \rightarrow E_M$.

Holonomy Axiom 5. ***(Development map)***
Let $(\mathfrak{h}, h)$ be a holonomy map for a gravitational field such that there is a map $\mathfrak{h}_M : \Pi_1^{\text{in}} M \rightarrow G$. Moreover, let $e_G : \Sigma \rightarrow G$ a map such that $e_G(v) = e_G$ for all $v \in \Sigma$. Let (K, k) be a groupoid morphism between the path groupoid $\mathcal{P}\Sigma$ over Σ and the intimate fundamental groupoid $\Pi_1^{\text{in}} M$ over M such that $K : \mathcal{P}^v_{\Sigma w} \rightarrow \Pi_1^{\text{in}} M^x_y$, $k(v) = x$ and $k(w) = y$.

There are two maps

$$\begin{aligned}
&\Lambda : \mathcal{P}_\Sigma \rightarrow \mathcal{P}^{s_E}_\Sigma E_M, \ \Lambda_G : \mathcal{P}_\Sigma \rightarrow \mathcal{P}^{s_G}_{\{e_G\}}(G) \text{ such that} \\
&\mathcal{P}^v_{\Sigma w} \ni \gamma \mapsto \Lambda(\gamma) := \langle \tilde{\gamma}_t, \Lambda_G(\gamma) \rangle
\end{aligned}$$

whenever $\mathfrak{h}(\gamma) := \Lambda_G(\gamma)(1)$, $\Lambda_G(\gamma)(0) = e_G(v)$ and $w = \tilde{\pi}(\langle \tilde{\gamma}_t(1), \mathfrak{h}(\gamma) \rangle)$. Then the holonomy map $\mathfrak{H}_\Lambda$ along a path γ in Σ of the frame bundle $E(\Sigma, \tilde{\pi}, G)$ is given by

$$\mathfrak{H}_\Lambda(\gamma) := \Lambda|_{s=1}(\gamma) = \langle \tilde{\gamma}_t(1), \mathfrak{h}(\gamma) \rangle$$

Precisely, the map $\tilde{\gamma}_t$ is specified by a contraction in path space $\mathcal{P}\Sigma$ such that

$$\tilde{\gamma}_t(s) := (K \circ \gamma)(\varrho_s(t)) \text{ where } \varrho_s : [0,1] \rightarrow [0,s], \quad t \mapsto \varrho_s(t) := ts$$

and

$$(K \circ \gamma \circ \varrho_s)(t)|_{s=0} = \mathbb{1}_{k(v)} \in \Pi_1^{\text{in}} M_{k(v)}^{k(v)}$$

The map $s \mapsto \tilde{\gamma}_t(s)$ is called ***development map of the path γ in*** *M.*

Moroever, there is a projection $T_{\mathcal{P}E} : \mathcal{P}E_M \to \mathcal{P}\Sigma$ such that

$$T_{\mathcal{P}E}(\Lambda(\gamma)) = \gamma$$

The path space $\mathcal{P}E_M$ over E_M contain a path $\Lambda(\gamma)(s) = \langle\tilde{\gamma}_t(s), g_1(s)\rangle$, where $g_1(s) := \Lambda_G(\gamma)(s)$. There is a concatenation of composable paths, which are contained in $\mathcal{P}E_M$. This operation is given by

$$(\Lambda(\gamma) \circ \Lambda(\gamma'))(s) := \langle(\tilde{\gamma}_t \circ (R_{(t_{\mathcal{P}M}(\tilde{\gamma}_t), g_1)}\tilde{\gamma}'_t))(s), \Lambda_G(\gamma \circ \gamma')(s)\rangle \tag{3.34}$$

whenever $(g_1 \cdot g_2)(s) := \Lambda_G(\gamma \circ \gamma')(s) = (\Lambda_G(\gamma) \circ \Lambda_G(\gamma'))(s)$. Hence it is true that

$$g_1 \cdot g_2 := (\Lambda_G(\gamma) \circ \Lambda_G(\gamma'))(1) = \Lambda_G(\gamma)(1) \cdot \Lambda_G(\gamma')(1) \tag{3.35}$$

Holonomy Axiom 6. ***(Smoothness condition)***
[16, p. 1207 f. Axioms G4 and G5]
Let $(\mathfrak{h}, h)$ be a holonomy map for a gravitational field such that there is a map $\mathfrak{h}_M : \Pi_1^{\text{in}} M \to G$. Moreover, let $e_G : \Sigma \to G$ be a map such that $e_G(v) = e_G$ for all $v \in \Sigma$. Let (K, k) be a groupoid morphism between the path groupoid $\mathcal{P}\Sigma$ over Σ and the intimate fundamental groupoid $\Pi_1^{\text{in}} M$ over M such that $K : \mathcal{P}_{\Sigma w}^v \to \Pi_1^{\text{in}} M_y^x$, $k(v) = x$ and $k(w) = y$.

If the path γ in $\mathcal{P}_\Sigma$ is smooth at $t_0 \in I$, then it is required that $\tilde{\gamma}_t(s) := (K \circ \gamma \circ \varrho_s)(t)$ and $\Lambda_G(\gamma)(s)$ is smooth at t_0 for every $s \in I$. Therefore, the map $K : \mathcal{P}\Sigma \to \Pi_1^{\text{in}} M$ is required to be smooth at t_0. Additionally, it is assumed that $k : \Sigma \to M$ is smooth at t_0, too.

Notice that, for a groupoid morphism (K, k) between the fundamental Lie groupoid over Σ and fundamental Lie groupoid over M the maps K and k are smooth maps.

Moreover for a loop α in $\mathcal{P}_{\Sigma v}^v$ the lifted path $t \mapsto \tilde{\alpha}_t(1)$ in $\mathcal{P}_{Mx}^x$, which defines the element $\langle(\tilde{\alpha}_t)(1), \mathfrak{h}(\alpha)\rangle \in \mathcal{K}_x^x$, is not necessarily a loop in M again. This follows from the fact that, the map $k : \Sigma \to M$ is not assumed to be bijective. The requirement that, the maps k and K are bijective, is related to the assumption that the manifold Σ has to be complete, w.o.w. $\text{Im}(K) = \mathcal{P}M$.

The last axiom (HAxiom5) and (HAxiom6) replaces the axioms (BAxiom2) and (BAxiom4).

To derive a frame one basically use the action R of $\{0\} \times H$ on $\mathcal{P}M$, which is presented in equation (3.33). Therefore, an element of $\mathcal{K}_M^v$ is mapped onto $\mathcal{P}M$ in the following way

$$I(\langle\tilde{\gamma}_t(1), \mathfrak{h}_M(\gamma)\rangle) := R_{(0, \mathfrak{h}_M(\gamma))}(\tilde{\gamma}_t(1)) = \tilde{\gamma}_t(1) \cdot \mathfrak{h}_M(\gamma) \tag{3.36}$$

Set $\tilde{\gamma}_t(1) = K(\gamma)(t)$.

Holonomy Axiom 7. ***(Tangency condition)***

For a given tangent vector at a path $\tilde{\gamma}$ in M a frame $F_{\mathfrak{h}_M(\gamma)} : M \to T_{\tilde{\pi}(\langle\tilde{\gamma}, \mathfrak{h}_M(\gamma)\rangle)}\Sigma$ is defined for $\langle\tilde{\gamma}, \mathfrak{h}_M(\gamma)\rangle \in E_M$ and $t_0 = \gamma(1)$ by

$$F_{\mathfrak{h}_M(\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \tilde{\gamma}_t(1) \cdot \mathfrak{h}_M(\gamma)\right) = F_{\mathfrak{h}_M(\gamma)}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} K(\gamma)(t) \cdot \mathfrak{h}_M(\gamma)\right) = \frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t}\Big|_{t=t_0}$$

Furthermore assume that, the path $\tilde{\gamma}$ in M for which the tangent vectors are non-zero, the corresponding path in Σ have non-zero tangent vectors, too.

Holonomy Axiom 8. ***(Tangency condition II)***
If for two paths $\gamma,\gamma'\in\mathcal{P}^v_{\Sigma w}$ such that $\gamma(a)=\gamma'(b)$ for $a,b\in[0,1]$ and

$$\frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t}\Big|_{t=a}=\frac{\mathrm{d}\,\gamma'(t)}{\mathrm{d}\,t}\Big|_{t=b}$$

hold, then it is required that, $\tilde{\gamma}(a)=\tilde{\gamma}'(b)$ and

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=a}\tilde{\gamma}_s(t)\cdot\Lambda_G(\gamma)=\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=b}\tilde{\gamma}'_s(t)\cdot\Lambda_G(\gamma')$$

yields for every $s\in I$. Furthermore it is assumed that, additivity of tangent vectors in $T\Sigma$ carry over to additivity in M.

Proposition 3.3.6. *Let all Holonomy Axioms (HAxiom1) - (HAxiom8) be satisfied.*

Then Λ is the path connection of the bundle E_M associated to a frame at v, which is given in (7).

The construction is generalised in the context of Lie groupoids. The intimate fundamental groupoid $\Pi_1^{\text{in}}M$ over a connected manifold M is not a Lie groupoid, whereas the fundamental groupoid $\Pi_1(M)$ is a Lie groupoid. Consequently, in the following considerations the path-homotopic equivalence (relativ to endpoints) is used. Hence recall the fundamental Lie groupoid $\Pi_1\Sigma$ on the base Σ.

Let (K,k) be a Lie groupoid morphism between the Lie groupoids $\Pi_1(\Sigma)$ over Σ and $\Pi_1(M)$ over M.

Then define the pullback manifold (refer to definition 3.1.37)

$$\Pi_1(M)*\Sigma=k^!\Pi_1(M):=\{(\tilde{\gamma},v)\in\Pi_1(M)\times\Sigma:s_{\Pi_1(M)}(\tilde{\gamma})=k(v)\}$$

and $K^!$ denotes the map

$$K^!:\Pi_1(\Sigma)\to k^!\Pi_1(M),\quad \gamma\mapsto(K(\gamma),s_{\mathcal{P}\Sigma}(\gamma))$$

The inverse of $K^!$ is given by $\mathfrak{k}:k^!\Pi_1(M)\to\Pi_1(\Sigma)$.

Hence the projection is a map $t_{\mathcal{P}\Sigma}\circ\mathfrak{k}:\Pi_1(M)*\Sigma\to\Sigma$, which is smooth.

Moreover in general there is the tangent bundle projection $p_{\Pi_1\Sigma}:T(\Pi_1\Sigma)\to\Pi_1\Sigma$ which is a morphism of Lie groupoids over $p_\Sigma:T\Sigma\to\Sigma$.

But if the tanget space is assumed to be trivial, then one additionally assumes that $K^!$ is a diffeomorphism. Remark that in this case, K is called action morphism.

There exists a natural right action of a group H of isometries on M. Consequently, the bundle $\Pi_1(M)*\Sigma\times H\to\Sigma$ can be constructed.

The path connection $\Lambda_M\circ K$ is defined on $\Pi_1(\Sigma)$, where Λ_M is the path-connection on $\Pi_1(M)$. The properties of the holonomy Haxiom (1) (Holonomy Group condition), Haxiom (3) (Reparametrisation independence) and Haxiom (7) (Tangency condition) is naturally implemented by the choice of the path connection.

Moreover the fundamental group $\pi_1(M,x)$ at $x\in M$ is a subgroup of the isometries given by H.

There is a general theorem of Mackenzie [66], which states that the path connection Λ constructed from $\Pi_1(M)$ in Haxiom (5) (Development map) is unique up to the Lie groupoid morphism (K,k).

3.3.3 Holonomy maps and transformations for a gauge theory

In this section the concept of holonomy maps given by Barrett is further generalised with the help of the ideas of Mackenzie. In particular the holonomy mappings map paths to elements of the gauge groupoid instead of elements of the structure group of a principal fibre bundle. This leads to a new formulation of the configuration space of Loop Quantum Gravity: the space of smooth connections correspond one-to-one to the space of holonomy maps for a gauge theory.

In the first subsection the definition of holonomy maps is considered for the certain case of a general gauge theory. Then gauge and diffeomorphism transformations on the holonomy groupoid associated to a path connection are studied explicity. Finally the new formulation of the configuration and momentum space is presented.

The holonomy maps and the holonomy groupoid for a gauge theory

Consider a principal bundle $P(\Sigma, G)$ and construct the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. Moreover recall

$$\begin{aligned}\mathcal{G}^v &= \{\langle u,p\rangle : \pi(u) = v\}\\ \mathcal{G}^v_v &= \{\langle u,p\rangle : \pi(u) = v = \pi(p)\}\\ &= \{\langle u\delta(p,u),u\rangle\}\end{aligned}$$

Then a path connection in the Lie groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$ is a map $\Lambda_u : \mathcal{P}\Sigma^v \to \mathcal{P}^{s_\mathcal{G}}_{\{\mathcal{G}^v_v\}}\mathcal{G}^v$ for fixed $u \in P$, where $\pi(u) = v$, which is given by

$$\Lambda_u : \gamma \mapsto \Lambda_u(\gamma) := \langle u, \tilde{\gamma}_t\rangle \text{ for } \gamma \in \mathcal{P}\Sigma^v$$

where the map $s \mapsto \tilde{\gamma}(ts) =: \tilde{\gamma}_t(s)$ is a lifted path in P of a path $t \mapsto \gamma(t)$ in Σ such that $\tilde{\gamma}(0) = u$ and $\tilde{\gamma}(t) = p(t)$, $\pi(p(t)) = \gamma(t)$. Rewrite $\Lambda(\gamma)(s) =: \Lambda(\gamma, s)$ and $\Lambda(\gamma, s) := \langle u, \tilde{\gamma}(ts)\rangle$. Hence for a fixed path γ the mapping $s \mapsto \Lambda(\gamma, s)$ maps from an intervall I to $\mathcal{G}^v$. The source and target maps satisfy

$$\begin{aligned}s_P(\langle u, \tilde{\gamma}(ts)\rangle) &:= (\pi\circ \mathrm{pr}_1)(\langle u, \tilde{\gamma}(ts)\rangle)\\ &= \pi(u) = v\end{aligned}$$

$$\begin{aligned}t_P(\langle u, \tilde{\gamma}(ts)\rangle) &:= (\pi\circ \mathrm{pr}_2)(\langle u, \tilde{\gamma}(ts)\rangle)\\ &= \pi(\tilde{\gamma}(ts)) = \gamma(ts) = (\gamma\circ\varrho_s)(t)\end{aligned}$$

where

$$\langle u,u\rangle = \mathbb{1}_v \text{ and } t_P(\langle u,u\rangle) = t_P(\mathbb{1}_v) = \pi(u) = v = s_P(\mathbb{1}_v) \tag{3.37}$$

Summarising the path-connection Λ depends on the choice of the path γ and the point $u \in P$.

The reversal $\Lambda(\gamma, s)^{\leftarrow}$ is given by

$$\Lambda(\gamma, s)^{\leftarrow} := \langle \tilde{\gamma}(t), u\rangle\langle u, \tilde{\gamma}(t(1-s))\rangle = \langle \tilde{\gamma}(t), \tilde{\gamma}(t(1-s))\rangle$$

The concatenation operation $\cdot$ is defined by

$$\begin{aligned}(\Lambda^1_u\cdot\Lambda^2_p)(s) &:= R_{\gamma_2}\circ\langle u, \tilde{\gamma}(t)(s)\rangle\cdot\Lambda^2_p(\gamma_2) = \langle u, (\tilde{\gamma}_1\circ\tilde{\gamma}_2)(ts)\rangle\langle p, \tilde{\gamma}_2(ts)\rangle\\ &= \begin{cases}\langle u, \tilde{\gamma}_1(2ts)\rangle\langle p, \tilde{\gamma}_2(ts)\rangle & \text{for } s\in[0, 1/2]\\ \langle u, \tilde{\gamma}_2(t(2s-1))\rangle\langle p, \tilde{\gamma}_2(ts)\rangle & \text{for } s\in[1/2, 1]\\ \langle u, \tilde{\gamma}_1(0)\rangle\langle p, \tilde{\gamma}_2(0)\rangle = \langle u,u\rangle\langle p,p\rangle & \text{for } s = 0\\ \langle u, \tilde{\gamma}_2(t)\rangle = \langle \tilde{\gamma}_1(0), \tilde{\gamma}_2(t)\rangle & \text{for } s = 1\end{cases}\end{aligned} \tag{3.38}$$

In general, the **holonomy map for a gauge theory** is a groupoid morphism presented by the map

$$\mathfrak{H}_\Lambda : \mathcal{P}\Sigma^v \to \mathcal{G}^v, \gamma \mapsto \mathfrak{H}_\Lambda(\gamma) := \Lambda|_{s=1}(\gamma) = \langle u, \tilde{\gamma}_t(1)\rangle \tag{3.39}$$

The **holonomy groupoid for a gauge theory** is given by

$$\mathrm{Hol}^P_\Lambda(\Sigma) := \{\mathfrak{H}_\Lambda(\gamma) : \gamma \in \mathcal{P}\Sigma\}$$

Moreover, for a loop $\alpha \in \mathcal{P}\Sigma^v_v$ it is true that

$$\Lambda(\alpha, s)|_{s=1} = \langle u, \tilde{\alpha}_t(1)\rangle = \langle u, \tilde{\alpha}(t)\rangle \tag{3.40}$$

where $\tilde{\alpha}_t(1) := (\tilde{\alpha}\circ\varrho_s)|_{s=1}(t) = \tilde{\alpha}(t)$ In fact, there is a unique element $\mathfrak{h}_\Lambda(\alpha) \in G$ such that

$$\langle u, \tilde{\alpha}(t)\rangle\langle u,u\rangle = \langle u\delta(\tilde{\alpha}(t), u), u\rangle \text{ for } \pi(\tilde{\alpha}(t)) = \pi(u) = v \tag{3.41}$$

where $\delta(\tilde{\alpha}(t), u) =: \mathfrak{h}_\Lambda(\alpha)$. Consequently, the holonomy maps are related to the holonomy maps $\mathfrak{h}_\Lambda : \mathcal{P}\Sigma^v_v \to G$.

The lift $\tilde{\gamma}(t)$ of a path $\gamma(t)$ in P is a diffeomorphism between the fibres P_v and $P_{\gamma(t)}$, i.e. the map $\tilde{\gamma}(t) : I \rightarrow \mathrm{Diff}(P_v, P_{\gamma(t)})$.

A change of the base point $u \in P$ transforms as follows

$$\begin{aligned}\Lambda_{uk}(\gamma, s) &= \langle uk, \tilde{\gamma}_k(ts)\rangle \\ &= \langle uk, \tilde{\gamma}_k(ts)k^{-1}k\rangle = \langle uk, \tilde{\gamma}(ts)k\rangle = R_k \circ \Lambda_u(s) \\ &= \langle u, \tilde{\gamma}(ts)\rangle = \Lambda_u(\gamma, s)\end{aligned} \tag{3.42}$$

where $\tilde{\gamma}_k(0) = uk$ and $\tilde{\gamma}(0) = u$ such that

$$\Lambda_{uk}(\gamma, 1)\mathbb{1}_v = \langle uk, \tilde{\gamma}(t)k\rangle\langle u, u\rangle = \langle u\,\mathrm{Ad}(k)(\mathfrak{h}_\Lambda(\gamma)), u\rangle = \langle u\mathfrak{h}_\Lambda(\gamma), u\rangle = \Lambda_u(\gamma, 1)\mathbb{1}_v \tag{3.43}$$

This means that the lifts $\tilde{\gamma}$ have to be G-compatible, w.o.w. $\tilde{\gamma}_k(ts)k^{-1} = \tilde{\gamma}_{kk^{-1}}(ts) = \tilde{\gamma}(ts)$ for all $k \in G$. Summarising the lifts $\tilde{\gamma}$ are contained in the set $\mathrm{Diff}_{G\text{-comp.}}(P_v, P_w)$ of diffeomorphism between the fibres P_v and $P_{\gamma(t)}$, which are G-compatible.

Transformations in the holonomy groupoid for a gauge theory

Let $\mathcal{G} := \frac{P\times P}{G}$ over Σ be the gauge groupoid. Then the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ has been derived in the last section from a holonomy map $\mathfrak{H}_\Lambda : P\Sigma \rightarrow \mathcal{G}$, which has been defined by

$$\mathfrak{H}_\Lambda(\gamma) = \langle u, \tilde{\gamma}_t(1)\rangle = \langle u, \tilde{\gamma}(t)\rangle$$

where $\tilde{\gamma}_t(1) := (\tilde{\gamma} \circ \varrho_s)|_{s=1}(t) = \tilde{\gamma}(t)$ for a lifted path $s \mapsto \tilde{\gamma}(ts)$ in P of a path $\gamma \in P\Sigma$ and $s_{P\Sigma}(\gamma) = v$, $\pi(v) = u = s_P(\tilde{\gamma})$.

Definition 3.3.7. *Let $\sigma(v) := \langle u, \varphi(u)\rangle$ defines a bisection σ of $\frac{P\times P}{G} \rightrightarrows \Sigma$ for a gauge and diffeomorphism transformation $\varphi(\varphi_0, \mathrm{id})$, where $\pi(\varphi(u)) = \varphi_0(\pi(u)) = \varphi_0(v)$ holds.*

Then a left action L_σ on the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is defined by

$$L_\sigma\mathfrak{H}_\Lambda(\gamma) := \sigma((t_P \circ \sigma)^{-1}(s_P(\mathfrak{H}_\Lambda(\gamma))))\mathfrak{H}_\Lambda(\gamma) = \langle u, \varphi(u)\rangle\mathfrak{H}_\Lambda(\gamma)$$

whenever $\mathfrak{H}_\Lambda(\gamma) \in \mathrm{Hol}^P_\Lambda(\Sigma)$.

This holds since $\sigma(\pi(u)) = \sigma(v) = \langle u, \varphi(u)\rangle$ and $(t_P \circ \sigma)(v) = \pi(\varphi(u)) = \varphi_0(v)$. Notice that, $(\pi \circ s_P)(L_\sigma\mathfrak{H}_\Lambda(\gamma)) = v$ and $t_P(L_\sigma\mathfrak{H}_\Lambda(\gamma)) = \tilde{\gamma}(1)$ yields.

For pure gauge transformations, w.o.w. if $\pi(\varphi(u)) = \pi(u)$ and $\varphi_0(v) = v$ is satisfied, derive

$$L_\sigma\mathfrak{H}_\Lambda(\gamma) = \langle u, \varphi(u)\rangle\langle u, \tilde{\gamma}_t(1)\rangle = \langle u\delta(\varphi(u), u), \tilde{\gamma}_t(1)\rangle \tag{3.44}$$

In particular pure gauge transformations are given by $\varphi(u) := ug$ for suitable g in G. Then since φ is required to satisfy $\varphi(uk) = \varphi(u)k$ for all $k \in G$, the element $g \in G$ have to be such that $[g, k] = 0$ for all $k \in G$. Denote the center of the group G by $\mathcal{Z}(G)$. Hence for pure gauge transformations there is an action L on $\frac{P\times P}{G} \rightrightarrows \Sigma$ defined by

$$\begin{aligned}&L : \Big(\frac{P\times P}{G}, (P \times \mathcal{Z}(G))\Big) \longrightarrow \frac{P\times P}{G};\\ &L_{(u_2,g)}(\langle p, u_1\rangle) := \begin{cases} \langle p, u_1\rangle * (u_2, g) = \langle p, u_1 g\rangle & \text{for } \pi(u_1) = \pi(u_2) = v \\ \langle p, u_1\rangle * (u_2, g) = \langle p, u_1\rangle & \text{for } \pi(u_1) \neq \pi(u_2)\end{cases}\end{aligned}$$

Therefore the left action on $\mathrm{Hol}^P_\Lambda(\Sigma)$ associated to a bisection σ, which is build from a pure gauge transformation, is presented by

$$\begin{aligned}&L_\sigma\mathfrak{H}_\Lambda(\gamma) = \langle u, \varphi(u)\rangle\langle u, \tilde{\gamma}_t(1)\rangle = (\langle u, u\rangle * (u, g))\langle u, \tilde{\gamma}_t(1)\rangle \\ &= \langle u, ug\rangle\langle u, \tilde{\gamma}_t(1)\rangle = \langle u\delta(ug, u), \tilde{\gamma}_t(1)\rangle = \langle ug^{-1}, \tilde{\gamma}_t(1)\rangle\end{aligned} \tag{3.45}$$

and it is true that

$$\begin{aligned} L_\sigma\mathfrak{H}_\Lambda(\gamma) &= \langle uk, ukg\rangle\langle u,\tilde{\gamma}_t(1)\rangle = \langle uk, ugk\rangle\langle u,\tilde{\gamma}_t(1)\rangle = \langle uk\delta(ugk,u),\tilde{\gamma}_t(1)\rangle \\ &= \langle ug^{-1},\tilde{\gamma}_t(1)\rangle \text{ for all } k\in G \end{aligned} \tag{3.46}$$

holds.

Since $\tilde{\gamma}_t(1)g =: \tilde{\gamma}_g(1)$ and $\tilde{\gamma}(0)g = ug$ holds, deduce

$$L_\sigma\mathfrak{H}_\Lambda(\gamma) = \langle u, ug\rangle\langle u,\tilde{\gamma}_t(1)\rangle = \langle u,\tilde{\gamma}_t(1)g\rangle = \langle u,\tilde{\gamma}_g(1)\rangle \tag{3.47}$$

Definition 3.3.8. *Define the action $L_{\sigma^{-1}}$ on $\mathrm{Hol}^P_\Lambda(\Sigma)$ for a bisection σ^{-1} of $\mathrm{Hol}^P_\Lambda(\Sigma)$ by*

$$\begin{aligned} L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma) &:= \sigma^{-1}(s_P(\mathfrak{H}_\Lambda(\gamma)))\mathfrak{H}_\Lambda(\gamma) = \langle\varphi(u),u\rangle\mathfrak{H}_\Lambda(\gamma) \\ &= \langle\varphi(u),\tilde{\gamma}_t(1)\rangle \end{aligned}$$

for every gauge and diffeomorphism transformation $\varphi(\varphi_0,\mathrm{id})$ and $\pi(u) = v$.

Notice that, $(\pi\circ s_P)(\mathfrak{H}_\Lambda(\gamma)) = v$, $(\pi\circ s_P)(L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)) = w$ holds, where $w := (\pi\circ\varphi)(u) = \varphi_0(v)$ is not necessarily equal to v and $t_P(L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)) = \tilde{\gamma}_t(1)$ is satisfied.

If $\pi(\tilde{\gamma}_t(1)) = \pi(u) = v$ is fulfilled, then

$$L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)\mathbb{1}_v = \langle\varphi(u),\tilde{\gamma}_t(1)\rangle\langle u,u\rangle = \langle\varphi(u)\delta(\tilde{\gamma}_t(1),u),u\rangle \tag{3.48}$$

and $\mathfrak{h}_\Lambda(\gamma) = \delta(\tilde{\gamma}_t(1),u)$ yields. If $\varphi(u) = uk$ for $k\in\mathcal{Z}(G)$ is satisfied, then it follows that,

$$L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)\mathbb{1}_v = \langle uk\mathfrak{h}_\Lambda(\gamma),u\rangle \tag{3.49}$$

holds.

In general the composition of paths transfer to multiplication of elements on $\frac{P\times P}{G}$:

$$\mathfrak{H}_\Lambda(\gamma_1)L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma_2) = \langle u,\tilde{\gamma}^1_t(1)\rangle\langle\varphi(u),\tilde{\gamma}^2_t(1)\rangle = \langle u\delta(\tilde{\gamma}^1_t(1),\varphi(u)),\tilde{\gamma}^2_t(1)\rangle \tag{3.50}$$

Moreover if $\gamma^1(1) = \gamma^2(0)$ and $\pi(\gamma^2(0)) = \pi(u) = v = \pi(\gamma^1(1)) = \pi(\gamma^1(0))$ is true, then compute

$$\begin{aligned} \mathfrak{H}_\Lambda(\gamma_1)\mathfrak{H}_\Lambda(\gamma_2) &= \langle u,\tilde{\gamma}^1_t(1)\rangle\langle u,\tilde{\gamma}^2_t(1)\rangle = \langle u\delta(\tilde{\gamma}^1_t(1),u),\tilde{\gamma}^2_t(1)\rangle \\ &= \langle u\mathfrak{h}_\Lambda(\gamma_1),\tilde{\gamma}^2_t(1)\rangle \end{aligned} \tag{3.51}$$

For a pure gauge transformation $\varphi(u) = ug$ for $g\in\mathcal{Z}(G)$ then derive

$$\mathfrak{H}_\Lambda(\gamma_1)L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma_2) = \langle u\delta(\tilde{\gamma}^1_t(1),ug),\tilde{\gamma}^2_t(1)\rangle = \langle ug\mathfrak{h}_\Lambda(\gamma_1),\tilde{\gamma}^2_t(1)\rangle \tag{3.52}$$

If $\pi(\tilde{\gamma}^2_t(1)) = \pi(u) = v$ is fulfilled, then finally calculate

$$\mathfrak{H}_\Lambda(\gamma_1)L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma_2)\mathbb{1}_v = \langle ug\mathfrak{h}_\Lambda(\gamma_1)\mathfrak{h}_\Lambda(\gamma_2),u\rangle \tag{3.53}$$

Notice that,

$$L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma_1)\mathfrak{H}_\Lambda(\gamma_2)\mathbb{1}_v = \langle u\mathfrak{h}_\Lambda(\gamma_1)g^{-1}\mathfrak{h}_\Lambda(\gamma_2),u\rangle \tag{3.54}$$

holds.

Recall $\langle\tilde{\gamma}_t(1)g,\varphi(\tilde{\gamma}_t(1))g\rangle = \langle\tilde{\gamma}_t(1),\varphi(\tilde{\gamma}_t(1))\rangle$ and $(\varphi_0\circ\pi)(\tilde{\gamma}_t(1)) = (\pi\circ\varphi)(\tilde{\gamma}_t(1))$.

Definition 3.3.9. *Let $\sigma(v) := \langle u,\varphi(u)\rangle$ defines a bisection σ of $\frac{P\times P}{G}\rightrightarrows\Sigma$ for a gauge and diffeomorphism transformation $\varphi(\varphi_0,\mathrm{id})$.*

Then a right action R_σ on the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is defined by

$$\begin{aligned} R_\sigma\mathfrak{H}_\Lambda(\gamma) &:= \mathfrak{H}_\Lambda(\gamma)\sigma(t_P(\mathfrak{H}_\Lambda(\gamma))) = \langle u,\tilde{\gamma}_t(1)\rangle\langle\tilde{\gamma}_t(1),\varphi(\tilde{\gamma}_t(1))\rangle \\ &= \langle u,\varphi(\tilde{\gamma}_t(1))\rangle \end{aligned}$$

Then $(s_P \circ \pi)(R_\sigma \mathfrak{H}_\Lambda(\gamma)) = u = \pi(v)$ and $t_P(R_\sigma\mathfrak{H}_\Lambda(\gamma)) = \varphi(\tilde{\gamma}_t(1))$ holds. Observe that

$$R_\sigma \mathfrak{H}_\Lambda(\gamma)\mathbb{1}_v = \langle u, \varphi(\tilde{\gamma}_t(1))\rangle\langle u,u\rangle = \langle u\delta(\varphi(\tilde{\gamma}_t(1)),u),u\rangle \tag{3.55}$$

is true. If pure gauge transformations are considered, i.e. $\varphi(\tilde{\gamma}_t(1)) = \tilde{\gamma}_t(1)g$ for $g \in \mathcal{Z}(G)$ holds, then derive

$$R_\sigma \mathfrak{H}_\Lambda(\gamma)\mathbb{1}_v = \langle u, \tilde{\gamma}_t(1)g\rangle\langle u,u\rangle = \langle u\delta(\tilde{\gamma}_t(1)g,u),u\rangle = \langle u\mathfrak{h}_\Lambda(\gamma)g^{-1},u\rangle \tag{3.56}$$

Moreover for each $k \in G$ observe

$$R_\sigma \mathfrak{H}_\Lambda(\gamma)\mathbb{1}_v = \langle uk, \tilde{\gamma}_t(1)gk\rangle\langle u,u\rangle = \langle u\,\mathrm{Ad}(k)(\mathfrak{h}_\Lambda(\gamma))g^{-1},u\rangle \tag{3.57}$$

whenever $g \in \mathcal{Z}(G)$.

If $\pi(\gamma_1) = \pi(u) = v$ is satisfied, then calculate

$$\begin{aligned}&\mathfrak{H}_\Lambda(\gamma_1)R_\sigma\mathfrak{H}_\Lambda(\gamma_2) = \langle u, \tilde{\gamma}^1_t(1)\rangle\langle u,\varphi(\tilde{\gamma}^2_t(1))\rangle = \langle u\delta(\tilde{\gamma}^1_t(1),u),\varphi(\tilde{\gamma}^2_t(1))\rangle\\ &= \langle u\mathfrak{h}_\Lambda(\gamma_1),\varphi(\tilde{\gamma}^2_t(1))\rangle\end{aligned} \tag{3.58}$$

Moreover if $\pi(\gamma_2) = \pi(u) = v$ and $\varphi(\tilde{\gamma}^2_t(1)) = \tilde{\gamma}^2_t(1)g$ yields for $g \in \mathcal{Z}(G)$, then derive

$$\mathfrak{H}_\Lambda(\gamma_1)R_\sigma\mathfrak{H}_\Lambda(\gamma_2)\mathbb{1}_v = \; = \langle u\mathfrak{h}_\Lambda(\gamma_1)\delta(\varphi(\tilde{\gamma}^2_t(1)),u),u\rangle = \langle u\mathfrak{h}_\Lambda(\gamma_1)\mathfrak{h}_\Lambda(\gamma_2)g^{-1},u\rangle \tag{3.59}$$

Definition 3.3.10. *Let $\sigma(v) := \langle u, \varphi(u)\rangle$ defines a bisection σ of $\frac{P\times P}{G} \rightrightarrows \Sigma$ for a gauge and diffeomorphism transformation $\varphi(\varphi_0, \mathrm{id})$, where $\pi(\varphi(u)) = \varphi_0(\pi(u)) = \varphi_0(v)$ holds.*

Then an inner action I_σ on the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is implemented by

$$\begin{aligned}I_\sigma(\mathfrak{H}_\Lambda(\gamma))) &:= R_\sigma(L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)) = \sigma^{-1}(s_P(\mathfrak{H}_\Lambda(\gamma)))\mathfrak{H}_\Lambda(\gamma)\sigma(t_P(\mathfrak{H}_\Lambda(\gamma)))\\ &= \langle\varphi(u),u\rangle\langle u,\tilde{\gamma}_t(1)\rangle\langle\tilde{\gamma}_t(1),\varphi(\tilde{\gamma}_t(1))\rangle\\ &= \langle\varphi(u),\tilde{\gamma}_t(1)\rangle\langle\tilde{\gamma}_t(1),\varphi(\tilde{\gamma}_t(1))\rangle = \langle\varphi(u),\varphi(\tilde{\gamma}_t(1))\rangle\end{aligned}$$

whenever $\mathfrak{H}_\Lambda(\gamma) \in \mathrm{Hol}^P_\Lambda(\Sigma)$.

If additionally $\varphi(u) = ug$ for $g \in \mathcal{Z}(G)$ holds, then compute

$$R_\sigma(L_{\sigma^{-1}}\mathfrak{H}_\Lambda(\gamma)) = \langle u, \varphi(\tilde{\gamma}_t(1))\rangle \tag{3.60}$$

Let $\langle u, \tilde{\gamma}\rangle \in \mathcal{PG}^v$ a path in the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$, where $\pi(u) = v$ and $\tilde{\gamma} \in \mathcal{P}P$. Let $\rho_{\tilde{\gamma}} : I \to G$ be a curve such that $\rho_{\tilde{\gamma}}(0) = e_G$, $\rho_{\tilde{\gamma}}(1) = g$ for $g \in G$ and $R_k(\rho_{\tilde{\gamma}}(s)) = \mathrm{Ad}(k)(\rho_{\tilde{\gamma}}(s))$ for all $k \in G$. Then for a fixed $s \in I$ concern the following action associated to a purely gauge transformation as a map

$$\begin{aligned}&* : \Big(\frac{P\times P}{G}, \frac{P\times G}{G}\Big) \longrightarrow \frac{P\times P}{G}, \text{ where}\\ &\langle u, \tilde{\gamma}_1(s)\rangle * \lfloor\tilde{\gamma}_2(s), \rho_{\tilde{\gamma}}(s)\rfloor = \begin{cases}\langle u, \tilde{\gamma}_t(s)\rho_{\tilde{\gamma}}(s)\rangle & \text{for } (\pi\circ t_P)(\tilde{\gamma}_1) = (\pi\circ t_P)(\tilde{\gamma}_2) = v\\ \langle u, \tilde{\gamma}_1(s)\rangle & \text{for } (\pi\circ t_P)(\tilde{\gamma}_1) \neq (\pi\circ t_P)(\tilde{\gamma}_2)\end{cases}\end{aligned}$$

Recall

$$\lfloor\tilde{\gamma}_t(s), \rho_{\tilde{\gamma}}(s)\rfloor = \lfloor\tilde{\gamma}_t(s)kk^{-1}, \rho_{\tilde{\gamma}}(s)\rfloor = \lfloor\tilde{\gamma}_t(s)k, \mathrm{Ad}(k^{-1})(\rho_{\tilde{\gamma}}(s))\rfloor$$

Then the action R of a bisection σ associated to a purely gauge transformation on the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$ is reformulated by

$$R_\sigma\langle u, \tilde{\gamma}_t(s)\rangle = \langle u, \varphi(\tilde{\gamma}_t(s))\rangle = \langle u, \tilde{\gamma}_t(s)\rangle * \lfloor\tilde{\gamma}_t(s), \rho_{\tilde{\gamma}}(s)\rfloor = \langle u, \tilde{\gamma}_t(s)\rho_{\tilde{\gamma}}(s)\rangle \tag{3.61}$$

whenever $\tilde{\gamma}_t(s) \in P$ for a fixed $s \in I$, $(\tilde{\gamma}_t(s), \rho_{\tilde{\gamma}}(s)) \in \frac{P\times G}{G}$ and such that

$$\begin{aligned}R_\sigma\langle uk, \tilde{\gamma}_t(s)k\rangle &= \langle uk, \varphi(\tilde{\gamma}_t(s))k\rangle = \langle uk, \varphi(\tilde{\gamma}_t(s)k)\rangle = \langle uk, \tilde{\gamma}_t(s)k\,\mathrm{Ad}(k^{-1})(\rho_{\tilde{\gamma}}(s))\rangle\\ &= \langle u, \tilde{\gamma}_t(s)\rho_{\tilde{\gamma}}(s)\rangle\end{aligned} \tag{3.62}$$

yields. Futhermore for an element $\langle u, \tilde{\gamma}_t(1)\rangle \in \mathrm{Hol}^P_\Lambda(\Sigma)$ it is true that,

$$\begin{aligned} R_\sigma \langle uk, \tilde{\gamma}_t(1)k\rangle \mathbb{1}_v &= \langle uk, \tilde{\gamma}_t(1)\rho_{\tilde{\gamma}}(1)k\rangle\langle u,u\rangle = \langle uk\delta(\tilde{\gamma}_t(1)\rho_{\tilde{\gamma}}(1)k, u)u\rangle \\ &= \langle u\,\mathrm{Ad}(k)(\mathfrak{h}_\Lambda(\gamma))\rho_{\tilde{\gamma}}(1)^{-1}, u\rangle \end{aligned} \tag{3.63}$$

holds such that

$$R_\sigma\langle uk, \tilde{\gamma}(0)k\rangle = \langle u,u\rangle \quad \forall k \in G \tag{3.64}$$

is true. Choose the map $\rho_{\tilde{\gamma}}$ such that

$$\rho_{\tilde{\gamma}}(s) := \exp(X_{\tilde{\gamma}_t(s)}) \text{ for } X_{\tilde{\gamma}_t(s)} \in \mathfrak{g} \text{ and } \mathrm{Ad}(g)\rho_{\tilde{\gamma}}(s) = \rho_{\tilde{\gamma}}(s) \tag{3.65}$$

for a fixed element $s \in I$ and where $\tilde{\gamma}_t(s) \in P$ is satisfied.

Let W be an open subset of Σ. Let $X_{\tilde{\gamma}_t(1)}$ be an element of the section $\Gamma(\frac{T_W P}{G})$, then there exists a family of local bisections $\exp(tX_{\tilde{\gamma}_t(1)})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\exp(\tau X_{\tilde{\gamma}_t(1)})\Big|_{\tau=0} = X_{\tilde{\gamma}_t(1)} \tag{3.66}$$

Therefore set

$$\rho_{\tilde{\gamma}}(1) := \exp(\tau X_{\tilde{\gamma}_t(1)}) \text{ for } X_{\tilde{\gamma}_t(1)} \in \mathfrak{g}, \tau \in \mathbb{R} \tag{3.67}$$

and require $\mathrm{Ad}(g^{-1})\rho_{\tilde{\gamma}}(1) = \rho_{\tilde{\gamma}}(1)$, where $\tilde{\gamma}_t(1) \in P$ and $\pi(\tilde{\gamma}_t(1)) = w \in \Sigma$.

Consequently there is a right action R_σ on $\mathrm{Hol}_\Lambda(\Gamma)$ associated to the bisection $\sigma = (\varphi, \mathrm{id}_\Sigma)$, where $\varphi(\tilde{\gamma}_t(s)) = \tilde{\gamma}_t(s)\rho_{\tilde{\gamma}}(s)$ for all paths $\tilde{\gamma}_t(s)$ in P. Observe that. a connection reform $\omega^{/G} : \frac{TP}{G} \rightarrow \frac{P\times\mathfrak{g}}{G}$ is a map. which satisfies

$$\omega^{/G}_{pg}(T(R_g)_p X_{\tilde{\gamma}_t(1)}) = \mathrm{Ad}(g^{-1})(\omega^{/G}_p(X_{\tilde{\gamma}_t(1)})) = \omega^{/G}_p(X_{\tilde{\gamma}_t(1)})$$

for $X_{\tilde{\gamma}_t(1)} \in T_pP$ and $p = \tilde{\gamma}_t(1)$.

3.3.3.1 A set of holonomy maps for a gauge theory

A smooth connection correspond one-to-one to a holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$. Assume that, the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is a transitive Lie subgroupoid of the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. Recall the vector bundle $T_{e_G}(G)$ for a Lie group G. The elements of $T_{e_G}(G)$ are the right invariant vector fields. Therefore these right invariant vector fields X correspond to generalised fluxes E, which are derivations on G.

The Lie algebroid associated to the holonomy Lie groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is given by

$$A(\mathrm{Hol}^P_\Lambda(\Sigma)) := \left\{ X \in \frac{TP}{G} : X - (\gamma_A \circ \pi_*)(X) \in \mathrm{Im}(\bar{R}_A) \right\}$$

The space $\breve{\mathcal{A}}_s$ of smooth connections is an affine space w.r.t. the vector space $\omega^1_{\mathrm{hor}}(\frac{P\times\mathfrak{g}}{G})$. The affine structure of the smooth connections is mirrored by the existence of a Lie groupoid morphism between the Lie groupoids $\mathrm{Hol}^P_\Lambda(\Sigma)$ and $\mathrm{Hol}^P_{\Lambda'}(\Sigma)$ for a fixed gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. This is studied in the next corollary.

Corollary 3.3.11. *Let γ_A and $\gamma_{A'}$ be two Lie algebroid connections on the transitive Lie algebroids $A(\mathrm{Hol}^P_\Lambda(\Sigma))$ and $A(\mathrm{Hol}^P_{\Lambda'}(\Sigma))$ associated to the holonomy Lie groupoids $\mathrm{Hol}^P_\Lambda(\Sigma)$ and $\mathrm{Hol}^P_{\Lambda'}(\Sigma)$.*

Then there exists a Lie algebroid morphism $\mathfrak{a}'$ between $A(\mathrm{Hol}^P_\Lambda(\Sigma))$ and $A(\mathrm{Hol}^P_{\Lambda'}(\Sigma))$. Moreover there is a Lie groupoid morphism $\mathfrak{m}'$ from $\mathrm{Hol}^P_\Lambda(\Sigma)$ to $\mathrm{Hol}^P_{\Lambda'}(\Sigma)$.

Proof : Consider the Lie algebroid $A(\mathrm{Hol}^P_\Lambda(\Sigma))$ and

$$A(\mathrm{Hol}^P_{\Lambda'}(\Sigma)) := \left\{ X \in \frac{TP}{G} : X - (\gamma_{A'} \circ \pi_*)(X) \in \mathrm{Im}(\bar{R}_{A'}) \right\}$$

Recall that $\gamma_{A'} := \gamma_A + j \circ l$ holds. Derive

$$\begin{aligned} &j(\bar{R}_A(X,Y)) - j(\bar{R}_{A'}(X,Y)) = \gamma_A[X,Y] - [\gamma_A X, \gamma_A Y] - \gamma_{A'}[X,Y] + [\gamma_{A'} X, \gamma_{A'} Y] \\ &= \gamma_A[X,Y] - [\gamma_A X, \gamma_A Y] - \gamma_A[X,Y] + (j \circ l)[X,Y] + [(\gamma_A + j \circ l)X, (\gamma_A + j \circ l)Y] \\ &= -[\gamma_A X, \gamma_A Y] + (j \circ l)[X,Y] + [\gamma_A X, \gamma_A Y] + [(j \circ l)X, (j \circ l)Y] \\ &= (j \circ l)[X,Y] + [(j \circ l)X, (j \circ l)Y] =: j(\bar{R}_l(X,Y)) \end{aligned}$$

Hence obtain

$$A(\text{Hol}^P_{\Lambda'}(\Sigma)) := \left\{ X \in \frac{TP}{G} : X - (\gamma_A \circ \pi_*)(X) - ((j \circ l) \circ \pi_*)(X) \in \text{Im}(\bar{R}_A + \bar{R}_l) \right\}$$

Consequently for $X \in A(\text{Hol}^P_\Lambda(\Sigma))$

$$\mathfrak{a}_l(X) = \mathfrak{a}_l((\gamma_A \circ \pi_*)(X)) = (\gamma_A \circ \pi_*)(X) + ((j \circ l) \circ \pi_*)(X) =: \mathfrak{a}'(X)$$

defines a Lie algebroid morphism $\mathfrak{a}'$ from $A(\text{Hol}^P_\Lambda(\Sigma))$ to $A(\text{Hol}^P_{\Lambda'}(\Sigma))$ over the same base Σ. This is verified by proving that $\mathfrak{a}_l$ is a vector bundle morphism such that the anchor preserving condition

$$a' = a \circ \mathfrak{a}' \tag{3.68}$$

and the bracket condition

$$\mathfrak{a}'([X,Y]) = [\mathfrak{a}'(X), \mathfrak{a}'(Y)] \tag{3.69}$$

are satisfied. Remember $j \circ \omega^{/G} + \gamma_A \circ \pi_* = \text{id}_{A\mathcal{G}}$ and consequently it is true that,

$$\gamma_A \circ \pi_* + (j \circ l) \circ \pi_* = j \circ \omega + j \circ \omega'$$

whenever $\omega, \omega' \in \Omega^1_{\text{basic}}(P, \mathfrak{g})^G$ yields. Then derive $j \circ \omega + j \circ \omega' = j \circ \tilde{\omega} \in \Omega^1_{\text{basic}}(P, \mathfrak{g})^G$.

Finally define a morphsim $\mathfrak{m}' : \text{Hol}^P_\Lambda(\Sigma) \longrightarrow \text{Hol}^P_{\Lambda'}(\Sigma)$ by $\mathfrak{a}' =: \mathfrak{m}'_*$ such that

$$\gamma_{A'} = \mathfrak{m}'_* \circ \gamma_A = \mathfrak{a}_l \circ \gamma_A$$

and

$$\Lambda' = \mathfrak{m}' \circ \Lambda$$

is fulfilled. Then $\mathfrak{m}'$ is a Lie groupoid morphism.

■

Notice that, the failure of an infinitesimal Lie algebroid connection γ_A to be a Lie algebroid morphism is given by the curvature, i.e.

$$j(\bar{R}_A(X,Y)) = \gamma_A[X,Y] - [\gamma_A X, \gamma_A Y]$$

whenever $X, Y \in A(\text{Hol}^P_\Lambda(\Sigma))$.

Definition 3.3.12. *Let $\breve{L}$ be the set of all maps $l : T\Sigma \longrightarrow \frac{P \times \mathfrak{g}}{G}$ such that $\gamma_{A'} = \gamma_A + j \circ l$ and $\gamma_A, \gamma_{A'}$ are right splittings of the Atiyah sequence. Set $\Lambda' := \Lambda_l$.*

Then the ***set of holonomy maps for a gauge theory*** *is defined by*

$$\text{Hol}^P_{\breve{\Lambda}}(\Sigma) := \{(\mathfrak{m}_l \circ \mathfrak{h}_\Lambda)(\gamma) \in \text{Hol}^P_{\Lambda_l}(\Sigma) : \gamma \in \mathcal{P}\Sigma, l \in \breve{L}, \Lambda \in \breve{\Lambda}\}$$

where $\mathcal{G} = \frac{P \times P}{G}$.

This set replaces the configuration space $\breve{\mathcal{A}}$ of smooth connections for a gauge theory. The set of holonomy maps for a gauge theory corresponds one-to-one to the set, which is given by

$$A(\text{Hol}^P_{\breve{\Lambda}}(\Sigma)) := \left\{ X \in \frac{TP}{G} : X - (\gamma_A \circ \pi_*)(X) - ((j \circ l) \circ \pi_*)(X) \in \text{Im}(\bar{R}_A + \bar{R}_l) \forall l \in \breve{L} \right\}$$

Summarising this set is derived from the Lie algebroids associated to holonomy Lie groupoids.

3.3.4 Holonomy maps for finite path groupoids, graph systems and transformations

In section 3.1.3 the concept of finite path groupoids for analytic paths has been given. Now in section 3.3.4.1 the Barrett theory is used for the development of the holonomy maps for finite path groupoids and finite graph systems in the context of (semi-)analytic paths. The concept of Barrett relates a connection directly with a holonomy map. The ideas are familar with those presented by Thiemann [104]. But for example the finite graph systems have not been studied before. Ashtekar and Lewandowski [8] have defined the analytic holonomy C^*-algebra, which they have based on a finite set of independent hoops. The hoops are generalised for path groupoids and the independence requirement is implemented by the concept of finite graph systems. The generalisation of the holonomy maps of Barrett in the context of Mackenzie implies a generalisation of the holonomy maps for finite path groupoids (and finite graph systems), too. Therefore the concept of germs is needed. This is taken into account in section 3.3.4.2, but is not used for the definition of the Weyl C^*-algebra later.

3.3.4.1 Holonomy maps for finite path groupoids

Groupoid morphisms for finite path groupoids

In general the important objects are groupoid morphisms. The definition is the following.

Definition 3.3.13. *Let* $\mathcal{G}_1 \overset{s_1}{\underset{t_1}{\rightrightarrows}} \mathcal{G}_1^0$, $\mathcal{G}_2 \overset{s_2}{\underset{t_2}{\rightrightarrows}} \mathcal{G}_2^0$ *be two arbitrary groupoids.*

A ***groupoid morphism*** *between two groupoids* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *consists of two maps* $\mathfrak{h} : \mathcal{G}_1 \to \mathcal{G}_2$ *and* $h : \mathcal{G}_1^0 \to \mathcal{G}_2^0$ *such that*

$(G1)$ $\quad \mathfrak{h}(\gamma \circ \gamma') = \mathfrak{h}(\gamma)\mathfrak{h}(\gamma')$ *for all* $(\gamma, \gamma') \in \mathcal{G}_1^{(2)}$

$(G2)$ $\quad s_2(\mathfrak{h}(\gamma)) = h(s_1(\gamma)), \quad t_2(\mathfrak{h}(\gamma)) = h(t_1(\gamma))$

A ***strong groupoid morphism*** *between two groupoids* $\mathcal{G}_1$ *and* $\mathcal{G}_2$*, additionally, satisfy*

(SG) $\quad$ *for every pair* $(\mathfrak{h}(\gamma), \mathfrak{h}(\gamma')) \in \mathcal{G}_2^{(2)}$ *it follows that* $(\gamma, \gamma') \in \mathcal{G}_1^{(2)}$

Let G be a Lie group, then G over e_G is a groupoid, where the group multiplication $\cdot : G^2 \to G$ is defined for all elements $g_1, g_2, g \in G$ such that $g_1 \cdot g_2 = g$. A groupoid morphism between a finite path groupoid $\mathcal{P}_\Gamma\Sigma$ to G is given by the maps

$$\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma\Sigma \to G, \quad h_\Gamma : V_\Gamma \to e_G$$

Clearly,

$$\begin{aligned} &\mathfrak{h}_\Gamma(\gamma \circ \gamma') = \mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma') \text{ for all } (\gamma, \gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)} \\ &s_G(\mathfrak{h}_\Gamma(\gamma)) = h_\Gamma(s_{\mathcal{P}_\Gamma\Sigma}(\gamma)), \quad t_G(\mathfrak{h}_\Gamma(\gamma)) = h_\Gamma(t_{\mathcal{P}_\Gamma\Sigma}(\gamma)) \end{aligned} \tag{3.70}$$

But for an arbitrary pair $(\mathfrak{h}_\Gamma(\gamma_1), \mathfrak{h}_\Gamma(\gamma_2)) =: (g_1, g_2) \in G^{(2)}$ it does not follows that $(\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma\Sigma^{(2)}$. Hence, $\mathfrak{h}_\Gamma$ is not a strong groupoid morphism.

Definition 3.3.14. *Let* $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ *be a finite path groupoid.*

Two paths γ *and* γ' *in* $\mathcal{P}_\Gamma\Sigma$ *have the* ***same-holonomy for all connections*** *iff*

$$\mathfrak{h}_\Gamma(\gamma) = \mathfrak{h}_\Gamma(\gamma') \text{ for all } (\mathfrak{h}_\Gamma, h_\Gamma) \text{ groupoid morphisms}$$
$$\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma\Sigma \to G, h : V_\Gamma \to \{e_G\}$$

Denote the relation by $\sim_{s.hol.}$.

Lemma 3.3.15. *The same-holonomy for all connections relation is an equivalence relation.*

Notice that, the quotient of the finite path groupoid and the same-holonomy relation for all connections replace the hoop group, which has been used in [8].

Definition 3.3.16. *Let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid modulo same-holonomy for all connections equivalence.*

*A **holonomy map for a finite path groupoid** $\mathcal{P}_\Gamma\Sigma$ over V_Γ is a groupoid morphism consisting of the maps $(\mathfrak{h}_\Gamma, h_\Gamma)$, where $\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma\Sigma \rightarrow G, h_\Gamma : V_\Gamma \rightarrow \{e_G\}$. The set of all holonomy maps is abbreviated by* $\text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$.

For a short notation observe the following. In further sections it is always assumed that the finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ is considered modulo same-holonomy for all connections equivalence although it will be not stated explicitly.

If one do not assume this fact, the same-holonomy equivalence relation for a connection A can be defined as follows. For two paths γ and γ' in $\mathcal{P}_\Gamma\Sigma$

$$\gamma \sim_{\text{s.hol.},A} \gamma' \quad \Leftrightarrow \quad \mathfrak{h}_\Gamma(\gamma) = \mathfrak{h}_\Gamma(\gamma') \text{ for a groupoid morphism } (\mathfrak{h}_\Gamma, h_\Gamma) \tag{3.71}$$

From the fact that the paths γ and γ' are the same-holonomy equivalent, i.o.w. $\gamma \sim_{\text{s.hol.},A} \gamma'$, it follows that $\mathfrak{h}_\Gamma(\gamma \circ {\gamma'}^{-1}) = e_G$. Therefore, $\gamma \circ {\gamma'}^{-1}$ is an element of the kernel $\ker(\mathfrak{h}_\Gamma)$ of the holonomy map $(\mathfrak{h}_\Gamma, h_\Gamma)$.

This imply another definition.

Definition 3.3.17. *Let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid and $\mathfrak{h}_\Gamma$ be a groupoid morphism from $\mathcal{P}_\Gamma\Sigma$ over V_Γ to G over $\{e_G\}$.*

*Then the **holonomy groupoid for a finite path groupoid** is defined by* $\text{Hol}_A(\mathcal{P}_\Gamma\Sigma) := \{\mathfrak{h}_\Gamma(\gamma) : \gamma \in \mathcal{P}_\Gamma\Sigma\}$ *and* $\text{Hol}^0_A(\mathcal{P}_\Gamma\Sigma) = \ker(\mathfrak{h}_\Gamma)$.

The groupoid $\text{Hoop}_A(\mathcal{P}_\Gamma\Sigma) := {}^{\text{Hol}_A(\mathcal{P}_\Gamma\Sigma)}/_{\text{Hol}^0_A(\mathcal{P}_\Gamma\Sigma)}$ *is called the **restricted holonomy groupoid for a finite path groupoid**.*

Discretised surface sets and localised paths

In the context of discretised surface sets and paths starting or ending at certain vertices the following definitions are useful.

Let $\tilde{\Gamma}$ be a graph and $\breve{S}_{\text{d}}$ be a discretised surface set. Then the subgraph of $\tilde{\Gamma}$ such that, this graph contains all edges of the graph $\tilde{\Gamma}$ that do not intersect with any vertex of the discretised surface set $\breve{S}_{\text{d}}$, is denoted by $\bar{\Gamma}$. The set of vertices of the graph $\bar{\Gamma}$ is given by $V_{\bar{\Gamma}}$. The subgraph $\tilde{\Gamma} \setminus \bar{\Gamma}$ is abbreviated by Γ. Then the set of groupoid morphisms $\text{Hom}(\mathcal{P}_\Gamma^{\breve{S}_{\text{d}}}\Sigma, G)$ is symbolised by $\mathcal{A}_\Gamma^{\text{d}}$. Respectively, the short hand notation for the set $\text{Hom}(\mathcal{P}^\Gamma_{\breve{S}_{\text{d}}}\Sigma, G)$ is given by $\mathcal{A}_{\text{d}}^\Gamma$ and the definition of $\mathcal{A}_{\bar{\Gamma}}$ is $\text{Hom}(\mathcal{P}_{\bar{\Gamma}}\Sigma, G)$.

Admissable maps and equivalent groupoid morphisms

Now consider a finite path groupoid morphism $(\mathfrak{h}_\Gamma, h_\Gamma)$ from a finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ to the groupoid G over $\{e_G\}$, which is contained in $\text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$.

Consider an arbitrary map $\mathfrak{g}_\Gamma : \mathcal{P}_\Gamma\Sigma \rightarrow G$. Then there is a groupoid morphism defined by

$$\mathfrak{G}_\Gamma(\gamma) := \mathfrak{g}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma)\mathfrak{g}_\Gamma(\gamma^{-1})^{-1} \text{ for all } \gamma \in \mathcal{P}_\Gamma\Sigma \tag{3.72}$$

if and only if

$$\mathfrak{G}_\Gamma(\gamma_1 \circ \gamma_2) = \mathfrak{G}_\Gamma(\gamma_1)\mathfrak{G}_\Gamma(\gamma_2) \text{ for all } (\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma\Sigma^{(2)}$$

holds. Then $\mathfrak{G}_\Gamma \in \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$.

Hence, for all $(\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma\Sigma^{(2)}$ it is necessary that

$$\begin{aligned}\mathfrak{G}_\Gamma(\gamma_1 \circ \gamma_2) &= \mathfrak{g}_\Gamma(\gamma_1 \circ \gamma_2)\mathfrak{h}_\Gamma(\gamma_1 \circ \gamma_2)\mathfrak{g}_\Gamma(\gamma_2^{-1} \circ \gamma_1^{-1})^{-1} \\ &= \mathfrak{g}_\Gamma(\gamma_1 \circ \gamma_2)\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_2)\mathfrak{g}_\Gamma(\gamma_2^{-1} \circ \gamma_1^{-1})^{-1} \\ &\stackrel{!}{=} \mathfrak{g}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{g}_\Gamma(\gamma_1^{-1})^{-1}\mathfrak{g}_\Gamma(\gamma_2)\mathfrak{h}_\Gamma(\gamma_2)\mathfrak{g}_\Gamma(\gamma_2^{-1})^{-1}\end{aligned}$$

is satisfied. Therefore, the map is required to fulfill

$$\begin{aligned}&\mathfrak{g}_\Gamma(\gamma_1) = \mathfrak{g}_\Gamma(\gamma_1 \circ \gamma_2),\ \mathfrak{g}_\Gamma(\gamma_2^{-1}) = \mathfrak{g}_\Gamma((\gamma_1 \circ \gamma_2)^{-1}) \text{ and}\\ &\mathfrak{g}_\Gamma(\gamma_1^{-1})^{-1}\mathfrak{g}_\Gamma(\gamma_2) = e_G \text{ for all } (\gamma_1, \gamma_2) \in \mathcal{P}_\Gamma\Sigma^{(2)} \text{ in particular,}\\ &\mathfrak{g}_\Gamma(\gamma^{-1})^{-1}\mathfrak{g}_\Gamma(\gamma) = e_G \text{ for all } (\gamma^{-1}, \gamma) \in \mathcal{P}_\Gamma\Sigma^{(2)}\end{aligned} \tag{3.73}$$

for every refinement $\gamma_1 \circ \gamma_2$ of each γ in $\mathcal{P}_\Gamma\Sigma$ and γ_1 being an initial segment of $\gamma_1 \circ \gamma_2$ and γ_2^{-1} an final segment of $(\gamma_1 \circ \gamma_2)^{-1}$. In comparison with Fleischhack's definition in [39, Def. 3.7] such maps are called admissible.

Definition 3.3.18. *The set of maps $\mathfrak{g}_\Gamma : \mathcal{P}_\Gamma\Sigma \to G$ satisfying* (3.73) *for all pairs of decomposable paths in $\mathcal{P}_\Gamma^{(2)}\Sigma$ is called the* ***set of admissible maps*** *and is denoted by* $\text{Map}^\text{A}(\mathcal{P}_\Gamma\Sigma, G)$.

Consider a map $g_\Gamma : V_\Gamma \to G$ such that

$$(g_\Gamma, \mathfrak{h}_\Gamma) \in \text{Map}(V_\Gamma, G) \times \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$$

which is also called a local gauge map. Then the map $\tilde{\mathfrak{G}}_\Gamma$ defined by

$$\tilde{\mathfrak{G}}_\Gamma(\gamma) := g_\Gamma(s(\gamma))\mathfrak{h}_\Gamma(\gamma)g_\Gamma(s(\gamma^{-1}))^{-1} \text{ for all } \gamma \in \mathcal{P}_\Gamma\Sigma \tag{3.74}$$

is a groupoid morphism. This is a result of the computation

$$\begin{aligned}\tilde{\mathfrak{G}}_\Gamma(\gamma_1\gamma_2) &= g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1\gamma_2)g_\Gamma(t(\gamma_2))^{-1}\\ &= g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}g_\Gamma(s(\gamma_2))\mathfrak{h}_\Gamma(\gamma_2)g_\Gamma(t(\gamma))^{-1}\end{aligned}$$

since $t(\gamma_1) = s(\gamma_2)$.

Definition 3.3.19. *Two groupoid morphisms $(\mathfrak{h}_\Gamma, h_\Gamma)$ and $(\mathfrak{G}_\Gamma, h_\Gamma)$, or respectively $(\tilde{\mathfrak{G}}_\Gamma, h_\Gamma)$, between the groupoids $\mathcal{P}_\Gamma$ over V_Γ and the groupoid G over $\{e_G\}$, which are defined for $(\mathfrak{g}_\Gamma, \mathfrak{h}_\Gamma) \in \text{Map}(\mathcal{P}_\Gamma\Sigma, G) \times \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$ by* (3.72)*, or for $(g_\Gamma, \mathfrak{h}_\Gamma) \in \text{Map}(V_\Gamma, G) \times \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$ by* (3.74)*, are said to be* ***similar or equivalent groupoid morphisms****.*

3.3.4.2 Holonomy maps for finite path groupoids along germs

The classical configuration space of Loop Quantum Gravity is the space of all smooth connections. Following the idea of Barrett [16] the configuration space is replaced by the set of all groupoid morphism between the finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ and the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. Mackenzie's path connection implies that, the tangent vectors for paths are required to satisfy some additional rules. Now the analytic category is taken into account. Therefore the path connection has to be generalised to analytic path connections. Finally if the concept of Barrett is generalised to Mackenzie's path connections in the analytic category, then the holonomy maps have to satisfy some additional conditions. Note that, this generalisation is not necessary for the construction of the Weyl C^*-algebra, which is defined in chapter 6 or the holonomy-flux cross-product *-algebras defined in chapter 8.

A **(semi-)analytic path connection** Λ satisfy the (semi-)analyticity condition instead of the smoothness (iii) in definition 3.2.1. The (semi-)analyticity condition states that for an analytic path γ the path connection $\Lambda(\gamma(t), s)$ is required to be analytic for each $s \in I$.

Analogous to the definition used in [103] the term germ can be introduced.

Lemma 3.3.20. *Let Γ be a graph which contains only (semi-)analytic paths in Σ.*

There is an equivalence class of paths $[\gamma_0]$ at $v \in V_\Gamma$ such that all paths starting at v are (semi-)analytic extensions or restrictions of γ_0. The equivalence class is called a ***germ at*** *v*

Define the **set of initial germ paths at** v, which is denoted by $\mathcal{P}_\Gamma\Sigma^{s(v)}$, as a set of germs at v such that all γ_k start at v for each $1 \leq k \leq n$. Hence, the set $\{\gamma_1, ..., \gamma_n\}$ contain only mutually non-composable but non-independent paths.

Clearly, the same-holonomy equivalence relation transferes to analytic path groupoids.

Definition 3.3.21. *Let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid modulo same-holonomy equivalence for all connections. The set of* ***finite path groupoid holonomies along germs*** *is defined by*

$$\mathcal{A}_\Gamma := \mathrm{Hom}(\mathcal{P}_\Gamma\Sigma, G) = \{(\mathfrak{h}_\Gamma, h_\Gamma)| \quad \mathfrak{h}_\Gamma : \mathcal{P}_\Gamma\Sigma \rightarrow G, h_\Gamma : V_\Gamma \rightarrow \{e_G\} \\ (\mathfrak{h}_\Gamma, h_\Gamma) \textit{ groupoid morphism } \}$$

The set of ***path groupoid holonomies along germs*** *is defined by*

$$\mathcal{A} := \mathrm{Hom}(\mathcal{P}, G) = \{(\mathfrak{h}, h)| \quad \mathfrak{h} : \mathcal{P} \rightarrow G, h : \Sigma \rightarrow \{e_G\} \\ (\mathfrak{h}, h) \textit{ groupoid morphism } \}$$

The morphisms $\mathfrak{H}_{\Gamma,\Lambda}$, which are related to path connections Λ such that the conditions of tangency (iv) and additivity (v) are satisfied, are called **tangent groupoid morphisms**.

In particular, for those germs of a paths γ_0 and γ_1, which start at v and which satisfy

$$\frac{\mathrm{d}\,\gamma_1(t)}{\mathrm{d}\,t}\Big|_{t=t_0} = \frac{\mathrm{d}\,\gamma_0(t)}{\mathrm{d}\,t}\Big|_{t=t_0} \tag{3.75}$$

where $\gamma_0(t_0) = v$, the holonomy groupoid morphism has to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\gamma}_1(t,1)\Big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{h}_{\Gamma,\Lambda}(\gamma_1)\Big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\mathfrak{h}_{\Gamma,\Lambda}(\gamma_0)\Big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\gamma}_0(t,1)\Big|_{t=t_0} \tag{3.76}$$

Observe that there is a bijective correspondence between the path connection and the curvature given by

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\gamma}(t,1)\Big|_{t=t_0} = T(R_{\tilde{\gamma}(t,1)})\left(\gamma_A\Big(\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma(t)\Big)\right)\Big|_{t=t_0} \tag{3.77}$$

Definition 3.3.22. *Let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid modulo same-holonomy equivalence for all connections. The* ***holonomy maps along tangent germs for a Lie groupoid*** *$\mathcal{G}$ are defined by*

$$\mathcal{A}_{\breve{\Lambda},\Gamma,\mathcal{G}} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma, \mathcal{G}) = \{(\mathfrak{h}_{\Gamma,\Lambda}, h_{\Gamma,\Lambda})| \quad \mathfrak{h}_{\Gamma,\Lambda} : \mathcal{P}_\Gamma\Sigma \rightarrow \mathcal{G}, h_{\Gamma,\Lambda} : V_\Gamma \rightarrow \mathcal{G}^0 \\ (\mathfrak{h}_{\Gamma,\Lambda}, h_{\Gamma,\Lambda}) \textit{ is a tangent groupoid morphism} \\ \textit{associated to a path connection } \Lambda \textit{ in } \breve{\Lambda}\}$$

where $\breve{\Lambda}$ is the set of (analytic) path connections.

For $\mathcal{G}$ is equivalent to the Lie group G the ***holonomy maps along tangent germs*** *are denoted by*

$$\mathcal{A}_{\breve{\Lambda},\Gamma} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma, G) = \{(\mathfrak{h}_{\Gamma,\Lambda}, h_{\Gamma,\Lambda})| \quad \mathfrak{h}_{\Gamma,\Lambda} : \mathcal{P}_\Gamma\Sigma \rightarrow G, h_{\Gamma,\Lambda} : V_\Gamma \rightarrow \{e_G\} \\ (\mathfrak{h}_{\Gamma,\Lambda}, h_{\Gamma,\Lambda}) \textit{ is a tangent groupoid morphism} \\ \textit{associated to a path connection } \Lambda \textit{ in } \breve{\Lambda}\}$$

The ***source and target pointed holonomy maps along tangent germs for a Lie groupoid*** *$\mathcal{G}$ are defined by*

$$\mathcal{A}^{s(v)}_{\breve{\Lambda},\Gamma,\mathcal{G}} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma^{s(v)}, \mathcal{G}) \textit{ and } \mathcal{A}^{\breve{\Lambda},\Gamma,\mathcal{G}}_{t(w)} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma_{t(v)}\mathcal{G}).$$

For a Lie group G the ***source and target pointed holonomy maps along tangent germs*** *are denoted by*

$$\mathcal{A}^{s(v)}_{\breve{\Lambda},\Gamma} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma^{s(v)}, G) \textit{ and } \mathcal{A}^{\breve{\Lambda},\Gamma}_{t(w)} := \mathrm{Hom}_{\breve{\Lambda}}(\mathcal{P}_\Gamma\Sigma_{t(v)}, G).$$

3.3.4.3 Holonomy maps for finite graph systems

Ashtekar and Lewandowski [8] have presented the loop decomposition into a finite set of independent hoops (in the analytic category). This structure is replaced by a graph, since a graph is a set of independent edges. Notice that, the set of hoops that is generated by a finite set of independent hoops, is generalised to the set of finite graph systems. A finite path groupoid is generated by the set of edges, which defines a graph Γ, but a set of elements of the path groupoid need not be a graph again. The appropriate notion for graphs constructed from sets of paths is the finite graph system, which is defined in section 3.1.3. Now the concept of holonomy maps is generalised for finite graph systems. Since the set, which is generated by a finite number of independent edges, contains paths that are composable, there are two possibilities to identify the image of the holonomy map for a finite graph system on a fixed graph with a subgroup of $G^{|\Gamma|}$. One way is to use the generating set of independend edges of a graph, which has been also used in [8]. On the other hand, it is also possible to identify each graph with a disconnected subgraph of a fixed graph, which is generated by a set of independent edges. Notice that the author of this dissertation implements two situations. One case is given by a set of paths that can be composed further and the other case is related to paths that are not composable. This is necessary for the definition of an action of the flux operators. Precisely the identification of the image of the holonomy maps along these paths is necessary to define a well-defined action of a flux element on the configuration space. This issue will be studied in remark 6.1.2 in section 6.1.

Identifications of configuration spaces

First of all, consider a graph Γ that is generated by the set $\{\gamma_1, ..., \gamma_N\}$ of edges. Then each subgraph of a graph Γ contain paths that are composition of edges in $\{\gamma_1, ..., \gamma_N\}$ or inverse edges. For example the following set $\Gamma' := \{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\}$ defines a subgraph of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Hence, there is a natural identification available.

Definition 3.3.23. *A subgraph Γ' of a graph Γ is always generated by a subset $\{\gamma_1, ..., \gamma_M\}$ of the generating set $\{\gamma_1, ..., \gamma_N\}$ of independent edges that generates the graph Γ. Hence each subgraph is identified with a subset of $\{\gamma_1^{\pm 1}, ..., \gamma_N^{\pm 1}\}$. This is called the* ***natural identification of subgraphs****.*

Example 3.3.1: For example, consider a subgraph $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, ..., \gamma_{M-1} \circ \gamma_M\}$, which is identified naturally with a set $\{\gamma_1, ..., \gamma_M\}$. The set $\{\gamma_1, ..., \gamma_M\}$ is a subset of $\{\gamma_1, ..., \gamma_N\}$ where $N = |\Gamma|$ and $M \leq N$.

Another example is given by the graph $\Gamma'' := \{\gamma_1, \gamma_2\}$ such that $\gamma_2 = \gamma_1' \circ \gamma_2'$, then Γ'' will be identified naturally with $\{\gamma_1, \gamma_1', \gamma_2'\}$. This set is a subset of $\{\gamma_1, \gamma_1', \gamma_2', \gamma_3, ..., \gamma_{N-1}\}$.

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Definition 3.3.24. *Let Γ be a graph, $\mathcal{P}_\Gamma$ be the finite graph system. Let $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ be a subgraph of Γ.*

A ***holonomy map for a finite graph system*** *$\mathcal{P}_\Gamma$ is a given by a pair of maps $(\mathfrak{h}_\Gamma, h_\Gamma)$ such that there exists a holonomy map[2] $(\mathfrak{h}_\Gamma, h_\Gamma)$ for the finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ and*

$$\begin{aligned} &\mathfrak{h}_\Gamma : \mathcal{P}_\Gamma \rightarrow G^{|\Gamma|}, \quad \mathfrak{h}_\Gamma(\{\gamma_1, ..., \gamma_M\}) = (\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_M), e_G, ..., e_G) \\ &h_\Gamma : V_\Gamma \rightarrow \{e_G\} \end{aligned}$$

The set of all holonomy maps for the finite graph system is denoted by $\mathrm{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|})$.

The image of a map $\mathfrak{h}_\Gamma$ on each subgraph Γ' of the graph Γ is given by

$$(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_M), e_G, ..., e_G)$$

is an element of $G^{|\Gamma|}$. The set of all images of maps on subgraphs of Γ is denoted by $\bar{\mathcal{A}}_\Gamma$.

The idea is now to study two different restrictions of the set $\mathcal{P}_\Gamma$ of subgraphs. For a short notation of a "set of holonomy maps for a certain restricted set of subgraphs of a graph" in this dissertation the following notions are introduced.

[2] In the work the holonomy map for a finite graph system and the holonomy map for a finite path groupoid is denoted by the same pair $(\mathfrak{h}_\Gamma, h_\Gamma)$.

Definition 3.3.25. *If the subset of all disconnected subgraphs of the finite graph system $\mathcal{P}_\Gamma$ is considered, then the restriction of $\bar{\mathcal{A}}_\Gamma$, which is identified with $G^{|\Gamma|}$ appropriately, is called the* ***non-standard identification of the configuration space****. If the subset of all natural identified subgraphs of the finite graph system $\mathcal{P}_\Gamma$ is considered, then the restriction of $\bar{\mathcal{A}}_\Gamma$, which is identified with $G^{|\Gamma|}$ appropriately, is called the* ***natural identification of the configuration space****.*

A comment on the non-standard identification of $\bar{\mathcal{A}}_\Gamma$ is the following. If $\Gamma' := \{\gamma_1 \circ \gamma_2\}$ and $\Gamma'' := \{\gamma_2\}$ are two subgraphs of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3\}$. The graph Γ' is a subgraph of Γ. Then evaluation of a map $\mathfrak{h}_\Gamma$ on a subgraph Γ' is given by

$$\mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1 \circ \gamma_2), \mathfrak{h}_\Gamma(s(\gamma_2)), \mathfrak{h}_\Gamma(s(\gamma_3))) = (\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_2), e_G, e_G) \in G^3$$

and the holonomy map of the subgraph Γ'' of Γ' is evaluated by

$$\mathfrak{h}_\Gamma(\Gamma'') = (\mathfrak{h}_\Gamma(s(\gamma_1)), \mathfrak{h}_\Gamma(s(\gamma_2))\mathfrak{h}_\Gamma(\gamma_2), \mathfrak{h}_\Gamma(s(\gamma_3))) = (\mathfrak{h}_\Gamma(\gamma_2), e_G, e_G) \in G^3$$

Example 3.3.2: Recall example 3.3.4.1. For example for a subgraph $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3 \circ \gamma_4, ..., \gamma_{M-1} \circ \gamma_M\}$, which is naturally identified with a set $\{\gamma_1, ..., \gamma_M\}$. Then the holonomy map is evaluated at Γ' such that

$$\mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1), \mathfrak{h}_\Gamma(\gamma_2), ..., \mathfrak{h}_\Gamma(\gamma_M), e_G, ..., e_G) \in G^N$$

where $N = |\Gamma|$. For example, let $\Gamma' := \{\gamma_1, \gamma_2\}$ such that $\gamma_2 = \gamma_1' \circ \gamma_2'$ and which is naturally identified with $\{\gamma_1, \gamma_1', \gamma_2'\}$. Hence,

$$\mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1), \mathfrak{h}_\Gamma(\gamma_1'), \mathfrak{h}_\Gamma(\gamma_2'), e_G, ..., e_G) \in G^N$$

is true.

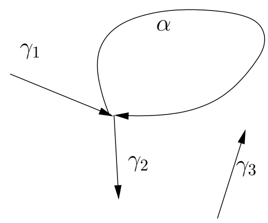
Another example is given by the disconnected graph $\Gamma' := \{\gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4\}$, which is a subgraph of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Then the non-standard identification is given by

$$\mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1 \circ \gamma_2 \circ \gamma_3), \mathfrak{h}_\Gamma(\gamma_4), e_G, e_G) \in G^4$$

If the natural identification would be used, then $\mathfrak{h}_\Gamma(\Gamma')$ is idenified with

$$(\mathfrak{h}_\Gamma(\gamma_1), \mathfrak{h}_\Gamma(\gamma_2), \mathfrak{h}_\Gamma(\gamma_3), \mathfrak{h}_\Gamma(\gamma_4)) \in G^4$$

Consider the following example. Let $\Gamma''' := \{\gamma_1, \alpha, \gamma_2, \gamma_3\}$ be a graph such that



Then notice the sets $\Gamma_1 := \{\gamma_1 \circ \alpha, \gamma_3\}$ and $\Gamma_2 := \{\gamma_1 \circ \alpha^{-1}, \gamma_3\}$. In the non-standard identification of the configuration space $\bar{\mathcal{A}}_{\Gamma'''}$ it is true that

$$\mathfrak{h}_{\Gamma'''}(\Gamma_1) = (\mathfrak{h}_{\Gamma'''}(\gamma_1 \circ \alpha), \mathfrak{h}_{\Gamma'''}(\gamma_3), e_G, e_G) \in G^4,$$
$$\mathfrak{h}_{\Gamma'''}(\Gamma_2) = (\mathfrak{h}_{\Gamma'''}(\gamma_1 \circ \alpha^{-1}), \mathfrak{h}_{\Gamma'''}(\gamma_3), e_G, e_G) \in G^4$$

Whereas in the natural identification of $\bar{\mathcal{A}}_{\Gamma'''}$

$$\mathfrak{h}_{\Gamma'''}(\Gamma_1) = (\mathfrak{h}_{\Gamma'''}(\gamma_1), \mathfrak{h}_{\Gamma'''}(\alpha), \mathfrak{h}_{\Gamma'''}(\gamma_3), e_G) \in G^4,$$
$$\mathfrak{h}_{\Gamma'''}(\Gamma_2) = (\mathfrak{h}_{\Gamma'''}(\gamma_1), \mathfrak{h}_{\Gamma'''}(\alpha^{-1}), \mathfrak{h}_{\Gamma'''}(\gamma_3), e_G) \in G^4$$

holds.

■

Discretised surface sets and localised paths

Recall the situation of discretised surfaces and paths starting or ending at these surfaces. Then the following abbreviations are useful. Let $\tilde{\Gamma}$ be a graph and $\breve{S}_{\rm d}$ be a discretised surface set. Then recall the subgraph $\bar{\Gamma}$ of $\tilde{\Gamma}$ and the subgraph Γ. Then one distinguishes between a set of disconnected subgraphs of Γ and $\bar{\Gamma}$.

If the subset of all disconnected subgraphs of the finite graph system $\mathcal{P}_\Gamma$, that contain paths starting (resp. ending) at $\breve{S}_{\rm d}$, is considered, then the restriction of $\bar{\mathcal{A}}^{\rm d}_\Gamma$ (resp. $\bar{\mathcal{A}}^\Gamma_{\rm d}$), which is identified with $G^{|\Gamma|}$ appropriately, is called the non-standard identification of the configuration space $\bar{\mathcal{A}}^{\rm d}_\Gamma$. In analogy if the subset of all disconnected subgraphs of the finite graph system $\mathcal{P}_{\bar{\Gamma}}$ is concerned, then the restriction of $\bar{\mathcal{A}}_{\bar{\Gamma}}$, which is identified with $G^{|\Gamma|}$ appropriately, is called non-standard, too.

The detailed structure of one component is presented by the product space

$$\bar{\mathcal{A}}^{\rm d}_\Gamma = \bigotimes_{i\in I} \bigotimes_{k=1,...,N^i_k} \bar{\mathcal{A}}^{\rm d}_{\gamma_{i,1}\circ ...\circ\gamma_{i,k}},$$

where I is the number of independent edges of Γ starting at a vertex $v \in \breve{S}_{\rm d}$, and N^i_k is the number of composed paths in $\mathcal{P}^{\breve{S}_{\rm d}}_\Gamma\Sigma$, which are of the form $\gamma_{i,1}$, $\gamma_{i,1}\circ\gamma_{i,2}$ and so on. Recognize that, $\bar{\mathcal{A}}^{\rm d}_\gamma$ denotes the set of images of the maps in $\mathrm{Hom}(\mathcal{P}^{\breve{S}_{\rm d}}_\Gamma\Sigma, G)$ for a fixed path γ in Γ. An element of $\bar{\mathcal{A}}^{\rm d}_\gamma$ is given by $(\mathfrak{h}(\gamma), e_G, ..., e_G)$, which is an element of $G^{|\Gamma|}$ by the non-standard identification.

Admissable maps and equivalent groupoid morphisms

Finally the equivalence class of similar or equivalent groupoid morphisms defined in definition 3.3.19 allows to define the following object. The set of images of all holonomy maps of a finite graph system modulo the similar or equivalent groupoid morphisms equivalence relation is denoted by $\bar{\mathcal{A}}_\Gamma/\bar{\mathfrak{G}}_\Gamma$.

3.3.4.4 Transformations in finite path groupoids and finite graph systems

The aim of this section is to clearify the graph changing operators in LQG framework and the role of finite diffeomorphisms in Σ. Therefore operations, which add, delete or transform paths, are introduced. In particular, translations in a finite path graph groupoid and in the groupoid G over $\{e_G\}$ are studied.

Transformations in finite path groupoid

Definition 3.3.26. *Let φ be a C^k-diffeomorphism on Σ, which maps surfaces into surfaces.*

Then let $(\Phi_\Gamma, \varphi_\Gamma)$ be a pair of bijective maps, where $\varphi|_{V_\Gamma} = \varphi_\Gamma$ and

$$\Phi_\Gamma : \mathcal{P}_\Gamma\Sigma \to \mathcal{P}_\Gamma\Sigma \text{ and } \varphi_\Gamma : V_\Gamma \to V_\Gamma \tag{3.78}$$

such that

$$(s\circ\Phi_\Gamma)(\gamma) = (\varphi_\Gamma\circ s)(\gamma), \quad (t\circ\Phi_\Gamma)(\gamma) = (\varphi_\Gamma\circ t)(\gamma) \text{ for all } \gamma\in\mathcal{P}_\Gamma\Sigma \tag{3.79}$$

holds such that $(\Phi_\Gamma, \varphi_\Gamma)$ defines a groupoid morphism.

*Call the pair $(\Phi_\Gamma, \varphi_\Gamma)$ a **path-diffeomorphism of a finite path groupoid** $\mathcal{P}_\Gamma\Sigma$ over V_Γ. Denote the set of finite path-diffeomorphisms by* $\mathrm{Diff}(\mathcal{P}_\Gamma\Sigma)$.

Notice that for $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ it is true that

$$\Phi_\Gamma(\gamma\circ\gamma') = \Phi_\Gamma(\gamma)\circ\Phi_\Gamma(\gamma') \tag{3.80}$$

requires that

$$(t \circ \Phi_\Gamma)(\gamma) = (s \circ \Phi_\Gamma)(\gamma') \tag{3.81}$$

Hence, from (3.80) and (3.81) it follows that $\Phi_\Gamma(\mathbb{1}_v) = \mathbb{1}_{\varphi_\Gamma(v)}$.

A path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$ can be lifted to $\text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$.
The pair $(\mathfrak{h}_\Gamma \circ \Phi_\Gamma, h_\Gamma \circ \varphi_\Gamma)$ defined by

$$\begin{aligned}
&\mathfrak{h}_\Gamma \circ \Phi_\Gamma : \mathcal{P}_\Gamma\Sigma \to G, \quad \gamma \mapsto (\mathfrak{h}_\Gamma \circ \Phi_\Gamma)(\gamma)\\
&h_\Gamma \circ \varphi_\Gamma : V_\Gamma \to \{e_G\}, \quad (h_\Gamma \circ \varphi_\Gamma)(v) = e_G
\end{aligned}$$

and

$$\begin{aligned}
&s_{\text{Hol}}((\mathfrak{h}_\Gamma \circ \Phi_\Gamma)(\gamma)) = (h_\Gamma \circ \varphi_\Gamma)(s(\gamma)) = e_G,\\
&t_{\text{Hol}}(\mathfrak{h}_\Gamma \circ \Phi_\Gamma(\gamma)) = (h_\Gamma \circ \varphi_\Gamma)(t(\gamma)) = e_G \text{ for all } \gamma \in \mathcal{P}_\Gamma\Sigma
\end{aligned}$$

where $(\mathfrak{h}_\Gamma, h_\Gamma) \in \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$ and a path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$, is a holonomy map for a finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ.

Definition 3.3.27. *A* ***left-translation in the finite path groupoid*** $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *at a vertex* v *is a map*

$$L_\theta : \mathcal{P}_\Gamma\Sigma^v \to \mathcal{P}_\Gamma\Sigma^w, \quad \gamma \mapsto L_\theta(\gamma) := \theta \circ \gamma$$

for some $\theta \in \mathcal{P}_\Gamma\Sigma^w_v$ *and all* $\gamma \in \mathcal{P}_\Gamma\Sigma^v$.

In analogy a right-translation R_θ and an inner-translation $I_{\theta,\theta'}$ in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ at a vertex v can be defined.

Remark 3.3.28. *Let* $(\Phi_\Gamma, \varphi_\Gamma)$ *be a path-diffeomorphism on a finite path groupoid* $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ*. Then a left-translation in the finite path groupoid* $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *at a vertex* v *is defined by a path-diffeomorphism* $(\Phi_\Gamma, \varphi_\Gamma)$ *and by the following object*

$$L_{\Phi_\Gamma} : \mathcal{P}_\Gamma\Sigma^v \to \mathcal{P}_\Gamma\Sigma^{\varphi_\Gamma(v)}, \quad \gamma \mapsto L_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma) \text{ for } \gamma \in \mathcal{P}_\Gamma\Sigma^v \tag{3.82}$$

Furthermore, a right-translation in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *at a vertex* v *is defined by a path-diffeomorphism* $(\Phi_\Gamma, \varphi_\Gamma)$ *and by the following object*

$$R_{\Phi_\Gamma} : \mathcal{P}_\Gamma\Sigma_v \to \mathcal{P}_\Gamma\Sigma_{\varphi_\Gamma(v)}, \quad \gamma \mapsto R_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma) \text{ for } \gamma \in \mathcal{P}_\Gamma\Sigma_v \tag{3.83}$$

Finally, an inner-translation in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *at the vertices* v *and* w *is defined by*

$$I_{\Phi_\Gamma} : \mathcal{P}_\Gamma\Sigma^v_w \to \mathcal{P}_\Gamma\Sigma^{\varphi_\Gamma(v)}_{\varphi_\Gamma(w)}, \quad \gamma \mapsto I_{\Phi_\Gamma}(\gamma) = \Phi_\Gamma(\gamma) \text{ for } \gamma \in \mathcal{P}_\Gamma\Sigma^v_w$$

where $(s \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(v)$ *and* $(t \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(w)$.

In the following considerations the right-translation in a finite path groupoid is focused, but there is a generalisation to left-translations and inner-translations.

Definition 3.3.29. *A* ***bisection of a finite path groupoid*** $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *is a map* $\sigma : V_\Gamma \to \mathcal{P}_\Gamma\Sigma$*, which is right-inverse to the map* $s : \mathcal{P}_\Gamma\Sigma \to V_\Gamma$ *(i.o.w.* $s \circ \sigma = \text{id}_{V_\Gamma}$*) and such that* $t \circ \sigma : V_\Gamma \to V_\Gamma$ *is a bijective map*[3]*. The set of bisections on* $\mathcal{P}_\Gamma\Sigma$ *over* V_Γ *is denoted* $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$.

Remark 3.3.30. *Discover that a bisection* $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ *defines a path-diffeomorphism* $(\varphi_\Gamma, \Phi_\Gamma) \in \text{Diff}(\mathcal{P}_\Gamma\Sigma)$*, where* $\varphi_\Gamma = t \circ \sigma$ *and* Φ_Γ *is given by the right-translation* $R_{\sigma(v)} : \mathcal{P}_\Gamma\Sigma_v \to \mathcal{P}_\Gamma\Sigma_{\varphi_\Gamma(v)}$ *in* $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$*, where* $R_{\sigma(v)}(\gamma) = \Phi_\Gamma(\gamma)$ *for all* $\gamma \in \mathcal{P}_\Gamma\Sigma_v$*, for a fixed* $v \in V_\Gamma$*. The right-translation is defined by*

$$R_{\sigma(v)}(\gamma) := \begin{cases} \gamma \circ \sigma(v) & v = t(\gamma) \\ \gamma \circ \mathbb{1}_{t(\gamma)} & v \neq t(\gamma) \end{cases} \tag{3.84}$$

whenever $t(\gamma)$ *is the target vertex of a non-trivial path* γ *in* Γ*. For a trivial path* $\mathbb{1}_v$ *the right-translation is defined by* $R_{\sigma(v)}(\mathbb{1}_v) = \mathbb{1}_{(t\circ\sigma)(v)}$ *and* $R_{\sigma(v)}(\mathbb{1}_w) = \mathbb{1}_w$ *whenever* $v \neq w$*. The right-translation* $R_{\sigma(v)}$ *is required to be bijective. Before this result is proven in lemma 3.3.33 notice the following considerations.*

[3]Note that in the infinite case of path groupoids an additional condition for the map $t \circ \sigma : \Sigma \to \Sigma$ has to be required. The map has to be a diffeomorphism. Observe that the map $t \circ \sigma$ defines the finite diffeomorphism $\varphi_\Gamma : V_\Gamma \to V_\Gamma$.

Note that $(R_{\sigma(v)}, t\circ\sigma)$ transfers to the holonomy map such that

$$\begin{aligned}(\mathfrak{h}_\Gamma \circ R_{\sigma(t(\gamma'))})(\gamma\circ\gamma') &= \mathfrak{h}_\Gamma(\gamma\circ\gamma'\circ\sigma(t(\gamma')))\\ &= \mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'\circ\sigma(t(\gamma')))\end{aligned} \tag{3.85}$$

is true. There is a bijective map between a right-translation $R_{\sigma(v)}:\mathcal{P}_\Gamma\Sigma_v\rightarrow\mathcal{P}_\Gamma\Sigma_{(t\circ\sigma)(v)}$ and a path-diffeomorphism $(\varphi_\Gamma,\Phi_\Gamma)$. In particular, observe that $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma_v)$ and $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma\Sigma_v)$. Simply speaking the path-diffeomorphism does not change the source and target vertex at the same time. The path-diffomorphism changes the target vertex by a (finite) diffeomorphism and, therefore, the path is transformed.

Bisections σ in a finite path groupoid can be transfered, likewise path-diffeomorphisms, to holonomy maps. The pair $(\mathfrak{h}_\Gamma\circ\Phi_\Gamma, h_\Gamma\circ\varphi_\Gamma)$ of the maps defines a pair of maps $(\mathfrak{h}_\Gamma\circ\Phi_\Gamma, h_\Gamma\circ\varphi_\Gamma)$ by

$$\mathfrak{h}_\Gamma\circ\Phi_\Gamma:\mathcal{P}_\Gamma\Sigma_v\rightarrow G \text{ and } h_\Gamma\circ\varphi_\Gamma: V_\Gamma\rightarrow\{e_G\} \tag{3.86}$$

which is a holonomy map for a finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ.

Lemma 3.3.31. *The set $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ of bisections on the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ forms a group.*

Proof : The group multiplication is given by

$$(\sigma*\sigma')(v)=\sigma'(v)\circ\sigma(t(\sigma'(v)))\text{ for } v\in V_\Gamma$$

whenever $\sigma'(v)\in\mathcal{P}_\Gamma\Sigma^v_{\varphi'_\Gamma(v)}$ and $\sigma(t(\sigma'(v)))\in\mathcal{P}_\Gamma\Sigma^{(t\circ\sigma')(v)}_{\varphi_\Gamma(v)}$.

Clearly, the group multiplication is associative. The unit id is equivalent to the object inclusion $v\mapsto\mathbb{1}_v$ of the groupoid $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$, where $\mathbb{1}_v$ is the constant loop at v, and the inversion is given by

$$\sigma^{-1}(v)=\sigma((t\circ\sigma)^{-1}(v))^{-1}\text{ for } v\in V_\Gamma$$

and

$$(\sigma*\sigma^{-1})(v)=\sigma(v)\circ\sigma((t\circ\sigma)^{-1}(v))^{-1}\text{ for } v\in V_\Gamma$$

■

The group property of bisections $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ carry over to holonomy maps. Using the group multiplication $\cdot$ of G conclude that

$$(\mathfrak{h}_\Gamma\circ R_{(\sigma*\sigma')(v)})(\mathbb{1}_v)=\mathfrak{h}_\Gamma\circ(R_{\sigma'(v)}\circ R_{\sigma(t(\sigma'(v)))})(\mathbb{1}_v)=\mathfrak{h}_\Gamma(\sigma'(v))\cdot\mathfrak{h}_\Gamma(\sigma(t(\sigma'(v))))\text{ for } v\in V_\Gamma$$

is true.

Remark 3.3.32. *Moreover, right-translations define path-diffeomorphisms, i.e. $R_{(\sigma)(v)}=\Phi_\Gamma$ and $\varphi_\Gamma=t\circ\sigma$ whenever $v\in V_\Gamma$. But for two bisections $\sigma_{\Gamma'},\breve\sigma_{\Gamma'}\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ the object $\sigma_{\Gamma'}(v)\circ\breve\sigma_{\Gamma'}(v)$ is not comparable with $(\sigma_{\Gamma'}*\breve\sigma_{\Gamma'})(v)$. Then for the composition $\Phi_1(\gamma)\circ\Phi_2(\gamma)$ there exists no path-diffeomorphism Φ such that $\Phi_1(\gamma)\circ\Phi_2(\gamma)=\Phi(\gamma)$ yields in general. Moreover, generally the object $\Phi_1(\gamma)\circ\Phi_2(\gamma')=\Phi(\gamma\circ\gamma')$ is not well-defined.*

But the following is defined

$$R_{(\sigma*\sigma')(v)}(\gamma)=\Phi'_\Gamma(\gamma)\circ\Phi_\Gamma(\mathbb{1}_{\varphi'_\Gamma(v)})=:(\Phi'_\Gamma*\Phi_\Gamma)(\gamma) \tag{3.87}$$

whenever $\gamma\in\mathcal{P}_\Gamma\Sigma_v$, $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma\Sigma_v)$ and $(\varphi'_\Gamma,\Phi'_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma\Sigma_{\varphi'_\Gamma(v)})$ are path-diffeomorphisms such that $\varphi_\Gamma=t\circ\sigma$, $\Phi_\Gamma=R_{\sigma(\varphi'_\Gamma(v))}$ and $\varphi'_\Gamma=t\circ\sigma'$, $\Phi'_\Gamma=R_{\sigma'(v)}$.

Moreover, for $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ and $\gamma'\in\mathcal{P}_\Gamma\Sigma_v$ it is true that

$$(\Phi'_\Gamma*\Phi_\Gamma)(\gamma\circ\gamma')=\Phi'_\Gamma(\gamma\circ\gamma')\circ\Phi_\Gamma(\mathbb{1}_{\varphi'_\Gamma(v)})=\Phi'_\Gamma(\gamma)\circ\Phi'_\Gamma(\gamma')\circ\Phi_\Gamma(\mathbb{1}_{\varphi'_\Gamma(v)})=\Phi'_\Gamma(\gamma)\circ(\Phi'_\Gamma*\Phi_\Gamma)(\gamma')$$

Then the following lemma easily follows.

Lemma 3.3.33. *Let σ be a bisection contained in $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ and $v\in V_\Gamma$.*
The pair $(R_{\sigma(v)},t\circ\sigma)$ of maps such that

$$\begin{aligned}R_{\sigma(v)}&:\mathcal{P}_\Gamma\Sigma_v\rightarrow\mathcal{P}_\Gamma\Sigma_{(t\circ\sigma)(v)}, & s\circ R_{\sigma(v)}&=(t\circ\sigma)\circ s\\ t\circ\sigma&:V_\Gamma\rightarrow V_\Gamma, & t\circ R_{\sigma(v)}&=(t\circ\sigma)\circ t\end{aligned}$$

defined in remark 3.3.30 is a path-diffeomorphism in $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$.

Proof : This follows easily from the derivation

$$R_{\sigma(t(\gamma'))}(\gamma\circ\gamma')=\gamma\circ\gamma'\circ\sigma(t(\gamma'))=R_{\sigma(t(\gamma'))}(\gamma)\circ R_{\sigma(t(\gamma'))}(\gamma')$$

$$\begin{aligned}R_{\sigma(t(\gamma))}(\mathbb{1}_{s(\gamma)}\circ\gamma)&=R_{\sigma(t(\gamma))}(\mathbb{1}_{s(\gamma)})\circ R_{\sigma(t(\gamma))}(\gamma)=\mathbb{1}_{s(\gamma)}\circ\gamma\circ\sigma(t(\gamma))\\ R_{\sigma(t(\gamma))}(\gamma\circ\mathbb{1}_{t(\gamma)})&=R_{\sigma(t(\gamma))}(\gamma)\circ R_{\sigma(t(\gamma))}(\mathbb{1}_{t(\gamma)})=\gamma\circ\sigma(t(\gamma))\circ\mathbb{1}_{(t\circ\sigma)(t(\gamma))}\end{aligned}$$

The inverse map satisfies

$$R^{-1}_{\sigma(v)}(\gamma\circ\sigma(v))=R_{\sigma^{-1}(v)}(\gamma\circ\sigma(v))=\gamma\circ\sigma(v)\circ\sigma^{-1}(v)=\gamma$$

whenever $v=t(\gamma)$,

$$R^{-1}_{\sigma(v)}(\gamma)=\gamma$$

whenever $v\neq t(\gamma)$ and

$$R^{-1}_{\sigma(v)}(\mathbb{1}_{(t\circ\sigma)(v)})=\mathbb{1}_v$$

Moreover, derive

$$(s\circ R_{\sigma(v)})(\gamma')=((t\circ\sigma)\circ s)(\gamma')$$

for all $\gamma'\in\mathcal{P}_\Gamma\Sigma_v$ and a fixed bisection $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$.

■

Notice that $L_{\sigma(v)}$ and $I_{\sigma(v)}$ similarly to the pair $(R_{\sigma(v)},t\circ\sigma)$ can be defined. Summarising, the pairs $(R_{\sigma(v)},t\circ\sigma)$, $(L_{\sigma(v)},t\circ\sigma)$ and $(I_{\sigma(v)},t\circ\sigma)$ for a bisection $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ are path-diffeomorphisms of a finite path groupoid $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$.

In general, a right-translation $(R_\sigma,t\circ\sigma)$ in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over Σ for a bisection $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ is defined by the bijective maps R_σ and $t\circ\sigma$, which are given by

$$\begin{aligned}R_\sigma&:\mathcal{P}_\Gamma\Sigma\rightarrow\mathcal{P}_\Gamma\Sigma, & s\circ R_\sigma&=s\\ t\circ\sigma&:V_\Gamma\rightarrow V_\Gamma, & t\circ R_\sigma&=(t\circ\sigma)\circ t\\ R_\sigma(\gamma)&:=\gamma\circ\sigma(t(\gamma))\quad\forall\gamma\in\mathcal{P}_\Gamma\Sigma;\quad R^{-1}_\sigma:=R_{\sigma^{-1}}\end{aligned}$$

For example, for a fixed suitable bisection σ the right-translation is $R_\sigma(\mathbb{1}_v)=\gamma$, then $R^{-1}_\sigma(\gamma)=\gamma\circ\gamma^{-1}=\mathbb{1}_v$ for $v=s(\gamma)$. Clearly, the right-translation $(R_\sigma,t\circ\sigma)$ is not a groupoid morphism in general.

Definition 3.3.34. *Define, for a given bisection σ in $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$, the* ***right-translation in the groupoid** G **over*** *$\{e_G\}$ through*

$$\begin{aligned}&\mathfrak{h}_\Gamma\circ R_\sigma:\mathcal{P}_\Gamma\Sigma\rightarrow G,\quad\gamma\mapsto(\mathfrak{h}_\Gamma\circ R_\sigma)(\gamma):=\mathfrak{h}_\Gamma(\gamma\circ\sigma(t(\gamma)))=\mathfrak{h}_\Gamma(\gamma)\cdot\mathfrak{h}_\Gamma(\sigma(t(\gamma)))\\ &h_\Gamma\circ t\circ\sigma:V_\Gamma\rightarrow e_G\end{aligned}$$

Furthermore, for a fixed $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ define the ***left-translation in the groupoid** G **over*** *$\{e_G\}$ by*

$$\begin{aligned}&\mathfrak{h}_\Gamma\circ L_\sigma:\mathcal{P}_\Gamma\Sigma\rightarrow G,\quad\gamma\mapsto\mathfrak{h}_\Gamma(\sigma((t\circ\sigma)^{-1}(t(\gamma)))\circ\gamma)=\mathfrak{h}_\Gamma(\sigma((t\circ\sigma)^{-1}(t(\gamma))))\cdot\mathfrak{h}_\Gamma(\gamma)\\ &h_\Gamma\circ t\circ\sigma:V_\Gamma\rightarrow e_G\end{aligned}$$

and the ***inner-translation in the groupoid** G **over*** *$\{e_G\}$*

$$\begin{aligned}&\mathfrak{h}_\Gamma\circ I_\sigma:\mathcal{P}_\Gamma\Sigma\rightarrow G,\quad\gamma\mapsto\mathfrak{h}_\Gamma(\sigma((t\circ\sigma)^{-1}(t(\gamma)))\circ\gamma\circ\sigma(t(\gamma)))=\mathfrak{h}_\Gamma(\sigma((t\circ\sigma)^{-1}(t(\gamma))))\cdot\mathfrak{h}_\Gamma(\gamma)\cdot\mathfrak{h}_\Gamma(\sigma(t(\gamma)))\\ &h_\Gamma\circ t\circ\sigma:V_\Gamma\rightarrow e_G\end{aligned}$$

such that $I_\sigma=L_{\sigma^{-1}}\circ R_\sigma$.

The pairs $(R_\sigma, t\circ\sigma)$ and $(L_\sigma, t\circ\sigma)$ are not groupoid morphisms. Whereas the pair $(I_\sigma, t\circ\sigma)$ is a groupoid morphism, since, for all pairs $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ such that $t(\gamma) = s(\gamma')$ it is true that $\sigma(t(\gamma)) \circ \sigma((t\circ\sigma)^{-1}(t(\gamma)))^{-1} = \mathbb{1}_{t(\gamma)}$. Notice that, in this situation $\sigma(t(\gamma)) = \sigma(t(\gamma\circ\gamma'))$ is satisfied.

Proposition 3.3.35. *The map $\sigma \mapsto R_\sigma$ is a group isomorphism, i.e. $R_{\sigma*\sigma'} = R_\sigma \circ R_{\sigma'}$. Where $\sigma \mapsto t\circ\sigma$ is a group isomorphism from $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ to the group of finite diffeomorphisms $\mathrm{Diff}(V_\Gamma)$ in a finite subset V_Γ of Σ.*

The maps $\sigma \mapsto L_\sigma$ and $\sigma \mapsto I_\sigma$ are group isomorphisms.

There is a generalisation of path-diffeomorphisms in the finite path groupoid, which coincide with the graphomorphism presented by Fleischhack in [39]. In this approach the diffeomorphism $\varphi : \Sigma \to \Sigma$ changes the source and target vertex of a path γ. Consequently, the path-diffeomorphism (Φ,φ), which implements the inner-translation I_Φ in the path groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$, is a graphomorphism in the context of Fleischhack. Some element of the set of graphomorphisms is directly related to a right-translation R_σ in the path groupoid. Precisely, for every $v\in\Sigma$ and $\sigma \in \mathfrak{B}(\mathcal{P}\Sigma)$ the pairs $(R_{\sigma(v)}, t\circ\sigma)$, $(L_{\sigma(v)}, t\circ\sigma)$ and $(I_{\sigma(v)}, t\circ\sigma)$ define graphomorphism. Furthermore, the right-translation $R_{\sigma(v)}$, the left-translation $L_{\sigma(v)}$ and the inner-translation $I_{\sigma(v)}$ are required to be bijective maps, and hence the maps cannot map non-trivial paths to trivial paths. This property restricts the set of all graphomorphism, which is generated by these translations. In particular, in this dissertation graph changing operations, which change the number of edges of a graph, will be studied. Hence, the left- or right-translation in a finite path groupoid will be used in the further development. Notice that in general, these objects do not define graphomorphism. Finally, notice that, in particular for the graphomorphism $(R_{\sigma(v)}, t\circ\sigma)$ and a holonomy map for the path groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$ a similar relation (3.85) holds. The last equation is fundamental for the construction of C^*-dynamical systems, which contain the analytic holonomy C^*-algebra restricted to a finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ and a point-norm continuous action of the finite path-diffeomorphism group $\mathrm{Diff}(V_\Gamma)$ on this algebra. Clearly, the right-, left- and inner-translations R_σ, L_σ and I_σ are constructed such that (3.85) generalises. But note that in the infinite case, considered by Fleischhack, the action of the bisections $\mathfrak{B}(\mathcal{P}\Sigma)$ are not point-norm continuous implemented. The advantage of the usage of bisections is that the map $\sigma \mapsto t\circ\sigma$ is a group morphism between the group $\mathfrak{B}(\mathcal{P}\Sigma)$ of bisections in $\mathcal{P}\Sigma \rightrightarrows \Sigma$ and the group $\mathrm{Diff}(\Sigma)$ of diffeomorphisms in Σ. Consequently, there is an action of the group of diffeomorphisms in Σ on the finite path groupoid, which can be used to define an action of the group of diffemorphisms in Σ on the analytic holonomy C^*-algebra.

Transformations in finite graph systems

To proceed, it is necessary to transfer the notion of bisections and right-translations to finite graph systems. A right-translation R_{σ_Γ} is a mapping that maps graphs to graphs. Each graph is a finite union of independent edges. This causes problems. Since, often the definition of right-translation in a finite graph system $\mathcal{P}_\Gamma$ is not well-defined for all bisections in the finite graph system and all graphs. For example, if the graph $\Gamma := \{\gamma_1,\gamma_2\}$ is disconnected and the bisection $\tilde\sigma$ in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ is defined by $\tilde\sigma(s(\gamma_1)) = \gamma_1$, $\tilde\sigma(s(\gamma_2)) = \gamma_2$, $\tilde\sigma(t(\gamma_1)) = \gamma_1^{-1}$ and $\tilde\sigma(t(\gamma_2)) = \gamma_2^{-1}$ where $V_\Gamma := \{s(\gamma_1), t(\gamma_1), s(\gamma_2), t(\gamma_2)\}$. Let $\mathbb{1}_\Gamma$ be the set given by the elements $\mathbb{1}_{s(\gamma_1)}$,$\mathbb{1}_{s(\gamma_2)}$,$\mathbb{1}_{t(\gamma_1)}$ and $\mathbb{1}_{t(\gamma_2)}$. Then notice that a bisection σ_Γ, which maps a set of vertices in V_Γ to a set of paths in $\mathcal{P}_\Gamma\Sigma$, is given for example by $\sigma_\Gamma(V_\Gamma) = \{\gamma_1,\gamma_2,\gamma_1^{-1},\gamma_2^{-1}\}$. In this case, the right-translation $R_{\sigma_\Gamma(V_\Gamma)}(\mathbb{1}_\Gamma)$ is equivalent to $\{\gamma_1,\gamma_2,\gamma_1^{-1},\gamma_2^{-1}\}$, which is not a set of independent edges and, hence, not a graph. Loosely speaking the graph-diffeomorphism acts on all vertices in the set V_Γ and, hence, implement four new edges. But a bisection σ_Γ, which maps a subset $V := \{s(\gamma_1), s(\gamma_2)\}$ of V_Γ to a set of paths, lead to a translation $R_{\sigma_\Gamma(V)}(\{\mathbb{1}_{s(\gamma_1)}, \mathbb{1}_{s(\gamma_2)}\}) = \{\gamma_1,\gamma_2\}$, which is indeed a graph. Set $\Gamma' := \{\gamma_1\}$ and $V' = \{s(\gamma_1)\}$. Then observe that for a restricted bisection, which maps a set V' of vertices in V_Γ to a set of paths in $\mathcal{P}_{\Gamma'}\Sigma$, the right-translation become $R_{\sigma_{\Gamma'}(V')}(\{\mathbb{1}_{s(\gamma_1)}\}) = \{\gamma_1\}$, which defines a graph, too. Notice that $\mathbb{1}_{s(\gamma_1)}$ is a subgraph of Γ'. Hence, in the simpliest case new edges are emerging. The next definition of the right-tranlation shows that composed paths arise, too.

Definition 3.3.36. *Let Γ be a graph, $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid and let $\mathcal{P}_\Gamma$ be a finite graph system. Moreover, the set V_Γ is given by $\{v_1,...,v_{2N}\}$.*

*A **bisection of a finite graph system** $\mathcal{P}_\Gamma$ is a map $\sigma_\Gamma : V_\Gamma \to \mathcal{P}_\Gamma$ such that there exists a bisection $\tilde\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ such that $\sigma_\Gamma(V) = \{\tilde\sigma(v_i) : v_i \in V\}$ whenever V is a subset of V_Γ.*

Define a restriction $\sigma_{\Gamma'} : V_{\Gamma'} \to \mathcal{P}_{\Gamma'}$ of a bisection σ_Γ in $\mathcal{P}_\Gamma$ by

$$\sigma_{\Gamma'}(V) := \{\tilde\sigma(w_k) : w_k \in V\}$$

for each subgraph Γ' of Γ and $V \subseteq V_{\Gamma'}$.

A ***right-translation in the finite graph system*** $\mathcal{P}_\Gamma$ *is a map* $R_{\sigma_{\Gamma'}} : \mathcal{P}_{\Gamma'} \to \mathcal{P}_{\Gamma'}$*, which is given by a bisection* $\sigma_{\Gamma'} : V_{\Gamma'} \to \mathcal{P}_{\Gamma'}$ *such that*

$$\begin{aligned} R_{\sigma_{\Gamma'}}(\Gamma'') &= R_{\sigma_{\Gamma'}}(\{\gamma''_1, ..., \gamma''_M, \mathbb{1}_{w_i} : w_i \in \{s(\gamma'_1), ..., s(\gamma'_K) \in V^s_{\Gamma'} : s(\gamma'_i) \neq s(\gamma'_j) \forall i \neq j\} \setminus V_{\Gamma''}\}) \\ &:= \left\{ \begin{array}{l} \gamma''_1, ..., \gamma''_j, \gamma''_{j+1} \circ \tilde{\sigma}(t(\gamma''_{j+1})), ..., \gamma''_M \circ \tilde{\sigma}(t(\gamma''_M)), \mathbb{1}_{w_i} \circ \tilde{\sigma}(w_i) : \\ w_i \in \{s(\gamma'_1), ..., s(\gamma'_K) \in V^s_{\Gamma'} : s(\gamma'_i) \neq s(\gamma'_j) \forall i \neq j\} \setminus V_{\Gamma''}, \quad t(\gamma''_i) \neq t(\gamma''_l) \quad \forall i \neq l; i, l \in [j+1, M] \end{array} \right\} \\ &= \Gamma''_\sigma \end{aligned}$$

where $\tilde{\sigma} \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$*,* $K := |\Gamma'|$ *and* $M := |\Gamma''|$*,* $V^s_{\Gamma'}$ *is the set of all source vertices of* Γ' *and such that* $\Gamma'' := \{\gamma''_1, ..., \gamma''_M\}$ *is a subgraph of* $\Gamma' := \{\gamma'_1, ..., \gamma'_K\}$ *and* Γ''_σ *is a subgraph of* Γ'*.*

Derive that for $\tilde{\sigma}(t(\gamma_i)) = \gamma_i^{-1}$ it is true that $(t \circ \tilde{\sigma})(s(\gamma_i^{-1})) = s(\gamma_i) = (t \circ \tilde{\sigma})(t(\gamma_i))$.

Example 3.3.3: Let Γ be a disconnected graph. Then for a bisection $\tilde{\sigma} \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ such that $\sigma(t(\gamma_i)) = \gamma_i^{-1}$ for all $1 \leq i \leq |\Gamma|$ it is true that

$$\begin{aligned} R_{\sigma_\Gamma}(\Gamma) &= \Big\{\gamma_1 \circ \tilde{\sigma}(t(\gamma_1)), ..., \gamma_N \circ \tilde{\sigma}(t(\gamma_N)), \mathbb{1}_{s(\gamma_1)} \circ \tilde{\sigma}(s(\gamma_1)), ..., \mathbb{1}_{s(\gamma_N)} \circ \tilde{\sigma}(s(\gamma_N))\Big\} \\ &= \{\mathbb{1}_{s(\gamma_1)}, ..., \mathbb{1}_{s(\gamma_N)}\} \end{aligned}$$

Set $\Gamma' := \{\gamma'_1, ..., \gamma'_M\}$ then derive

$$\begin{aligned} R_{\sigma_\Gamma}(\Gamma') &= \Big\{\gamma'_1 \circ \tilde{\sigma}(t(\gamma'_1)), ..., \gamma'_M \circ \tilde{\sigma}(t(\gamma'_M)), \mathbb{1}_{s(\gamma_1)} \circ \tilde{\sigma}(s(\gamma_1)), ..., \mathbb{1}_{s(\gamma_{N-M})} \circ \tilde{\sigma}(s(\gamma_{N-M}))\Big\} \\ &= \{\mathbb{1}_{s(\gamma'_1)}, ..., \mathbb{1}_{s(\gamma'_M)}, \gamma_1, ..., \gamma_{N-M}\} \end{aligned}$$

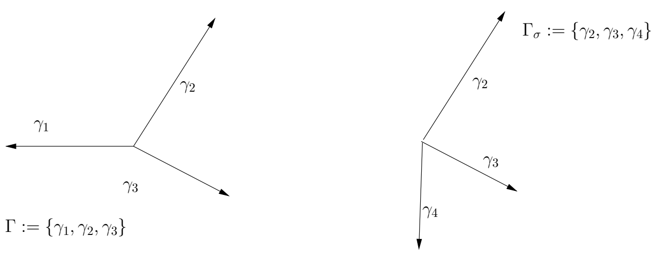
if $\Gamma = \Gamma' \cup \{\gamma_1, ..., \gamma_{N-M}\}$.

■

To understand the definition of the right-translation notice the following problem.

Problem 3.3.1: Consider a subgraph Γ of $\tilde{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, a map $\tilde{\sigma} : V_{\tilde{\Gamma}} \to \mathcal{P}_{\tilde{\Gamma}}\Sigma$ and a translation

$$R_{\sigma_{\tilde{\Gamma}}}(\Gamma) = \{\gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ \mathbb{1}_{t(\gamma_2)}, \gamma_3 \circ \mathbb{1}_{t(\gamma_3)}, \mathbb{1}_{s(\gamma_1)} \circ \gamma_4\} =: \Gamma_\sigma$$



Notice that, the map σ maps $t(\gamma_1) \mapsto s(\gamma_1)$, $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $s(\gamma_1) \mapsto t(\gamma_4)$. Then the map $\tilde{\sigma}$ is not a bisection in the finite path groupoid $\mathcal{P}_{\tilde{\Gamma}}\Sigma$ over $V_{\tilde{\Gamma}}$ and does not define a right-translation $R_{\sigma_{\tilde{\Gamma}}}$ in the finite graph system $\mathcal{P}_{\tilde{\Gamma}}$.

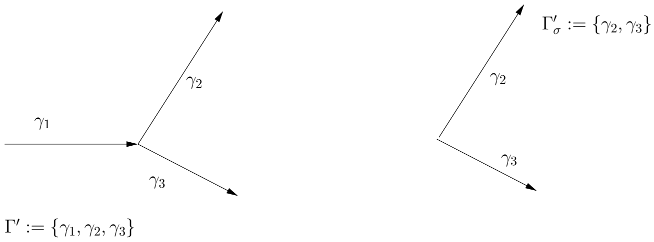
This is a general problem. For every bisection $\tilde{\sigma}$ in a finite path groupoid such that a graph $\Gamma := \{\gamma\}$ is translated to $\{\gamma \circ \tilde{\sigma}(t(\gamma), \tilde{\sigma}(s(\gamma))\}$. Hence, either such translations in the graph system are excluded or the definition of the bisections has to be restricted to maps such that the map $t \circ \tilde{\sigma}$ is not bijective. Clearly, the restriction of the right-translation such that Γ is mapped to $\{\gamma \circ \tilde{\sigma}(t(\gamma), \mathbb{1}_{s(\gamma)}\}$ imply that a simple path orientation transformation is not implemented by a right-translation.

Furthermore, there is an ambiguity for graph containing to paths γ_1 and γ_2 such that $t(\gamma_1) = t(\gamma_2)$. Since, in this case a bisection σ, which maps $t(\gamma_1)$ to $t(\gamma_3)$, the right-translation is $\{\gamma_1 \circ \gamma_3, \gamma_2 \circ \gamma_3\}$, which is not a graph anymore.

■

Example 3.3.4: Otherwise, there is for example a subgraph Γ' of $\tilde{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and a bisection $\tilde{\sigma}_{\tilde{\Gamma}}$ such that

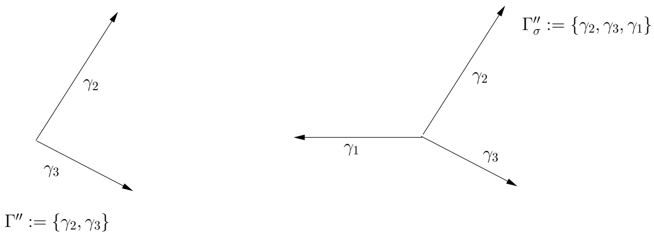
$$\Gamma'_\sigma := \{\gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ \mathbb{1}_{s(\gamma_2)}, \gamma_3 \circ \mathbb{1}_{s(\gamma_3)}\}$$



Notice that, $t(\gamma_1) \mapsto s(\gamma_1)$, $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $t(\gamma_4) \mapsto t(\gamma_4)$. Hence, the the map $\tilde{\sigma}_{\tilde{\Gamma}} : V_{\tilde{\Gamma}} \to \mathcal{P}_{\tilde{\Gamma}}\Sigma$ is bijective map and, consequently, a bisection. The bisection $\sigma_{\tilde{\Gamma}}$ in the graph system $\mathcal{P}_{\tilde{\Gamma}}$ defines a right-translation $R_{\sigma_{\tilde{\Gamma}}}$ in $\mathcal{P}_{\tilde{\Gamma}}$.

Moreover, for a subgraph $\Gamma'' := \{\gamma_2, \gamma_3\}$ of the graph $\breve{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3\}$ there exists a map $\sigma_{\breve{\Gamma}} : V_{\breve{\Gamma}} \to \mathcal{P}_{\breve{\Gamma}}$ such that

$$R_{\sigma_{\breve{\Gamma}}}(\Gamma'') = \{\gamma_2, \gamma_3, \tilde{\sigma}(s(\gamma_1))\} = \{\gamma_2, \gamma_3, \gamma_1\}$$



where $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $s(\gamma_1) \mapsto t(\gamma_1)$. Consequently, in this example the map $\tilde{\sigma}_{\breve{\Gamma}}$ is a bisection, which defines a right-translation in $\mathcal{P}_{\breve{\Gamma}}$, too.

Note that for a graph Γ such that $\tilde{\Gamma}$ and $\breve{\Gamma}$ are subgraphs. Hence, the bisection $\sigma_{\tilde{\Gamma}}$ extends to a bisection σ in $\mathcal{P}_\Gamma$ and $\sigma_{\breve{\Gamma}}$ extends to a bisection $\breve{\sigma}$ in $\mathcal{P}_\Gamma$.

■

Moreover, the bisections of a finite graph system can be transfered, analogously, to bisections of a finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ to the group $G^{|\Gamma|}$. Let $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and $(\mathfrak{h}_\Gamma, h_\Gamma) \in \text{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|})$. Thus, there are two maps

$$\mathfrak{h}_\Gamma \circ R_\sigma : \mathcal{P}_\Gamma \to G^{|\Gamma|} \text{ and } h_\Gamma \circ (t \circ \sigma) : V_\Gamma \to \{e_G\} \tag{3.88}$$

The pair of maps is called holonomy map for a finite graph system if σ is suitable.

Now, a similar right-translation in a finite graph system in comparison to the right-translation $R_{\sigma(v)}$ in a finite path groupoid is studied. Let $\sigma_{\Gamma'} : V_{\Gamma'} \to \mathcal{P}_{\Gamma'}$ be a restriction of $\sigma_\Gamma \in \mathfrak{B}(\mathcal{P}_\Gamma)$. Moreover, let V be a subset of $V_{\Gamma'}$, let Γ'' be a subgraph of Γ' and Γ''' be a subgraph of Γ''. Then a right-translation

$$\begin{aligned}
&R_{\sigma_{\Gamma'}(V)}(\Gamma'')\\
&:= \begin{cases} R_{\sigma_{\Gamma'}}(\{\gamma''_1, ..., \gamma''_M, \mathbb{1}_{w_i} : w_i \in \{s(\gamma'_1), ..., s(\gamma'_K) \in V_{\Gamma'} : s(\gamma'_i) \neq s(\gamma'_j) \forall i \neq j\} \setminus V_{\Gamma''}\} & : V_{\Gamma''} \subset V \\ R_{\sigma_{\Gamma'}}(\{\gamma''_1, ..., \gamma'''_N, \mathbb{1}_{w_i} : w_i \in \{s(\gamma'_1), ..., s(\gamma'_K) \in V_{\Gamma''} : s(\gamma'_i) \neq s(\gamma'_j) \forall i \neq j\} \setminus V_{\Gamma'''}\}) & \\ \quad \cup \{\mathbb{1}_{x_i} : x_i \in V \setminus V_{\Gamma''}\} \cup \{\Gamma'' \setminus \Gamma'''\} & : V_{\Gamma''} \not\subset V, V_{\Gamma'''} \subset V \end{cases}
\end{aligned}$$

Loosely speaking, the action of a path-diffeomorphism is somehow localised on a fixed vertex set V.

For example note that for a subgraph $\Gamma' := \{\gamma\circ\gamma'\}$ of $\Gamma := \{\gamma,\gamma'\}$ and a subset $V := \{t(\gamma')\}$ of V_Γ, it is true that

$$(\mathfrak{h}_\Gamma\circ R_{\sigma_\Gamma(V)})(\gamma\circ\gamma') = (\mathfrak{h}_\Gamma\circ R_{\sigma_\Gamma(V)})(\gamma)\cdot(\mathfrak{h}_\Gamma\circ R_{\sigma_\Gamma(V)})(\gamma') = \mathfrak{h}_\Gamma(\gamma)\cdot(\mathfrak{h}_\Gamma\circ R_{\sigma_\Gamma(V)})(\gamma') = \mathfrak{h}_\Gamma(\gamma\circ\gamma'\circ\sigma(t(\gamma')))$$

whenever $\sigma_\Gamma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$. For a special bisection $\breve{\sigma}_\Gamma$, it is true that

$$(\mathfrak{h}_\Gamma\circ R_{\breve{\sigma}_\Gamma})(\gamma) = \mathfrak{h}_\Gamma(\gamma\circ\gamma') = (\mathfrak{h}_\Gamma\circ R_{\breve{\sigma}_\Gamma})(\gamma)\cdot(\mathfrak{h}_\Gamma\circ R_{\breve{\sigma}_\Gamma})(\gamma')$$

holds, whenever $\breve{\sigma}_\Gamma\in\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$, $\breve{\sigma}_\Gamma(t(\gamma')) = \mathbb{1}_{t(\gamma')}$ and $\breve{\sigma}_\Gamma(t(\gamma)) = \gamma'$. Let $\tilde{\sigma}$ be the bisection in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ that defines the bisection $\breve{\sigma}$ in $\mathcal{P}_\Gamma$. Then the last statement is true, since, $R_{\breve{\sigma}_\Gamma}(\gamma') = \gamma'\circ{\gamma'}^{-1}$ requires $\tilde{\sigma}_\Gamma : t(\gamma')\mapsto s(\gamma')$ and $R_{\breve{\sigma}_\Gamma}(\gamma) = \gamma\circ\gamma'$ needs $\tilde{\sigma}_\Gamma : t(\gamma)\mapsto t(\gamma')$, where $s(\gamma') = t(\gamma)$. Then $R_{\breve{\sigma}_\Gamma}(\gamma)$ and $R_{\sigma_\Gamma(t(\gamma'))}(\gamma)$ coincide if $\breve{\sigma}_\Gamma(t(\gamma)) = \sigma_\Gamma(t(\gamma))$ and $\breve{\sigma}_\Gamma(t(\gamma')) = \mathbb{1}_{t(\gamma')}$.

Problem 3.3.2: Let Γ' be a subgraph of the graph Γ, σ_Γ be a bisection in $\mathcal{P}_\Gamma$, $\sigma_{\Gamma'} : V_{\Gamma'}\to\mathcal{P}_{\Gamma'}$ be a restriction of $\sigma_\Gamma\in\mathfrak{B}(\mathcal{P}_\Gamma)$. Moreover, let V be a subset of $V_{\Gamma'}$, let $\Gamma'' := \{\gamma\circ\gamma'\}$ be a subgraph of Γ'. Let $(\gamma,\gamma')\in\mathcal{P}_{\Gamma'}\Sigma^{(2)}$.

Then even for a suitable bisection $\sigma_{\Gamma'}$ in $\mathcal{P}_\Gamma$, it follows that

$$R_{\sigma_{\Gamma'}(V)}(\gamma\circ\gamma') \neq R_{\sigma_{\Gamma'}(V)}(\gamma)\circ R_{\sigma_{\Gamma'}(V)}(\gamma') \quad (3.89)$$

This is a general problem. In comparison with problem 3.1.3.1 the multiplication map $\circ$ is not well-defined and, hence,

$$R_{\sigma_{\Gamma'}(V)}(\gamma)\circ R_{\sigma_{\Gamma'}(V)}(\gamma')$$

is not well-defined. Recognize that $R_{\sigma_{\Gamma'}(V)} : \mathcal{P}_\Gamma\to\mathcal{P}_\Gamma$.

Consequently, in general it is not true that

$$(\mathfrak{h}_\Gamma\circ R_{\sigma_{\Gamma'}(V)})(\gamma\circ\gamma') = \mathfrak{h}_\Gamma(R_{\sigma_{\Gamma'}(V)}(\gamma)\circ R_{\sigma_{\Gamma'}(V)}(\gamma')) = (\mathfrak{h}_\Gamma\circ R_{\sigma_{\Gamma'}(V)})(\gamma)\cdot(\mathfrak{h}_\Gamma\circ R_{\sigma_{\Gamma'}(V)})(\gamma') \quad (3.90)$$

yields.

▪

With no doubt, the left-tranlation $L_{\sigma_{\Gamma'}}$ and the inner autmorphisms $I_{\sigma_{\Gamma'}}$ in a finite graph system $\mathcal{P}_\Gamma$ for every $\Gamma'\in\mathcal{P}_\Gamma$ can be defined.

Definition 3.3.37. *Let $\sigma_\Gamma\in\mathfrak{B}(\mathcal{P}_\Gamma)$ be a bisection in the finite graph system $\mathcal{P}_\Gamma$. Let $R_{\sigma_\Gamma(V)}$ be a right-tranlation, where V is a subset of V_Γ.*

*Then the pair $(\Phi_\Gamma,\varphi_\Gamma)$ defined by $\Phi_\Gamma = R_{\sigma_\Gamma(V)}$ (or, respectively, $\Phi_\Gamma = L_{\sigma_\Gamma(V)}$, or $\Phi_\Gamma = I_{\sigma_\Gamma(V)}$) for a subset $V\subseteq V_\Gamma$ and $\varphi_\Gamma = t\circ\sigma_\Gamma$ is called a **graph-diffeomorphism of a finite graph system**. Denote the set of finite graph-diffeomorphisms by* $\mathrm{Diff}(\mathcal{P}_\Gamma)$.

Let Γ' be a subgraph of Γ and $\sigma_{\Gamma'}$ be a restriction of bisection σ_Γ in $\mathcal{P}_\Gamma$. Then for example another graph-diffeomorphism $(\Phi_{\Gamma'},\varphi_{\Gamma'})$ in $\mathrm{Diff}(\mathcal{P}_\Gamma)$ can be defined by $\Phi_{\Gamma'} = R_{\sigma_{\Gamma'}(V)}$ for a subset $V\subseteq V_{\Gamma'}$ and $\varphi_{\Gamma'} = t\circ\sigma_{\Gamma'}$.

Remembering that the set of bisections of a finite path groupoid form a group (refer 3.3.31) one may ask if the bisections of a finite graph system form a group, too.

Proposition 3.3.38. *The set of bisections $\mathfrak{B}(\mathcal{P}_\Gamma)$ in a finite graph system $\mathcal{P}_\Gamma$ form a group.*

Proof : Let Γ be a graph and let V_Γ be equivalent to the set $\{v_1,...,v_{2N}\}$.

First two different multiplication operations are studied. The studies are comparable with the results of the definition 3.3.36 of a right-translation in a finite graph system. The easiest multiplication operation is given by $*_1$, which is defined by

$$(\sigma *_1 \sigma')(V_\Gamma) := \{(\tilde{\sigma}*\tilde{\sigma}')(v_1),...,(\tilde{\sigma}*\tilde{\sigma}')(v_{2N}) : v_i\in V_\Gamma\}$$

where $*$ denotes the multiplication of bisections on the finite path groupoid $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$. Notice that this operation is not well-defined in general. In comparison with the definition of the right-translation in a finite graph system one has to take care. First, the set of vertices doesn't contain any vertices twice, the map σ in the finite path system is bijective, the mapping σ maps each set to a set of vertices containing no vertices twice and the situation in problem 3.3.1 has to be avoided.

Fix a bisection $\tilde{\sigma}$ in a finite path groupoid $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$. Let $V_{\sigma'}$ be a subset of V_Γ where $\Gamma := \{\gamma_1, ..., \gamma_N\}$ and for each v_i in $V_{\sigma'}$ it is true that $v_i \neq v_j$ and $v_i \neq (t \circ \tilde{\sigma}')(v_j)$ for all $i \neq j$. Define the set $V_{\sigma,\sigma'}$ to be equal to a subset of the set of all vertices $\{v_k \in V_{\sigma'} : 1 \leq k \leq 2N\}$ such that each pair (v_i, v_j) of vertices in $V_{\sigma,\sigma'}$ satisfies $(t \circ (\tilde{\sigma} * \tilde{\sigma}'))(v_i) \neq (t \circ \tilde{\sigma}')(v_j)$ and $(t \circ \tilde{\sigma}')(v_i) \neq (t \circ \tilde{\sigma}')(v_j)$ for all $i \neq j$. Define

$$W_{\sigma,\sigma'} := \Big\{w_i \in \{V_\sigma \cap V_{\sigma'}\} \setminus V_{\sigma,\sigma'} : (t \circ \tilde{\sigma})(w_j) \neq (t \circ \tilde{\sigma}')(w_i) \quad \forall i \neq j, \quad 1 \leq i, j \leq l\Big\}$$

The set $V_{\sigma,\sigma',\breve{\sigma}}$ is a subset of all vertices $\{v_k \in V_{\sigma,\sigma'} : 1 \leq k \leq 2N\}$ such that each pair (v_i, v_j) of vertices in $V_{\sigma,\sigma',\breve{\sigma}}$ satisfies $(t \circ (\breve{\sigma} * \tilde{\sigma} * \tilde{\sigma}'))(v_i) \neq (t \circ (\tilde{\sigma} * \tilde{\sigma}'))(v_j)$ and $(t \circ (\tilde{\sigma} * \tilde{\sigma}'))(v_i) \neq (t \circ (\tilde{\sigma} * \tilde{\sigma}'))(v_j)$ for all $i \neq j$.

Consequently, define a second multiplication on $\mathfrak{B}(\mathcal{P}_\Gamma)$ similarly to the operation $*_1$. This is done by the following definition. Set

$$\begin{aligned}(\sigma *_2 \sigma')(V_{\sigma'}) := &\{(\tilde{\sigma} * \sigma')(v_1), ..., (\tilde{\sigma} * \sigma')(v_k) : v_1, ..., v_k \in V_{\sigma,\sigma'}, 1 \leq k \leq 2N\} \\ &\cup \{\tilde{\sigma}(w_1), \sigma'(w_1) ..., \tilde{\sigma}(w_l), \sigma'(w_l) : w_1, ..., w_l \in W_{\sigma,\sigma'}, 1 \leq l \leq 2N\} \\ &\cup \{\mathbb{1}_{p_1}, ..., \mathbb{1}_{p_n} : p_1, ..., p_n \in V_{\sigma'} \setminus \{V_{\sigma,\sigma'} \cup W_{\sigma,\sigma'}\}, 1 \leq n \leq 2N\}\end{aligned}$$

Hence, the inverse is supposed to be $\sigma^{-1}(V_\Gamma) = \sigma((t \circ \sigma)^{-1}(V_\Gamma))^{-1}$ such that

$$\begin{aligned}(\sigma *_2 \sigma^{-1})(V_{\sigma^{-1}}) = &\{(\tilde{\sigma} * \tilde{\sigma}^{-1})(v_1), ..., (\tilde{\sigma} * \tilde{\sigma}^{-1})(v_{2N}) : v_i \in V_{\sigma,\sigma^{-1}}\} \\ &\cup \Big\{\tilde{\sigma}(w_1), \sigma'(w_1)^{-1} ..., \tilde{\sigma}(w_l), \sigma'(w_l)^{-1} : w_1, ..., w_l \in W_{\sigma,\sigma^{-1}}, 1 \leq l \leq 2N\Big\} \\ &\cup \Big\{\mathbb{1}_{p_1}, ..., \mathbb{1}_{p_n} : p_1, ..., p_n \in V_{\sigma'} \setminus \{V_{\sigma,\sigma^{-1}} \cup W_{\sigma,\sigma^{-1}}\}, 1 \leq n \leq 2N\Big\}\end{aligned}$$

■

Notice that the problem 3.3.2 can be solved by a multiplication operation $\circ_2$, which can be defined similarly to $*_2$. Hence, the equality of (3.89) is available and, hence, (3.90) is true. Furthermore, a similar remark to 3.87 can be done.

Example 3.3.5: Now, consider the following example. Set $\Gamma' := \{\gamma_1, \gamma_3\}$, let $\Gamma := \{\gamma_1, \gamma_2, \gamma_3\}$ and $V_\Gamma := \{s(\gamma_1), t(\gamma_1), s(\gamma_2), t(\gamma_2), s(\gamma_3), t(\gamma_3) : s(\gamma_i) \neq s(\gamma_j), t(\gamma_i) \neq t(\gamma_j)\ \forall i \neq j\}$.
Set V be equal to $\{s(\gamma_1), s(\gamma_2), s(\gamma_3)\}$. Take two maps σ and σ' such that $\sigma'(V) = \{\gamma_1, \gamma_3\}$, $\sigma(V) = \{\gamma_2\}$, where $(t \circ \tilde{\sigma})(s(\gamma_3)) = t(\gamma_3)$, $\tilde{\sigma}'(s(\gamma_3)) = \gamma_3$, $\tilde{\sigma}'(s(\gamma_1)) = \gamma_1$ and $\tilde{\sigma}(t(\gamma_3)) = \gamma_2$. Then $s(\gamma_3) \in V_{\sigma_{\Gamma'}, \sigma'_{\Gamma'}}$ and $s(\gamma_1) \in W_{\sigma_{\Gamma'}, \sigma'_{\Gamma'}}$. Derive

$$(\sigma *_1 \sigma')(V) = \{\gamma_3 \circ \gamma_2, \gamma_1\}$$

Then conclude that

$$(\sigma *_2 \sigma')(V_\Gamma) = \{\gamma_3 \circ \gamma_2, \gamma_1\}$$

holds. Notice that

$$(\sigma *_2 \sigma')(V) \neq (\sigma' *_2 \sigma)(V) = \{\gamma_2, \gamma_1, \gamma_3\}$$

is true. Finally, one obtains

$$(\sigma *_2 \sigma^{-1})(V_\Gamma) = \{\gamma_3 \circ \gamma_3^{-1}, \gamma_1 \circ \gamma_1^{-1}\} = \{\mathbb{1}_{s(\gamma_3)}, \mathbb{1}_{s(\gamma_1)}\}$$

Let $\sigma'(V_\Gamma) = \{\gamma_1, \gamma_3\}$ and $\breve{\sigma}(V_\Gamma) = \{\gamma_2, \gamma_4\}$. Then

$$(\breve{\sigma} *_1 \sigma')(V_\Gamma) = \{\gamma_3 \circ \gamma_2, \gamma_1\}$$

and

$$(\breve{\sigma} *_2 \sigma')(V_\Gamma) = \{\gamma_3 \circ \gamma_2, \gamma_1, \gamma_4\}$$

yield.

Furthermore, assume supplementary that $t(\gamma_3) = t(\gamma_1)$ holds. Then calculate the product of the maps σ and σ' is

$$(\sigma *_1 \sigma')(V) = \{\gamma_3 \circ \gamma_2, \gamma_1 \circ \gamma_2\} \notin \mathcal{P}_\Gamma$$

and

$$(\sigma *_2 \sigma')(V_\Gamma) = \{\mathbb{1}_{t(\gamma_1)}, \mathbb{1}_{t(\gamma_3)}\} \in \mathcal{P}_\Gamma$$

■

The group structure of $\mathfrak{B}(\mathcal{P}_\Gamma)$ transferes to G. Let $\tilde{\sigma}$ be a bisection in the finite path groupoid $\mathcal{P}_\Gamma\Sigma \overset{s}{\underset{t}{\rightrightarrows}} V_\Gamma$, which defines a bisection σ in $\mathcal{P}_\Gamma$ and let $\tilde{\sigma}'$ be a bisection in $\mathcal{P}_\Gamma\Sigma \overset{s}{\underset{t}{\rightrightarrows}} V_\Gamma$, which defines another bisection σ' in $\mathcal{P}_\Gamma$. Let $V_{\sigma,\sigma'}$ be equal to V_Γ, then derive

$$\begin{aligned}
&\mathfrak{h}_\Gamma\left((\sigma *_2 \sigma')(V_\Gamma)\right) = \{\mathfrak{h}_\Gamma((\tilde{\sigma} * \sigma')(v_1)), ..., \mathfrak{h}_\Gamma((\tilde{\sigma} * \sigma')(v_{2N}))\}\\
&= \mathfrak{h}_\Gamma(\sigma'(V_\Gamma) \circ \sigma(t(\sigma'(V_\Gamma)))) = \{\mathfrak{h}_\Gamma(\sigma'(v) \circ \tilde{\sigma}(t(\sigma'(v_1)))), ..., \mathfrak{h}_\Gamma(\sigma'(v_N) \circ \tilde{\sigma}(t(\sigma'(v_N))))\}\\
&= \{\mathfrak{h}_\Gamma(\sigma'(v))\mathfrak{h}_\Gamma(\tilde{\sigma}(t(\sigma'(v_1)))), ..., \mathfrak{h}_\Gamma(\sigma'(v_N))\mathfrak{h}_\Gamma(\tilde{\sigma}(t(\sigma'(v_N))))\}\\
&= \mathfrak{h}_\Gamma(\sigma'(V_\Gamma))\mathfrak{h}_\Gamma(\sigma(V_\Gamma))
\end{aligned} \tag{3.91}$$

Consequently, the right-translation in the finite product $G^{|\Gamma|}$ can be defined.

Definition 3.3.39. *Let $\sigma_{\Gamma'}$ be in $\mathfrak{B}(\mathcal{P}_\Gamma)$, Γ' a subgraph of Γ, Γ'' a subgraph of Γ' and $R_{\sigma_{\Gamma'}}$ a right-translation, $L_{\sigma_{\Gamma'}}$ a left-translation and $I_{\sigma_{\Gamma'}}$ an inner-translation in $\mathcal{P}_\Gamma$.*

Then the ***right-translation in the finite product*** $G^{|\Gamma|}$ *is given by*

$$\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma'}} : \mathcal{P}_\Gamma \to G^{|\Gamma|}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma'}})(\Gamma'')$$

Furthermore, define the ***left-translation in the finite product*** $G^{|\Gamma|}$ *by*

$$\mathfrak{h}_\Gamma \circ L_{\sigma_{\Gamma'}} : \mathcal{P}_\Gamma \to G^{|\Gamma|}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ L_{\sigma_{\Gamma'}})(\Gamma'')$$

and the ***inner-translation in the finite product*** $G^{|\Gamma|}$

$$\mathfrak{h}_\Gamma \circ I_{\sigma_{\Gamma'}} : \mathcal{P}_\Gamma \to G^{|\Gamma|}, \quad \Gamma'' \mapsto (\mathfrak{h}_\Gamma \circ I_{\sigma_{\Gamma'}})(\Gamma'')$$

such that $I_{\sigma_{\Gamma'}} = L_{\sigma_{\Gamma'}^{-1}} \circ R_{\sigma_{\Gamma'}}$.

Lemma 3.3.40. *It is true that $R_{\sigma_{\Gamma'} *_2 \sigma'_{\Gamma'}} = R_{\sigma_{\Gamma'}} \circ R_{\sigma'_{\Gamma'}}$, $L_{\sigma_{\Gamma'} *_2 \sigma'_{\Gamma'}} = L_{\sigma_{\Gamma'}} \circ L_{\sigma'_{\Gamma'}}$ and $I_{\sigma_{\Gamma'} *_2 \sigma'_{\Gamma'}} = I_{\sigma_{\Gamma'}} \circ I_{\sigma'_{\Gamma'}}$ for all bisections $\sigma_{\Gamma'}$ and $\sigma'_{\Gamma'}$ in $\mathfrak{B}(\mathcal{P}_\Gamma)$.*

There is an action of $\mathfrak{B}(\mathcal{P}_\Gamma)$ on $G^{|\Gamma|}$ by

$$(\zeta_{\sigma_{\Gamma'}} \circ \mathfrak{h}_\Gamma)(\Gamma'') := (\mathfrak{h}_\Gamma \circ R_{\sigma_{\Gamma'}})(\Gamma'')$$

whenever $\sigma_{\Gamma'} \in \mathfrak{B}(\mathcal{P}_\Gamma)$, $\Gamma'' \in \mathcal{P}_{\Gamma'}$ and $\Gamma' \in \mathcal{P}_\Gamma$. Then for another $\breve{\sigma} \in \mathfrak{B}(\mathcal{P}_\Gamma)$ it is true that

$$((\zeta_{\breve{\sigma}_{\Gamma'}} \circ \zeta_{\sigma_{\Gamma'}}) \circ \mathfrak{h}_\Gamma)(\Gamma'') = (\mathfrak{h}_\Gamma \circ R_{\breve{\sigma} *_2 \sigma_{\Gamma'}})(\Gamma'') = (\zeta_{\breve{\sigma}_{\Gamma'} *_2 \sigma_{\Gamma'}} \circ \mathfrak{h}_\Gamma)(\Gamma'')$$

Recall that the map $\tilde{\sigma} \mapsto t \circ \tilde{\sigma}$ is a group isomorphism between the group of bisections $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ and the group $\text{Diff}(V_\Gamma)$ of finite diffeomorphisms in V_Γ. Therefore, if the graphs $\Gamma' = \Gamma''$ contain only the path γ, then the action $\zeta_{\sigma_{\Gamma'}}$ is equivalent to an action of the finite diffeomorphism group $\text{Diff}(V_\Gamma)$. Loosely speaking, the graph-diffeomorphisms $(R_{\sigma_{\Gamma'}(V)}, t \circ \sigma_{\Gamma'})$ on a subgraph Γ'' of Γ' transform graphs and respect the graph structure of Γ'. The diffeomorphism $t \circ \tilde{\sigma}$ in the finite path groupoid only implements the finite diffeomorphism in Σ, but it doesn't adopt any path groupoid or graph preserving structure. Summarising the bisections of a finite graph system respect the graph structure and implement the finite diffeomorphisms in Σ. There is another reason why the group of bisections is more fiundamental than the path- or graph-diffeomorphism group. In section 6.1 the concept of C^*-dynamical systems is studied. It turns out that there are three different C^*-dynamical systems, each is build from the analytic holonomy C^*-algebra and a point-norm continuous action of the group of bisections of a finite graph system. The actions are implemented by one of the three translations, i.e. the left-, right- or inner-translation in the finite product $G^{|\Gamma|}$. Furthermore, the actions can be related to each other by an unitary 1-cocycle. Hence, the automorphisms are exterior equivalent [109, Def.:2.66]. In this case, the full information is contained in the C^*-dynamical systems constructed from the right-translation in a finite graph system. Moreover, the path- (or graph-) diffeomorphisms define particular right-, left- or inner-translations associated to suitable bisections. This is why in the introduction and later in section 6 the path- (or graph-) diffeomorphism group are often indentified with the group of bisections in a finite path groupoid (or graph system).

Finally, the left or right-translations in a finite path groupoid can be studied in the context of natural or non-standard identification of the configuration space. This new concept leads to two different notions of diffeomorphism-invariant states. The actions of path- and graph-diffeomorphism and the concepts of natural or non-standard identification of the configuration space was not used in the context of LQG before.

Transformations and discretised surface sets

Now restricted sets of bisections are concerned. Consider a finite set of paths starting at a discretised surface. The idea is to define a bisection σ such that the map $t \circ \sigma$ preserves the set $\breve{S}_{\mathrm{d}}$ of discretised surfaces and each path of the certain set of paths composed with the bisection σ at the target vertex of this path is again a path that start at a discretised surface. The definition follows.

Define the set $V^{\breve{S}_{\mathrm{d}}}$, which contains all target vertices of paths in $\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}\Sigma$. Note that, the base point v with respect to $\mathcal{P}_\Gamma\Sigma^v$ is the source vertex of all paths in $\mathcal{P}_\Gamma\Sigma^v$ and all paths in $\mathcal{P}_\Gamma\Sigma^v$ are contained in $\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}\Sigma$ for all $v \in \breve{S}_{\mathrm{d}}$. Denote the set of bisections, which are bijective maps from the set $V^{\breve{S}_{\mathrm{d}}}$ to paths in $\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}\Sigma$, by $\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}\Sigma)$. On the level of graphs the restricted set of bisections in a graph system $\mathcal{P}_\Gamma$ is denoted by $\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$. Denote the set of graph-diffeomorphisms, which are defined by a bisection in $\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$ and the right-translation R_σ, by the term $\mathrm{Diff}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$.

3.3.5 Holonomy maps for path groupoids

In the smooth category the holonomy maps are constructed from the thin or intimate fundamental groupoid. Following the concept of Barrett the set $\breve{\mathcal{A}}_s$ of smooth connections is replaced by all continuous groupoid morphisms from a (thin/intimate) fundamental groupoid or generally a (algebraic) path groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$ to the Lie group G.

Definition 3.3.41. *Let $\mathcal{P}\Sigma \rightrightarrows \Sigma$ be a path groupoid over Σ.*

Two paths γ and γ' in $\mathcal{P}\Sigma$ have the ***same-holonomy for all smooth connections*** *iff*

$$\begin{aligned}\mathfrak{h}(\gamma) = \mathfrak{h}(\gamma') \text{ for all } (\mathfrak{h}, h) \text{ continuous groupoid morphisms}\\ \mathfrak{h} : \mathcal{P}\Sigma \rightarrow G, h : \Sigma \rightarrow \{e_G\}\end{aligned}$$

Lemma 3.3.42. *The same-holonomy for all smooth connections relation is an equivalence relation.*

Definition 3.3.43. *Let $\mathcal{P}\Sigma \rightrightarrows \Sigma$ be a path groupoid over Σ modulo same-holonomy for all smooth connections equivalence.*

A ***holonomy map for a path groupoid*** *is a continuous groupoid morphism $(\mathfrak{h}, h)$ where $\mathfrak{h} : \mathcal{P}\Sigma \rightarrow G, h : \Sigma \rightarrow \{e_G\}$.*

Definition 3.3.44. *Let $\mathcal{P}\Sigma \rightrightarrows \Sigma$ be a path groupoid over Σ modulo same-holonomy for all smooth connections equivalence. Moroever, let Γ be a* ***finite set of smooth paths*** *in $\mathcal{P}\Sigma$ with $|\Gamma|$ paths. A* ***finite system*** *$\mathcal{P}_\Gamma\Sigma$* ***of paths*** *is given by a finite number of sets of smooth paths, such that every finite set contains paths, which are compositions of the smooth paths or their inverses contained in Γ.*

The ***set of holonomy maps for a path groupoid*** *$\mathcal{P}\Sigma \rightrightarrows \Sigma$ is defined by*

$$\begin{aligned}\mathcal{A}_s := \operatorname{Hom}_{\breve{\mathcal{A}}_s}(\mathcal{P}\Sigma, G) = \{(\mathfrak{h}, h)| \quad & \mathfrak{h} : \mathcal{P}\Sigma \rightarrow G, h : \Sigma \rightarrow \{e_G\} \text{ such that}\\ & (\mathfrak{h}, h) \text{ is a holonomy map for the path groupoid } \mathcal{P}\Sigma \rightrightarrows \Sigma\}\end{aligned}$$

where $\breve{\mathcal{A}}_s$ is the set of all smooth connections.

The ***set of holonomy maps for a finite system*** *$\mathcal{P}_\Gamma\Sigma$* ***of paths*** *is defined by*

$$\begin{aligned}\mathcal{A}_s^\Gamma := \operatorname{Hom}_{\breve{\mathcal{A}}_s}(\mathcal{P}_\Gamma\Sigma, G^{|\Gamma|}) = \Big\{(\mathfrak{h}_\Gamma, h_\Gamma)| \quad & \exists \text{ holonomy map } (\mathfrak{h}, h) \text{ for a path groupoid } \mathcal{P}\Sigma \rightrightarrows \Sigma\\ & \mathfrak{h} : \mathcal{P}_\Gamma\Sigma \rightarrow G, h : \Sigma \rightarrow \{e_G\}\\ & \text{such that } \mathfrak{h}_\Gamma(\Gamma) = (\mathfrak{h}(\gamma_1), ..., \mathfrak{h}(\gamma_N))\\ & \forall \Gamma = \{\gamma_1, ..., \gamma_N\}\Big\}\end{aligned}$$

3.4 Classical and quantum flux variables

3.4.1 The classical flux variables

In the following it is assumed that, all paths and the manifold Σ are smooth. Consider the bundle $\mathcal{A}^*(\Sigma) := \mathcal{A}(T^*\Sigma)$, which is defined by a total space $\wedge^k T^*\Sigma$ minus the zero section for a fixed positive real number k, the basis manifold Σ with dimension D and the natural projection π. Hence, this is a principal $\mathbb{C}^*$-bundle over Σ, whose fibre at x is the k-fold antisymmetric tensor product of $T^*_x\Sigma$ and where $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. The bundle is denoted by $\mathcal{A}^*(\Sigma)(\Sigma, \mathbb{C}^*, \pi)$ in general. Similarly, the bundle $\mathcal{A}(\Sigma) := \mathcal{A}(T\Sigma)$ over Σ can be defined. Moreover, consider the bundle $\mathcal{A}^*(\Sigma) \otimes \underline{\mathfrak{g}}$, where $\underline{\mathfrak{g}}$ is the trivial bundle over Σ with fibre being equal to the Lie algebra $\mathfrak{g}$, and the bundle $\mathcal{A}^*(\Sigma) \otimes \underline{\mathfrak{g}}^*$, where $\underline{\mathfrak{g}}^*$ denotes a trivial bundle over Σ with fibre given by the dual $\mathfrak{g}^*$ of the Lie algebra. Then the sections of the bundle $\mathcal{A}^*(\Sigma) \otimes \underline{\mathfrak{g}}$ are called k-forms on Σ with values in $\mathfrak{g}$, whereas the sections of $\mathcal{A}^*(\Sigma) \otimes \underline{\mathfrak{g}}^*$ are called k-forms on Σ with values in $\mathfrak{g}^*$.

Choose a local coordinate system $(x^1, ..., x^D)$ and a basis $(\tau_1, .., \tau_d)$ of the Lie algebra $\mathfrak{g}$. Let S be a $(D-1)$-dimensional hypersurface in Σ. The canonical variables are one-forms $A_i = A_a \,\mathrm{d}\, x^a \otimes \tau_i$ on Σ with values in $\mathfrak{g}$, and vector densities E^i of weight one with values in $\mathfrak{g}^*$. The vector densities are precisely given by $E^i = E^a \partial_a \otimes \tau^{*i}$, where $\partial_a \in \mathcal{A}(T_x\Sigma)$ and τ^{*i} is an element of the basis of the dual $\mathfrak{g}^*$. On the other hand, E^i can be replaced by a pseudo $(D-1)$-form $\tilde{E}^i$ given as follows $\tilde{E}^i = \frac{1}{(D-1)!} E^a \epsilon_{aa_1\cdots a_{D-1}} \,\mathrm{d}\, x^{a_1} \wedge \cdots \wedge \mathrm{d}\, x^{a_{D-1}} \otimes \tau^{*i}$. Then there is an integral of $\tilde{E}^i$ over S defined by

$$E(S, f^S) := \int_S f^S_i \tilde{E}^i,$$

where the smearing functions $f^S : S \to \mathfrak{g}$ has compact support.

Let γ be a path that intersects a surface S in the target vertex of the path and lies below with respect to the surface orientation. Set $v = S \cap \gamma$. Then the Poisson bracket is given by

$$\left\{A_i(v), E^j(v)\right\} = i\delta^j_i$$

In terms of the integral $E(S, f^S)$ the Poisson bracket reads

$$\left\{A_a(v), E(S, f^S)\right\} = -\iota(\gamma, S) f^S_i(v)\tau^i \; A_a(v) \tag{3.92}$$

where the intersection function ι is defined by $\iota(\gamma, S) := \{0, \pm 1\}$ such that $\iota(\gamma, S) = -1$ if $S \cap \gamma = t(\gamma)$, $\iota(\gamma, S) = 1$ if $S \cap \gamma = s(\gamma)$ and $\iota(\gamma, S) = 0$ if $S \cap \gamma = \{\varnothing\}$ or $\gamma \subset S$ for a path γ. Notice that only in the case of an intersection in the (source or) target vertex of the path the Poisson bracket is non-trivial. The classical flux variable $E(S, f_S)$ associated to a surface S depends on the intersection behavior of the path γ and the surface S.

3.4.2 The Lie algebra-valued quantum flux operators associated to surfaces and graphs

The Lie algebra-valued quantum flux operators

The quantum analogue of a classical connection $A_a(v)$ is given by the holonomy along a path γ and is denoted by $\mathfrak{h}(\gamma)$. The quantum flux operator $E_S(\gamma)$, which replaces the classical flux variable $E(S, f^S)$, is given by a map E_S from a graph to the Lie algebra $\mathfrak{g}$. Let Exp be the exponential map from the Lie algebra $\mathfrak{g}$ to G and set $U_t(E_S(\gamma)) := \mathrm{Exp}(tE_S(\gamma))$. Then the quantum flux operator $E_S(\gamma)$ and the quantum holonomies $\mathfrak{h}(\gamma)$ satisfy the following canonical commutator relation

$$E_S(\gamma)\mathfrak{h}(\gamma) = i \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0} U_t(E_S(\gamma))\mathfrak{h}(\gamma)$$

where γ is a path that intersects the surface S in the target vertex of the path and lies below with respect to the surface orientation of S.

In this section different definitions of the quantum flux operator, which is associated to a fixed surface S, are presented. For example the quantum flux operator E_S is defined to be a map from a graph Γ to a direct sum

$\mathfrak{g} \oplus \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ associated to the Lie group G. This is related to the fact that, one distinguishes between paths that are ingoing and paths that are outgoing with resepect to the surface orientation of S. If there are no intersection points of the surface S and the source or target vertex of a path γ_i of a graph Γ, then the map maps the path γ_i to zero in both entries. For different surfaces or for a fixed surface different maps refer to different quantum flux operators. Furthermore, the quantum flux operators are also defined as maps form the graph Γ to direct sum $\mathcal{E} \oplus \mathcal{E}$ of the universal enveloping algebra $\mathcal{E}$ of $\mathfrak{g}$.

Definition 3.4.1. *Let $\breve{S}$ be a finite set $\{S_i\}$ of surfaces in Σ, which is closed under a flip of orientation of the surfaces. Let Γ be a graph such that each path in Γ satisfies one of the following conditions*

- *the path intersects each surface in $\breve{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path intersects each surface in $\breve{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path intersects each surface in $\breve{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path does not intersect any surface S contained in $\breve{S}$.*

Then define the intersection functions $\iota_L : \breve{S} \times \Gamma \rightarrow \{\pm 1, 0\}$ such that

$$\iota_L(S,\gamma) := \begin{cases} 1 & \text{for a path } \gamma \text{ lying above and outgoing w.r.t. } S \\ -1 & \text{for a path } \gamma \text{ lying below and outgoing w.r.t. } S \\ 0 & \text{the path } \gamma \text{ is not outgoing w.r.t. } S \end{cases}$$

and the intersection functions $\iota_R : \breve{S} \times \Gamma \rightarrow \{\pm 1, 0\}$ such that

$$\iota_L(S,\gamma) := \begin{cases} -1 & \text{for a path } \gamma' \text{ lying above and ingoing w.r.t. } S \\ 1 & \text{for a path } \gamma' \text{ lying below and ingoing w.r.t. } S \\ 0 & \text{the path } \gamma' \text{ is not ingoing w.r.t. } S \end{cases}$$

whenever $S \in \breve{S}$ and $\gamma \in \Gamma$.

Define a map $\sigma_L : \breve{S} \rightarrow \mathfrak{g}$ such that

$$\sigma_L(S) = \sigma_L(S^{-1})$$

whenever $S \in \breve{S}$ and S^{-1} is the surface S with reversed orientation. Denote the set of such maps by $\breve{\sigma}_L$. Respectively, the map $\sigma_R : \breve{S} \rightarrow \mathfrak{g}$ such that

$$\sigma_R(S) = \sigma_R(S^{-1})$$

whenever $S \in \breve{S}$. Denote the set of such maps by $\breve{\sigma}_R$. Moreover, there is a map $\sigma_L \times \sigma_R : \breve{S} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ such that

$$(\sigma_L, \sigma_R)(S) = (\sigma_L, \sigma_R)(S^{-1})$$

whenever $S \in \breve{S}$. Denote the set of such maps by $\breve{\sigma}$.

*Finally, define the **Lie algebra-valued quantum flux set for paths***

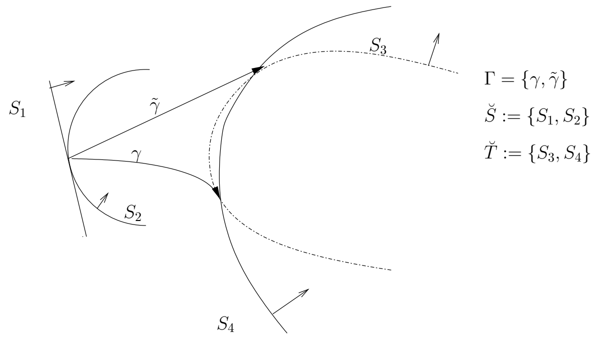
$$g_{\breve{S},\Gamma} := \bigcup_{\sigma_L \times \sigma_R \in \breve{\sigma}} \bigcup_{S \in \breve{S}} \Big\{ (E^L, E^R) \in \mathrm{Map}(\Gamma, \mathfrak{g} \oplus \mathfrak{g}) : \quad (E^L, E^R)(\gamma) := (\iota_L(S,\gamma)\sigma_L(S), \iota_R(S,\gamma)\sigma_R(S)) \Big\}$$

where $\mathrm{Map}(\Gamma, \mathfrak{g} \oplus \mathfrak{g})$ *is the set of all maps from the graph Γ to the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ of Lie algebras.*

Observe that, $(\iota_L \times \iota_R)(S^{-1}, \gamma) = (-\iota_L \times -\iota_R)(S, \gamma)$ for every $\gamma \in \Gamma$ holds.

Remark that the condition $E^L(\gamma) = E^R(\gamma^{-1})$ is not required.

Example 3.4.1: Analyse the following example. Consider a graph Γ and two disjoint surface sets $\breve{S}$ and $\breve{T}$.



Then the elements of $\mathfrak{g}_{\breve{S},\Gamma}$ are for example given by the maps $E_i^L \times E_i^R$ for $i = 1, 2$ such that

$$\begin{aligned}
E_1(\gamma) &:= (E_1^L, E_1^R)(\gamma) = (\iota_L(S_1,\gamma)\sigma_L(S_1), \iota_R(S_1,\gamma)\sigma_R(S_1)) = (X_1, 0)\\
E_1(\tilde{\gamma}) &:= (E_1^L, E_1^R)(\tilde{\gamma}) = (\iota_L(S_1,\tilde{\gamma})\sigma_L(S_1), \iota_R(S_1,\tilde{\gamma})\sigma_R(S_1)) = (X_1, 0)\\
E_2(\gamma) &:= (E_2^L, E_2^R)(\gamma) = (\iota_L(S_2,\gamma)\sigma_L(S_2), \iota_R(S_2,\gamma)\sigma_R(S_2)) = (X_2, 0)\\
E_2(\tilde{\gamma}) &:= (E_2^L, E_2^R)(\tilde{\gamma}) = (\iota_L(S_2,\tilde{\gamma})\sigma_L(S_2), \iota_R(S_2,\tilde{\gamma})\sigma_R(S_2)) = (X_2, 0)\\
E_3(\gamma) &:= (E_3^L, E_3^R)(\gamma) = (\iota_L(S_3,\gamma)\sigma_L(S_3), \iota_R(S_3,\gamma)\sigma_R(S_3)) = (0, -Y_3)\\
E_3(\tilde{\gamma}) &:= (E_3^L, E_3^R)(\tilde{\gamma}) = (\iota_L(S_3,\tilde{\gamma})\sigma_L(S_3), \iota_R(S_3,\tilde{\gamma})\sigma_R(S_3)) = (0, -Y_3)\\
E_4(\gamma) &:= (E_4^L, E_4^R)(\gamma) = (\iota_L(S_4,\gamma)\sigma_L(S_4), \iota_R(S_4,\gamma)\sigma_R(S_4)) = (0, Y_4)\\
E_4(\tilde{\gamma}) &:= (E_4^L, E_4^R)(\tilde{\gamma}) = (\iota_L(S_4,\tilde{\gamma})\sigma_L(S_4), \iota_R(S_4,\tilde{\gamma})\sigma_R(S_4)) = (0, Y_4)
\end{aligned}$$

This example shows that, the surfaces $\{S_1, S_2\}$ are similar, whereas the surfaces $\{T_1, T_2\}$ produce different signatures for different paths. Moreover, the set of surfaces are chosen such that one component of the direct sum is always zero.

∎

For a particular surface set $\breve{S}$, the set

$$\bigcup_{\sigma_L\times\sigma_R\in\breve{\sigma}}\ \bigcup_{S\in\breve{S}} \Big\{(E^L, E^R) \in \text{Map}(\Gamma, \mathfrak{g}\oplus\mathfrak{g}) : \quad (E^L, E^R)(\gamma) := (\iota_L(S,\gamma)\sigma_L(S), 0)\Big\}$$

is identified with

$$\bigcup_{\sigma_L\in\breve{\sigma}_L}\ \bigcup_{S\in\breve{S}} \Big\{E \in \text{Map}(\Gamma, \mathfrak{g}) : \quad E(\gamma) := \iota_L(S,\gamma)\sigma_L(S)\Big\}$$

The same is observed for another surface set $\breve{T}$ and the set $\mathfrak{g}_{\breve{T},\Gamma}$ is identifiable with

$$\bigcup_{\sigma_R\in\breve{\sigma}_R}\ \bigcup_{T\in\breve{T}} \Big\{E \in \text{Map}(\Gamma, \mathfrak{g}) : \quad E(\gamma) := \iota_R(T,\gamma)\sigma_R(T)\Big\}$$

The intersection behavoir of paths and surfaces plays a fundamental role in the definition of the quantum flux operator. There are exceptional configurations of surfaces and paths in a graph. One of them is the following.

Definition 3.4.2. *A surface S has the* ***surface intersection property for a graph*** Γ *iff the surface intersects each path of* Γ *once in the source or target vertex of the path and there are no other intersection points of S and the path.*

This is for example the case for the surface S_1 or the surface S_3, which are presented in example 3.4.1. Notice that in general, for the surface S there are N intersection points with N paths of the graph. In the example the evaluated map $E_1(\gamma) = (X_1, 0) = E_1(\tilde{\gamma})$ for $\gamma, \tilde{\gamma} \in \Gamma$ if the surface S_1 is considered.

The property of a path lying above or below is not important for the definition of the surface intersection property for a surface. This indicates that, the surface S_4 in the example 3.4.1 has the surface intersection property, too.

Let a surface S does not have the surface intersection property for a graph Γ, which contains only one path γ. Then for example the path γ intersects the surface S in the source and target vertices such that the path lies above the surface S. Then the map $E^L \times E^R$ is evaluated for the path γ by

$$(E^L \times E^R)(\gamma) = (X, -Y)$$

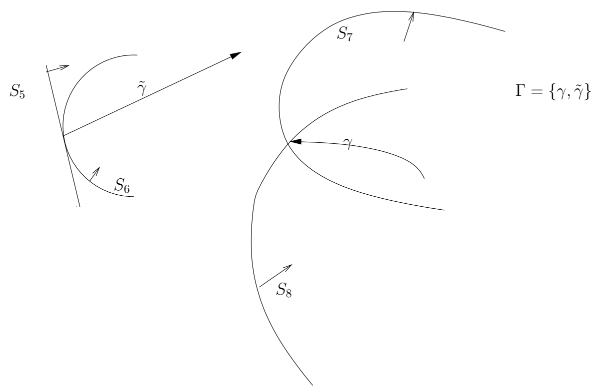
Hence, simply speaking the surface intersection property reduces the components of the map $E^L \times E^R$, but for different paths to different components.

Now, consider a bunch of sets of surfaces such that for each surface there is only one intersection point.

Definition 3.4.3. *A set $\breve{S}$ of N surfaces has the* ***surface intersection property for a graph*** *Γ with N independent edges iff it contains only surfaces, for which each path γ_i of a graph Γ intersects each surface S_i only once in the same source or target vertex of the path γ_i, there are no other intersection points of each path γ_i and each surface in $\breve{S}$, and there is no other path γ_j that intersects the surface S_i for $i \neq j$ where $1 \leq i, j \leq N$.*

Then for example consider the following configuration.

Example 3.4.2:



The sets $\{S_6, S_7\}$ or $\{S_5, S_8\}$ have the surface intersection property for the graph Γ. The images of a map E is

$$E_5(\tilde{\gamma}) = (X_5, 0), \quad E_8(\gamma) = (0, Y_8)$$

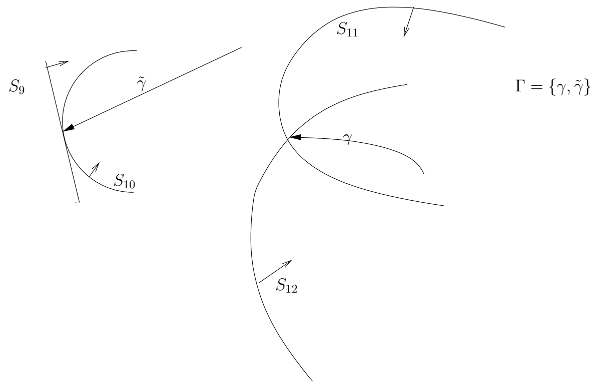
■

Note that, simply speaking the property indicates that each map reduces to a component of $E^L \times E^R$ but for different surfaces the map reduces to E^L or E^R.

A set of surfaces that has the surface intersection property for a graph can be further specialised by restricting the choice to paths lying ingoing and below with respect to the surface orientations.

Definition 3.4.4. *A set $\breve{S}$ of N surfaces has the* ***simple surface intersection property for a graph*** *Γ with N independent edges iff it contain only surfaces, for which each path γ_i of a graph Γ intersects only one surface S_i only once in the target vertex of the path γ_i, the path γ_i lies above and there are no other intersection points of each path γ_i and each surface in $\breve{S}$.*

Example 3.4.3: Consider the following example.



The sets $\{S_9, S_{11}\}$ or $\{S_{10}, S_{12}\}$ have the simple surface intersection property for the graph Γ. Calculate

$$E_9(\tilde{\gamma}) = (0, -Y_9), \quad E_{11}(\gamma) = (0, -Y_{11})$$

■

In this case the set $\mathfrak{g}_{\breve{S},\Gamma}$ reduces to

$$\bigcup_{\sigma_R \in \breve{\sigma}_R} \bigcup_{S \in \breve{S}} \Big\{ E \in \text{Map}(\Gamma, \mathfrak{g}) : \quad E(\gamma) := -\sigma_R(S) \text{ for } \gamma \cap S = t(\gamma) \Big\}$$

Notice that, the set $\Gamma \cap \breve{S} = \{t(\gamma_i)\}$ for a surface $S_i \in \breve{S}$ and $\gamma_i \cap S_j \cap S_i = \{\varnothing\}$ for a path γ_i in Γ and $i \neq j$.

On the other hand, the set of surfaces can be such that each path of a graph intersect all surfaces of the set in the same vertex. This contradicts the assumption that, each path of a graph intersects only one surface once.

Definition 3.4.5. *Let Γ be a graph that contains no loops.*

*A set $\breve{S}$ of surfaces has the **same surface intersection property for a graph** Γ iff each path γ_i in Γ intersects with all surfaces of $\breve{S}$ in the same source vertex $v_i \in V_\Gamma$ $(i = 1, .., N)$, all paths are outgoing and lie below each surface $S \in \breve{S}$ and there are no other intersection points of each path γ_i and each surface in $\breve{S}$.*

*A surface set $\breve{S}$ has the **same right surface intersection property for a graph** Γ iff each path γ_i in Γ intersects with all surfaces of $\breve{S}$ in the same target vertex $v_i \in V_\Gamma$ $(i = 1, .., N)$, all paths are ingoing and lie above each surface $S \in \breve{S}$ and there are no other intersection points of each path γ_i and each surface in $\breve{S}$.*

Recall the example 3.4.1. Then the set $\{S_1, S_2\}$ has the same surface intersection property for the graph Γ.

Then the set $\mathfrak{g}_{\breve{S},\Gamma}$ reduces to

$$\bigcup_{\sigma_L \in \breve{\sigma}_L} \bigcup_{S \in \breve{S}} \Big\{ E \in \text{Map}(\Gamma, \mathfrak{g}) : \quad E(\gamma) := -\sigma_L(S) \text{ for } \gamma \cap S = s(\gamma) \Big\}$$

Notice that, $\gamma \cap S_1 \cap ... \cap S_N = s(\gamma)$ for a path γ in Γ, whereas $\Gamma \cap \breve{S} = \{s(\gamma_i)\}_{1 \leq i \leq N}$. Clearly, $\Gamma \cap S_i = s(\gamma_i)$ holds for a surface S_i in $\breve{S}$.

Simply speaking the physical intution is that fluxes associated to different surfaces should act on the same path.

A very special configuration is the following.

Definition 3.4.6. *A set $\breve{S}$ of surfaces has the **same surface intersection property for a graph** Γ **containing only loops** iff each loop γ_i in Γ intersects with all surfaces of $\breve{S}$ in the same vertices $s(\gamma_i) = t(\gamma_i)$ in V_Γ $(i = 1, .., N)$, all loops lie below each surface $S \in \breve{S}$ and there are no other intersection points of each loop in Γ and each surface in $\breve{S}$.*

Notice that, both properties can be restated for other surface and path configurations. Hence, a surface set can have the simple or same surface intersection property for paths that are outgoing and lie above (or ingoing and below, or outgoing and below). The important fact is related to the question if the intersection vertices are the same for all surfaces or not.

In section 3.1.3 the concept of finite graph systems is introduced. The following remark shows that, the properties simply generalises to this new structure.

Remark 3.4.7. *A set $\breve{S}$ has the surface intersection property for a finite graph system $\mathcal{P}_\Gamma$ iff the set $\breve{S}$ has the surface intersection property for each subgraph of Γ and Γ.*

A set $\breve{S}$ has the same surface intersection property for a finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ associated to a graph Γ (with no loops) iff the set $\breve{S}$ has the same surface intersection property the graph Γ.

A set $\breve{S}$ has the simple surface intersection property for a finite orientation preserved[4] *graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ associated to a graph Γ iff the set $\breve{S}$ has the simple surface intersection property for the graph Γ.*

Definition 3.4.8. *Let $\breve{S}$ be a surface set and Γ be a graph such that the only intersections of the graph and each surface in $\breve{S}$ are contained in the vertex set V_Γ.*

Then the set of images $\{E(\gamma) : E \in \mathrm{g}_{\breve{S},\Gamma}\}$ of flux maps for a fixed path γ in Γ is denoted by $\bar{\mathrm{g}}_{\breve{S},\gamma}$.

Proposition 3.4.9. *Let $\breve{S}$ be a set of surfaces and Γ be a fixed graph (with no loops) such that the set $\breve{S}$ has the same surface intersection property for a graph Γ. Moreover, let $\breve{T}$ be a set of surfaces and Γ be a fixed graph such that the set $\breve{T}$ has the simple surface intersection property for a graph Γ.*

Then the set $\bar{\mathrm{g}}_{\breve{S},\gamma}$ is equipped with a structure, which is induced from the Lie algebra structure of $\mathfrak{g}$, such that it forms a Lie algebra. The the set $\bar{\mathrm{g}}_{\breve{T},\gamma}$ is equipped with a structure to form a Lie algebra, too.

Proof : **Step 1: linear space over $\mathbb{C}$**
Consider a path γ in Γ that lies above and ingoing w.r.t. the surface orientation of each surface S in $\breve{S}$ and ingoing and above with respect to T. Then there is a map E_S such that

$$E_S(\gamma) = -X$$

There exists an operation + given by the map $s : \bar{\mathrm{g}}_{\breve{S},\gamma} \times \bar{\mathrm{g}}_{\breve{S},\gamma} \rightarrow \bar{\mathrm{g}}_{\breve{S},\gamma}$ such that

$$(E_1^L(\gamma), E_2^L(\gamma)) \mapsto s(E_1^L(\gamma), E_2^L(\gamma)) := E_1^L(\gamma) + E_2^L(\gamma) = -\sigma_L^1(S_1) - \sigma_L^2(S_2) = -\sigma_L^3([S])$$

since $\sigma_L^i \in \breve{\sigma}_L$ and where $[S]$ denotes an arbitrary representative of the set $\breve{S}$. Respectively it is defined

$$(E_1^L(\gamma), E_2^L(\gamma)) \mapsto s(E_1^L(\gamma), E_2^L(\gamma)) := E_1^L(\gamma) + E_2^L(\gamma) = -\sigma_R^1(T) - \sigma_R^2(T) = -\sigma_R^3(T)$$

whenever $\sigma_R^i \in \breve{\sigma}_R$ and $T \in \breve{T}$. There is an inverse

$$E(\gamma) - E(\gamma) = X - X = 0$$

and a null element

$$E(\gamma) + E_0(\gamma) = X$$

whenever $E_0(\gamma) = -\sigma_L(S) = 0$. Notice the following map

$$\bar{\mathrm{g}}_{\breve{S},\gamma} \times \bar{\mathrm{g}}_{\breve{S},\gamma'} \ni (E_1(\gamma), E_2(\gamma')) \mapsto E_1(\gamma) \dot{+} E_2(\gamma') \in \mathfrak{g} \tag{3.93}$$

is not considered, since, this map is not well-defined. One can show easily that $(\bar{\mathrm{g}}_{\breve{S},\gamma}, +)$ is an additive group. The scalar multiplication is defined by

$$\lambda \cdot E(\gamma) = \lambda X$$

[4]Let $\breve{S}$ be equal to S. Then notice that, the property of all graphs being orientation preserved subgraphs is necessary, since, for a subgraph $\Gamma' := \{\gamma'\}$ of Γ the graph $\{\gamma'^{-1}\}$ is a subgraph of Γ, too. Consequently, if there is a surface S intersecting a path γ' such that γ' is ingoing and lies above, then S intersects the path γ'^{-1} such that γ'^{-1} is outgoing and lies above. This implies that, the surface S cannot have the same surface intersection property for each subgraph of Γ.

for all $\lambda \in \mathbb{C}$ and $X \in \mathfrak{g}$. Finally, prove that $(\bar{\mathfrak{g}}_{\breve{S},\gamma}, +)$ is a linear space over $\mathbb{C}$.

Step 2: Lie bracket is defined by the Lie bracket of the Lie algebra $\mathfrak{g}$ and

$$[E_1(\gamma), E_2(\gamma)] := [X_1, X_2]$$

for $E_1(\gamma), E_2(\gamma) \in \bar{\mathfrak{g}}_{\breve{S},\gamma}$ and $\gamma \in \Gamma$.

■

If a surface set $\breve{S}$ does not have the same or simple surface intersection property for the graph Γ, then the surface set can be decomposed into several sets and the graph Γ can be decomposed into a set of subgraphs. Then for each modified surface set there is a subgraph such that required condition is fulfilled.

Definition 3.4.10. *Let $\breve{S}$ a set of surfaces and Γ be a fixed graph (with no loops) such that the set $\breve{S}$ has the same (or simple) surface intersection property for a graph Γ.*

The universal enveloping Lie algebra of the Lie algebra $\bar{\mathfrak{g}}_{\breve{S},\gamma}$ of fluxes for paths of a path γ in Γ and all surfaces in $\breve{S}$ is called the ***universal enveloping flux algebra $\bar{\mathbb{E}}_{\breve{S},\gamma}$ associated to a path and a finite set of surfaces****.*

For a detailed construction of the universal enveloping flux algebra associated to a surface set and a path refer to the section 8.2.1 and definition 8.2.6.

Now, the definitions are rewritten for finite orientation preserved graph systems.

Definition 3.4.11. *Let $\breve{S}$ be a surface set and Γ be a graph such that the only intersections of the graph and each surface in $\breve{S}$ are contained in the vertex set V_Γ. $\mathcal{P}_\Gamma$ denotes the finite graph system associated to Γ. Let $\mathcal{E}$ be the universal Lie enveloping algebra of $\mathfrak{g}$.*

Define the set of ***Lie algebra-valued quantum flux operators for graphs***

$$\begin{aligned}\mathfrak{g}_{\breve{S},\Gamma} := \bigcup_{\sigma_L\times\sigma_R\in\breve{\sigma}} \bigcup_{S\in\breve{S}} \Big\{ E_{S,\Gamma} \in \text{Map}(\mathcal{P}_\Gamma, \bigoplus_{|E_\Gamma|}\mathfrak{g} \oplus \bigoplus_{|E_\Gamma|}\mathfrak{g}) : \quad & E_{S,\Gamma} := E_S \times ... \times E_S \\ & \text{where } E_S(\gamma) := (\iota_L(\gamma,S)\sigma_L(S), \iota_R(\gamma,S)\sigma_R(S)), \\ & E_S \in \mathfrak{g}_{\breve{S},\Gamma}, S \in \breve{S}, \gamma \in \Gamma \Big\}\end{aligned}$$

Moreover, define

$$\begin{aligned}\mathcal{E}_{\breve{S},\Gamma} := \bigcup_{\sigma_L\times\sigma_R\in\breve{\sigma}} \bigcup_{S\in\breve{S}} \Big\{ E_{S,\Gamma} \in \text{Map}(\mathcal{P}_\Gamma, \bigoplus_{|E_\Gamma|}\mathcal{E} \oplus \bigoplus_{|E_\Gamma|}\mathcal{E}) : \quad & E_{S,\Gamma} := E_S \times ... \times E_S \\ & \text{where } E_S(\gamma) := (\iota_L(\gamma,S)\sigma_L(S), \iota_R(\gamma,S)\sigma_R(S)), \\ & E_S \in \mathcal{E}_{\breve{S},\Gamma}, S \in \breve{S}, \gamma \in \Gamma \Big\}\end{aligned}$$

The set of all images of the linear hull of all maps in $\mathfrak{g}_{\breve{S},\Gamma}$ for a fixed surface set $\breve{S}$ and a fixed graph Γ is denoted by $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$. The set of all images of the linear hull of all maps in $\mathfrak{g}_{\breve{S},\Gamma}$ for a fixed surface set $\breve{S}$ and a fixed subgraph Γ' of Γ is denoted by $\bar{\mathfrak{g}}_{\breve{S},\Gamma'\leq\Gamma}$.

Note that, the set of Lie algebra-valued quantum flux operators for graphs can be generalised for the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$. This follows from the fact that each element of the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$ is a graph.

Proposition 3.4.12. *Let $\breve{S}$ be a set of surfaces and $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be a finite orientation preserved graph system such that the set $\breve{S}$ has the same surface intersection property for a graph Γ (with no loops).*

The set $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ forms a Lie algebra and is called the ***Lie flux algebra associated to a graph and a finite surface set****. The* ***universal enveloping flux algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ associated to a graph and a finite surface set*** *is the enveloping Lie algebra of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$.*

Proof. This follows from the observation that $\mathfrak{g}_{\breve{S},\Gamma}$ is identified with

$$\bigcup_{\sigma_L\in\breve{\sigma}_L}\bigcup_{S\in\breve{S}}\Big\{E_{S,\Gamma}\in\text{Map}(\mathcal{P}^{\mathfrak{o}}_\Gamma,\bigoplus_{|E_\Gamma|}\mathfrak{g}):\quad E_{S,\Gamma}:=E_S\times ...\times E_S$$
$$\text{where } E_S(\gamma):=-\sigma_L(S),E_S\in\mathfrak{g}_{\breve{S},\Gamma},S\in\breve{S},\gamma\in\Gamma\Big\}$$

and the addition operation

$$\begin{aligned}E^1_{S_1,\Gamma}(\Gamma)+E^2_{S_2,\Gamma}(\Gamma)&:=\big(E^1_{S_1}(\gamma_1)+E^2_{S_2}(\gamma_1),...,E^1_{S_1}(\gamma_N)+E^2_{S_2}(\gamma_N)\big)\\&=(-\sigma^1_L(S_1)-\sigma^2_L(S_2),...,-\sigma^1_L(S_1)-\sigma^2_L(S_2))\\&=(E^3_{[S]}(\gamma_1),...,E^3_{[S]}(\gamma_N))\end{aligned}$$

whenever $\Gamma:=\gamma_1,...,\gamma_N$. □

Notice that indeed it is true that,

$$\mathfrak{g}_{\breve{S},\Gamma}=\mathfrak{g}_{S_i,\Gamma}$$

yields for every $S_i\in\breve{S}$. The more general definition is due to physical arguments.

Proposition 3.4.13. *Let $\breve{T}$ be a set of surfaces and $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be a finite orientation preserved graph system such that the set $\breve{T}$ has the simple surface intersection property for Γ.*

The set $\bar{\mathfrak{g}}_{\breve{T},\Gamma}$ forms a Lie algebra.

Notice this follows from the fact that, $\mathfrak{g}_{\breve{T},\Gamma}$ reduces to

$$\bigcup_{\sigma_L\in\breve{\sigma}_L}\Big\{E_{\breve{T},\Gamma}\in\text{Map}(\mathcal{P}^{\mathfrak{o}}_\Gamma,\bigoplus_{|E_\Gamma|}\mathfrak{g}):\quad E_{\breve{T},\Gamma}:=E_{T_1}\times ...\times E_{T_N}$$
$$\text{where } E_{T_i}(\gamma_i):=-\sigma_L(T_i),E_S\in\mathfrak{g}_{\breve{S},\Gamma},T_i\in\breve{T},$$
$$\gamma_i\cap T_i=t(\gamma_i),\gamma\in\Gamma\Big\}$$

since,

$$\begin{aligned}E_{S_1,\Gamma}(\Gamma)+...+E_{S_N,\Gamma}(\Gamma)&=(E_{T_1}(\gamma_1),0,...,0))+(0,E_{T_2}(\gamma_2),0,...,0))+...+(0,...,0,E_{T_N}(\gamma_N)))\\&=(E_{T_1}(\gamma_1),...,E_{T_N}(\gamma_N))=:E_{\breve{T},\Gamma}(\Gamma)\end{aligned}$$

If it is additionally required that $E^L_S(\gamma_1)=-E^R_S(\gamma_2)$ holds, then actions of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ on a configuration space have to be very carefully implemented. This will be deeply analysed in section 6.1.

The Lie flux algebra and the universal enveloping flux algebra for the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$ and a fixed suitable surface set $\breve{S}$ is denoted by $\bar{\mathfrak{g}}_{\breve{S}}$ and $\bar{\mathcal{E}}_{\breve{S}}$.

The discretised and localised quantum flux operator

Now, consider a restriction of the quantum flux operators to discretised surfaces and graphs. Notice that, the Lie algebra-valued quantum flux operator usually distinguishes between paths, which are lying below, and paths, which are lying above the surface in a surface set. For simplicity in this dissertation the case of outgoing paths that lie below is considered only. With no doubt, the second case can be defined analogously. The discretised surfaces do not allow to distinguish between paths lying above or below with respect to a surface orientation of a surface. Hence in this situation the discretised surface set has to be associated to a set of surfaces with a surface orientation. Summarising the cases below or above are not treated in the context of discretised surfaces. In this way, the intersection functions of definition 3.4.1 are maps such that $\iota_L:\breve{S}_{\text{d}}\times\Gamma\to\{0,-1\}$ and $\iota_R:\breve{S}_{\text{d}}\times\Gamma\to\{0,1\}$.

Definition 3.4.14. *Let $\breve{S}_{\rm d}$ be a set of discretised surfaces, which is constructed from a set $\breve{S}$ of surfaces, and let Γ be a graph. Let $\{\Gamma_i\}_{i=1,..\infty}$ be an inductive family of graphs such that for every graph Γ_i the intersection points of a surface set $\breve{S}_{\rm d}$ and the graph Γ_i are vertices of V_{Γ_i}. Denote the set of intersections of a graph Γ_i and a discretised surface set $\breve{S}_{\rm d}$ by $i_S(\{\Gamma_i\})$.*

Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Furthermore, assume that, the set $\breve{S}$ is chosen such that

(i) for each graph of the family the surface set $\breve{S}$ has the same surface intersection property,

(ii) the inductive structure preserves the same surface intersection property[5] for $\breve{S}$ and

(iii) each surface in $\breve{S}$ intersects the inductive limit Γ_∞ a finite or an infinite number of vertices.

Then $E_{S_{\rm d}}(\Gamma)^+E_{S_{\rm d}}(\Gamma)$ denote the (Lie algebra-valued) ***discretised quantum flux operator associated to a surface*** *$S_{\rm d}$* ***and a graph*** *Γ such that $S_{\rm d}\cap\Gamma$ is a subset of the set of vertices V_Γ and $E_{S_{\rm d}}\in\mathfrak{g}_{\breve{S}_{\rm d},\Gamma}$.*

The (Lie algebra-valued) ***discretised and localised quantum flux operator*** *$\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})$* ***associated to a surface*** *$S_{\rm d}$* ***and an inductive family of graphs*** *$\{\Gamma_i\}_{i=1,..\infty}$ is defined by the difference operator*

$$\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1}) := E_{S_{\rm d}}(\Gamma_{i+1})^+E_{S_{\rm d}}(\Gamma_{i+1}) - E_{S_{\rm d}}(\Gamma_i)^+E_{S_{\rm d}}(\Gamma_i)$$

for $E_{S_{\rm d}}(\Gamma_i)\in\bar{\mathfrak{g}}_{\breve{S}_{\rm d},\Gamma_i}$ and $E_{S_{\rm d}}(\Gamma_{i+1})\in\bar{\mathfrak{g}}_{\breve{S}_{\rm d},\Gamma_{i+1}}$ such that

(i) $\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})$ is non-trivial only for intersections of the surfaces in $\breve{S}$ and the graph Γ_{i+1} in vertices contained in the set $i_S(\{\Gamma_{i+1}\})\setminus i_S(\{\Gamma_i\})$ and

(ii) $(\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1}))^+ = \tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})$ yields.

The set of such discretised and localised quantum flux operators associated to a graph Γ, for example given by $\tilde{E}_{S_{\rm d}}(\Gamma)^+\tilde{E}_{S_{\rm d}}(\Gamma)$, is denoted by $\bar{\mathfrak{g}}^{\rm loc}_{\breve{S}_{\rm d},\Gamma}$ and called the ***localised Lie flux algebra associated to a discretised surface set and a graph****. The set of such discretised and localised quantum flux operator associated to an inductive family of graphs $\{\Gamma_i\}_{i=1,..\infty}$, for example given by $\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})$, is denoted by $\bar{\mathfrak{g}}^{\rm loc}_{\breve{S}_{\rm d}}$ and called the* ***localised Lie flux algebra associated to a discretised surface set*** *(and an inductive family of graphs).*

The ***localised enveloping flux algebra*** *$\mathcal{E}^{\rm loc}_{\breve{S}_{\rm d},\Gamma}$* ***associated to a discretised surface set and a graph*** *is given by the enveloping algebra of the localised Lie flux algebra associated to a discretised surface set $\breve{S}_{\rm d}$ and the graph Γ.*

Finally, the ***localised enveloping flux algebra*** *$\mathcal{E}^{\rm loc}_{\breve{S}_{\rm d}}$* ***associated to a discretised surface set*** *(and an inductive family of graphs) is given by the enveloping algebra of the localised Lie flux algebra associated to a discretised surface set $\breve{S}_{\rm d}$.*

If the situation of all paths are ingoing w.r.t all surfaces in a set $\breve{S}$, then the localised Lie flux algebra associated to a discretised surface set $\breve{S}_{\rm d}$ associated to $\breve{S}$ and an inductive family of graphs is denoted by $\bar{\mathfrak{g}}^{\breve{S}_{\rm d}}_{\rm loc}$.

3.4.3 The group-valued quantum flux operators associated to surfaces and graphs

On the other hand, the exponentiated fluxes can be encoded in a flux set associated to surfaces. In the following considerations G is, therefore, assumed to be a locally compact group.

Definition 3.4.15. *Let $\breve{S}$ be a finite set $\{S_i\}$ of surfaces in Σ, which is closed under a flip of orientation of the surfaces. Let Γ be a graph such that each path in Γ satisfies one of the following conditions*

· *the path intersects each surface in $\breve{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*

[5] In particular, a graph Γ', which has the same intersection surface property for $\breve{S}$, has the same intersection behavior for $\breve{S}$ if Γ' is considered as a subgraph of a graph Γ, too.

- *the path intersects each surface in $\breve{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path intersects each surface in $\breve{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path does not intersect any surface S contained in $\breve{S}$.*

Finally, let $\mathcal{P}_\Gamma$ denote the finite graph system associated to Γ.

Define a map $o_L : \breve{S} \rightarrow G$ such that

$$o_L(S) = o_L(S^{-1})$$

whenever $S \in \breve{S}$ and S^{-1} is the surface S with reversed orientation. Denote the set of such maps by $\breve{o}_L$. Respectively, the map $o_R : \breve{S} \rightarrow G$ such that

$$o_R(S) = o_R(S^{-1})$$

whenever $S \in \breve{S}$. Denote the set of such maps by $\breve{o}_R$. Moreover, there is a map $o_L \times o_R : \breve{S} \rightarrow G \times G$ such that

$$(o_L, o_R)(S) = (o_L, o_R)(S^{-1})$$

whenever $S \in \breve{S}$. Denote the set of such maps by $\breve{o}$.

*Then define the **group-valued quantum flux set for paths***

$$\mathbb{G}_{\breve{S},\Gamma} := \bigcup_{o_L \times o_R \in \breve{o}} \bigcup_{S \in \breve{S}} \Big\{ (\rho^L, \rho^R) \in \text{Map}(\Gamma, G \times G) : \quad (\rho^L, \rho^R)(\gamma) := (o_L(S)^{\iota_L(S,\gamma)}, o_R(S)^{\iota_R(S,\gamma)}) \Big\}$$

where $\text{Map}(\Gamma, G \times G)$ is the set of all maps from the graph Γ to the direct product $G \times G$.

*Define the **set of group-valued quantum flux operators for graphs***

$$\begin{aligned} G_{\breve{S},\Gamma} := \bigcup_{o_L \times o_R \in \breve{o}} \bigcup_{S \in \breve{S}} \Big\{ \rho_{S,\Gamma} \in \text{Map}(\mathcal{P}_\Gamma^{\mathfrak{o}}, G^{|\Gamma|} \times G^{|\Gamma|}) : \quad & \rho_{S,\Gamma} := \rho_S \times ... \times \rho_S \\ & \text{where } \rho_S(\gamma) := (o_L(S)^{\iota_L(\gamma,S)}, o_R(S)^{\iota_R(\gamma,S)}), \\ & \rho_S \in \mathbb{G}_{\breve{S},\Gamma}, S \in \breve{S}, \gamma \in \Gamma \Big\} \end{aligned}$$

Notice if H is a closed subgroup of G, then $H_{\breve{S},\Gamma}$ can be defined in analogy to $G_{\breve{S},\Gamma}$. In particular, if the group H is replaced by the center $\mathcal{Z}(G)$ of the group G, then the set $\mathbb{G}_{\breve{S},\Gamma}$ is replaced by $\mathcal{Z}(\mathbb{G}_{\breve{S},\gamma})_{\breve{S},\Gamma}$ and $G_{\breve{S},\Gamma}$ is changed to $\mathcal{Z}_{\breve{S},\Gamma}$.

There is a generalisation of the set of group-valued quantum flux operators for the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$.

Definition 3.4.16. *The set of all images of maps in $\mathbb{G}_{\breve{S},\Gamma}$ for a fixed surface set $\breve{S}$ and a fixed path γ in Γ is denoted by $\bar{\mathbb{G}}_{\breve{S},\gamma}$.*

The set of all finite products of images of maps in $G_{\breve{S},\Gamma}$ for a fixed surface set $\breve{S}$ and a fixed graph Γ is denoted by $\bar{G}_{\breve{S},\Gamma}$.

The product $\cdot$ on $\bar{G}_{\breve{S},\Gamma}$ is given by

$$\begin{aligned} \rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma) &= (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), ..., \rho_{S_1}(\gamma_N) \cdot \rho_{S_2}(\gamma_N)) \\ &= (o_L(S_1)^{-1} o_L(S_2)^{-1}, ..., o_L(S_2)^{-1} o_L(S_1)^{-1}) = ((o_L(S_2) o_L(S_1))^{-1}, ..., (o_L(S_2) o_L(S_1))^{-1}) \\ &= \rho_{S_3,\Gamma}(\Gamma) \end{aligned}$$

Definition 3.4.17. *Let S be a surface and Γ be a graph such that the only intersections of the graph and the surface in S are contained in the vertex set V_Γ. Moreover, let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid associated to Γ.*

Then define the set for a fixed surface S by

$$\mathrm{Map}_S(\mathcal{P}_\Gamma\Sigma, G\times G) := \bigcup_{o_L\times o_R\in\breve{o}}\bigcup_{S\in\breve{S}}\Big\{(\rho^L,\rho^R)\in\mathrm{Map}(\mathcal{P}_\Gamma\Sigma, G\times G):\quad (\rho^L,\rho^R)(\gamma):=(o_L(S)^{\iota_L(S,\gamma)}, o_R(S)^{\iota_R(S,\gamma)})\Big\}$$

Proposition 3.4.18. *Let $\breve{S}$ a set of surfaces and Γ be a fixed graph, which contains no loops, such that the set $\breve{S}$ has the same surface intersection property for the graph Γ.*

The set $\bar{\mathbb{G}}_{\breve{S},\gamma}$ has the structure of a group.

The group $\bar{\mathbb{G}}_{\breve{S},\gamma}$ is called the ***flux group associated to a path and a finite set of surfaces****.*

Proof : This follows easily from the observation that in this case $\mathbb{G}_{\breve{S},\gamma}$ reduces to

$$\bigcup_{o_L\in\breve{o}_L}\bigcup_{S\in\breve{S}}\Big\{\rho^L\in\mathrm{Map}(\Gamma,G):\quad \rho^L(\gamma):=o_L(S)^{-1}\text{ for }\gamma\cap S=s(\gamma)\Big\}$$

There always exists a map $\rho^L_{S,3}\in\mathbb{G}_{\breve{S},\gamma}$ such that the following equation defines a multiplication operation

$$\rho^L_{S,1}(\gamma)\cdot\rho^L_{S,2}(\gamma)=g_1g_2:=\rho^L_{S,3}(\gamma)\in\bar{\mathbb{G}}_{\breve{S},\gamma}$$

with inverse $(\rho^L_S(\gamma))^{-1}$ such that

$$\rho^L_S(\gamma)\cdot(\rho^L_S(\gamma))^{-1}=(\rho^L_S(\gamma))^{-1}\cdot\rho^L_S(\gamma)=e_G\quad\forall\gamma\in\Gamma$$

■

Notice that for a loop α an element $\rho_S(\alpha)\in\bar{\mathbb{G}}_{\breve{S},\gamma}$ can be defined by

$$\rho_S(\alpha):=(\rho^L_S\times\rho^R_S)(\alpha)=(g,h)\in G^2$$

In the case of a path γ' that intersects a surface S in the source and target vertex there is also an element $\rho_S(\gamma')\in\bar{\mathbb{G}}_{\breve{S},\gamma}$ can be defined by

$$\rho_S(\gamma'):=(\rho^L_S\times\rho^R_S)(\gamma')=(g,h)\in G^2$$

Proposition 3.4.19. *Let $\breve{S}$ be a set of surfaces and Γ be a fixed graph, which contains no loops, such that the set $\breve{S}$ has the same surface intersection property for the graph Γ. Let $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be a finite orientation preserved graph system such that the set $\breve{S}$.*

The set $\bar{G}_{\breve{S},\Gamma}$ has the structure of a group.

The set $\bar{G}_{\breve{S},\Gamma}$ is called the ***flux group associated to a graph and a finite set of surfaces****.*

If instead of G the center $\mathcal{Z}(G)$ of G is used, then the set $\bar{G}_{\breve{S},\Gamma}$ is replaced by the **commutative flux group $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$ associated to a graph and a finite set of surfaces**.

Proof : This follows from the observation that $G_{\breve{S},\Gamma}$ can be identified with

$$\bigcup_{\sigma_L\in\breve{\sigma}_L}\bigcup_{S\in\breve{S}}\Big\{\rho_{S,\Gamma}\in\mathrm{Map}(\mathcal{P}^{\mathfrak{o}}_\Gamma, G^{|E_\Gamma|}):\quad \rho_{S,\Gamma}:=\rho_S\times\ldots\times\rho_S$$
$$\text{where }\rho_S(\gamma):=o_L(S)^{-1},\rho_S\in\mathbb{G}_{\breve{S},\Gamma}, S\in\breve{S},\gamma\in\Gamma\Big\}$$

Let $\breve{S}$ be a surface set having the same intersection property for a fixed graph $\Gamma := \{\gamma_1, ..., \gamma_N\}$. Then for two surfaces S_1, S_2 contained in $\breve{S}$ one can define

$$\begin{aligned}\rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma) &= (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), ..., \rho_{S_1}(\gamma_N) \cdot \rho_{S_2}(\gamma_N))\\ &= (g_{S_1}, ..., g_{S_1}) \cdot (g_{S_2}, ..., g_{S_2}) = (g_{S_1}g_{S_2}, ..., g_{S_1}g_{S_2})\end{aligned}$$

where $\Gamma = \{\gamma_1, ..., \gamma_N\}$. Note that since the maps o_L are arbitrary maps from $\breve{S}$ to G, it is assumed that the maps $o_L(S_i) := g_{S_i}^{-1} \in G$ for $i = 1, 2$. Clearly, this is related to in this particular case of the graph Γ.

The inverse operation is given by

$$(\rho_{S,\Gamma}(\Gamma))^{-1} = ((\rho_S(\gamma_1))^{-1}, ..., (\rho_S(\gamma_N))^{-1})$$

where $N = |\Gamma|$ and $\rho_S \in \mathbb{G}_{\breve{S},\gamma}$ for $S \in \breve{S}$. Since, it is true that

$$\begin{aligned}\rho_{S,\Gamma}(\Gamma) \cdot \rho_{S,\Gamma}(\Gamma)^{-1} &= (g_S, ..., g_S) \cdot (g_S^{-1}, ..., g_S^{-1})\\ &= (\rho_S(\gamma_1) \cdot \rho_S(\gamma_1)^{-1}, ..., \rho_S(\gamma_N) \cdot \rho_S(\gamma_N)^{-1})\\ &= (g_S g_S^{-1}, ..., g_S g_S^{-1}) = (e_G, ..., e_G)\end{aligned}$$

■

Notice that it is not defined that

$$\begin{aligned}\rho_{S_1,\Gamma}(\Gamma) \bullet_R \rho_{S_2,\Gamma}(\Gamma) &= (o_L(S_2)^{-1}o_L(S_1)^{-1}, ..., o_L(S_2)^{-1}o_L(S_1)^{-1}) = ((o_L(S_1)o_L(S_2))^{-1}, ..., (o_L(S_1)o_L(S_2))^{-1})\\ &= \rho_{S_3,\Gamma}(\Gamma)\end{aligned}$$

Moreover, observe that if all subgraphs of a finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$ are naturally identified, then $\bar{G}_{\breve{S},\Gamma'\leq\Gamma}$ is a subgroup of $\bar{G}_{\breve{S},\Gamma}$ for all subgraphs Γ' in $\mathcal{P}^{\mathfrak{o}}_\Gamma$. If G is assumed to be a compact Lie group, then the flux group $\bar{G}_{\breve{S},\Gamma}$ is called the Lie flux group.

The flux group for the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$ and a finite set of surfaces is denoted by $\bar{G}_{\breve{S}}$. This follows from the fact that each element of the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$ is a graph.

There is another group if another surface set is considered.

Proposition 3.4.20. *Let $\breve{T}$ be a set of surfaces and Γ be a fixed graph such that the set $\breve{T}$ has the simple surface intersection property for the graph Γ. Let $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be a finite orientation preserved graph system.*

The set $\bar{G}_{\breve{T},\Gamma}$ has the structure of a group.

The same arguments using the identification of $\bar{G}_{\breve{T},\Gamma}$ with

$$\bigcup_{\sigma_R\in\breve{\sigma}_R} \Big\{\rho_{T,\Gamma} \in \mathrm{Map}(\mathcal{P}^{\mathfrak{o}}_\Gamma, G^{|E_\Gamma|}) : \quad \rho_{\breve{T},\Gamma} := \rho_{T_1} \times ... \times \rho_{T_N}$$
$$\text{where } \rho_{T_i}(\gamma) := o_R(T_i)^{-1}, \rho_{T_i} \in \mathbb{G}_{\breve{T},\Gamma}, T_i \in \breve{T}, \gamma \in \Gamma\Big\}$$

which is given by

$$\begin{aligned}\rho_{T_1,\Gamma}(\Gamma) \cdot ... \cdot \rho_{T_N,\Gamma}(\Gamma) &= (\rho_{T_1}(\gamma_1)e_G, e_G, ..., e_G) \cdot (e_G, \rho_{T_2}(\gamma_2)e_G, e_G, ..., e_G) \cdot \cdot ... \cdot (e_G, ..., e_G, \rho_{T_N}(\gamma_N)e_G)\\ &= (\rho^1_{T_1}(\gamma_1), ..., \rho^1_{T_N}(\gamma_N)) = (g_1, ..., g_N) \in G^N\\ &=: \rho_{\breve{T},\Gamma}(\Gamma)\end{aligned}$$

Then the multiplication operation is presented by

$$\begin{aligned}\rho^1_{\breve{T},\Gamma}(\Gamma) \cdot \rho^2_{\breve{T},\Gamma}(\Gamma) &= (\rho^1_{T_1}(\gamma_1) \cdot \rho^2_{T_1}(\gamma_1), ..., \rho^1_{T_N}(\gamma_N) \cdot \rho^2_{T_N}(\gamma_N))\\ &= (g_{1,1}, ..., g_{1,N}) \cdot (g_{2,1}, ..., g_{2,N}) = (g_{1,1}g_{2,1}, ..., g_{1,N}g_{2,N}) \in G^N\end{aligned}$$

where $\Gamma = \{\gamma_1, ..., \gamma_N\}$.

It is also possible that the fluxes are located only in a vertex and do not depend on ingoing or outgoing, above or below orientation properties.

Definition 3.4.21. *Let $\mathcal{P}_\Gamma$ be a finite graph groupoid associated to a graph Γ and let N be the number of edges of the graph Γ.*

Define the set of maps

$$G_\Gamma^{\text{loc}} := \Big\{ \boldsymbol{g}_\Gamma \in \text{Map}(\mathcal{P}_\Gamma, G^{|\Gamma|}) : \boldsymbol{g}_\Gamma := g_\Gamma^1 \circ s \times ... \times g_\Gamma^N \circ s \\ g_\Gamma^i \in \text{Map}(\Gamma, G) \Big\}$$

Then $\bar{G}_\Gamma^{\text{loc}}$ is the set of all images of maps in G_Γ^{loc} for all graphs in $\mathcal{P}_\Gamma$ and $\bar{G}_\Gamma^{\text{loc}}$ is called the ***local flux group associated to a finite graph system****.*

Finally, let $\breve{S}_{\text{d}}$ be a discretised surface set associated to a surface set $\breve{S}$ and G be a connected compact Lie group. Then assume that $\bar{G}_{\breve{S}_{\text{d}},\Gamma}$ (resp. $\bar{G}_{\breve{S}_{\text{d}}}$) denotes the Lie flux group associated to Lie flux algebra $\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}},\Gamma}$ (resp. $\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}}}$).

3.4.4 The group-valued quantum flux operators associated to surfaces and finite path groupoids

Recall the set of admissible maps $\text{Map}^{\text{A}}(\mathcal{P}_\Gamma\Sigma, G)$ presented in definition 3.3.18.

Definition 3.4.22. *Let $\breve{S}$ be a finite set of surfaces which is closed under a flip of orientation of the surfaces. Let $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ be a finite path groupoid associated to a graph Γ such that each path in $\mathcal{P}_\Gamma\Sigma$ satisfies one of the following conditions*

- *the path intersects each surface in $\breve{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path intersects each surface in $\breve{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path intersects each surface in $\breve{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\breve{S}$,*
- *the path does not intersect any surface S contained in $\breve{S}$.*

Finally, let $\mathcal{P}_\Gamma$ denote the finite graph system associated to Γ.

Then the ***set of admissible maps associated to a graph and surfaces*** *$\breve{S}$ are defined by*

$$\mathbb{G}^{\text{A}}_{\breve{S},\Gamma} := \bigcup_{o_L\times o_R\in\breve{o}} \bigcup_{S\in\breve{S}} \Big\{ (\varrho^L,\varrho^R) \in \text{Map}^{\text{A}}(\mathcal{P}_\Gamma\Sigma, G\times G) : \quad (\varrho^L,\varrho^R)(\gamma) := (o_L(S)^{\iota_L(S,\gamma)}, o_R(S)^{\iota_R(S,\gamma)}) \Big\}$$

Define the ***set of admissible maps associated to a finite graph system and surfaces*** *$\breve{S}$ is presented by*

$$G^{\text{A}}_{\breve{S},\Gamma} := \bigcup_{o_L\times o_R\in\breve{o}} \bigcup_{S\in\breve{S}} \Big\{ \varrho_{S,\Gamma} \in \text{Map}^{\text{A}}(\mathcal{P}_\Gamma, G^{|\Gamma|}\times G^{|\Gamma|}) : \quad \varrho_{S,\Gamma} := \varrho_S \times ... \times \varrho_S \\ \text{where } \varrho_S(\gamma_i) = (o_L(S)^{\iota_L(S,\gamma)}, o_R(S)^{\iota_R(S,\gamma_i)}) \\ \varrho_S \in \mathbb{G}^{\text{A}}_{\breve{S},\Gamma}, S\in\breve{S}, \gamma_i\in\Gamma', \Gamma'\in\mathcal{P}_\Gamma \Big\}.$$

Observe that these maps have the following properties. For all elements of $\mathcal{P}_\Gamma\Sigma_v$ (or $\mathcal{P}_\Gamma\Sigma^v$) that intersect the surface S only in their target (or source) vertex v the maps ϱ^L_S (or ϱ^R_S) in $\mathbb{G}^{\text{A}}_{\breve{S},\Gamma}$ satisfies

$$\begin{aligned} &\varrho^L_S(\gamma) = \varrho^L_S(\gamma\circ\gamma') = \varrho^L_S(\gamma'') = g_{S,L} \quad \forall \gamma,\gamma\circ\gamma',\gamma''\in\mathcal{P}_\Gamma\Sigma_v \text{ and } v = s(\gamma) = S\cap\gamma \\ &\varrho^L_S({\gamma'}^{-1}) = \varrho^L_S((\gamma\circ\gamma')^{-1}) = k_{S,L} \quad \forall {\gamma'}^{-1},(\gamma\circ\gamma')^{-1}\in\mathcal{P}_\Gamma\Sigma_v \text{ and } v = t(\gamma') = S\cap{\gamma'}^{-1} \end{aligned} \tag{3.94}$$

Furthermore, for paths γ and γ' that compose and intersect S in the common vertex $t(\gamma) = s(\gamma')$ it is true that

$$(\varrho_S^R(\gamma^{-1}))^{-1}\varrho_S^L(\gamma') = e_G \qquad \text{and} \qquad (\varrho_S^R(\gamma))\varrho_S^L({\gamma'}^{-1})^{-1} = e_G \tag{3.95}$$

whenever $(\gamma, \gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ and for all maps $(\varrho_S^L, \varrho_S^R) \in \mathbb{G}_{\breve{S},\Gamma}^{\mathrm{A}}$.

In both definitions of the sets $\mathbb{G}_{\breve{S},\Gamma}$ or $\mathbb{G}_{\breve{S},\Gamma}^{\mathrm{A}}$ of maps, there is a mapping ρ_S or, respectively, ϱ_S, which maps all paths in $\mathcal{P}_\Gamma\Sigma^v$ to one element X_S, i.e. $\rho_S(\gamma) = \rho_S(\gamma\circ\gamma')$ for all $\gamma, \gamma\circ\gamma' \in \mathcal{P}_\Gamma\Sigma^v$ where $v = s(\gamma)$. But the equalities (3.95) are required only for maps in $\mathbb{G}_{\breve{S},\Gamma}^{\mathrm{A}}$.

Notice that if the group G is replaced by the center $\mathcal{Z}(G)$ of the group G, then the set $\mathbb{G}_{\breve{S},\Gamma}^{\mathrm{A}}$ is replaced by $\mathcal{Z}(\mathbb{G}_{\breve{S},\gamma})_{\breve{S},\Gamma}^{\mathrm{A}}$ and $G_{\breve{S},\Gamma}^{\mathrm{A}}$ is changed to $\mathcal{Z}_{\breve{S},\Gamma}^{\mathrm{A}}$.

Part II

Quantum algebras of Loop Quantum Gravity and Cosmology

Chapter 4

The quantum algebras of Loop Quantum Cosmology

4.1 The algebra of almost periodic functions

The mathematical concept of Loop Quantum Cosmology has been discovered for example by Ashtekar, Bojowald and Lewandowski in [5] or Velhinho in [107]. In this dissertation the algebra of quantum observables in Loop Quantum Cosmology is rewritten in terms of a twisted group and a transformation group algebra. In the next paragraphs different algebras constructed from the quantum configuration variables are presented. In this work the configuration space of LQC is identified with the abelian locally compact group $\mathbb{R}_{\text{d}}$ ($\mathbb{R}$ with discrete topology).

In the preliminary section 1.4.1 an isomorphism is presented between the group algebra $\mathcal{C}^*(\mathbb{R}_{\text{d}})$ of the discretised real line $\mathbb{R}_{\text{d}}$ and the algebra $C_0(\hat{\mathbb{R}}_{\text{d}})$ of continuous functions on the dual $\hat{\mathbb{R}}_{\text{d}}$ of $\mathbb{R}_{\text{d}}$ vanishing at infinity.

In the context of the abelian locally compact group $\mathbb{R}$ the dual group $\hat{\mathbb{R}}$ is equivalent to $\mathbb{R}$. Clearly, the isomorphism between $\mathbb{R}$ and $\mathbb{R}$ is given by $\mathbb{R} \ni s \mapsto \gamma_s \in \mathbb{R}$ where $\gamma_s(t) = \exp(ist)$. By Pontryagin duality, the group $\mathbb{R}$ can be regarded as the set of all group homomorphisms $\mathbb{R} \to \mathbb{T}$ that are continuous in the usual compact topology of $\mathbb{R}$, whereas $b\mathbb{R}$ is the group of all not necessary continuous group homomorphisms. The group $b\mathbb{R}$ is called Bohr compactification of $\mathbb{R}$ and the topology on $b\mathbb{R}$ is the pointwise convergence topology which is weaker than the topology of $\mathbb{R}$.

Equipp $\mathbb{R}$ with a locally convex Hausdorff topology. Then there is a commutative C^*-algebra of almost periodic functions $\text{AP}(\mathbb{R}')$, where $\mathbb{R}'$ is the topological dual of $\mathbb{R}$. Consider all functions of the form

$$f(x)(\xi) := \exp(i\xi(x)) \quad \xi \in \mathbb{R}'_{\text{d}}, x \in \mathbb{R}_{\text{d}} \tag{4.1}$$

which are linearly independent. The linear hull $\{f(x) : x \in \mathbb{R}\}$ of those elements form a commutative *-algebra with pointwise multiplication, complex conjugation and supremum norm. The Weyl relations are

$$f(x)f(y) = f(x+y), \quad f(x)^* = f(-x) \quad \forall x, y \in \mathbb{R}_{\text{d}} \tag{4.2}$$

and the completion w.r.t. the supremum norm is equal to the commutative C^*-algebra $AP(\mathbb{R}')$. Moreover the algebra of continuous complex-valued functions $C(b\mathbb{R}')$ is defined on the compactification of $\mathbb{R}'$. Finally, there is an *-isomorphism between $\text{AP}(\mathbb{R}')$ and $C(b\mathbb{R}')$.

Summarising, there are the following isomorphisms between commutative C^*-algebras

$$\text{AP}(\mathbb{R}') \simeq C(b\mathbb{R}') \simeq C(b\mathbb{R})$$

where the last *-isomorphism can be deduced from the next observation. The embedding of $\mathbb{R}'$ in the character group $\hat{\mathbb{R}}$ extends continuously to a continuous group isomorphism between $b\mathbb{R}'$ and $\hat{\mathbb{R}}$ ([33, Chapter 16. §2]). Consequently the compactification of the dual group $\mathbb{R}'$ is independent of the chosen locally convex Hausdorff topology.

Summarising, there are the following isomorphisms between commutative C^*-algebras

$$\mathrm{AP}(\mathbb{R}') \simeq C(b\mathbb{R}') \simeq C(b\mathbb{R}) \simeq C^*(\mathbb{R}_{\mathrm{d}}) \tag{4.3}$$

The last isomorphism is implemented by a generalised Fourier transfromation. The convolution algebra $C^*(\mathbb{R}_{\mathrm{d}})$ and Weyl algebras over the symplectic space $\mathbb{R}_{\mathrm{d}}$ are analysed in detail in section 7.4.

A generalisation for topological groups

In general, there is a covariant functor between the category of topological groups and the category of continuous homomorphisms. Let G be a topological group and β be a unique up to isomorphisms continuous homomorphism from G to the Bohr compactification of G, which is usually denoted by $\beta(G)$. Then for every other compact Hausdorff topological group K and a map $f : G \longrightarrow K$, there exists a unique continuous homomorphism $\beta(f) : \beta(G) \longrightarrow K$ such that $f = \beta(f) \circ \beta$.

A bounded continuous complex-valued function f on G is uniformly almost periodic iff there exists a continuous function $F : \beta(G) \longrightarrow \mathbb{C}$ such that $f = F \circ \beta$. If f is uniformly almost periodic, then there exists a unique mean $M(f) = \int_{\beta(G)} F(h)\,\mathrm{d}\,\mu(h)$.

Then the almost periodic function algebra over a topological group can be defined. The development of this structure is analysed in section 5.

4.2 Weyl algebras over pre-symplectic spaces and Weyl algebra of LQC

Let (X, σ) be a pre-symplectic space, i.o.w. a real vector space X equipped with a $\mathbb{R}$-bilinear antisymmetric map $\sigma : X \times X \longrightarrow \mathbb{R}$ such that $\sigma(x, y) = \overline{\sigma(y, x)}$ for $x, y \in X$. Consider the Weyl elements which are defined by all unitaries, i.o.w. maps W form X to unitary operators on a Hilbert space, such that

$$W(x)W(y) = \exp\Big(- \frac{i}{2}\sigma(x, y)\Big) W(x + y), \quad W^*(x) = W(-x) \tag{4.4}$$

for $x, y \in X$ and where W is a projective unitary representation of the additive group $\mathbb{R}$ and $\lambda \mapsto W(\lambda x)$ is weakly continuous. Certainly, the multiplication of Weyl elements carry over to an addition of X homomorphically up to a phase factor given by the bicharacter $\exp(-\frac{i}{2}\sigma(.,.))$.

Denote by $W(X)$ the set of linearly independent Weyl elements $W(x)$, $x \in X$ and denote by $\mathbf{W}(X, \sigma)$ the vector space of all finite complex linear combinations of $W(x)$, $x \in X$. With the *-involution the algebra $\mathbf{W}(X, \sigma)$ is a *-algebra and is called the Weyl *-algebra. Observe that, the Weyl *-algebra is generated only by unitaries on a Hilbert space $\mathcal{H}$ with a norm $\|.\|_2$ and completed with respect to this norm a C^*-algebra $\mathsf{W}(X)$.

Equipp the Weyl *-algebra with the enveloping norm, which is defned by the map

$$\mathbf{W}(X, \sigma) \ni A \mapsto \|A\|^2 := \sup\{\|\pi(A)\| : \pi \text{ is a representation of } \mathbf{W}(X, \sigma) \text{ on a Hilbert space}\}$$

such that $\mathbf{W}(X, \sigma)$ completed w.r.t. this norm is a C^*-algebra, which is symbolised by $\mathcal{W}(X, \sigma)$. One can show that there is an isomorphism between the unital Weyl C^*-algebra $\mathcal{W}(X, \sigma)$ and the twisted group algebra $C^*_\sigma(X)$.

Clearly, for $\sigma = 0$ on X the Weyl C^*-algebra is commutative and isomorphic to the C^*-algebra of continuous functions over the the spectrum of $\mathcal{W}(X, \sigma)$.

Summarising, in Loop Quantum Cosmology the quantum algebra of configuration variables is given by the group C^*-algebra $\mathcal{C}^*(\mathbb{R}_{\mathrm{d}})$ and the Weyl C^*-algebra of configuration and momentum variables can be defined as the C^*-algebra $\mathcal{W}(\mathbb{R}_{\mathrm{d}}, \sigma)$. Furthermore the algebra $C^*(\mathbb{R}_{\mathrm{d}})$ is isomorphic to $C(\beta\mathbb{R})$.

Finally, notice that the usual Weyl algebra in Quantum mechanics (refer to the section 1.3.1.2) and the state

$$\omega_0(v_p) = \begin{cases} 0 & \text{if } p \neq 0 \\ 1 & \text{if } p = 0 \end{cases}$$

is considered. Then the GNS-representation associated to ω_0 is called the polymer representation of this Weyl C^*-algebra.

Chapter 5

The smooth holonomy C^*-algebra

5.1 The algebra of almost periodic functions on the loop group

The ideas of a construction of the quantum algebra of holonomy variables w.r.t. smooth paths and connections have been presented by Ashtekar and Isham [7]. The authors have investigated the abelian loop group $\mathrm{LG}(v)$ at a base point $v \in \Sigma$, which is a topological group if it is equipped with the Barrett topology. In this section the C^*-algebra of almost periodic functions on $\mathrm{LG}(v)$ is developed. This C^*-algebra can be identified with the C^*-algebra of all continuous complex functions on a compact group G, whose continuous irreducible representations are one-to-one with the finite-dimensional irreducible representations of $\mathrm{LG}(v)$. The compact group G is called the associated compact group for $\mathrm{LG}(v)$. The underlying mathematical theory is presented in the book [33] of Dixmier. In this section the ideas of Ashtekar and Isham are rewritten in the context of Dixmier and are further generalised in this framework.

Definition 5.1.1. *Let H be a topological group.*

*The **associated compact group** G **for** H is uniquely determined up to isomorphisms by the property that there exists a continuous morphism $\beta : H \longrightarrow G$ such that for every compact group K and every continuous morphism $\beta' : H \longrightarrow K$ there exists a unique continuous morphism $\alpha : H \longrightarrow K$ and $\beta' = \alpha \circ \beta$.*

Let G be the associated compact group to $\mathrm{LG}(v)$ and $\mathfrak{h} : \mathrm{LG}(v) \longrightarrow G$ be a continuous morphism.

Let F be an element of the vector space $C_b(\mathrm{LG}(v))$ of bounded continuous complex-valued function on $\mathrm{LG}(v)$ completed w.r.t. the uniform norm. Then the following conditions are equivalent:

(i) The set of left translates $\{F(\gamma \circ \gamma_i) : \gamma_i \in \mathrm{LG}(v)\}$ is precompact in $C_b(\mathrm{LG}(v))$.

(ii) There exists a continuous complex-valued function f on G such that $F = f \circ \mathfrak{h}$ where $\mathfrak{h} : \mathrm{LG}(v) \longrightarrow G$ is a continuous morphism.

(iii) F is the uniform limit over $\hat{G}$ of linear combinations of coefficients $T^i_{s,j}$ of finite-dimensional irreducible continuous unitary representations π_s of G, i.o.w.

$$F(\alpha) = \sum_{\pi_s \in \hat{G}} T^i_{s,j}(\mathfrak{h}(\alpha))$$

where $\hat{G}$ is the set of all finite-dimensional irreducible continuous unitary representations of G.

Definition 5.1.2. *The *-algebra $AP(\mathrm{LG}(v))$ of almost periodic functions on $\mathrm{LG}(v)$ is defined by all functions of the form*

$$F(\alpha) = (f \circ \mathfrak{h})(\alpha) \text{ for } \mathfrak{h} \in \mathrm{Hom}_{\breve{\mathcal{A}}_s}(\mathrm{LG}(v), G), f \in C(G) \text{ and where } \alpha \in \mathrm{LG}(v)$$
$$F^*(\alpha) = \bar{F}(\alpha^{-1})$$

equipped with convolution multiplication.

A function $F \in AP(\mathrm{LG}(v))$ has the following property

$$F(\alpha \circ \beta) = \sum_{\pi_s \in \hat{G}} (\dim \pi_s)^{-1} \sum_n T^i_{s,n}(\mathfrak{h}(\alpha)) T^n_{s,j}(\mathfrak{h}(\beta)) \tag{5.1}$$

Notice that, this is different to the assumptions, which have been used by Ashtekar and Isham in [7], for the construction of their C^*-algebra. In particular, the *-algebra $AP(LG(v))$ is equipped with the pointwise multipliction operation.

There exists a unique mean

$$\omega_{AP}(F) = \int_G (f \circ \mathfrak{h})(\alpha) \,\mathrm{d}\,\mu(\mathfrak{h}(\alpha)) \tag{5.2}$$

such that

$$\begin{aligned}\omega_{AP}(F) &= \int_G (f \circ \mathfrak{h})(\alpha \circ \beta) \,\mathrm{d}\,\mu(\mathfrak{h}(\alpha \circ \beta)) = \int_G (f \circ \mathfrak{h})(\gamma \circ \alpha) \,\mathrm{d}\,\mu(\mathfrak{h}(\gamma \circ \alpha)) \\ &= \int_G (f \circ \mathfrak{h})(\gamma \circ \alpha \circ \beta) \,\mathrm{d}\,\mu(\mathfrak{h}(\gamma \circ \alpha \circ \beta))\end{aligned} \tag{5.3}$$

for $F \in AP(\mathrm{LG}(v))$. For N-different base points $v_1, .., v_N$ and a N-tuple of loops $(\alpha_1, ..., \alpha_N)$ with $s(\alpha_k) = v_k$ there exists paths β_k for all $1 \leq k \leq N$ such that $s(\beta) = v$ for a fixed $v \in \Sigma$, $(\alpha'_1, ..., \alpha'_N) = (\beta_1 \circ \alpha_1 \circ \beta_1^{-1}, ..., \beta_N \circ \alpha_N \circ \beta_N^{-1})$ and an almost periodic function on the product $\times_{k=1}^N \mathrm{LG}(v)$ is represented by

$$F(\alpha'_1, ..., \alpha'_N) = \sum_{k=1}^N \sum_{\pi_s \in \hat{G}} T^i_{s,j}(\mathfrak{h}(\beta_k \circ \alpha_k \circ \beta_k^{-1})) \tag{5.4}$$

for all $\alpha_k \in LG(v_k)$ and $1 \leq k \leq N$.

For two almost periodic functions F, G the mean ω_{AP} defines an inner product

$$\omega_{AP}(FG) = \langle F, G \rangle_{AP}$$

such that $AP(\mathrm{LG}(v))$ completed w.r.t. that norm is a Hilbert space, canonically isomorphic to the Hilbert space completion of the vector space $C(G)$ of continuous functions on G endowed with pointwise multiplication and with inner product $\langle f, g \rangle_2 = \int_G \mathrm{d}\,\mu f g$.

Definition 5.1.3. *The* C^****-algebra*** $\mathrm{Cyl}(\mathrm{LG}(v))$ ***of almost periodic functions on the loop group*** $\mathrm{LG}(v)$ *is given by the completition w.r.t. the supremum norm of the* **-algebra* $AP(\mathrm{LG}(v))$*, which is equipped with pointwise multiplication, the complex conjugation as inversion and supremum norm.*

The C^*-algebra $\mathrm{Cyl}(\mathrm{LG}(v))$ is commutative and unital. Now, it is necessary to analyse what is the concrete associated group for the loop group $\mathrm{LG}(v)$.

In general it is quite difficult to find the associated group to an arbitrary topological group. If H is a commutative locally compact group and $\hat{H}$ is the dual locally compact group, then the associated group of H can be constructed as follows. Let K be the commutative compact group whose dual is given by the group $\hat{H}_{\mathrm{d}}$ with discrete topology. Using the Pontryagin duality, it is possible to identify H with all continuous homomorphisms from $\hat{H}$ to $\mathbb{T}$ and $\hat{H}_{\mathrm{d}}$ with all homomorphisms $\hat{H}$ to $\mathbb{T}$. Then K is the associated group to H and $\mathfrak{h} : H \longrightarrow K$ is the continuous morphism. In the context of the loop group $\mathrm{LG}(v)$ the associated group is not known.

5.2 The cylindrical function C^*-algebra for path groupoids

In the context of quantum gravity, the idea is to consider the continuous groupoid morphisms $\mathfrak{h}$ from the thin fundamental groupoid, or generally from a path groupoid $\mathcal{P}\Sigma \rightrightarrows \Sigma$, which generalise the topological loop group $\mathrm{LG}(v)$, to a given suitable compact Lie group G. Then the smooth holonomy algebra can be defined in analogy to the algebra of almost periodic functions on the loop group as follows.

Definition 5.2.1. *Let $\mathcal{P}\Sigma \rightrightarrows \Sigma$ be a path groupoid over Σ. Let $\{\Gamma_i\}$ be an inductive family of finite sets of smooth paths in $\mathcal{P}\Sigma$, each set Γ_i contain $|\Gamma_i|$ paths. The inductive limit of the family is given by Γ_∞. Furthermore let $\{\mathcal{P}_{\Gamma_i}\Sigma\}$ be an inductive family of finite systems of paths with $\mathcal{P}_\infty\Sigma$ being the inductive limit system of paths.*

Then the *-algebra of cylindrical functions $f \in \mathrm{Cyl}^0(\mathcal{P}_\infty\Sigma)$ is given by all elements of the form

$$\begin{aligned} f(\Gamma_i) &= (f_{\Gamma_i} \circ \mathfrak{h}_{\Gamma_i})(\Gamma_i) \text{ for } \mathfrak{h}_{\Gamma_i} \in \mathrm{Hom}_{\breve{\mathcal{A}}_s}(\mathcal{P}_{\Gamma_i}\Sigma, G^{|\Gamma_i|}) \\ &\qquad \text{where } \Gamma_i \in \mathcal{P}_{\Gamma_i}\Sigma, f_{\Gamma_i} \in C(G^{|\Gamma_i|}) \\ f^* &= \bar{f} \end{aligned}$$

equipped with pointwise multiplication and sup*-norm.* $\mathrm{Cyl}^0(\mathcal{P}_\infty\Sigma)$ *completed in* sup*-norm is a commutative unital C^*-algebra and it is called the* ***smooth holonomy C^*-algebra*** $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$ ***associated to a path groupoid*** $\mathcal{P}\Sigma \rightrightarrows \Sigma$.

Recall that $\mathfrak{h}_\Gamma(\Gamma) := (\mathfrak{h}(\gamma_1), ..., \mathfrak{h}(\gamma_N))$ where $\mathfrak{h} \in \mathcal{A}_s$. This means that $\mathfrak{h}(\gamma_i)$ is the holonomy along the path γ_i and the holonomy $\mathfrak{h}$ is in one-to-one correpondence to a smooth connection A in $\breve{\mathcal{A}}_s$. A function in $\mathrm{Cyl}^0(\mathcal{P}_\infty\Sigma)$ is of the form

$$f(\Gamma) = \sum_{k=1}^{|\Gamma|} \sum_{\pi_{s,\gamma_i} \in \hat{G}} T^i_{s,j}(\mathfrak{h}(\gamma_k))$$

where $\hat{G}$ is the set of all finite-dimensional irreducible continuous unitary representations of G.

Moreover each $\mathfrak{h}_{\Gamma_i} \in \mathrm{Hom}_{\breve{\mathcal{A}}_s}(\mathcal{P}_{\Gamma_i}\Sigma, G^{|\Gamma_i|})$ defines a nonzero linear multiplicative functional on $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$, since,

$$\mathfrak{h}_{\Gamma_i}(f)(\Gamma_i) = (f_{\Gamma_i} \circ \mathfrak{h}_{\Gamma_i})(\Gamma_i)$$

where $\mathfrak{h}_\Gamma(\Gamma) := (\mathfrak{h}(\gamma_1), ..., \mathfrak{h}(\gamma_N))$.

Since, $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$ is a unital commutative C^*-algebra the spectrum is a compact Hausdorff space, which is denoted by $\bar{\mathcal{A}}_s$. The space $\bar{\mathcal{A}}_s$ is called the **space of generalised connections**. Consequently the C^*-algebra $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$ is isomorphic to $C(\bar{\mathcal{A}}_s)$. Moreover the restriction of $\mathrm{Cyl}(\mathcal{P}_\infty\Sigma)$ to a set $\mathcal{P}_\Gamma\Sigma$ of paths lead to the C^*-subalgebra $C(\bar{\mathcal{A}}_s^\Gamma)$. Then the algebra $C(\bar{\mathcal{A}}_s)$ can be understood as the inductive limit C^*-algebra of an inductive family $\{(C(\bar{\mathcal{A}}_s^\Gamma), \beta_{\Gamma,\Gamma'}) : \mathcal{P}_\Gamma\Sigma \leq \mathcal{P}_{\Gamma'}\Sigma\}$ of C^*-algebras. Equivalently, due to the commutativity of the C^*-algebras, this correponds to a projective limit $\bar{\mathcal{A}}_s$ of the family $\{\bar{\mathcal{A}}^\Gamma\}$ of configuration spaces. The space $\bar{\mathcal{A}}_s^\Gamma$ consists of all not necessarily continuous holonomy maps, since every $\mathfrak{h}_\Gamma \in \mathrm{Hom}(\mathcal{P}_\Gamma\Sigma, G^{|\Gamma|})$ define a nonzero linear multiplicative functional on $\mathrm{Cyl}(\mathcal{P}_\Gamma\Sigma)$. Clearly, there is a unique continuous morphism $\pi_\Gamma : G^{|\Gamma|} \rightarrow \bar{\mathcal{A}}_s^\Gamma$ such that $\mathfrak{h}'_\Gamma = \pi_\Gamma \circ \mathfrak{h}_\Gamma$ and $\mathfrak{h}' : \mathcal{P}_\Gamma\Sigma \rightarrow \bar{\mathcal{A}}_s^\Gamma$. Then notice

$$f(\Gamma_i) = (f_{\Gamma_i} \circ \pi_{\Gamma_i})(\mathfrak{h}_{\Gamma_i}(\Gamma_i))$$

where $\mathfrak{h}_{\Gamma_i} \in \mathrm{Hom}_{\breve{\mathcal{A}}_s}(\mathcal{P}_{\Gamma_i}\Sigma, G^{|\Gamma_i|})$ and $\Gamma_i \in \mathcal{P}_{\Gamma_i}\Sigma$ and for $f_{\Gamma_i} \in C(\bar{\mathcal{A}}_s^\Gamma)$.

The concept of projective spaces in the Loop Quantum Gravity framework has been introduced by Ashtekar and Lewandowski [9] and [10].

Projective limit structure on the configuration space

Recall the inductive family $\{\Gamma_i\}$ of finite sets of smooth paths in $\mathcal{P}\Sigma$. The inductive limit of the family is given by Γ_∞. Furthermore let $\{\mathcal{P}_{\Gamma_i}\Sigma\}$ be an inductive family of finite systems of paths with $\mathcal{P}_\infty\Sigma$ being the inductive limit system of paths. Consequently there is projective family $\{(\bar{\mathcal{A}}_s^\Gamma, \bar{\pi}_{\Gamma,\Gamma'}) \mid \mathcal{P}_{\Gamma'}\Sigma \geq \mathcal{P}_\Gamma\Sigma\}$ where $\bar{\pi}_{\Gamma,\Gamma'} : \bar{\mathcal{A}}_s^\Gamma \rightarrow \bar{\mathcal{A}}_s^{\Gamma'}$ are surjective maps and for which the following consistency condition is satisfied.

For all pairs $(\mathcal{P}_{\Gamma'}\Sigma, \mathcal{P}_\Gamma\Sigma)$ of two sets of paths such that $\mathcal{P}_{\Gamma'}\Sigma \geq \mathcal{P}_\Gamma\Sigma$ it is true that for all elements $\Gamma \in \mathcal{P}_\Gamma\Sigma$ the following conditions

$$\begin{aligned} &\bar{\pi}_{\Gamma,\Gamma'}(\mathfrak{h}_\Gamma)(\Gamma) = \mathfrak{h}_{\Gamma'}(\Gamma) \text{ for all } \mathfrak{h}_\Gamma \in \bar{\mathcal{A}}_s^\Gamma \text{ and } \mathfrak{h}_{\Gamma'} \in \bar{\mathcal{A}}_s^{\Gamma'}, \\ &(\bar{\pi}_{\Gamma',\Gamma''} \circ \bar{\pi}_{\Gamma,\Gamma'})(\mathfrak{h}_\Gamma)(\Gamma) = \mathfrak{h}_{\Gamma''}(\Gamma) \text{ for all } \mathcal{P}_{\Gamma''}\Sigma \geq \mathcal{P}_{\Gamma'}\Sigma \geq \mathcal{P}_\Gamma\Sigma \end{aligned} \tag{5.5}$$

holds.

The projective limit is equivalent to the compact space $\bar{\mathcal{A}}_s$.

In LQG it is assumed that $\bar{\mathcal{A}}_s$ is identifiable with $\text{Hom}(\mathcal{P}\Sigma, G)$. Consequently set $\bar{\mathcal{A}}_s^\Gamma$ equal to $\text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$. The set $\text{Hom}(\mathcal{P}\Sigma, G)$ establish the name of generalised connections, since a non-continuous groupoid morphism is not associated to a smooth connection.

Inductive limit structure on the smooth holonomy C^*-algebra

Concern the directed family of (unital) commutative C^*-algebras $\{(C(\bar{\mathcal{A}}_s^\Gamma), \beta_{\Gamma,\Gamma'}) : \quad \mathcal{P}_{\Gamma'}\Sigma \geq \mathcal{P}_\Gamma\Sigma\}$ where $\beta_{\Gamma,\Gamma'} : C(\bar{\mathcal{A}}_s^\Gamma) \longrightarrow C(\bar{\mathcal{A}}_s^{\Gamma'})$ is an injective (unital) *-homomorphisms such that

$$\beta_{\Gamma,\Gamma''}(f_\Gamma) = (\beta_{\Gamma',\Gamma''} \circ \beta_{\Gamma,\Gamma'})(f_\Gamma) \text{ for all } \mathcal{P}_{\Gamma''}\Sigma \geq \mathcal{P}_{\Gamma'}\Sigma \geq \mathcal{P}_\Gamma\Sigma \tag{5.6}$$

The (unital) *-homomorphisms $\beta_{\Gamma,\Gamma'}$ are isometries, since

$$\sup_{\mathcal{A}_s^{\Gamma'}} | \beta_{\Gamma,\Gamma'}(f_\Gamma) | = \sup_{\mathcal{A}_s^\Gamma} | f_\Gamma | \quad \text{for all } f_\Gamma \in C(\bar{\mathcal{A}}_s^\Gamma) \tag{5.7}$$

Projective limit on the state space of the holonomy C^*-algebra

Let $\mathcal{S}(C(\bar{\mathcal{A}}_s^\Gamma))$ be the state space of the holonomy algebra $C(\bar{\mathcal{A}}_s^\Gamma)$. Then if ω is a state on $C(\bar{\mathcal{A}}_s)$, then ω restricted to $C(\bar{\mathcal{A}}_s^\Gamma)$ defines a state ω_Γ on $C(\bar{\mathcal{A}}_s^\Gamma)$. Moreover then for all sets $\mathcal{P}_\Gamma\Sigma, \mathcal{P}_{\Gamma'}\Sigma$ such that $\mathcal{P}_\Gamma\Sigma \leq \mathcal{P}_{\Gamma'}\Sigma$, there is a conjugate map $\beta^*_{\Gamma,\Gamma'} : C(\bar{\mathcal{A}}_s^\Gamma) \longrightarrow C(\bar{\mathcal{A}}_s^{\Gamma'})$ such that

$$\begin{aligned} &\omega_{\Gamma'} = \beta^*_{\Gamma,\Gamma'}\omega_\Gamma \\ &\beta^*_{\Gamma,\Gamma''} = \beta^*_{\Gamma,\Gamma'} \circ \beta^*_{\Gamma',\Gamma''} \text{ if } \mathcal{P}_\Gamma\Sigma \leq \mathcal{P}_{\Gamma'}\Sigma \leq \mathcal{P}_{\Gamma''}\Sigma \end{aligned} \tag{5.8}$$

Conversely, an inductive system $\{(\omega_\Gamma, \beta^*_{\Gamma,\Gamma'}) : \beta^*_{\Gamma,\Gamma'} : \mathcal{A}_s^{\Gamma'} \rightarrow \mathcal{A}_s^\Gamma, \mathcal{P}_\Gamma\Sigma \geq \mathcal{P}_{\Gamma'}\Sigma\}$ of states which statisfies the condition (5.8) defines a state on $C(\bar{\mathcal{A}}_s)$. In fact, the state space $\mathcal{S}(C(\bar{\mathcal{A}}_s))$ of $C(\bar{\mathcal{A}}_s)$ is homeomorphic to the projective limit of the state space $\mathcal{S}(C(\bar{\mathcal{A}}_s^\Gamma))$ of $C(\bar{\mathcal{A}}_s^\Gamma)$. Since, $\text{Cyl}(\mathcal{P}_\Gamma\Sigma)$ is a commutative unital C^*-algebra the spectrum $\bar{\mathcal{A}}_s$ is the projective limit of a family of spectra $\bar{\mathcal{A}}_s$ of $C(\bar{\mathcal{A}}_s^\Gamma)$. Hence the projective limit of bounded positive Radon measures replaces the projective limit of states spaces.

Cylindrical function algebra over the configuration space $\breve{\mathcal{A}}_s$ of smooth connections

A short comment on the cylindrical function algebra over the configuration space $\breve{\mathcal{A}}_s$ of smooth conections is given in the next paragraph.

The *-algebra $\text{Cyl}(\breve{\mathcal{A}}_s)$ of cylindrical functions on the configuration space $\breve{\mathcal{A}}_s$ is defined by all functions of the form

$$\begin{aligned} F(\mathfrak{h}) &= (f \circ \pi)(\mathfrak{h}) \text{ for } \text{ where } \pi : \breve{\mathcal{A}}_s \rightarrow \bar{\mathcal{A}}_s^L \text{ is a continuous morphism} \\ &\quad \text{for } f \in C(\bar{\mathcal{A}}_s^L) \\ F^* &= \bar{F} \end{aligned}$$

Assume that, the space of smooth connections $\breve{\mathcal{A}}_s$ is identified with the set $\text{Hom}_{\breve{\mathcal{A}}_s}(\text{LG}(v), G)$ of all continuous holonomy maps from the abelian loop group $\text{LG}(v)$ at $v \in \Sigma$ to G with respect to the Barrett topology on $\text{LG}(v)$. Assume that, the loop group $\text{LG}(v)$ and the dual group of $\text{LG}(v)$ are equal. Then similar to the Pontryagin duality, the space $\breve{\mathcal{A}}_s$ is required to be equal to the Pontryagin dual of $\hat{\mathcal{A}}_s$, where $\breve{\mathcal{A}}_s$ denotes the space of smooth connections and $\hat{\mathcal{A}}_s$ is the dual space. Then the space $\bar{\mathcal{A}}_s^L$ of generalised connection is equivalent to the set $\text{Hom}(\text{LG}(v), G)$ of all holonomy maps. In comparison with the usual Pontryagin duality for abelian locally compact groups, the space $\breve{\mathcal{A}}_s$ is required to be the dual of $\text{LG}(v)$ and $\bar{\mathcal{A}}_s^L$ is the compactification of $\text{LG}(v)$. Note that, this contradicts the assumption that $\text{LG}(v)$ and the dual group of $\text{LG}(v)$ coincide. Hence $\breve{\mathcal{A}}_s$ and $\text{Hom}_{\breve{\mathcal{A}}_s}(\text{LG}(v), G)$ cannot be easily identified. Furthermore this indicate that the C^*-algebras $\text{Cyl}(\text{LG}(v))$ and $\text{Cyl}(\breve{\bar{\mathcal{A}}}_s)$ are not isomorphic. Moreover the cylindrical function algebra $\text{Cyl}(\breve{\mathcal{A}}_s)$ is not isomorphic to the C^*- algebra $C(\bar{\mathcal{A}}_s^L)$ of continuous function on the compact quantum configuration space $\bar{\mathcal{A}}_s^L$.

Finally, observe that in LQG literature G is chosen to be $U(1)$ or $SU(2)$ and hence the holonomy algebras have been contructed by Ashtekar and Isham [7] through a set of functions and relations between them.

5.3 The modified Wilson C^*-algebra

Ashtekar and Isham [7] have developed the holonomy C^*-algebra by a set of Wilson functions. They assume the Lie group G to be equivalent to $U(1)$, $SU(2)$ or $SL(2,\mathbb{C})$. In particular, for $SL(2,\mathbb{C})$ the Mandelstam relations are used to define relation between Wilson functions. For their construction of the Wilson *-algebra refer to the article of Ashtekar and Isham. In this section another modificated definition is presented. Notice only that for the derivation of the algebra the loop group $\mathrm{LG}(v)$ is equipped in [7, p.11] with an addition $+$ and a product $\cdot$ such that

$$\alpha \cdot \beta = \alpha \circ \beta + \alpha \circ \beta^{-1} \tag{5.9}$$

holds. In this way, $\mathrm{LG}(v)$ is a real vector space.

In the following a slightly modified Wilson algebra is defined, which is influenced by the definition of a group algebra.

Definition 5.3.1. *Let* $\mathrm{LG}(v)$ *be the abelian topological loop group at* v*. Equipp* $\mathrm{LG}(v)$ *with a discrete additive topology and denote the group by* $\mathrm{LG}(v)_{\mathrm{d}}$*. Let* G *be* $\mathbb{T}$*.*

Then the ***modified Wilson*** ****-algebra*** $\mathbf{W}(LG_{\mathrm{d}}(v))$ *is generated by all complex-valued functions* F *on the group* G *with finite support and such that*

(i) $F(\alpha) := a\,\mathrm{tr}(\mathfrak{h}(\alpha))$ *for* $a \in \mathbb{C}$, $\alpha \in LG_{\mathrm{d}}(v)$ *and where* $\mathfrak{h} : LG_{\mathrm{d}}(v) \longrightarrow G$ *is a continuous holonomy map for the abelian group* $LG_{\mathrm{d}}(v)$*,*

(ii) $F(\gamma) = \sum_{\substack{\alpha,\beta\in LG_{\mathrm{d}}(v);\\ \alpha\circ\beta=\gamma}} F(\alpha)F(\beta)$, $\gamma \in LG_{\mathrm{d}}(v)$*,*

(iii) $F^*(\alpha) = \overline{F(\alpha^{-1})}$, $\alpha \in LG_{\mathrm{d}}(v)$ *and*

(iv) $(F_1 + F_2)(\alpha) = F_1(\alpha) + F_2(\alpha)$ *for* $F_1, F_2 \in \mathbf{W}(LG_{\mathrm{d}}(v))$

and supremum norm.

Observe that, even $F(\alpha) = F(\alpha^{-1})$ holds.

Notice that, Ashtekar and Isham have not required a discrete topology on $\mathrm{LG}(v)$ and the convolution product (ii). They have chosen G to be equal to $SU(2)$ or $Sl(2,\mathbb{C})$. Then their Wilson *-algebra has been equipped with the multiplication operation $\cdot$, which is defined by

$$F(\alpha) \cdot F(\beta) := \frac{1}{2}\left(F(\alpha \circ \beta) + F(\alpha \circ \beta^{-1})\right)$$

the inversion * defined by $F^*(\alpha) = \overline{F(\alpha)}$ and sup-norm. Hence Ashtekar and Isham have followed the construction of a usual Weyl algebra, which was presented in simple examples in the preliminary section 1.3.

Proposition 5.3.2. *The modified Wilson* **-algebra* $\mathbf{W}(LG_{\mathrm{d}}(v))$ *equipped with supremum norm is a unital commutative Banach* **-algebra. The completion of* $\mathbf{W}(LG_{\mathrm{d}}(v))$[1] *w.r.t an universal norm becomes a* C^**-algebra* $\mathcal{W}(LG_{\mathrm{d}}(v))$*, which is called the* ***modified Wilson*** C^****-algebra for loops****.*

Then recall the algebra of almost periodic functions on an abelian locally compact group presented in section 4.1. Then the author of this dissertation suggest that the modified Wilson algebra $\mathcal{W}(LG_{\mathrm{d}}(v))$ is isomorphic to the center of the group algebra $C^*(LG_{\mathrm{d}}(v))$. The arguments for this conjecture is presented in section 8.1. Furthermore the relation of the C^*-algebra $\mathrm{Cyl}(LG_{\mathrm{d}}(v))$ and the group algebra $C^*(LG_{\mathrm{d}}(v))$ should be studied further, since in the example of almost periodic functions on $\mathbb{R}_{\mathrm{d}}$ there is an isomorphism between this algebra and the group algebra $C^*(\mathbb{R}_{\mathrm{d}})$. Refer to section 4.1 for this example. The evidence of the conjectures are hard to find, since $\mathrm{LG}(v)$ is neither a finite nor a abelian locally compact group. Therefore, a Pontryagin duality is not simply given. Furthermore there is no argument for an isomorphsim between $LG_{\mathrm{d}}(v)$ and the Pontryagin dual of $LG_{\mathrm{d}}(v)$.

[1] modulo the two-sided self-adjoint ideal of the *-algebra defined by $I = \{F : \|F\|_2 = 0\}$ where $\|.\|_2$ denotes the $L^2(\bar{\mathcal{A}}_s, \mu)$-norm.

The Pontryagin duality can be considered for example for a vector space V and its dual vector space $\widehat{V}$. The duality in this example is given by the isomorphism between the dual of $\widehat{V}$ and V. Or, generally, the dual $\widehat{G}$ of a locally compact abelian Hausdorff topological group $(G, \circ)$ is a group and $(G, \circ)$ can be identified with the set of all continuous group homomorphisms from the abelian character group (the Pontryagin dual) of the group G to $\mathbb{T}$. There is no isomorphism between the dual group $\widehat{G}$ and G in general. If the vector space V is finite-dimensional, then V is isomorphic to $\widehat{V}$. But, the endomorphism algebra of an infinite dimensional vector space and its dual vector space are isomorphic. Hence the same holds for finite groups. Or, equivalently the group algebra $C^*(\widehat{G})$ is isomorphic to $C_0(G)$.

Furthermore the author of this work tried to define a Wilson *-algebra by a certain Hopf *-algebra. For a finite abelian topological group G with discrete topology the Hopf *-algebra $\mathbb{C}(G)$ of complex-valued functions on the group G can be defined. Then one can show that there is a dual Hopf *-algebra constructed from the function algebra $K(G)$. Refer to the appendix 12.2.2.1 for a detailed mathematical theory. This concept can be further generalised to quantum groups, which are defined by a locally compact group G. Then for example Woronowicz and Napiórkowski have argued in [115] that the quantum group $(C^*(G), \triangle)$ determines the structure of G uniquely. Hence the quantum group is the Pontryagin dual of a non-commutative locally compact group G. But the arguments cannot be used to define the Pontryagin dual of $LG(v)$.

Chapter 6

The analytic holonomy C^*-algebra and Weyl C^*-algebra

The LQG-viewpoint

In LQG, the algebra of holonomy variables has been introduced by Ashtekar and Lewandowski [8]. The analytic holonomy algebra is given by a commutative C^*-algebra $C(\bar{\mathcal{A}})$, which is an inductive limit of a family $\{C(\bar{\mathcal{A}}_\Gamma)\}$ of commutative C^*-algebras associated to graphs. The inductive limit of C^*-algebras corresponds to a projective limit of a family $\{\bar{\mathcal{A}}_\Gamma\}$ of configuration spaces. Due to the Tychnov-theorem the inductive limit space $\bar{\mathcal{A}}$, which is constructed from the compact Hausdorff spaces, is a compact Hausdorff space, too. Each configuration space $\bar{\mathcal{A}}_\Gamma$ is identified with $G^{|\Gamma|}$ where G is the structure group of a principal fibre bundle. Usually this group is chosen to be a compact Lie group. On the configuration space $\bar{\mathcal{A}}_\Gamma$ associated to each graph Γ there exists a Haar measure μ_Γ. The consistent family $\{\mu_\Gamma\}$ of measures defines a measure μ_{AL} on $\bar{\mathcal{A}}$. Homeomorphisms on the compact Hausdorff space $\bar{\mathcal{A}}$ leaving the measure μ_{AL} invariant corresponds one-to-one to unitary operators $U(g)$ for elements g, which are contained in the structure group G. These unitary operators implement the fluxes associated to a surface S. In particular, measure preserving transformations associated to a graph Γ define $G^{|\Gamma|}$-invariant states on the C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$. For a detailed investigation of this construction in the context of compact Hausdorff spaces and measures refer to Marolf and Mourão [67] for graphs containing only analytic loops and Fleischhack [36] for general index sets. A study of the interplay of the projective structure of the configuration space and the inductive structure of the C^*-algebra has been given by Ashtekar and Lewandowski [10], Fleischhack [37] or Velhinho [105, 106].

Although several other diffeomorphism invariant states on $C(\bar{\mathcal{A}})$ have been available due to Baez [15, 14] or Ashtekar and Lewandowski [10, 9], only states which are $G^{|\Gamma|}$-invariant will allow to extend the quantum algebra by the flux operators for a surface S. This question has been analysed for example by Sahlmann in [83]. In the context of Weyl algebras constructed from holonomies and quantum flux operators, which are exponentiated Lie algebra-valued operators, the first attempts have been presented by Sahlmann and Thiemann [86]. The Weyl algebra of holonomies and (exponentiated) quantum fluxes, which are introduced by particular pull-backs of homeomorphisms on the configuration space, has been constructed by Fleischhack [39]. The developement of the Weyl algebra is related to transformation groups associated to a flux group and the configuration space. First attempts in this direction has been presented by Velhinho [108]. The irreducibility of the Weyl C^*-algebra has been studied first by Sahlmann and Thiemann [85]. Fleischhack has proved in [39] irreducibility and under some technical assumptions that there is only one irreducible and diffeomorphism-invariant representation of his Weyl C^*-algebra on the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$. For a short overview refer to Fleischhack [38]. In comparison with the Weyl algebra presented in this dissertation, Fleischhack has considered more general stratified objects, instead of $D-1$-dimensional surfaces in a D-dimensional manifold only, for the construction of his Weyl C^*-algebra.

The new viewpoint

The analytic holonomy C^*-algebra $\mathfrak{A}_\Gamma$ associated to a graph Γ is isomorphic to a norm-closed *-subalgebra of the concrete C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ of bounded operators on a Hilbert space $\mathcal{H}_\Gamma$. In general for every C^*-algebra $\mathfrak{A}$ there exists a Hilbert space $\mathcal{H}$ and a linear isomorphism Φ from $\mathfrak{A}$ onto a norm-closed *-subalgebra of $\mathcal{L}(\mathcal{H})$. This is the content of the second Gelfand-Naĭmark theorem. In particular, each commutative C^*-algebra is isomorphic to the algebra of continuous functions on a suitable configuration space. The analytic holonomy C^*-algebra is, therefore isomorphic to $C(\bar{\mathcal{A}}_\Gamma)$. In contrast to usual LQG literature in this work the inductive limit analytic holonomy algebra $C(\bar{\mathcal{A}})$ is derived from an inductive limit of finite graph systems.

In section 6.1, a lot of different actions of flux groups associated to suitable surfaces and graphs on each analytic holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ are analsed. Each C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ and an action α of a flux group $\bar{G}_{\breve{S},\Gamma}$ associated to a surface set $\breve{S}$ define a C^*-dynamical system. The set of actions, which defines C^*-dynamical systems in the C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ of bounded operators on a Hilbert space, is denoted by $\mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))$.

There exists a representation of the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ on the Hilbert space $\mathcal{H}_\Gamma = L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ as a multiplication operator

$$\Phi_M(f_\Gamma)\psi_\Gamma = f_\Gamma\psi_\Gamma \qquad (6.1)$$

and a set $\mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ of representations of flux groups associated to suitable surface sets on the C^*-algebra $\mathcal{K}(\mathcal{H}_\Gamma)$ of compact operators on a Hilbert space $\mathcal{H}_\Gamma$.

The quantum flux operators are group-valued operators depending on the intersection behavior of a surface and a path. For particular surfaces and graphs, these objects form a group. The left (or right) regular representations of the flux group define the unitary quantum flux operators on a Hilbert space, which are also called Weyl elements. The Weyl elements generate a C^*-algebra. The representation Φ_M and a left or right regular representation $U_{S,\Gamma}$ of the flux group $\bar{G}_{S,\Gamma}$ associated to a fixed surface S and a graph Γ forms a covariant representation $(\Phi_M, U_{S,\Gamma})$ of a C^*-dynamical system $(C(\bar{\mathcal{A}}_\Gamma), \bar{G}_{S,\Gamma}, \alpha)$.

The configuration space $\bar{\mathcal{A}}_\Gamma$ of generalised connections restricted to a finite graph system $\mathcal{P}_\Gamma$ is identified naturally or in a non-standard way with a product $G^{|\Gamma|}$ of a compact Lie group G. In this two cases, an action of a graph changing operation and the actions of the flux group associated to surfaces and graphs on the analytic holonomy C^*-algebra do not commute in general. The graph changing action has to preserve the structure of the Weyl elements and consequently the action is required to map quantum flux operators associated to a surface set $\breve{S}$ to quantum fluxes associated to a surface set $\breve{S}$ again. Or more generally Weyl elements are required to be mapped to Weyl elements. The problems is solved by using actions of the group $\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma)$ of surface-orientation-preserving bisections of a finite graph system on the analytic holonomy C^*-algebra and the C^*-algebra of the Weyl elements. This issue is treated in section 6.2. There is an argument for the use of graph systems instead of graphs only. One would like to define graph changing automorphisms on the analytic holonomy C^*-algebra associated to a graph. Therefore the algebra is required to depend on sets of graphs. Otherwise the actions could only be defined on the limit C^*-algebra. This can be done, but in this situation the interplay with the action of the quantum flux group, which are naturally associated to graphs and surfaces, become more difficult. Clearly, this is finally a matter of taste of the author.

The Weyl elements and the continuous functions of the analytic holonomy algebra generate the Weyl C^*-algebra $\mathsf{Weyl}(\breve{S},\Gamma)$ associated to a graph and surfaces, this is presented in section 6.3. Then actions of the group $\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma)$ of surface-orientation-preserving bisections of a finite graph system on the commutative Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\breve{S},\Gamma)$ for a surface set $\breve{S}$ and a finite graph system $\mathcal{P}_\Gamma$ are studied. The quantum flux operators are associated to exceptional sets of surfaces such that they form a group. Therefore the sets are very restrictive. The set $\mathbb{S}_{\mathcal{Z}}$ contain all possible surface sets such that each surface set $\breve{S}$ in $\mathbb{S}_{\mathcal{Z}}$ define a group $\bar{G}_{\breve{S},\Gamma}$. Note that, the set $\bar{G}_{\mathbb{S}_{\mathcal{Z}},\Gamma}$ is not required to form a group. Then the commutative Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ for surfaces and a finite graph system $\mathcal{P}_\Gamma$ is defined. In section 6.3 it is shown that, there exists a $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$- and $\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma)$-invariant state $\omega^\Gamma_{M,\mathfrak{B}}$ on the Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$, where $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$ is the commutative flux group derived from the center of the group G.

Finally the inductive limit C^*-algebra of the inductive family $\{\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)\}$ of commutative Weyl C^*-algebras is called the commutative Weyl C^*-algebra for surfaces and is denoted by $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}})$. This algebra is presented in section 6.3. The inductive family $\{\Gamma_i\}$ of graphs defines the flux group $\bar{G}_{\breve{S}}$ associated to a fixed surface set $\breve{S}$ and where Γ_∞ is the inductive limit graph. In section 6.4, it is proven that there exists a unique pure state $\omega_{M,\mathfrak{B}}$ on $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$, which is $\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_{\Gamma_i})$- for each graph Γ_i and $\bar{\mathcal{Z}}_{\breve{S}}$-invariant if the configuration space is identified in non-standard way.

At the end of this section in 6.5 the holonomy-flux von Neumann algebra is defined and the issue of KMS-states is sudied in this context. In particular it is shown that, the von Neumann algebra is non-standard. On the other hand a KMS theory is also available for the Weyl C^*-algebra. But it turns out that there are no KMS states for this particular C^*-algebra. Finally a short overview about the issue of time avarages in the context of the Weyl C^*-algebra is given in 6.5.

6.1 Dynamical systems of actions of the flux group on the analytic holonomy C^*-algebra

The analytic holonomy C^*-algebra for finite graph systems

First there are inductive structures on the graphs and on the system of graphs. The inductive limit Γ_∞ of an inductive family of graphs is a graph, which consists of an infinite countable number of paths. The inductive limit $\mathcal{P}_{\Gamma_\infty}$ of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems contains an infinite number of subgraphs of Γ_∞, each of them is a finite set of arbitrary independent paths in Σ.

In this dissertation the analytic holonomy algebra $C(\bar{\mathcal{A}}_\Gamma)$ restricted to a graph system $\mathcal{P}_\Gamma$ is given by the set $C(\bar{\mathcal{A}}_\Gamma)$ of continuous functions on $\bar{\mathcal{A}}_\Gamma$, pointwise multiplication, complex conjugation and the completion is taken with respect to the sup-norm. The **analytic holonomy C^*-algebra** $C(\bar{\mathcal{A}})$ is given by the inductive limit of the family of unital commutative C^*-algebras $\{(C(\bar{\mathcal{A}}_{\Gamma_i}), \beta_{\Gamma_i,\Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i,j \in \mathbb{N}\}$ for an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems, where $\beta_{\Gamma_i,\Gamma_j}$ is a unit-preserving injective *-homomorphism from the analytic holonomy algebra $C(\bar{\mathcal{A}}_{\Gamma_i})$ to $C(\bar{\mathcal{A}}_{\Gamma_j})$. The maps β_Γ satisfy the consistency conditions

$$\beta_{\Gamma,\Gamma''} = \beta_{\Gamma,\Gamma'} \circ \beta_{\Gamma',\Gamma''}$$

whenever $\mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma''}$. The configuration space $\bar{\mathcal{A}}_{\Gamma_i}$ associated to a graph Γ_i is derived from the set of all holonomy maps from the finite graph system $\mathcal{P}_{\Gamma_i}$ to the product group $G^{|\Gamma_i|}$. Recall the notion of natural or non-standard identification of the configuration space $\bar{\mathcal{A}}_\Gamma$, which is presented in section 3.3.4.3. The elements of $\bar{\mathcal{A}}_\Gamma$ are identified naturally or in a non-standard way with $G^{|\Gamma|}$ by the evaluation of the holonomy map for a subset of the finite graph system $\mathcal{P}_\Gamma$. Simply speaking the choice of the identification is a matter of the labeling of the configuration variables. In this work the identifications are needed for the definition of graph changing automorphisms acting on the analytic holonomy C^*-algebra.

An element of the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}})$ is of the form

$$f = f_{\Gamma_i} \circ \pi_{\Gamma_i} = \beta_{\Gamma_i} \circ f_{\Gamma_i}$$

where $f \in C(\bar{\mathcal{A}})$, $\pi_{\Gamma_i} : \bar{\mathcal{A}} \to \bar{\mathcal{A}}_{\Gamma_i}$, $f_{\Gamma_i} \in C(G^{|\Gamma_i|})$ and the map $\beta_{\Gamma_i} : C(\bar{\mathcal{A}}_{\Gamma_i}) \to C(\bar{\mathcal{A}})$ is an unit-preserving injective *-homomorphisms. Furthermore the maps $\beta_\Gamma : C(\bar{\mathcal{A}}_\Gamma) \longrightarrow C(\bar{\mathcal{A}})$ are isometries, since

$$\|f\| = \|\beta_{\Gamma_i} f_{\Gamma_i}\| = \sup |f_{\Gamma_i}|$$

whenever $f \in C(\bar{\mathcal{A}})$ and $f_{\Gamma_i} \in C(\bar{\mathcal{A}}_{\Gamma_i})$ for all graphs Γ_i. A detailed analysis of the construction is given at the beginning of section 6.2.

The idea is to define actions of groups on the C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ of continuous functions on the compact Hausdorff space $\bar{\mathcal{A}}_\Gamma$ associated to a graph Γ, which can be extended to actions on the inductive limit algebra $C(\bar{\mathcal{A}})$.

Group actions on the configuration space

Let Γ be a graph, $\mathcal{P}_\Gamma$ be the associated finite graph system. Assume that, the subgraphs in a finite graph system $\mathcal{P}_\Gamma$ are identified naturally and hence the configuration space $\bar{\mathcal{A}}_\Gamma$ is identified in the natural way with $G^{|\Gamma|}$.

Then there is a group action

$$G^{|\Gamma|} \times \bar{\mathcal{A}}_\Gamma \ni ((g_1, .., g_N), (\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))) \mapsto (g_1 \mathfrak{h}_\Gamma(\gamma_1), ..., g_N \mathfrak{h}_\Gamma(\gamma_N)) \in \bar{\mathcal{A}}_\Gamma$$

of a finite product of a compact group G on the compact Hausdorff space $\bar{\mathcal{A}}_\Gamma$ where $N = |\Gamma|$. For each $\mathbf{g} := (g_1, ..., g_N) \in G^{|\Gamma|}$ the map $L(\mathbf{g})$ given by

$$\bar{\mathcal{A}}_\Gamma \ni (\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) \mapsto L(\mathbf{g})(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) := (g_1 \mathfrak{h}_\Gamma(\gamma_1), ..., g_N \mathfrak{h}_\Gamma(\gamma_N)) \in \bar{\mathcal{A}}_\Gamma$$

is a homeomorphism $L(\mathbf{g}) : \bar{\mathcal{A}}_\Gamma \longrightarrow \bar{\mathcal{A}}_\Gamma$. Moreover

$$L(\mathbf{g})(L(\mathbf{h})(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))) = (L(\mathbf{gh}))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))$$

for all $\mathbf{g}, \mathbf{h} \in G^{|\Gamma|}$ and $(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) \in \bar{\mathcal{A}}_\Gamma$ yields. Clearly, there is a right action presented by the map $R(\mathbf{g})$, which is defined by

$$\bar{\mathcal{A}}_\Gamma \ni (\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) \mapsto R(\mathbf{g})(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) := (\mathfrak{h}_\Gamma(\gamma_1)g_1^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)g_N^{-1}) \in \bar{\mathcal{A}}_\Gamma$$

such that

$$\begin{aligned} R(\mathbf{gh})(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) &= (\mathfrak{h}_\Gamma(\gamma_1)(g_1h_1)^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)(g_Nh_N)^{-1}) \\ &= (\mathfrak{h}_\Gamma(\gamma_1)h_1^{-1}g_1^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)h_N^{-1}g_N^{-1}) \\ &= R(\mathbf{g})(R(\mathbf{h})(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))) \end{aligned}$$

Consider a finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ associated to a graph Γ and a finite set of surfaces $\breve{S}$ such that the set $\breve{S}$ has the same surface intersection property for the graph Γ. Then the flux group $\bar{G}_{\breve{S},\Gamma}$ is a subgroup of $G^{|\Gamma|}$. Since, each subgraph Γ' of Γ, for example, consists only paths that intersect each surface in $\breve{S}$ in the source vertex of the path and lie above. The evaluation of a map $\rho_{S,\Gamma}$ in $G_{\breve{S},\Gamma'\leq\Gamma}$ for a subgraph Γ' in $\mathcal{P}_\Gamma^{\mathfrak{o}}$ is given by $\rho_{S,\Gamma}(\Gamma') = (\rho_S(\gamma_1), ..., \rho_S(\gamma_M))$. The element $\rho_{S,\Gamma}(\Gamma')$ is contained in $\bar{G}_{\breve{S},\Gamma'\leq\Gamma}$. Furthermore the element $(\rho_S(\gamma_1), ..., \rho_S(\gamma_M), e_G, ..., e_G)$ is contained in $G_{\breve{S},\Gamma}$.

Consider a graph $\Gamma := \{\gamma_1, ..., \gamma_N\}$ and a subgraph $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ of Γ, a finite graph system $\mathcal{P}_\Gamma$ and a finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ exists. Then there is a surfaces set $\breve{S}$, which has the same surface intersection property for Γ. Moreover assume that $\Gamma' \in \mathcal{P}_\Gamma^{\mathfrak{o}}$. Then for a map $\rho_{S,\Gamma} \in G_{\breve{S},\Gamma}$ there exists a left action $L : \bar{G}_{S,\Gamma} \to \bar{\mathcal{A}}_\Gamma$, which is given by

$$\begin{aligned} L(\rho_{S,\Gamma}(\Gamma))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) &= L(\rho_S(\gamma_1), ..., \rho_S(\gamma_N))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) \\ &:= (\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1), ..., \rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) \end{aligned} \tag{6.2}$$

and which defines a homeomorphism on $\bar{\mathcal{A}}_\Gamma$. Certainly, if the surface set $\breve{S}$ has the same surface intersection property for Γ, then there is a right action R of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. This action R is of the form

$$\begin{aligned} &R(\rho_{S,\Gamma}(\Gamma'))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) = R(\rho_S(\gamma_1), ..., \rho_S(\gamma_M))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) \\ &:= (\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_S(\gamma_M)^{-1}, \mathfrak{h}_\Gamma(\gamma_{M+1}), ..., \mathfrak{h}_\Gamma(\gamma_N)) \end{aligned} \tag{6.3}$$

for $\rho_{S,\Gamma}(\Gamma') \in \bar{G}_{S,\Gamma}$. This action defines a homeomorphism of $\bar{\mathcal{A}}_\Gamma$, too. Notice that, the flux operator given by $\rho_{S,\Gamma}(\Gamma')$ is for example restricted to a subgraph Γ', whereas the holonomies are computated on the whole graph Γ. Mathematically, this is well-defined. Physically, the flux operators are somehow localised on a subgraph.

Transformation groups

Observe the actions defined above are directly related to the concept of a transformation groups.

Definition 6.1.1. *A **transformation group** is given by a locally compact group G and a locally compact space X if there is a continuous map $G \times X \ni (g, x) \mapsto gx \in X$ and there is a homeomorphism $\xi_L : x \mapsto gx$ for $g \in G$ and $x \in X$ such that $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.*

Observe that, (G, X) form also an transformation group if there is a continuous map $G \times X \ni (g, x) \mapsto xg^{-1} \in X$ and there is a homeomorphism $\xi_R : x \mapsto xg^{-1}$ for $g \in G$ and $x \in X$ such that $(xg^{-1})h^{-1} = x(hg)^{-1}$ for all $g, h \in G$ and $x \in X$.

Clearly the set $\text{Homeo}(X)$ of all homeomorphisms of X is much larger than the set of all homeomorphisms that defines the transformation group (G, X). For example let $\xi_g \in \text{Homeo}(X)$ such that $\xi_g(x) = gx$ for a fixed $g \in G$, $x \in X$ and set $\xi^g_{\mathbb{Z}} : n \mapsto g^n x \in X$ for $n \in \mathbb{Z}$ then $\xi^g_{\mathbb{Z}}$ is a homeomorphism of X, too.

Moreover there exists a measure μ on X such that the transormation group (G, X) defines an action of G on X, which is measure preserving.

Consequently for a fixed suitable surface set $\breve{S}$ and a fixed finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ the pair $(\bar{G}_{\breve{S},\Gamma},\bar{\mathcal{A}}_\Gamma)$ is a transformation group. Clearly, there are many different transformation groups $(\bar{G}_{\breve{S},\Gamma},\bar{\mathcal{A}}_\Gamma)$ associated to surfaces and finite graph systems. The new Weyl algebra is constructed from all those transformation groups refering to the flux group $\bar{G}_{\breve{S},\Gamma}$ and the configuration space $\bar{\mathcal{A}}_\Gamma$. The transformation group $(\mathbb{Z},\bar{\mathcal{A}}_\Gamma)$ is not considered. The Weyl algebra of Fleischhack in [39] is generated by a suitable set of homeomorphisms of $\bar{\mathcal{A}}$ and consequently a suitable set of homeomorphisms of $\bar{\mathcal{A}}_\Gamma$. The suitable set of homeomorphisms of $\bar{\mathcal{A}}$ depends on a surface, its orientation and some additional function.

Group actions on the holonomy C^*-algebra for a fixed graph system and surfaces

Let $\breve{S}$ be a suitable surface set for a finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ and the object $\bar{G}_{\breve{S},\Gamma}$ is the flux group. Then, equivalently to a group action of $\bar{G}_{\breve{S},\Gamma}$ on the configuration space $\bar{\mathcal{A}}_\Gamma$ an action α of $\bar{G}_{\breve{S},\Gamma}$ on the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ associated to a graph Γ can be studied. The action is for example of the form

$$(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma)) := f_\Gamma(L(\rho_{S,\Gamma}(\Gamma))(\mathfrak{h}_\Gamma(\Gamma)))$$

where $\rho_{S,\Gamma}\in G_{\breve{S},\Gamma}$ and $f_\Gamma\in C(\bar{\mathcal{A}}_\Gamma)$.

Notice that, there is a state on $C(\bar{\mathcal{A}}_\Gamma)$, which is $\bar{G}_{\breve{S},\Gamma}$-invariant. In this section a bunch of different actions of this form are constructed.

Before the investigations start, the following remark on the nature of the definition of the flux operators has to be done. In section 3.4 the flux operators are constructed from maps, which map a graph Γ to the structure group G. If the flux operators would be defined by groupoid morphisms between the finite path groupoid $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$ and the groupoid G over $\{e_G\}$ then the following difficulties arise.

Remark 6.1.2. *Let* $\mathrm{Hom}_S(\mathcal{P}_\Gamma\Sigma,G)$ *be the set of groupoid morphisms between the finite path groupoid* $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$ *and the groupoid* G *over* $\{e_G\}$ *associated to a surface* S *such that each groupoid morphism* p_S *is an element of* $\mathrm{Map}_S(\mathcal{P}_\Gamma\Sigma,G)$. *Then every groupoid morphism* p_S *in* $\mathrm{Hom}_S(\mathcal{P}_\Gamma\Sigma,G)$ *satisfies*

$$p_S(\gamma^{-1})=p_S(\gamma)^{-1},\quad p_S(\gamma\circ\gamma')=p_S(\gamma)p_S(\gamma')\quad\forall\gamma\in\mathcal{P}_\Gamma\Sigma,(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$$

and the maps p_S *have the special structure of* $\mathrm{Map}_S(\mathcal{P}_\Gamma\Sigma,G)$, *which implements the intersection behavior of the paths of* Γ *and the surface* S. *Observe that, for a surface* S *and a path* $\gamma\circ\gamma'$ *that intersects* S *only in* $s(\gamma)$ *the maps* p_S *satisfy* $p_S(\gamma\circ\gamma')=p_S(\gamma)$, *since* $p_S(\gamma')=e_G$.

Due to the specific structure of the groupoid homomorphisms $\mathfrak{h}_\Gamma:\mathcal{P}_\Gamma\Sigma\longrightarrow G$ *there is in general no groupoid morphism* $\mathfrak{H}$ *defined by*

$$\mathrm{Hom}(\mathcal{P}_\Gamma\Sigma,G)\ni\mathfrak{h}_\Gamma(\gamma)\mapsto\mathfrak{H}(\gamma):=p_S(\gamma)\mathfrak{h}_\Gamma(\gamma)\notin\mathrm{Hom}(\mathcal{P}_\Gamma\Sigma,G)$$

for $p_S\in\mathrm{Hom}_S(\mathcal{P}_\Gamma\Sigma,G)$. *Let* $\Gamma=\{\gamma,\gamma'\}$, *then* $\gamma,\gamma'\in\mathcal{P}_\Gamma\Sigma$ *and assume that* $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$. *Then this can be shown by the computation*

$$\begin{aligned}&\mathfrak{H}(\gamma\circ\gamma')=p_S(\gamma\circ\gamma')\mathfrak{h}_\Gamma(\gamma\circ\gamma')=p_S(\gamma)p_S(\gamma')\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma')\\&\neq p_S(\gamma)\mathfrak{h}_\Gamma(\gamma)p_S(\gamma')\mathfrak{h}_\Gamma(\gamma')\\&=\mathfrak{H}(\gamma)\mathfrak{H}(\gamma')\end{aligned}\tag{6.4}$$

for $p_S\in\mathrm{Hom}_S(\mathcal{P}_\Gamma\Sigma,G)$.

The equality holds for all $p_S\in\mathrm{Map}_S(\mathcal{P}_\Gamma\Sigma,\mathcal{Z}(G))$, *where* $\mathcal{Z}(G)$ *is the center of the group* G. *This indicate that for* $p_S\in\mathrm{Hom}_S(\mathcal{P}_\Gamma\Sigma,\mathcal{Z}(G))$ *the following properties are true*

(i) $p_S(\gamma')\mathfrak{h}_\Gamma(\gamma)=\mathfrak{h}_\Gamma(\gamma)p_S(\gamma')$ *for all paths* γ *and* γ' *contained in* $\mathcal{P}_\Gamma\Sigma$,

(ii) $p_S(\gamma^{-1})=p_S(\gamma)^{-1}$ *and* $p_S(\gamma\circ\gamma')=p_S(\gamma)p_S(\gamma')$ *for all paths* $\gamma,\gamma'\in\mathcal{P}_\Gamma\Sigma$,

(iii) $p_S(\gamma)=p_S(\gamma'')$ *for all* $\gamma,\gamma''\in\mathcal{P}_\Gamma\Sigma^v$ *where* $v=s(\gamma)=s(\gamma'')$,

(iv) $p_S(\gamma\circ\gamma^{-1})=e_G$ *for each path* γ *in* $\mathcal{P}_\Gamma\Sigma$ *and*

(v) $p_S(\gamma) = e_G$ *if* γ *in* $\mathcal{P}_\Gamma\Sigma$ *does not intersect* S *in the source or target vertex.*

Moreover consider $\mathcal{Z}_{S,\Gamma}$ *to be the the set*

$$\begin{aligned}\mathcal{Z}_{S,\Gamma} := \{p_{S,\Gamma} \in \operatorname{Hom}_S(\mathcal{P}_\Gamma, \mathcal{Z}(G)^{|\Gamma|}) : \quad & \exists\, p_S \in \operatorname{Hom}_S(\mathcal{P}_\Gamma\Sigma, \mathcal{Z}(G)) \text{ s.t.}\\ & p_{S,\Gamma}(\Gamma') = (p_S(\gamma_1), ..., p_S(\gamma_M))\\ & \forall \Gamma' \in \mathcal{P}_\Gamma\}\end{aligned}$$

and $\mathcal{Z}_{\breve{S},\Gamma} := \times_{S\in\breve{S}}\mathcal{Z}_{S,\Gamma}$.

Remark 6.1.3. *Otherwise, in section 3.4 the definition 3.4.22 of the set of maps* $\mathbb{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ *associated to a set of surfaces* $\breve{S}$ *is presented. For a pair of maps* ϱ^L_S *and* ϱ^R_S *in* $\mathbb{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ *and for each surface* S *in* $\breve{S}$ *it is true that*

(vii) $S \cap \gamma = \{s(\gamma), t(\gamma)\}$ *for all* $\gamma \in \mathcal{P}_\Gamma\Sigma$,

(viii) $\varrho^L_S(\gamma\circ\gamma') = \varrho^L_S(\gamma)$, $\varrho^R_S(\gamma\circ\gamma') = \varrho^R_S(\gamma)$ *for all* $\gamma, \gamma\circ\gamma' \in \mathcal{P}_\Gamma\Sigma^v$ *and* $S\cap\gamma\circ\gamma' = \{s(\gamma)\}$,

(ix) $\varrho^L_S((\gamma\circ\gamma')^{-1}) = \varrho^L_S(\gamma'^{-1})$, $\varrho^R_S((\gamma\circ\gamma')^{-1}) = \varrho^R_S(\gamma'^{-1})$ *for all* $\gamma'^{-1}\circ\gamma^{-1} \in \mathcal{P}_\Gamma\Sigma^v$ *and* $S\cap\gamma'^{-1}\circ\gamma^{-1} = \{t(\gamma'^{-1})\}$,

(x) $\varrho^L_S(\gamma)^{-1} = \varrho^L_S(\gamma^{-1})$, $\varrho^R_S(\gamma)^{-1} = \varrho^R_S(\gamma^{-1})$ *for* $\gamma \in \mathcal{P}_\Gamma\Sigma$,

(xi) $\varrho^L_S(\gamma) = \varrho^L_S(\gamma'')$, $\varrho^R_S(\gamma) = \varrho^R_S(\gamma'')$ *for all* $\gamma, \gamma'' \in \mathcal{P}_\Gamma\Sigma^v$ *where* $v = s(\gamma) = s(\gamma'')$,

(xii) $\varrho^R_S(\gamma^{-1})^{-1}\varrho^L_S(\gamma') = e_G$ *for all* $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$

(xiii) $\varrho^L_S(\gamma\circ\gamma^{-1}) = e_G$, $\varrho^R_S(\gamma\circ\gamma^{-1}) = e_G$ *for a path* γ *in* $\mathcal{P}_\Gamma\Sigma$ *and*

(xiv) $\varrho^L_S(\gamma) = e_G$, $\varrho^R_S(\gamma) = e_G$ *if* γ *in* $\mathcal{P}_\Gamma\Sigma$ *does not intersect* S *in the source or target vertex.*

Then for example, for a path γ *that intersects a fixed surface* S *in the source vertex* $s(\gamma)$ *the map* $\mathfrak{G}^L_\Gamma$ *defined by*

$$\operatorname{Hom}(\mathcal{P}_\Gamma\Sigma, G) \ni \mathfrak{h}_\Gamma(\gamma) \mapsto \mathfrak{G}^L_\Gamma(\gamma) := \varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma) \notin \operatorname{Hom}(\mathcal{P}_\Gamma\Sigma, G) \tag{6.5}$$

is not a groupoid morphism. This can be verified by

$$\begin{aligned}\mathfrak{G}^L_\Gamma(\gamma\circ\gamma') &= \varrho^L_S(\gamma\circ\gamma')\mathfrak{h}_\Gamma(\gamma\circ\gamma') = \varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma)\varrho^L_S(\gamma^{-1})^{-1}\varrho^L_S(\gamma')\mathfrak{h}_\Gamma(\gamma')\\ &\neq \mathfrak{G}^L_\Gamma(\gamma)\mathfrak{G}^L_\Gamma(\gamma') = \varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma)\varrho^L_S(\gamma')\mathfrak{h}_\Gamma(\gamma')\end{aligned}$$

Otherwise, if the path γ *intersects the surface* S *in the target vertex* $t(\gamma)$ *then the map* $\mathfrak{G}^R_\Gamma$ *given by*

$$\mathfrak{h}_\Gamma(\gamma) \mapsto \mathfrak{G}^R_\Gamma(\gamma) := \mathfrak{h}_\Gamma(\gamma)\rho^R_S(\gamma^{-1})^{-1}$$

is not a groupoid morphism, too.

Recall in definition 3.3.19 a groupoid morphism $\mathfrak{G}_\Gamma$ *was defined for a path* γ *that intersects the surface* S *in the source vertex* $s(\gamma)$ *and the target* $t(\gamma)$ *by*

$$\mathfrak{h}_\Gamma(\gamma) \mapsto \mathfrak{G}_\Gamma(\gamma) := \varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma)\varrho^R_S(\gamma^{-1})^{-1}$$

Notice that, there is a difference between $\operatorname{Hom}_S(\mathcal{P}_\Gamma\Sigma, \mathcal{Z}(G))$ *and* $\mathbb{G}^{\mathrm{A}}_{\breve{S},\Gamma}$. *For example in condition (i), which solve the problem* (6.4) *of the groupoid multiplication by using the center of* G *and condition (ii) for maps in* $\operatorname{Hom}_S(\mathcal{P}_\Gamma\Sigma, \mathcal{Z}(G))$. *Otherwise, the maps in* $\mathbb{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ *satisfy, in particularly, the condition (xiii) for composable paths.*

Identify the subgraph Γ' *naturally with the subset* $\{\gamma_1, .., \gamma_M\}$ *of the set of generators of* Γ. *After the evaluation of the maps in* $G^{\mathrm{A}}_{S,\Gamma}$ *for this subgraph, the element* $\varrho_{S,\Gamma}(\Gamma') = (\varrho_S(\gamma_1), ..., \varrho_S(\gamma_M))$ *is contained in the set* $\bar{G}^{\mathrm{A}}_{S,\Gamma}$. *Then the following actions of* $\bar{G}^{\mathrm{A}}_{S,\Gamma}$ *on the holonomy* C^**-algebra* $C(\bar{\mathcal{A}}_\Gamma)$ *can be defined for a subgraph* $\Gamma' := \{\gamma\}$ *of* Γ

$$\begin{aligned}(\alpha^{\mathrm{A},l}_L(\varrho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &:= f_\Gamma(\varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma))\\ (\alpha^{\mathrm{A},l}_R(\varrho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho^R_S(\gamma^{-1})^{-1})\\ (\alpha^{\mathrm{A},l}_{L,R}(\varrho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &:= f_\Gamma(\varrho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma)\varrho^R_S(\gamma^{-1})^{-1})\end{aligned} \tag{6.6}$$

whenever S *and* γ *are suitable (w.r.t. their intersection vertex and the behavoir of the path w.r.t. the surface orientation of* S*),* $\varrho_{S,\Gamma} \in \bar{G}^{\mathrm{A}}_{S,\Gamma}$ *and* $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$. *These actions are analysed further in the next section.*

A dynamical system of actions of the flux group on the analytic holonomy algebra for a fixed finite graph system and surfaces

First restrict the surface and graph configuration to the simpliest case. Consider a surface S, which has the same surface intersection property for a graph Γ. This equivalent to consider a surface S that intersects each path of Γ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of S. Furthermore there are no other intersection points of the surface S with paths of the graph Γ.

In this section representations and actions of the flux group $\bar{G}_{\breve{S},\Gamma}$ in the C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ are studied instead of analysing group actions on the configuration space or transformation groups. These representations are maps from the flux group $\bar{G}_{\breve{S},\Gamma}$ to the multiplier algebra of the C^*-algebra having several properties presented in the appendix 12.2.4.4 and 12.2.4.7. Equivalently to representations, actions of the flux group $\bar{G}_{S,\Gamma}$ on C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ instead of unitary multipliers can be studied. In the following different actions on the analytic holonomy algebra are investigated first.

Moreover for a more general framework it is assumed that, G is a locally compact unimodular group. Therefore the configuration space $\bar{\mathcal{A}}_\Gamma$ for a finite graph system $\mathcal{P}_\Gamma$ associated to a graph Γ is a locally compact Hausdorff space. Let $\mathfrak{A}_\Gamma$ be the quantum algebra generated by the configuration variables. Assume that, $\mathfrak{A}_\Gamma$ is isomorphic to $C_0(\bar{\mathcal{A}}_\Gamma)$. Notice that, the elements of $\bar{\mathcal{A}}_\Gamma$ are identified with $G^{|\Gamma|}$ by the natural identification and the evaluation of the holonomy map $\mathfrak{h}_\Gamma$ for a finite graph system $\mathcal{P}_\Gamma$ on a subgraph Γ' of Γ is $\mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_M), e_G, ..., e_G)$ an element in $G^{|\Gamma|}$.

An important property of actions on commutative C^*-algebras is the the following.

Definition 6.1.4. *Let $\mathfrak{A}$ be a commutative C^*-algebra isometrically isomorphic to $C_0(X)$ where X is a locally compact space, G be an arbitrary group and α be an automorphism of $\mathfrak{A}$.*

Then the action α of G on $\mathfrak{A}$ is ***automorphic*** *if the following conditions are satisfied*

(i) $\alpha(gh)(f) = \alpha(g)(\alpha(h)(f))$ *for any* $f \in \mathfrak{A}$, $g, h \in G$

(ii) $\alpha(g)(f_1 f_2) = \alpha(g)(f_1)\alpha(g)(f_2)$ *for any* $f_1, f_2 \in \mathfrak{A}$, $g \in G$

(iii) $\alpha(g)(f^*) = \alpha(g)(f)^*$ *for any* $f \in \mathfrak{A}$, $g \in G$

In this work the flux operators w.r.t. a surface S are implemented as group actions of $\bar{G}_{\breve{S},\Gamma}$ on the configuration space $\bar{\mathcal{A}}_\Gamma$, or equivalently as group actions on C^*-algebras. Consequently for each surface set and graph system configuration an action on the holonomy algebra for a suitable finite graph system can be defined. Therefore investigate the following actions on the holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$, where the configuration space $\bar{\mathcal{A}}_\Gamma$ is identified with $G^{|\Gamma|}$ naturally.

Lemma 6.1.5. *Let Γ be a graph and $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be the finite orientation preserved graph system associated to Γ. Furthermore let S be a fixed surface in Σ such that S intersects each path of Γ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of S. There are no other intersection points of the surface S with paths of the graph Γ.*

Let $\bar{G}_{\breve{S},\Gamma}$ denote the flux group and $\bar{\mathcal{A}}_\Gamma$ denote the configuration space for the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$, where all elements of $\mathcal{P}^{\mathfrak{o}}_\Gamma$ are identified in the natural way with a subset of the set of generators of Γ.

Then there is an action α of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ defined by

$$(\alpha(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) := f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1), ..., \rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))$$

for $\rho_{S,\Gamma} \in G_{\breve{S},\Gamma}$ and $\rho_S \in \mathbb{G}_{S,\gamma}$, which is automorphic.

Proof : Observe

$$\begin{aligned}(\alpha(\rho_{S,\Gamma}(\Gamma)\tilde{\rho}_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) &= f_\Gamma(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1), ..., \rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) \\ &= (\alpha(\rho_{S,\Gamma}(\Gamma))(\alpha(\tilde{\rho}_{S,\Gamma}(\Gamma))f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

Since the multiplication between two functions in $C_0(\bar{\mathcal{A}}_\Gamma)$ is pointwise:

$$\begin{aligned}&\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))f'_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)))\\&=\alpha(\rho_{S,\Gamma}(\Gamma))(\tilde{f}_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&=\tilde{f}_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))f'_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=(\alpha(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))(\alpha(\rho_{S,\Gamma}(\Gamma))f'_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

Finally,

$$\begin{aligned}(\alpha(\rho_{S,\Gamma}(\Gamma))f^*_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))&=\overline{f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))}\\&=(\alpha(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))^*\end{aligned}$$

■

In particular, there is a map $\tilde{J}:\bar{\mathbb{G}}_{\breve{S},\Gamma}\longrightarrow\bar{\mathbb{G}}_{\breve{S},\Gamma}$, $J:\rho_S(\gamma)\mapsto\rho_{S^{-1}}(\gamma)$. Then for $\rho_S(\gamma)=g$, where the surface S and the path γ are suitable and $\breve{S}:=\{S,S^{-1}\}$, the map satisfies $(\tilde{J}(\rho_S))(\gamma)=\rho_{S^{-1}}(\gamma)=g^{-1}$. With no doubt, there is a general map $J:\bar{G}_{\breve{S},\Gamma}\longrightarrow\bar{G}_{\breve{S},\Gamma}$, $J:\rho_{S,\Gamma}(\Gamma)\mapsto\rho_{S^{-1},\Gamma}(\Gamma)$. Then derive

$$\begin{aligned}&(\alpha(\rho_{S,\Gamma}(\Gamma)\big(J(\rho_{S,\Gamma})(\Gamma)\big))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\rho_S(\gamma_1)\rho_{S^{-1}}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\rho_{S^{-1}}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(gg^{-1}\mathfrak{h}_\Gamma(\gamma_1),...,gg^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.7}$$

It is necessary that the action is defined for a finite orientation preserved graph system, since the following observation can be made. Let Γ be equivalent to a path γ. Then it is true that $\mathfrak{h}_\Gamma(\gamma^{-1})=(\mathfrak{h}_\Gamma(\gamma))^{-1}$ and consequently

$$(\alpha(\rho_{S,\Gamma}(\Gamma))f_\Gamma)((\mathfrak{h}_\Gamma(\gamma))^{-1})=f_\Gamma(\rho^L_S(\gamma)(\mathfrak{h}_\Gamma(\gamma))^{-1})=f_\Gamma(\rho^L_S(\gamma)\mathfrak{h}_\Gamma(\gamma^{-1}))$$

for $\rho_{S,\Gamma}\in G_{\breve{S},\Gamma}$ and $\rho^L_S\in\mathbb{G}_{S,\gamma}$. Although the holonomy is evaluated for the path γ^{-1} the action is defined by the left action L. Later it is analysed when a left action L can be transfered to a right action R. The right action is given by

$$(\alpha(\rho_{S,\Gamma^{-1}}(\Gamma^{-1}))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma^{-1}))=f_\Gamma(R(\rho^R_S(\gamma^{-1}))(\mathfrak{h}_\Gamma(\gamma^{-1})))=f_\Gamma(\mathfrak{h}_\Gamma(\gamma^{-1})\rho^R_S(\gamma^{-1})^{-1})$$

for $\rho_{S,\Gamma^{-1}}\in G_{\breve{S},\Gamma^{-1}}$ and $\rho^R_S\in\mathbb{G}_{S,\gamma^{-1}}$.

Furthermore automorphic action has another interesting property which allows to speak about C^*-dynamical systems.

Corollary 6.1.6. *Let Γ be a graph and $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be the finite orientation preserved graph system associated to Γ. Furthermore let S be a fixed surface in Σ such that S intersects each path of Γ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of S. There are no other intersection points of the surface S with paths of the graph Γ.*

Let $\bar{G}_{\breve{S},\Gamma}$ denote the flux group and $\bar{\mathcal{A}}_\Gamma$ denote the configuration space for the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$, where all elements of $\mathcal{P}^{\mathfrak{o}}_\Gamma$ are identified in the natural way with a subset of the set of generators of Γ, where all elements of $\mathcal{P}^{\mathfrak{o}}_\Gamma$ are identified in the natural way with a subset of the set of generators of Γ.

*The triple $(\bar{G}_{S,\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha)$ consisting of a locally compact groups $\bar{G}_{S,\Gamma}$, a C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ and an automorphic action α of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ such that for each $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$ the function $\bar{G}_{S,\Gamma}\ni\rho_{S,\Gamma}(\Gamma)\mapsto\|\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)\|$ is continuous[1], is a C^*-**dynamical system for a finite orientation preserved graph system associated to a graph** Γ.*

[1] I.o.w. for every $f_\Gamma\in\mathfrak{A}_\Gamma$ the map $\alpha:\rho_{S,\Gamma}(\Gamma)\mapsto\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)$ is a continuous map from the $\bar{G}_{S,\Gamma}$-open set topology on $\bar{G}_{S,\Gamma}$ to the norm topology on $\mathfrak{A}_\Gamma$ (α is point-norm continuous).

Proof : Let $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\Gamma := \{\gamma\}$ then for a fixed suitable surface S it is true that

$$\begin{aligned}&\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)} \|\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma) - f_\Gamma\|\\ &= \lim_{\rho_S(\gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\gamma)} \|f_\Gamma(L(\rho_{S,\Gamma}(\gamma))(\mathfrak{h}_\Gamma(\gamma))) - f_\Gamma(\mathfrak{h}_\Gamma(\gamma))\|\\ &= 0\end{aligned}$$

for $\rho_{S,\Gamma}, \mathrm{id}_{S,\Gamma} \in G_{S,\Gamma}$ and $\rho_S, \mathrm{id}_S \in \mathbb{G}_{S,\gamma}$, if $\rho_S(\gamma) = g \in G$ and $\mathrm{id}_S(\gamma) = e_G$ for all $\gamma \in \Gamma$ yields.

■

Observe if $\mathfrak{A}_\Gamma$ is a C^*-subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ of bounded operators on a Hilbert space $\mathcal{H}_\Gamma$, then $\mathfrak{A}_\Gamma$ is non-degenerately represented on $\mathcal{H}_\Gamma$ if the inclusion map of $\mathfrak{A}_\Gamma$ into $\mathcal{L}(\mathcal{H}_\Gamma)$ is a non-degenerate representation of $\mathfrak{A}_\Gamma$. Set $\mathcal{H}_\Gamma$ be equal to $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. Then the multiplication representation Φ_M of $C_0(\bar{\mathcal{A}}_\Gamma)$ in $\mathcal{H}_\Gamma$ defined by

$$\Phi_M(f_\Gamma)\psi_\Gamma = f_\Gamma \cdot \psi_\Gamma \text{ for } \psi_\Gamma \in \mathcal{H}_\Gamma \text{ and } f_\Gamma \in \mathfrak{A}_\Gamma \tag{6.8}$$

is non-degenerate.

Definition 6.1.7. *Let S be a fixed surface in Σ, Γ be a graph, $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be the finite orientation preserved graph system associated to Γ such that the surface S has the surface intersection property for a finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$. Let $\bar{G}_{\breve{S},\Gamma}$ denote the flux group and $\bar{\mathcal{A}}_\Gamma$ denote the configuration space for the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$, where all elements of $\mathcal{P}^{\mathfrak{o}}_\Gamma$ are identified in the natural way with a subset of the set of generators of Γ.*

*A **covariant representation** of the C^*-dynamical system $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ in a C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ consists of a pair (Φ_M, U) where Φ_M is a non-degenerate representation of $C_0(\bar{\mathcal{A}}_\Gamma)$ on a Hilbert space $\mathcal{H}_\Gamma$ (i.e. $\Phi_M \in \mathrm{Mor}(C_0(\bar{\mathcal{A}}_\Gamma), \mathcal{L}(\mathcal{H}_\Gamma))$) and U is a (strongly continuous) unitary representation of $\bar{G}_{S,\Gamma}$ on $\mathcal{H}_\Gamma$ such that for all $\rho_{S,\Gamma}(\Gamma) \in \bar{G}_{S,\Gamma}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ it is true that*

$$U(\rho_{S,\Gamma}(\Gamma))\Phi_M(f_\Gamma)U^*(\rho_{S,\Gamma}(\Gamma)) = \Phi_M(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)) \tag{6.9}$$

*The pair (Φ_M, U) is also called a **covariant Hilbert space representation of the C^*-dynamical system** $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$.*

There is a covariant representation of $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ with respect to the automorphic action α of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by M and the unitary operator U, which is a map $U : \bar{G}_{S,\Gamma} \longrightarrow U(\mathcal{H}_\Gamma)$, where $U(\mathcal{H}_\Gamma)$ is the unitary group of $\mathcal{L}(\mathcal{H}_\Gamma)$, since,

$$\begin{aligned}U(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma))^*\psi_\Gamma(\mathfrak{h}_\Gamma(\Gamma)) &= U(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma)^{-1})\psi_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\\ &= U(\rho_{S,\Gamma}(\Gamma))\psi_\Gamma(L(\rho_{S,\Gamma}(\Gamma)^{-1})(\mathfrak{h}_\Gamma(\Gamma)))\\ &= \psi_\Gamma(L(\rho_{S,\Gamma}(\Gamma)\rho_{S,\Gamma}(\Gamma)^{-1})(\mathfrak{h}_\Gamma(\Gamma)))\\ &= \psi_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\end{aligned} \tag{6.10}$$

for $\psi_\Gamma \in \mathcal{H}_\Gamma$ and which satisfies (6.9). Notice that, $U(\rho_{S,\Gamma}(\Gamma))^* = U(\rho_{S^{-1},\Gamma}(\Gamma))$.

In the following it is often assumed that strongly continuous unitary representations of the flux group on a Hilbert space is a representation of the group on the C^*-algebra of compact operators on the Hilbert space. Therefore, this relation is explicity given in the following lemma.

Lemma 6.1.8. *Let (Φ_M, U) is a covariant representation U of a C^*-dynamical system $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$. Then the strongly continuous unitary representation of $\bar{G}_{S,\Gamma}$ on $\mathcal{H}_\Gamma$ is a representation of the group $\bar{G}_{S,\Gamma}$ on the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}_\Gamma$, i.e. $U \in \mathrm{Rep}(\bar{G}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$.*

Proof : To show that $U \in \mathrm{Rep}(\bar{G}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$, consider an element $|\psi_\Gamma\rangle\langle\phi_\Gamma|$ of $\mathcal{K}(\mathcal{H}_\Gamma)$ then for an strongly continuous unitary representation U_L, i.e.

$$\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)} \|U(\rho_{S,\Gamma}(\Gamma))\psi_\Gamma - \psi_\Gamma\|_\Gamma = 0 \tag{6.11}$$

an operator $K_{\rho_{S,\Gamma}(\Gamma)}$ can be defined by

$$\langle\phi_\Gamma,\varphi_\Gamma\rangle\left(U(\rho_{S,\Gamma}(\gamma))|\psi_\Gamma\rangle - |\psi_\Gamma\rangle\right) = \left(U(\rho_{S,\Gamma}(\Gamma))|\psi_\Gamma\rangle\langle\phi_\Gamma| - |\psi_\Gamma\rangle\langle\phi_\Gamma|\right)|\varphi_\Gamma\rangle := K_{\rho_{S,\Gamma}(\Gamma)}|\varphi_\Gamma\rangle$$

such that

$$\begin{aligned}&\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\|K_{\rho_{S,\Gamma}(\Gamma)}\|_\Gamma = \lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\sup_{\|\varphi_\Gamma\|_\Gamma=1}\|K_{\rho_{S,\Gamma}(\Gamma)}\varphi_\Gamma\|_\Gamma\\ &= \lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\sup_{\|\varphi_\Gamma\|_\Gamma=1}\|\langle\phi_\Gamma,\varphi_\Gamma\rangle\left(U(\rho_{S,\Gamma}(\Gamma))\psi_\Gamma - \psi_\Gamma\right)\|_\Gamma\\ &= \lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\|\phi_\Gamma\|_\Gamma\|U(\rho_{S,\Gamma}(\Gamma))\psi_\Gamma - \psi_\Gamma\|_\Gamma = 0\end{aligned} \tag{6.12}$$

Now observe that any compact operator K can be represented in Hilbert-Schmidt representation such that for two orthonormal bases $\{\phi^i_\Gamma\}$ and $\{\psi^i_\Gamma\}$ in the infinite-dimensional Hilbert space $\mathcal{H}_\Gamma$ there is a monoton decreasing sequence of non-negative numbers λ_i such that

$$K\varphi_\Gamma := \sum_{i=1}^M \lambda_i|\psi^i_\Gamma\rangle\langle\phi^i_\Gamma|\varphi_\Gamma\rangle = \sum_{i=1}^M \lambda_i\langle\phi^i_\Gamma,\varphi_\Gamma\rangle|\psi^i_\Gamma\rangle \tag{6.13}$$

Conclude

$$\tilde{K}_{\rho_{S,\Gamma}(\Gamma)}|\varphi_\Gamma\rangle = \sum_{i=1}^M \lambda_i\left(U(\rho_{S,\Gamma}(\Gamma))|\psi^i_\Gamma\rangle\langle\phi^i_\Gamma| - |\psi^i_\Gamma\rangle\langle\phi^i_\Gamma|\right)|\varphi_\Gamma\rangle$$

and

$$\begin{aligned}&\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\sup_{\|\varphi_\Gamma\|_\Gamma=1}\|\tilde{K}_{\rho_{S,\Gamma}(\Gamma)}\varphi_\Gamma\|_\Gamma\\ &= \sum_{i=1}^M|\lambda_i|^2\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\sup_{\|\varphi_\Gamma\|_\Gamma=1}\|\langle\phi^i_\Gamma,\varphi_\Gamma\rangle\left(U(\rho_{S,\Gamma}(\Gamma))\psi^i_\Gamma - \psi^i_\Gamma\right)\|_\Gamma\\ &= \sum_{i=1}^M|\lambda_i|^2\lim_{\rho_{S,\Gamma}(\Gamma)\longrightarrow \mathrm{id}_{S,\Gamma}(\Gamma)}\|\phi^i_\Gamma\|_\Gamma\|U(\rho_{S,\Gamma}(\Gamma))\psi^i_\Gamma - \psi^i_\Gamma\|_\Gamma = 0\end{aligned}$$

■

Proposition 6.1.9. *Let Γ be a graph and $\mathcal{P}^{\mathfrak{o}}_\Gamma$ be the finite orientation preserved graph system associated to Γ. Furthermore let S be a fixed surface in Σ such that S intersects each path of Γ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of S. There are no other intersection points of the surface S with paths of the graph Γ.*

Let $\bar{G}_{\breve{S},\Gamma}$ denote the flux group and $\bar{\mathcal{A}}_\Gamma$ denote the configuration space for a identified in the natural way finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$. Moreover let $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ be a C^*-dynamical system, and (M,U) a covariant representation of $(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ on a Hilbert space $\mathcal{H}_\Gamma$.

Then there exists a GNS-triple $(\mathcal{H}_\Gamma, \Phi_M, \Omega_\Gamma)$ where Ω_Γ is the cyclic vector for Φ_M on $\mathcal{H}_\Gamma$. Moreover the associated GNS-state ω^Γ_M on $C_0(\bar{\mathcal{A}}_\Gamma)$ is $\bar{G}_{S,\Gamma}$-invariant, i.e.

$$\omega^\Gamma_M(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)) = \omega^\Gamma_M(f_\Gamma) := \langle\Omega_\Gamma, \Phi_M(f_\Gamma)\Omega_\Gamma\rangle_\Gamma$$

for all $\rho_{S,\Gamma}\in G_{S,\Gamma}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

In general automorphic actions on C^*-algebras define C^*-dynamical systems, since they are related to covariant representations of groups on C^*-algebras (refer to appendix 12.2.4.7). This is connected to the definition of inner automorphisms.

Definition 6.1.10. *Let G be group and $\mathfrak{A}$ be a C^*-algebra.*

*An automorphic action α of a group G on $\mathfrak{A}$ is called **inner**, if there is a representation of the group G on $\mathfrak{A}$, i.e. $U\in\mathrm{Rep}(G,\mathfrak{A})$ such that*

$$\alpha_g(A) = U(g)AU(g)^*$$

whenever $A\in\mathfrak{A}$ and $g\in G$. Otherwise, α is called outer.

But since the holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ for a finite graph system is commutative there is only one inner automorphic action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by the trivial one.

Due to the different intersection behavior of surfaces and paths there are a lot of different automorphic actions on the holonomy C^*-algebra for finite graph systems.

Dynamical systems of the analytic holonomy algebra and actions of the flux goup w.r.t. different graph and surface configurations

A graph is a set of paths. Each path and a surface S have a specific intersection behavior. In general a graph does not contain only paths that are ingoing and lie above w.r.t. the surface orientation of S. In this section different actions for general graphs are studied. In the interesting configurations the corresponding actions on the analytic holonomy algebra turn out to be automorphic and point-norm continuous. There are only few configurations, which have to be excluded.

Purely left or right actions of the flux group

In the construction of dynamical systems the following surface and graph configurations play a particular role. This implies that, particular actions, which are for example purely left or right actions of a group on a C^*-algebra, are analysed.

During the whole section the set $\bar{G}_{\breve{S},\Gamma}$ and the multiplication operation

$$(\rho_x^{S_1}(\gamma_1),...,\rho_x^{S_N}(\gamma_N))\cdot(\tilde{\rho}_x^{S_1}(\gamma_1),...,\tilde{\rho}_x^{S_N}(\gamma_N)) = (\rho_x^{S_1}(\gamma_1)\tilde{\rho}_x^{S_1}(\gamma_1),...,\rho_x^{S_N}(\gamma_N)\tilde{\rho}_x^{S_N}(\gamma_N))$$

where x is equal to L or R and $\breve{S} = \{S_i\}_{1\leq i\leq|\Gamma|}$, is considered. If $\breve{S}$ contains for example only one surface, then $S_i = S$ for all $1 \leq i \leq |\Gamma|$. The right group multiplication is explicity assumed in this context. If both left and right multiplication would be used, the set $\bar{G}_{\breve{A},\Gamma}$ does not form a group and consequently there are problems in the definition of automorphic actions, which are stated in the problem 6.1.1.

For a summary recall the definition of the last section.

Lemma 6.1.11. *Let Γ be a graph and $\mathcal{P}_\Gamma^{\mathfrak{o}}$ be the finite orientation preserved graph system associated to Γ. Furthermore let S be a fixed surface in Σ such that S intersects each path of Γ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of S. There are no other intersection points of the surface S with paths of the graph Γ.*

Then redefine the action

$$\begin{aligned}(\alpha_{\overleftarrow{L}}^1(\rho_{S,\Gamma}^1)f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.14}$$

for $\rho_{S,\Gamma}^1 = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (g_S,...,g_S)$, $\rho_{S,\Gamma}^1 \in \bar{G}_{S,\Gamma}$ such that $\rho_S \in \mathbb{G}_{S,\gamma}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha_{\overleftarrow{L}}^1$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

For a simplification of the following considerations allways assume that there exists a finite orientation preserved graph system $\mathcal{P}_\Gamma^{\mathfrak{o}}$ associated to Γ. Furthermore there are no other intersection points of the surface S with paths of the graph Γ except the intersections, which are required in the different lemmata.

Lemma 6.1.12. *Let only the path γ_N in Γ intersect in (source) vertex of the set V_Γ with a surface S such that γ_N is outgoing and lies above the surface S.*

Then define an action

$$\begin{aligned}(\alpha_{\overrightarrow{L}}^{1,1}(\rho_{S,\Gamma}^{1,1})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.15}$$

for $\rho_{S,\Gamma}^{1,1} \in \bar{G}_{\breve{S},\Gamma}$ and $\rho_S \in \mathbb{G}_{S,\gamma}$ and $\rho_{S,\Gamma}^{1,1} = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (e_G,...,e_G,g_S^{-1})$.

Then the action $\alpha_{\overrightarrow{L}}^{1,1}$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

Lemma 6.1.13. *Let Γ be a graph given by $\{\gamma_1,...,\gamma_N\}$. Moreover let only the paths $\gamma_1,...,\gamma_{1,N-1}$ intersect in (source) vertices of the set V_Γ with a surface S such that all paths are outgoing and lie below.*

Then define the action

$$\begin{aligned}(\alpha^{1,N-1}_{\overleftarrow{L}}(\rho^{1,N-1}_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S\mathfrak{h}_\Gamma(\gamma_{1,N-1}),\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.16}$$

for $\rho^{1,N-1}_{S,\Gamma}\in\bar{G}_{S,\Gamma}$ and $\rho_S\in\mathbb{G}_{S,\gamma}$ and $\rho^{1,N-1}_{S,\Gamma}=(\rho_S(\gamma_1),...,\rho_S(\gamma_N))=(g_S,...,g_S,e_G)$.

The action $\alpha^{1,N-1}_{\overleftarrow{L}}$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

In the following an action is defined, which does not lead to a point-norm continuous automorphic action on the analytic holonomy algebra.

Lemma 6.1.14. *Let Γ be a graph given by $\{\gamma_1,...,\gamma_N\}$. Moreover let all paths intersect in (source) vertices of the set V_Γ with a surface S such that all paths $\gamma_1,...,\gamma_{N-1}$ are outgoing and lie below, γ_N is outgoing and lies above the surface S. There are no other intersection point of paths and the surface S.*

Then define the action

$$\begin{aligned}(\alpha^1_L(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.17}$$

for $\rho^1_{S,\Gamma}\in G_{\breve{S},\Gamma}$ and $\rho_S\in\mathbb{G}_{S,\gamma}$ and $\rho^1_{S,\Gamma}=(\rho_S(\gamma_1),...,\rho_S(\gamma_N))=(g_S,...,g_S^{-1})$.

The action α^1_L of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and the action is not point-norm continuous.

Proof : The crucial property of the action α being an automorphism yields:

$$\begin{aligned}&(\alpha^1_L(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S^{-1}\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^1_L(\rho^1_{S,\Gamma}(\alpha^1_L(\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

The problem for point-norm continuity is given by

$$\begin{aligned}&\lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}}\|\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma})(f_\Gamma)-f_\Gamma\|\\ &=\lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}}\|f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))-f_\Gamma(\mathfrak{h}_\Gamma(\gamma))\|\neq 0\end{aligned}$$

for $\rho^1_{S,\Gamma},\mathrm{id}^1_{S,\Gamma}\in\bar{G}_{S,\Gamma}$, since there is no sequence

$$\rho^1_{S,\Gamma}(\Gamma)=(\rho_S(\gamma_1),...,\rho_S(\gamma_N))=(g_S,...,g_S^{-1})$$

such that this sequence converge to $\mathrm{id}^1_{S,\Gamma}(\Gamma)=(\mathrm{id}_S(\gamma_1),...,\mathrm{id}_S(\gamma_N))$ for $\rho_S,\mathrm{id}_S\in\mathbb{G}_{S,\gamma}$ and $\mathrm{id}_S(\gamma)=e_G$ for all $\gamma\in\Gamma$

■

Notice that, in this configuration the paths γ_i for each $i\in[1,N]$ and γ_N cannot be composed. One can always decompose the graph into two subgraphs.

Clearly, the action α^1_L can be composed by other actions defined above. For example derive

$$\begin{aligned}&(\alpha^1_L(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S^{-1}\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^{1,N-1}_{\overleftarrow{L}}(\rho^{1,N-1}_{S,\Gamma}\tilde{\rho}^{1,N-1}_{S,\Gamma})\alpha^{1,1}_{\overrightarrow{L}}(\rho^{1,1}_{S,\Gamma}\tilde{\rho}^{1,1}_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^{1,1}_{\overrightarrow{L}}(\rho^{1,1}_{S,\Gamma}\tilde{\rho}^{1,1}_{S,\Gamma})\alpha^{1,N-1}_{\overleftarrow{L}}(\rho^{1,N-1}_{S,\Gamma}\tilde{\rho}^{1,N-1}_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.18}$$

where $\alpha^{1,N-1}_{\overleftarrow{L}}(\rho^{1,N-1}_{S,\Gamma})$ defined in lemma 6.1.13 and $\alpha^{1,1}_{\overrightarrow{L}}(\rho^{1,1}_{S,\Gamma})$ is defined in lemma 6.1.12. Hence if the graph is devided into to parts $\{\gamma_1,...,\gamma_{1,N-1}\}$ and γ_N, then consider two disjoint actions of two graphs. This can be done in the following way.

Lemma 6.1.15. *Let $\Gamma := \{\gamma_1,...,\gamma_N\}$ be a graph. The paths $\{\gamma_1,...,\gamma_{1,N-1}\}$ intersect in their source vertices with a surface S such that the paths are outgoing and lie below the surface S. The path γ_N in Γ intersect in the source vertex with the surface S' such that γ_N is outgoing and lies above the surface S'. There are no other intersection point of paths and the surface S and S'.*

Then the action defined by

$$\begin{aligned}(\alpha^2_L(\rho_{S,\Gamma}(\Gamma),\rho_{S',\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_{S'}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1),...,g^{-1}_{S'}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.19}$$

for $\rho^1_{\breve{S},\Gamma}\in\bar{G}_{\breve{S},\Gamma}$ and $\rho_S,\rho_{S'}\in\mathbb{G}_{\breve{S},\gamma}$ and $\rho^1_{\breve{S},\Gamma} = (\rho_S(\gamma_1),...,\rho_{S'}(\gamma_N)) = (g_S,...,g^{-1}_{S'})$ where $g_S\neq g_{S'}$.

The action α^2_L of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

For some elements $\rho_S,\tilde{\rho}_S,...,\breve{\rho}_S\in\mathbb{G}_{S,\gamma}$ set $\rho_S(\gamma)=g_S$, $\tilde{\rho}_S(\gamma)=\tilde{g}_S$ and so on. Recall that for a connected semisimple Lie group G the following is true. For each $g\in G$ there exists a finite number of elements $g_S,\tilde{g}_S,...\breve{g}_S,\breve{g}_S$ such that $g=g_S\tilde{g}_S...\breve{g}_S\breve{g}_S$ and $e_G=\breve{g}_S\breve{g}_S...\tilde{g}_Sg_S$. Hence it is true that

$$g_S\tilde{g}_S...\breve{g}_Sg^{-1}_S\tilde{g}^{-1}_S...\breve{g}^{-1}_S = g_S\tilde{g}_S...\breve{g}_S$$

This was pointed out by Fleischhack in [40]. This is indeed possible if G is assumed to be equivalent to $SU(2)$. Consequently for an element $g=g^{-1}_S\tilde{g}^{-1}_S$ in G the following statement yields

$$\begin{aligned}&(\alpha^1_L(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,g^{-1}_S\tilde{g}^{-1}_S\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S\tilde{g}_Sg^{-1}_S\tilde{g}^{-1}_S\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,g^{-1}_S\tilde{g}^{-1}_S\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})\alpha^{1,1}_{\overrightarrow{L}}(\rho^{1,1}_{S,\Gamma}\tilde{\rho}^{1,1}_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.20}$$

where $\alpha^{1,1}_{\overrightarrow{L}}$ is defined in lemma 6.1.12 and $\alpha^1_{\overleftarrow{L}}$ defined in lemma 6.1.11. But the action α^1_L is not point-norm continuous. Therefore, consider a composition of two actions $\alpha^1_{\overleftarrow{L}}$ w.r.t. a surface S and $\alpha^{1,1}_{\overrightarrow{L}}$ w.r.t. a surface S', which is given by

$$(\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma})\alpha^{1,1}_{\overrightarrow{L}}(\tilde{\rho}^{1,1}_{S',\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) = f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1),...,\tilde{g}^{-1}_{S'}\mathfrak{h}_\Gamma(\gamma_N)) \tag{6.21}$$

Notice there is a special case of the action α^1_L. Derive

$$\begin{aligned}&(\alpha^1_L(\rho^1_{S,\Gamma}\rho^1_{S^{-1},\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) = (\alpha^1_L(\rho^1_{S,\Gamma}(\rho^1_{S,\Gamma})^{-1})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\rho_S(\gamma_1)\rho_S(\gamma_1)^{-1}\mathfrak{h}_\Gamma(\gamma_1),...,\rho_S(\gamma_N)\rho_S(\gamma_N)^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(g^{-1}_Sg_S\mathfrak{h}_\Gamma(\gamma_1),...,g_Sg^{-1}_S\mathfrak{h}_\Gamma(\gamma_N)) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha^1_L(\rho^1_{S,\Gamma}(J(\rho^1_{S,\Gamma})))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.22}$$

where $J:\bar{G}_{\breve{S},\Gamma}\longrightarrow\bar{G}_{\breve{S},\Gamma}$ is the linear operator such that $\rho_{S,\Gamma}(\Gamma)\mapsto\rho_{S^{-1},\Gamma}(\Gamma)$.

Lemma 6.1.16. *Let all paths in $\Gamma := \{\gamma_1,...,\gamma_N\}$ intersect in vertices of the set V_Γ with a surface S_N such that $\gamma_1,...,\gamma_N$ are outgoing and lie below the surface S_N. Let the paths $\gamma_1,...,\gamma_{N-1}$ in Γ intersect in vertices of the set V_Γ with a surface S_{N-1} such that $\gamma_1,...,\gamma_{1,N-1}$ are outgoing and lie below the surface S_{N-1}. The same is true for a surface S_{N-2} and paths $\{\gamma_1,...,\gamma_{N-2}\}$, and so on, til S_1 and $\{\gamma_1\}$. There are no other intersections between the paths and surfaces $S_1,...,S_{N-1}$ and S_N. Moreover let each path γ_i in Γ intersect in vertices of the set V_Γ with a surface $S_{1,i}$ such that γ_i is outgoing and lie below the surface S_i for $i=1,...,N$. There are no other intersections* *between the paths in Γ and surfaces $S_{1,1},...,S_{1,N}$. The surfaces S_N and $S_{1,N}$ coincide. The set $\breve{S} := \{S_{1,i}\}_{1\leq i\leq N}$ has the simple surface intersection property for Γ.*

The action for two different maps $\rho_{S_{N-1},\Gamma}$ in $G_{S_{N-1},\Gamma}$ and $\tilde{\rho}_{S_N,\Gamma}$ in $G_{S_N,\Gamma}$, such that there is an action of $\bar{G}_{S_{1,N-1},\Gamma}\times \bar{G}_{S_N,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\begin{aligned}&(\alpha^2_{\overleftarrow{L}}(\rho_{S_{N-1},\Gamma},\tilde{\rho}_{S_N,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&:= f_\Gamma(\rho_{S_{N-1}}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),\rho_{S_{N-1}}(\gamma_2)\mathfrak{h}_\Gamma(\gamma_2),...,\rho_{S_{N-1}}(\gamma_{N-1})\mathfrak{h}_\Gamma(\gamma_{N-1}),\tilde{\rho}_{S_N}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&= f_\Gamma(g_{S_{N-1}}\mathfrak{h}_\Gamma(\gamma_1),g_{S_{N-1}}\mathfrak{h}_\Gamma(\gamma_2),...,g_{S_{N-1}}\mathfrak{h}_\Gamma(\gamma_{N-1}),h_{S_N}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.23}$$

The action for $(N-1)$-different maps $\rho_{S_2,\Gamma}$ in $G_{S_2,\Gamma}$, $\tilde{\rho}_{S_{1,3},\Gamma}$ in $G_{S_{1,3},\Gamma}$ til $\breve{\rho}_{S_{1,N-2},\Gamma}$ in $G_{S_{1,N-2},\Gamma}$, such that there is an action of $\bar{G}_{S_2,\Gamma}\times\bar{G}_{S_{1,3},\Gamma}\times\cdots\times\bar{G}_{S_{1,N-2},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\begin{aligned}&(\alpha^{N-1}_{\overleftarrow{L}}((\rho_{S_2,\Gamma},\rho_{S_2,\Gamma},\tilde{\rho}_{S_{1,3},\Gamma}...,\breve{\rho}_{S_{1,N-1},\Gamma}))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&:= f_\Gamma(\rho_{S_2}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),\rho_{S_2}(\gamma_2)\mathfrak{h}_\Gamma(\gamma_2),\tilde{\rho}_{S_{1,3}}(\gamma_3)\mathfrak{h}_\Gamma(\gamma_3),...,\breve{\rho}_{1,S_{N-2}}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&= f_\Gamma(g_{S_2}\mathfrak{h}_\Gamma(\gamma_1),g_{S_2}\mathfrak{h}_\Gamma(\gamma_2),k_{S_{1,3}}\mathfrak{h}_\Gamma(\gamma_3),...,h_{S_{1,N-2}}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.24}$$

whenever $N>5$.

Respectively, the action of N-different maps is equivalent to an action of $\bar{G}_{S_{1,1},\Gamma}\times\cdots\times\bar{G}_{S_{1,N},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, which is defined by

$$\begin{aligned}&(\alpha^N_{\overleftarrow{L}}(\rho_{S_{1,1},\Gamma}(\Gamma),...,\breve{\rho}_{S_{1,N},\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&:= f_\Gamma(\rho_{S_{1,1}}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_{S_{1,N}}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&= f_\Gamma(g_{S_{1,1}}\mathfrak{h}_\Gamma(\gamma_1),...,h_{S_{1,N}}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.25}$$

Then the actions $\alpha^2_{\overleftarrow{L}}$, ..., $\alpha^{1,N-1}_{\overleftarrow{L}}$ and $\alpha^N_{\overleftarrow{L}}$ of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ are automorphic and point-norm continuous actions.

Lemma 6.1.17. *Let all paths of a graph Γ intersect in vertices of the set V_Γ with a surface S such that all paths are ingoing and lie above the surface. Moreover let the set $\breve{S} := \{S_{1,i}\}_{1\leq i\leq N}$ has the simple surface intersection property for Γ.*

Then there is an action such that

$$\begin{aligned}&(\alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma})f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&\quad:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)\rho_S(\gamma_N)^{-1})\\&\quad= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)g_S^{-1})\end{aligned}\tag{6.26}$$

This action can be changed such that

$$\begin{aligned}&(\alpha_N^{\overleftarrow{R}}(\rho^N_{\breve{S},\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&\quad:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_{1,1}}(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)\rho_{S_{1,N}}(\gamma_N)^{-1})\\&\quad= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g^{-1}_{S_{1,1}},...,\mathfrak{h}_\Gamma(\gamma_N)h^{-1}_{S_{1,N}})\end{aligned}\tag{6.27}$$

Then the actions $\alpha_1^{\overleftarrow{R}}$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$,... and $\alpha_N^{\overleftarrow{R}}$ of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ are automorphic and point-norm continuous actions.

Proof : The action is an automorphism on $C_0(\bar{\mathcal{A}}_\Gamma)$, since,

$$\begin{aligned}&(\alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma})(\alpha_1^{\overleftarrow{R}}(\tilde{\rho}^1_{S,\Gamma})f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1))^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)(\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N))^{-1})\\&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde{g}_S^{-1}g_S^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)\tilde{g}_S^{-1}g_S^{-1})\end{aligned}$$

■

Lemma 6.1.18. *Let all paths intersect in their target vertices contained in the set V_Γ with a surface S such that all paths are ingoing and lie below the surface S.*

Then there is an action such that

$$\begin{aligned}&(\alpha_1^{\overrightarrow{R}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &\quad := f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S^{-1}(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)\rho_S^{-1}(\gamma_N))\\ &\quad = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S,...,\mathfrak{h}_\Gamma(\gamma_N)g_S)\end{aligned}\tag{6.28}$$

holds.

Then the action $\alpha_1^{\overrightarrow{R}}$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is an automorphic and point-norm continuous action.

Proof. The action is an automorphism on $C_0(\bar{\mathcal{A}}_\Gamma)$, since,

$$\begin{aligned}&(\alpha_1^{\overrightarrow{R}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1))^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)(\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N))^{-1})\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde{g}_S g_S,...,\mathfrak{h}_\Gamma(\gamma_N)\tilde{g}_S g_S)\\ &= (\alpha_1^{\overrightarrow{R}}(\rho^1_{S,\Gamma})(\alpha_1^{\overrightarrow{R}}(\tilde{\rho}^1_{S,\Gamma})f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.29}$$

□

The actions in the last paragraphs have been constructed such that there is a always decomposition of left and right structures. For example if graphs are considered such that all paths have the same intersection behavior w.r.t a fixed surface set $\breve{S}$, then for elements of a finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$ an action is defined. On the other hand for every graph Γ in $\mathcal{P}^{\mathfrak{o}}_\Gamma$ there always exists a graph Γ^{-1}, which referes to the set $\{\gamma_1^{-1},...,\gamma_N^{-1}\}$, which is obviously not an element of $\mathcal{P}^{\mathfrak{o}}_\Gamma$. But this graph of reversed path orientations forms a second finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma^{-1}}$. Moreover there is an action of the corresponding flux group $\bar{G}_{\breve{S},\Gamma^{-1}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, where the configuration space is constructed from the finite graph groupoid $\mathcal{P}_\Gamma$. Recall that, $\mathfrak{h}_\Gamma(\gamma^{-1}) = \mathfrak{h}_\Gamma(\gamma)^{-1}$ yields for an arbitrary $\gamma\in\mathcal{P}_\Gamma\Sigma$. Hence it is easy to verify that

$$\begin{aligned}&(\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &\quad = f_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &(\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)^{-1})\\ &\quad = f_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)^{-1},...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)^{-1})\\ &\quad = f_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1}))\\ &(\alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma^{-1}}))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\mathfrak{h}_\Gamma(\gamma_N^{-1}))\\ &\quad = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1^{-1})\rho^R_S(\gamma_1^{-1})^{-1},...,\mathfrak{h}_\Gamma(\gamma_N^{-1})\rho^R_S(\gamma_N^{-1})^{-1})\\ &\quad = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)^{-1}\rho^R_S(\gamma_1^{-1})^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)^{-1}\rho^R_S(\gamma_N^{-1})^{-1})\end{aligned}\tag{6.30}$$

where ρ^L_S and ρ^R_S denote the maps in $\mathbb{G}_{\breve{S},\Gamma}$.

Definition 6.1.19. *Define the map $I: C_0(\bar{\mathcal{A}}_\Gamma)\rightarrow C_0(\bar{\mathcal{A}}_\Gamma)$*

$$I: f_\Gamma\mapsto \breve{f}_\Gamma,\ \text{ where } \breve{f}_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N) := f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)^{-1})$$

such that $I^2 = \mathbb{1}$ where $\mathbb{1}$ is the identical automorphism on $C_0(\bar{\mathcal{A}}_\Gamma)$.

Then one can deduce

$$\begin{aligned}&I(\alpha^1_{\overleftarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)^{-1})\\ &\quad = (If_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)^{-1},...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)^{-1})\\ &\quad = f_\Gamma((\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)^{-1})^{-1},...,(\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)^{-1})^{-1})\\ &\quad = (f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1^{-1})\rho^L_S(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N^{-1})\rho^L_S(\gamma_N)^{-1})\\ &= (\alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma^{-1}})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\mathfrak{h}_\Gamma(\gamma_N^{-1}))\end{aligned}\tag{6.31}$$

if $\rho_S^L(\gamma_i) = \rho_S^R(\gamma_i^{-1})$ for all $i = 1,...,N$, ρ_S^L and ρ_S^R are maps in $\mathbb{G}_{\breve{S},\Gamma}$. Consequently the actions satisfy

$$\begin{aligned} I\alpha_{\overleftarrow{L}}^1(\rho^1_{S,\Gamma})If_\Gamma &= \alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma^{-1}})If_\Gamma \\ I\alpha_{\overleftarrow{L}}^1(\rho^1_{S,\Gamma})f_{\Gamma^{-1}} &= \alpha_1^{\overleftarrow{R}}(\rho^1_{S,\Gamma^{-1}})f_{\Gamma^{-1}} \end{aligned} \tag{6.32}$$

Notice that, if if $\rho_S^L(\gamma_i)^{-1} = \rho_{S^{-1}}^R(\gamma_i^{-1})$ for all $i = 1,...,N$, ρ_S^L and ρ_S^R are maps in $\mathbb{G}_{\breve{S},\Gamma}$ then

$$I\alpha_{\overleftarrow{L}}^1(J(\rho^1_{S,\Gamma}))f_{\Gamma^{-1}} = \alpha_1^{\overleftarrow{R}}(\rho^1_{S^{-1},\Gamma^{-1}})f_{\Gamma^{-1}} \tag{6.33}$$

where J is the map $J : \bar{G}_{\breve{S},\Gamma} \longrightarrow \bar{G}_{\breve{S},\Gamma}$, $J : \rho_{S,\Gamma}(\Gamma) \mapsto \rho_{S^{-1},\Gamma}(\Gamma)$ where $\breve{S} := \{S, S^{-1}\}$.

Lemma 6.1.20. *Let $S_1,...,S_N$ form a set $\breve{S}$, where the set $\breve{S}$ has the simple surface intersection property for a graph Γ, let $\mathcal{P}_\Gamma^{\mathfrak{o}}$ be a finite orientation preserved graph system for Γ.*

Then there is an action of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\begin{aligned} \alpha_{\overleftarrow{R}}(\rho_{\breve{S},\Gamma}(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma')) &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_1}(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_M)\rho_{S_M}(\gamma_M)^{-1}) \\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_{S_1},...,\mathfrak{h}_\Gamma(\gamma_M)g_{S_M}) \end{aligned} \tag{6.34}$$

whenever $\Gamma' := \{\gamma_1,...,\gamma_M\}$ is an element of $\mathcal{P}_\Gamma^{\mathfrak{o}}$, for all $\rho_{\breve{S},\Gamma} \in G_{\breve{S},\Gamma}$. This action is point-norm continuous and automorphic.

Notice that, in this case of a suitbale surface set $\breve{S}$ instead of $\bar{G}_{\breve{S},\Gamma}$ one can use $\times_{i=1}^N \bar{G}_{S_i,\Gamma}$, equivalently.

Left and right actions of the flux group

Left and right actions are defined on the same level for some configurations of the surfaces and paths. Therefore, recall the maps contained in the set $\mathbb{G}_{\breve{S},\Gamma}$ with left multiplication operation, which decomposes into $\rho_S^L \times \rho_S^R$.

Lemma 6.1.21. *Let all paths in Γ intersect in vertices of the set V_Γ with a surface S such that $\gamma_1,...,\gamma_{1,N-1}$ are ingoing paths and lie above the surface S, whereas γ_N is a outgoing path lying below w.r.t. the surface orientation of S.*

Then the action defined by

$$\begin{aligned} (\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S^R(\gamma_1)^{-1},...,\rho_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) \\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S^{-1},...,g_S\mathfrak{h}_\Gamma(\gamma_N)) \end{aligned} \tag{6.35}$$

for $\rho^1_{S,\Gamma} = (\rho_S^R(\gamma_1),...,\rho_S^L(\gamma_N)) = (g_S,...,g_S)$, $\rho^1_{S,\Gamma} \in \bar{G}_{S,\Gamma}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}$ of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

Proof : The first action $\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}$ is an automorphism on $C_0(\bar{\mathcal{A}}_\Gamma)$, since

$$\begin{aligned} &(\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) \\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S^R(\gamma_1)\tilde{\rho}_S^R(\gamma_1))^{-1},...,\rho_S^L(\gamma_N)\tilde{\rho}_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) \\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde{g}_S^{-1}g_S^{-1},...,g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_N)) \\ &= (\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})(\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}\tilde{\rho}^1_{S,\Gamma})f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) \end{aligned}$$

yields.

To show that the action $\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}$ is point-norm continuous, calculate

$$\begin{aligned} &\lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}} \|\alpha_{\overleftarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})(f_\Gamma) - f_\Gamma\| \\ &= \lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}} \|f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S^R(\gamma_1)^{-1},...,\rho_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) - f_\Gamma(\mathfrak{h}_\Gamma(\gamma))\| = 0 \end{aligned}$$

for $\rho^1_{S,\Gamma},\mathrm{id}^1_{S,\Gamma}\in\bar G_{S,\Gamma}$,

$$\begin{aligned}\rho^1_{S,\Gamma}(\Gamma)&=(\mathrm{id}^L_S\times\rho^R_S(\gamma_1),...,\rho^L_S(\gamma_N))\times\mathrm{id}^R_S)=(g_S\times e_G,....,e_G\times g_S),\\ \mathrm{id}^1_{S,\Gamma}(\Gamma)&=(\mathrm{id}_S(\gamma_1)\times\mathrm{id}_S(\gamma_1),...,\mathrm{id}_S(\gamma_N)\times\mathrm{id}_S(\gamma_1))\end{aligned}$$

$\rho_S,\mathrm{id}_S\in\mathbb{G}_{S,\gamma}$ and $\mathrm{id}_S(\gamma)=e_G$ for all $\gamma\in\Gamma$.

■

Lemma 6.1.22. *Let $\{S_i\}_{1\leq i\leq N}$ be equivalent to a surface set $\breve S$. Furthermore let each path γ_i in Γ intersect in one vertex of the set V_Γ with a surface S_i and there are no other intersections with any other surface. In particular, for $i=1,...,N-1$ each the path γ_i is an ingoing path and lie above the surface S_i, whereas γ_N is a outgoing path lying below w.r.t. the surface orientation of S_N.*

Then the action of $\bar G_{\breve S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is

$$\begin{aligned}(\alpha^{\overleftarrow{R},N}_{\overleftarrow{L}}(\rho^N_{\breve S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))&:=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho^R_{S_1}(\gamma_1)^{-1},...,\rho^L_{S_N}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g^{-1}_{S_1},...,g_{S_N}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.36}$$

for $\rho^N_{\breve S,\Gamma}=(\rho^R_{S_1}(\gamma_1),...,\rho^L_{S_N}(\gamma_N))=(g_{S_1},...,g_{S_N})$, $\rho^N_{\breve S,\Gamma}\in\breve G_{\breve S,\Gamma}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha^{\overleftarrow{R},N}_{\overleftarrow{L}}$ of $\bar G_{\breve S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

Lemma 6.1.23. *Let all paths in Γ intersect in vertices of the set V_Γ with a surface S such that $\gamma_1,...,\gamma_{1,N-1}$ are ingoing and lie below the surface S, whereas γ_N is a outgoing path lying above w.r.t. the surface orientation of S.*

Then the action is presented by

$$\begin{aligned}(\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))&:=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho^R_S(\gamma_1)^{-1},...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S,...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.37}$$

for $\rho^1_{S,\Gamma}=(\rho^R_S(\gamma_1),...,\rho^L_S(\gamma_N))=(g_S^{-1},...,g_S^{-1})$, $\rho^1_{S,\Gamma}\in\bar G_{S,\Gamma}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}$ is an automorphic and point-norm continuous action of $\bar G_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$.

Proof :

$$\begin{aligned}&(\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma}\tilde\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S(\gamma_1)\tilde\rho_S(\gamma_1))^{-1},...,\rho_S(\gamma_1)\tilde\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde g_S g_S,...,g_S^{-1}\tilde g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\&=(\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma})(\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}(\tilde\rho^1_{S,\Gamma})))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

■

One can easily verify that an equivalent action can be defined for a whole suitable surface set $\breve S$.

Lemma 6.1.24. *Let all paths in Γ intersect in source or target vertices contained in the set V_Γ with a surface S and a set of surfaces $\breve S$ such that all surfaces in $\breve S$ are subsets of S with same surface orientation and the paths $\gamma_1,...,\gamma_{1,N-1}$ are ingoing and lie below the surface S and hence w.r.t. all surfaces in $\breve S$, whereas γ_N is a outgoing path lying above w.r.t. the surface orientation of S and hence all surfaces in $\breve S$.*

Then there is an action given by

$$\begin{aligned}(\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))&:=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho^R_S(\gamma_1)^{-1},...,\rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S,...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.38}$$

for $\rho^1_{S,\Gamma}=(\rho^R_S(\gamma_1),...,\rho^L_S(\gamma_N))=(g_S^{-1},...,g_S^{-1})$, $\rho^1_{S,\Gamma}\in\bar G_{\breve S,\Gamma}$, a surface S in $\breve S$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha^{\overrightarrow{R},1}_{\overrightarrow{L}}$ of $\bar G_{\breve S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

Proof : Let S, S' in $\breve{S}$, then derive

$$\begin{aligned}(\alpha_{\overrightarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S',\Gamma}\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S^{-1}(\gamma_1)\rho_{S'}^{-1}(\gamma_1),...,\rho_{S'}(\gamma_N)\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_Sg_{S'},...,g_{S'}^{-1}g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\ &= (\alpha_{\overrightarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S',\Gamma})(\alpha_{\overrightarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S,\Gamma})f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

■

Lemma 6.1.25. *Let all paths in Γ intersect in source or target vertices contained in the set V_Γ with a surface S and a set of surfaces $\breve{S}$ such that all surfaces in $\breve{S}$ are subsets of S with same surface orientation and the paths $\gamma_1,...,\gamma_{1,N-1}$ are ingoing and lie above the surface S and hence w.r.t. all surfaces in $\breve{S}$, whereas γ_N is a outgoing path lying above w.r.t. the surface orientation of S and hence all surfaces in $\breve{S}$.*

Then there is an action given by

$$\begin{aligned}(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S^R(\gamma_1)^{-1},...,\rho_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S^{-1},...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.39}$$

for $\rho^1_{S,\Gamma} = (\rho_S^R(\gamma_1),...,\rho_S^L(\gamma_N)) = (g_S,...,g_S^{-1})$, $\rho^1_{S,\Gamma}\in\bar{G}_{\breve{S},\Gamma}$, a surface S in $\breve{S}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is automorphic and point-norm continuous.

Proof : It is true that

$$\begin{aligned}&(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1))^{-1},...,\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde{g}_S^{-1}g_S^{-1},...,g_S^{-1}\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\ &= \Big(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})\big(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\tilde{\rho}^1_{S,\Gamma})(f_\Gamma)\big)\Big)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

■

Let $\breve{S}^{-1}$ be the set of orientation reversed surfaces and $S^{-1}\in\breve{S}^{-1}$. Notice that, if the surface orientation of S is reversed, then there is another action $\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}$ of $\bar{G}_{\breve{S}^{-1},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, which is presented by

$$\begin{aligned}(\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S^{-1},\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho^R_{S^{-1}}(\gamma_1)^{-1},...,\rho^L_{S^{-1}}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S,...,g_S\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.40}$$

for $\rho^1_{S^{-1},\Gamma} = (\rho^R_{S^{-1}}(\gamma_1),...,\rho^L_{S^{-1}}(\gamma_N)) = (g_S^{-1},...,g_S)$ and $g_S^{-1} = g_{S^{-1}}$, $\rho^1_{S^{-1},\Gamma}\in\bar{G}_{\breve{S}^{-1},\Gamma}$, a surface S in $\breve{S}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$. This action is also automorphic, since the following computation can be done.

$$\begin{aligned}&(\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S^{-1},\Gamma}\tilde{\rho}^1_{S^{-1},\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_{S^{-1}}(\gamma_1)\tilde{\rho}_{S^{-1}}(\gamma_1))^{-1},...,\rho_{S^{-1}}(\gamma_N)\tilde{\rho}_{S^{-1}}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\tilde{g}_Sg_S,...,g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_N))\\ &= \Big(\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S^{-1},\Gamma})\big(\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}(\tilde{\rho}^1_{S^{-1},\Gamma})(f_\Gamma)\big)\Big)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned} \tag{6.41}$$

Lemma 6.1.26. *Let all paths in Γ intersect in source and target vertices of the set V_Γ with a surface S. Devide each path γ_i into two paths such that there is one segment γ_i' such that lying above and is ingoing w.r.t. the surface S, and the other segment γ_i'' lies above and is outgoing w.r.t. the surface S.*

Then define an action

$$\begin{aligned}(\beta_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho_S^L(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)\rho_S^R(\gamma_1)^{-1},...,\rho_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)\rho_S^R(\gamma_N)^{-1})\\ &= f_\Gamma(g_S^{-1}\mathfrak{h}_\Gamma(\gamma_1)g_S^{-1},...,g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N)g_S^{-1})\end{aligned} \tag{6.42}$$

for $\rho^1_{S,\Gamma} \in \bar{G}_{S,\Gamma}$, $\rho^1_{S,\Gamma} = (\rho^L_S(\gamma_1) \times \rho^R_S(\gamma_1), ..., \rho^L_S(\gamma_N) \times \rho^R_S(\gamma_N)) = (g_S^{-1} \times g_S, ..., g_S^{-1} \times g_S)$ *and* $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\beta^{\overleftarrow{R},1}_{\overrightarrow{L}}$ *of* $\bar{G}_{S,\Gamma}$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$ *is automorphic and point-norm continuous.*

Proof : To verify that the action is point-norm continuous, derive

$$\begin{aligned}&\lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}} \|\beta^{\overleftarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma})(f_\Gamma) - f_\Gamma\|\\ &= \lim_{\rho^1_{S,\Gamma}\longrightarrow \mathrm{id}^1_{S,\Gamma}} \|f_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)\rho^R_S(\gamma_1)^{-1}, ..., \rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)\rho^R_S(\gamma_N)^{-1}) - f_\Gamma(\mathfrak{h}_\Gamma(\gamma))\| = 0\end{aligned}$$

for $\rho^1_{S,\Gamma}, \mathrm{id}^1_{S,\Gamma} \in \bar{G}_{S,\Gamma}$,

$$\begin{aligned}\rho^1_{S,\Gamma}(\Gamma) &= ((\rho^L_S \times \rho^R_S)(\gamma_1), ..., (\rho^L_S \times \rho^R_S)(\gamma_N)) = (g_S^{-1} \times g_S, ..., g_S^{-1} \times g_S),\\ \mathrm{id}^1_{S,\Gamma}(\Gamma) &= (\mathrm{id}_S(\gamma_1) \times \mathrm{id}_S(\gamma_1), ..., \mathrm{id}_S(\gamma_N) \times \mathrm{id}_S(\gamma_1))\end{aligned}$$

$\rho_S, \mathrm{id}_S \in \mathbb{G}_{S,\gamma}$ and $\mathrm{id}_S(\gamma) = e_G$ for all $\gamma \in \Gamma$.

■

Lemma 6.1.27. *Let all paths in* Γ *intersect in source and target vertices of the set* V_Γ *with a surface* S. *Devide each path* γ_i *into two paths such that there is one segment* γ'_i *such that lying below (or above) and is ingoing w.r.t. the surface* S *and the other segment* γ''_i *lies above (or below) and is outgoing w.r.t. the surface* S.

Then define an action

$$\begin{aligned}(\beta^{\overrightarrow{R},1}_{\overrightarrow{L}}(\rho^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) &:= f_\Gamma(\rho^L_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)\rho^R_S(\gamma_1)^{-1}, ..., \rho^L_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)\rho^R_S(\gamma_N)^{-1})\\ &= f_\Gamma(g_S^{-1}\mathfrak{h}_\Gamma(\gamma_1)g_S, ..., g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N)g_S)\end{aligned} \tag{6.43}$$

for $\rho^1_{S,\Gamma} \in \bar{G}_{S,\Gamma}$,

$$\rho^1_{S,\Gamma} = (\rho^L_S(\gamma_1) \times \rho^R_S(\gamma_1), ..., \rho^L_S(\gamma_N) \times \rho^R_S(\gamma_N)) = (g_S^{-1} \times g_S^{-1}, ..., g_S^{-1} \times g_S^{-1})$$

and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Then the action $\beta^{\overrightarrow{R},1}_{\overrightarrow{L}}$ *of* $\bar{G}_{S,\Gamma}$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$ *is automorphic and point-norm continuous.*

To generalise the actions from graphs to finite graph systems study the following observations.

Lemma 6.1.28. *Let all paths in* Γ' *intersect in their source vertices contained in the set* $V_{\Gamma'}$ *with a surface* S *such that all paths lie below the surface* S. *Let* Γ'' *be a graph such that there is no intersection of the surface* S *and any path of* Γ''. *Moreover let* Γ' *and* Γ'' *be disjoint, i.e. all paths* $\gamma_i \neq \gamma'_j$ *for all* $i = 1, .., N$ *and* $j = 1, ..., N'$ *are disjoint. Let* $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ *for* $M \in \mathbb{N}$ *fixed be a subgraph of* $\Gamma''' := \{\gamma_1, ..., \gamma_K\}$ *for* $K \in \mathbb{N}$ *and* Γ''' *be a subgraph of* Γ. *Finally, let* Γ', Γ'' *and* Γ''' *be elements of* $\mathcal{P}^{\mathfrak{o}}_\Gamma$.

Then the action α *of* $\bar{G}_{\breve{S},\Gamma'\leq\Gamma}$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$ *satisfies*

$$(\alpha(\rho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'')) = f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'')) \tag{6.44}$$

whenever $\rho_{S,\Gamma} \in G_{S,\Gamma}$, $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ *and where* $\Gamma' \leq \Gamma$, $\Gamma'' \leq \Gamma$ *and* $\Gamma' \cap \Gamma'' = \{\varnothing\}$.

There is an action given by

$$\begin{aligned}(\alpha(\rho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma''')) &:= f_\Gamma(\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1), ..., \rho_S(\gamma_M)\mathfrak{h}_\Gamma(\gamma_M), ..., \mathfrak{h}_\Gamma(\gamma_K))\\ &= f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1), ..., g_S\mathfrak{h}_\Gamma(\gamma_M), ..., \mathfrak{h}_\Gamma(\gamma_K))\end{aligned}$$

which is automorphic and point-norm continuous.

Consequently the actions defined above can be easily generalised to finite orientation preserved graph systems.

Problem 6.1.1: Observe that, there is a problem if the following actions[2] on $\bar{\mathcal{A}}_\Gamma$ for a surface S are defined. Assume that, the set $\bar{G}_{\breve{S},\Gamma}$ is equipped with a left and right multiplication on the same time. This could be the case if a surface S is considered such that S intersects the paths $\gamma_1,...,\gamma_{N-1}$ in the source vertices, hence, the paths lie outgoing, and the paths lie below. Furthermore S intersects the path γ_N such that the path lies outgoing and above. Then a left action, which is defined by the set $\bar{G}_{\breve{S},\Gamma}$ with a left multiplication for the paths $\gamma_1,...,\gamma_{N-1}$ and a right multiplication for the path γ_N, is not automorphic. This can be verified by the following computation

$$\begin{aligned}
&(\alpha^1_{\underrightarrow{L},\underleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\
&= f_\Gamma(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\tilde{\rho}_S(\gamma_N)\rho_S(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))\\
&= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,\tilde{g}_S g_S\mathfrak{h}_\Gamma(\gamma_N))\\
&\neq (\alpha^1_{\underrightarrow{L},\underleftarrow{L}}(\rho^1_{S,\Gamma})(\alpha^1_{\underrightarrow{L},\underleftarrow{L}}(\tilde{\rho}^1_{S,\Gamma}f_\Gamma(\mathfrak{h}_\Gamma))))(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\end{aligned} \tag{6.45}$$

This is the reason why all paths that lie outgoing w.r.t. a surface S are defined by left actions of $\bar{G}_{\breve{S},\Gamma}$ with a left multiplication on $\bar{\mathcal{A}}_\Gamma$. Hence this problem is absent by definition of the actions, which are stated before this problem. A left action of $\bar{G}_{\breve{S},\Gamma}$ with a left multiplication on $\bar{\mathcal{A}}_\Gamma$ is called the **left action** of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. Whereas a left action of $\bar{G}_{\breve{S},\Gamma}$ with a right multiplication on $\bar{\mathcal{A}}_\Gamma$ is called the **inverse left action** of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. A right action of $\bar{G}_{\breve{S},\Gamma}$ with a left multiplication on $\bar{\mathcal{A}}_\Gamma$ is called the **right action** of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. Whereas a right action of $\bar{G}_{\breve{S},\Gamma}$ with a right multiplication on $\bar{\mathcal{A}}_\Gamma$ is called the **inverse right action** of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. Hence the action defined in equation (6.45) corresponds to a left action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the paths $\gamma_1,...,\gamma_{1,N-1}$ and a left inverse action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the path γ_N.

Recognize that there is no homeomorphism H on $\bar{\mathcal{A}}_\Gamma$ correponding to a group action of $\bar{G}_{S,\Gamma}$ with a left and right multiplication on $\bar{\mathcal{A}}_\Gamma$ defined in (6.45), i.o.w.

$$\begin{aligned}
H(g_S\tilde{g}_S)(\mathfrak{h}_\Gamma(\gamma_1),...\mathfrak{h}_\Gamma(\gamma_N))) &= (g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,\tilde{g}_S^{-1}g_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))\\
&\neq H(g_S)(H(\tilde{g}_S)(\mathfrak{h}_\Gamma(\gamma_1),...\mathfrak{h}_\Gamma(\gamma_N)))\\
H(g_S)(H(\tilde{g}_S)(\mathfrak{h}_\Gamma(\gamma_1),...\mathfrak{h}_\Gamma(\gamma_N))) &= H(g_S)((\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma_N)))\\
&= (g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,g_S^{-1}\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma_N))
\end{aligned} \tag{6.46}$$

In the same way the problem occur if there is a inverse right action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the paths $\gamma_1,...,\gamma_{1,N-1}$ intersecting a surface S such that they are ingoing and lie above, and a right action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the path γ_N intersecting S such that the path is ingoing and lie below, is studied. Then it is true that

$$\begin{aligned}
&(\alpha^1_{\underrightarrow{R},\underleftarrow{R}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)(\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N))^{-1})\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S^{-1}\tilde{g}_S^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)\tilde{g}_S^{-1}g_S^{-1})\\
&\neq (\alpha^1_{\underrightarrow{R},\underleftarrow{R}}(\rho^1_{S,\Gamma})(\alpha^1_{\underrightarrow{R},\underleftarrow{R}}(\tilde{\rho}^1_{S,\Gamma}f_\Gamma(\mathfrak{h}_\Gamma))))(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\end{aligned} \tag{6.47}$$

yields.

There is also a problem if there is a left action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the paths $\gamma_1,...,\gamma_{1,N-1}$, which intersect a surface S such that they are outgoing and lie above, and a inverse right group action on $\bar{\mathcal{A}}_\Gamma$ for the path γ_N lying ingoing and below is considered. Then derive

$$\begin{aligned}
&(\hat{\alpha}^{\overrightarrow{R},1}_{\underleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\
&= f_\Gamma(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)(\tilde{\rho}_S(\gamma_N)\rho_S(\gamma_N))^{-1})\\
&= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)\tilde{g}_S^{-1}g_S^{-1})\\
&\neq \Big(\hat{\alpha}^{\overrightarrow{R},1}_{\underleftarrow{L}}(\rho^1_{S,\Gamma})\big(\hat{\alpha}^{\overrightarrow{R},1}_{\underleftarrow{L}}(\tilde{\rho}^1_{S,\Gamma}f_\Gamma)\big)\Big)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\end{aligned} \tag{6.48}$$

[2]Notice that, there is a difference between a left action of a group on a space and a left action of a group on a C^*-algebra.

Finally, there is also a problem if there is a left inverse action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the paths $\gamma_1,...,\gamma_{1,N-1}$ lying outgoing and below, and a right action of $\bar{G}_{\breve{S},\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$ for the path γ_N lying ingoing and above is considered. In this case,

$$\begin{aligned}&(\hat{\alpha}_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\&=f_\Gamma(\tilde{\rho}_S(\gamma_1)\rho_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)(\rho_S(\gamma_N)\tilde{\rho}_S(\gamma_N))^{-1})\\&=f_\Gamma(\tilde{g}_Sg_S\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)\tilde{g}_S^{-1}g_S^{-1})\\&\neq\Big(\hat{\alpha}_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho^1_{S,\Gamma})\big(\hat{\alpha}_{\overrightarrow{L}}^{\overleftarrow{R},1}(\tilde{\rho}^1_{S,\Gamma})(f_\Gamma)\big)\Big)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{6.49}$$

holds. All these problems are excluded by the definition of the actions. Since, for example there is an action defined by $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ instead of the action $\hat{\alpha}_{\overrightarrow{L}}^{\overleftarrow{R},1}$ given in (6.49). Respectively, there is an action defined by $\alpha_{\overleftarrow{L}}^{\overrightarrow{R},1}$ instead of the action $\hat{\alpha}_{\overleftarrow{L}}^{\overrightarrow{R},1}$ given in (6.48).

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Non-standard identification of the configuration space

At the beginning of these considerations one assumes that, the subgraphs of the finite graph system are identified naturally. If instead the non-standard identification of the configuration space is used, the following observation can be made.

If the set $\text{Hom}(\mathcal{P}_\Gamma,G^{|\Gamma|})$ of holonomy maps for a finite graph system $\mathcal{P}_\Gamma$ is considered and the configuration space is identified with G^N by the non-standard way. Then there exists a situation such that $\mathfrak{h}_\Gamma(\gamma\circ\gamma')=\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma')$ for arbitrary $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ holds. There is an additional action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ derivable.

First observe that for all paths of Γ intersecting S in their source vertices and all paths lie below one concludes that

$$\begin{aligned}&(\alpha_{\overleftarrow{L}}(\rho_{S,\Gamma}(\Gamma))\circ\alpha^{\overleftarrow{R}}(\rho_{S,\Gamma^{-1}}(\Gamma^{-1}))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\mathfrak{h}_\Gamma(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1}))\\&=f_\Gamma(\rho_S^L(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1})\rho_S^R(\gamma_1^{-1}),...,\rho_S^L(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1})\rho_S^R(\gamma_N)^{-1})\\&=f_\Gamma(g_S\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1})g_S^{-1},...,g_S\mathfrak{h}_\Gamma(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1})g_S^{-1})\\&=f_\Gamma(e_G,...,e_G)\end{aligned}\tag{6.50}$$

whenever $\rho_{S,\Gamma}\in G_{S,\Gamma}$, $\rho_{S,\Gamma^{-1}}\in G_{S,\Gamma^{-1}}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$, where Γ^{-1} referes to the set $\{\gamma_1^{-1},...,\gamma_N^{-1}\}$ and it is assumed that $\rho_S^L(\gamma_i)=\rho_S^R(\gamma_i^{-1})=g_S$ for all $\gamma_i\in\Gamma$ yields. In this case there is an action $\alpha_{\overleftarrow{L}}^{\overleftarrow{R}}$ of $\bar{G}_{S,|\Gamma|}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ defined by

$$\begin{aligned}&(\alpha_{\overleftarrow{L}}^{\overleftarrow{R}}(\rho_S(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\mathfrak{h}_\Gamma(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1}))\\&:=(\alpha_{\overleftarrow{L}}(\rho_{S,\Gamma}(\Gamma))\circ\alpha^{\overleftarrow{R}}(\rho_{S,\Gamma^{-1}}(\Gamma^{-1}))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1^{-1}),...,\mathfrak{h}_\Gamma(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N^{-1}))\end{aligned}\tag{6.51}$$

Moreover for a graph Γ consisting of two paths γ and γ' such that γ and S intersect in the target vertex $t(\gamma)$ of γ and lie above and, resepectively, γ' and S intersect in $s(\gamma')$ such that γ' lie above and outgoing. Then $(\gamma,\gamma')\in\mathcal{P}_\Gamma^{(2)}\Sigma$. Set $\Gamma=\{\gamma,\gamma'\}$, $\Gamma':=\{\gamma\circ\gamma'\}$ and assume $S\cap\{\gamma,\gamma'\}=\{t(\gamma)\}$. Then there is an action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\begin{aligned}(\alpha_{\overleftarrow{L}}^{\overleftarrow{R}}(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma))&=(\alpha_{\overleftarrow{L}}^{\overleftarrow{R}}(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma),\mathfrak{h}_\Gamma(\gamma'))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\rho_S^R(\gamma)^{-1},\rho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\&=f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_S^{-1},g_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\end{aligned}\tag{6.52}$$

where it is assumed that $\rho_S^L(\gamma)=\rho_S^R(\gamma')^{-1}=g_S$ for all $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$. For the definition of an action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ whenever the configuration space is identified with G^N in the non-standard way, it is necessary to define a the following map.

Let $D_S : C_0(\bar{\mathcal{A}}_\Gamma) \longrightarrow C_0(\bar{\mathcal{A}}_\Gamma)$ be a map such that

$$(D_S f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)g, h\mathfrak{h}_\Gamma(\gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma)gh\mathfrak{h}_\Gamma(\gamma'))$$

whenever $g, h \in G$ and if $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ and $S \cap \{\gamma,\gamma'\} = \{t(\gamma)\}$. There is an ambiguity, to define the inverse of D_S, since it is possible that

$$(D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)gh\mathfrak{h}_\Gamma(\gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma), gh\mathfrak{h}_\Gamma(\gamma')) \text{ for } g \in G$$
$$\text{or}$$
$$(D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)gh\mathfrak{h}_\Gamma(\gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g, h\mathfrak{h}_\Gamma(\gamma'))$$
$$\text{or}$$
$$(D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)gh\mathfrak{h}_\Gamma(\gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma)gh, \mathfrak{h}_\Gamma(\gamma'))$$

whenever $g, h \in G$. Recall that $\mathfrak{h}_\Gamma(\gamma) = h$ is an element of G. Let $g \in \mathcal{Z}(G)$ and $\mathfrak{h}_\Gamma(\gamma) \neq \mathfrak{h}_\Gamma(\gamma')$. Then there always exists a groupoid morphism $\tilde{\mathfrak{h}}_\Gamma$ such that for a pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ it is true that $\tilde{\mathfrak{h}}_\Gamma(\gamma\circ\gamma') = \mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma')$ and $\tilde{\mathfrak{h}}_\Gamma(\gamma) = \tilde{\mathfrak{h}}_\Gamma(\gamma')$. The define

$$(D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)g^2\mathfrak{h}_\Gamma(\gamma')) = f_\Gamma(\tilde{\mathfrak{h}}_\Gamma(\gamma)g, g\tilde{\mathfrak{h}}_\Gamma(\gamma'))$$

is well-defined. The reason is given by the following property. Since, for example for $\mathfrak{h}_\Gamma(\gamma) = h = \mathfrak{h}_\Gamma(\gamma')$ and $g \in \mathcal{Z}(G)$, it is true that $hg(hg) = hggh = (hg)^2$.

More generally, one can consider all $g \in G$ such that there exists a $k \in G$ and $\mathfrak{h}_\Gamma(\gamma)gg\mathfrak{h}_\Gamma(\gamma') = k^2$ for all $\mathfrak{h}_\Gamma(\gamma), \mathfrak{h}_\Gamma(\gamma') \in \bar{\mathcal{A}}_\Gamma$ for a fixed pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$.

Definition 6.1.29. *Let S be a surface and Γ be a graph, which consists of two paths γ and γ' such that γ and S intersect in the target vertex $t(\gamma)$ of γ and lie above and, resepectively, γ' and S intersect in $s(\gamma')$ such that γ' lie above and outgoing.*

Then define the action of $\bar{\mathcal{Z}}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ by

$$(D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\rho_S^R(\gamma)^{-1}\rho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))$$

whenever $\rho_{S,\Gamma}(\Gamma) \in \bar{\mathcal{Z}}_{S,\Gamma}$ and it is assumed that

$$\begin{aligned}&\rho_S^L(\gamma) = \rho_S^R(\gamma')^{-1} = g_S^{-1} \text{ for } \rho_S^L, \rho_S^R \in \mathcal{Z}(\mathbb{G}_{\breve{S},\gamma})_{S,\Gamma},\\ &\mathfrak{h}_\Gamma(\gamma) = \mathfrak{h}_\Gamma(\gamma') \text{ for } \mathfrak{h}_\Gamma \in \mathrm{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|})\end{aligned} \tag{6.53}$$

for the pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ such that $\Gamma' := \{\gamma\circ\gamma'\}$. The action is redefined by

$$(D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := (\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))$$

Notice that, the action is computed in the following way

$$\begin{aligned}(D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &= (D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\ &= (D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma), \mathfrak{h}_\Gamma(\gamma'))\\ &= (D_S f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\rho_S^R(\gamma)^{-1}, \rho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\ &= (D_S f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)g_S^{-1}, g_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_S^{-1}g_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\end{aligned}$$

whenever $\rho_{S,\Gamma}(\Gamma) \in \bar{\mathcal{Z}}_{S,\Gamma}$ and it is assumed that $\rho_S^L(\gamma) = \rho_S^R(\gamma')^{-1} = g_S^{-1}$ for the pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ such that $\Gamma' := \{\gamma\circ\gamma'\}$.

Finally, derive that for this action it is true that

$$(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma')\hat{\rho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) = (\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma'))\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\hat{\rho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) \tag{6.54}$$

whenever $\hat{\rho}_{S,\Gamma}, \rho_{S,\Gamma} \in \mathcal{Z}_{S,\Gamma}$ holds. Hence the action $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ is automorphic. One can show that the action $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ is point-norm continuous.

But otherwise, if the graph is changed only slightly, then recognize the following.

Definition 6.1.30. *Let S be a surface and Γ' be a graph, which is given by the composed path of a path γ and γ' such that γ and S intersects in the target vertex $t(\gamma)$ of γ and lies below and, resepectively, γ' and S intersects in $s(\gamma')$ such that γ' lies above and outgoing. Then $(\gamma,\gamma')\in\mathcal{P}_\Gamma^{(2)}\Sigma$.*

The action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ in this case is presented by

$$\begin{aligned}(D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\rho_S^R(\gamma)^{-1}\rho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_Sg_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma\circ\gamma'))\end{aligned}$$

whenever $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho_{S,\Gamma}\in G_{S,\Gamma}$ and it is assumed that

$$\begin{aligned}&\rho_S^L(\gamma)=\rho_S^R(\gamma')=g_S^{-1}\text{ for }\rho_S^L,\rho_S^R\in\mathcal{Z}(\mathbb{G}_{\breve{S},\gamma})_{S,\Gamma},\\ &\mathfrak{h}_\Gamma(\gamma)=\mathfrak{h}_\Gamma(\gamma')\text{ for }\mathfrak{h}_\Gamma\in\operatorname{Hom}(\mathcal{P}_\Gamma,G^{|\Gamma|})\end{aligned}\tag{6.55}$$

for the pair $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ such that $\Gamma':=\{\gamma\circ\gamma'\}$. Set

$$(D_S\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')):=(\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}(\rho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))$$

Since, in this case

$$(D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)gg^{-1}\mathfrak{h}_\Gamma(\gamma'))=f_\Gamma(\mathfrak{h}_\Gamma(\gamma),\mathfrak{h}_\Gamma(\gamma'))\text{ for }g\in G$$

is well-defined. Clearly, this action can be restricted to those maps that are elements of $\mathcal{Z}_{S,\Gamma}$. The action $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ is automorphic and point-norm continuous.

Notice that, there are two actions $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ and $\alpha_{\overrightarrow{L}}^{\overleftarrow{R},1}$ of $\bar{\mathcal{Z}}_{S,\Gamma}$ are restricted by the requirement (6.53) or (6.55). The actions depend on the orientation of both paths of the pair $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ w.r.t. the surface orientation of S. Clearly the actions can be generalised to graphs containg a set of pairs $\{(\gamma,\gamma')\}$ of paths in $\mathcal{P}_\Gamma\Sigma^{(2)}$.

In section 3.3.4.1 admissible maps associated to surfaces have been introduced. Then the actions of the fluxes w.r.t. admissible maps are studied in the next paragraphs.

Definition 6.1.31. *Let $\Gamma=\{\gamma,\gamma'\}$, $\mathcal{P}_\Gamma$ be a finite graph systems, $\Gamma':=\{\gamma\circ\gamma'\}\in\mathcal{P}_\Gamma$ and assume $S\cap\{\gamma,\gamma'\}=\{t(\gamma)\}$ for two paths γ and γ' intersecting S in $v=s(\gamma')=t(\gamma)$ such that the path γ lies below and the other above the surface orientation.*

Then there exists a trivial action of $\bar{G}_{S,\Gamma}^{\mathrm{A}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, which is defined by

$$(D_S\overleftarrow{\alpha}^{\mathrm{A},l}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')):=f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})^{-1}\varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))$$

such that

$$\begin{aligned}&(D_S\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\ &=f_\Gamma(\mathfrak{h}_\Gamma(\gamma)(\varrho_S^R(\gamma^{-1})\hat{\varrho}_S^R(\gamma^{-1}))^{-1}\varrho_S^L(\gamma')\hat{\varrho}_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\ &=f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\hat{g}_S^{-1}g_S^{-1}g_S\hat{g}_S\mathfrak{h}_\Gamma(\gamma'))\\ &=f_\Gamma(\mathfrak{h}_\Gamma(\gamma\circ\gamma'))\end{aligned}$$

whenever $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$, $\varrho_{S,\Gamma},\hat{\varrho}_{S,\Gamma}\in G_{S,\Gamma}^{\mathrm{A}}$ and it is assumed that

$$\mathfrak{h}_\Gamma(\gamma)=\mathfrak{h}_\Gamma(\gamma')\text{ for }\mathfrak{h}_\Gamma\in\operatorname{Hom}(\mathcal{P}_\Gamma,G^{|\Gamma|})\tag{6.56}$$

for a pair $(\gamma,\gamma')\in\mathcal{P}_\Gamma^{(2)}$. Set

$$(D_S\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')):=(\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))$$

whenever $\varrho_{S,\Gamma}\in G_{S,\Gamma}^{\mathrm{A}}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Notice by definition it is true that $\varrho_S^R(\gamma^{-1}) = \varrho_S^L(\gamma') = g_S$ for all pairs $(\gamma,\gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$ and all maps $\varrho_S^L \times \varrho_S^R \in \mathbb{G}_{S,\Gamma}^{\mathrm{A}}$. Derive that

$$\begin{aligned}(D_S \overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &= (D_S \overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma)) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma),\mathfrak{h}_\Gamma(\gamma'))\\ &= (D_S f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})^{-1}, \varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})^{-1}\varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\end{aligned}$$

and conclude

$$(\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma')\overleftarrow{\alpha}^{\mathrm{A},l}(\hat{\varrho}_{S,\Gamma}(\Gamma')f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) \quad = (\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))\hat{\varrho}_{S,\Gamma}(\Gamma')f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))$$

whenever $\varrho_{S,\Gamma}, \hat{\varrho}_{S,\Gamma} \in G_{S,\Gamma}^{\mathrm{A}}$, $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Now, change the graph only a little and derive the following action.

Definition 6.1.32. *Let S be a surface and Γ'' be a subgraph of Γ, which is equivalent to the composition of the path γ, which is ingoing and lies above, and γ', which is outgoing and lies below.*

Then the action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is presented by

$$(D_S \overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) \quad := f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})\varrho_S^L(\gamma')^{-1}\mathfrak{h}_\Gamma(\gamma'))$$

such that

$$\begin{aligned}&(D_S \overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\ &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})\hat{\varrho}_S^R(\gamma^{-1})(\varrho_S^L(\gamma')\hat{\varrho}_S^L(\gamma'))^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_S\hat{g}_S\hat{g}_S^{-1}g_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma\circ\gamma'))\end{aligned}$$

whenever $\varrho_{S,\Gamma}, \hat{\varrho}_{S,\Gamma} \in G_{S,\Gamma}^{\mathrm{A}}$, $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and it is assumed that

$$\mathfrak{h}_\Gamma(\gamma) = \mathfrak{h}_\Gamma(\gamma') \text{ for } \mathfrak{h}_\Gamma \in \mathrm{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|}) \tag{6.57}$$

for a pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma^{(2)}$. Set

$$(D_S \overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'')) := (\overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma''))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma''))$$

whenever $\varrho_{S,\Gamma} \in G_{S,\Gamma}^{\mathrm{A}}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Definition 6.1.33. *For a subgraph Γ', which contains the composed path of an ingoing path γ that lies below and an outgoing path γ' that lies below, the action is given by*

$$(D_S \overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma')) := f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})^{-1}\varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))$$

such that

$$\begin{aligned}&(D_S \overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)(\hat{\varrho}_S^R(\gamma^{-1})\varrho_S^R(\gamma^{-1}))^{-1}\hat{\varrho}_S^L(\gamma')\varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_S^{-1}\hat{g}_S^{-1}\hat{g}_S g_S\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma\circ\gamma'))\end{aligned}$$

whenever $\varrho_{S,\Gamma}, \hat{\varrho}_{S,\Gamma} \in G_{S,\Gamma}^{\mathrm{A}}$, $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and it is assumed that

$$\mathfrak{h}_\Gamma(\gamma) = \mathfrak{h}_\Gamma(\gamma') \text{ for } \mathfrak{h}_\Gamma \in \mathrm{Hom}(\mathcal{P}_\Gamma, G^{|\Gamma|}) \tag{6.58}$$

for a pair $(\gamma,\gamma') \in \mathcal{P}_\Gamma^{(2)}$. Set

$$(D_S \overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1} f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := (\overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))$$

whenever $\varrho_{S,\Gamma} \in G_{S,\Gamma}^{\mathrm{A}}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Definition 6.1.34. *If the subgraph Γ'' is equivalent to the composed path of a path γ, which is ingoing and lies above, and a path γ', which is outgoing and lies above, the action is presented by*

$$(D_S\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) \quad := f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S^R(\gamma^{-1})\varrho_S^L(\gamma')^{-1}\mathfrak{h}_\Gamma(\gamma'))$$

such that

$$\begin{aligned}&(D_S\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\ &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\hat{\varrho}_S^R(\gamma^{-1})\varrho_S^R(\gamma^{-1})(\hat{\varrho}_S^L(\gamma')\varrho_S^L(\gamma'))^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\hat{g}_Sg_Sg_S^{-1}\hat{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma\circ\gamma'))\end{aligned}$$

whenever $\varrho_{S,\Gamma},\hat{\varrho}_{S,\Gamma}\in G^{\mathrm{A}}_{S,\Gamma}$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$. Set

$$(D_S\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'')) := (\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma''))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma''))$$

whenever $\varrho_{S,\Gamma}\in G^{\mathrm{A}}_{S,\Gamma}$, $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$ and it is assumed that

$$\mathfrak{h}_\Gamma(\gamma)=\mathfrak{h}_\Gamma(\gamma') \text{ for } \mathfrak{h}_\Gamma\in\mathrm{Hom}(\mathcal{P}_\Gamma,G^{|\Gamma|}) \tag{6.59}$$

for a pair $(\gamma,\gamma')\in\mathcal{P}_\Gamma^{(2)}$.

There is also an action of $\bar{G}^{\mathrm{A}}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ if a path γ can be decomposed into $\gamma'\circ\gamma''$ such that γ' intersect the surface S in the source and target vertex $s(\gamma')$, $t(\gamma')$ and lie below S, the path γ'' intersect the surface S in the source and target vertex $s(\gamma'')$, $t(\gamma'')$ and lie above S. Then there is an action

$$\begin{aligned}&(D_S\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma')\mathfrak{h}_\Gamma(\gamma''))\\ &:= f_\Gamma(\varrho_S^L(\gamma')\hat{\varrho}_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma')(\varrho_S^R(\gamma'^{-1})\hat{\varrho}_S^R(\gamma'^{-1}))^{-1}\varrho_S^L(\gamma'')\hat{\varrho}_S^L(\gamma'')\mathfrak{h}_\Gamma(\gamma'')(\varrho_S^R(\gamma'')\hat{\varrho}_S^R(\gamma''))^{-1})\\ &= f_\Gamma(g_S\hat{g}_S\mathfrak{h}_\Gamma(\gamma')\hat{g}_S^{-1}g_S^{-1}g_S\hat{g}_S\mathfrak{h}_\Gamma(\gamma'')\hat{g}_S^{-1}g_S^{-1})\\ &= f_\Gamma(g_S\hat{g}_S\mathfrak{h}_\Gamma(\gamma'\circ\gamma'')\hat{g}_S^{-1}g_S^{-1})\end{aligned} \tag{6.60}$$

Or consider a path γ, which can be decomposed into $\gamma'\circ\gamma''$ such that γ' intersect the surface S in the source and target vertex $s(\gamma')$, $t(\gamma')$ and lie below S, the path γ'' intersect the surface S in the source and target vertex $s(\gamma'')$, $t(\gamma'')$ and lie below S. Then there is an action

$$\begin{aligned}&(D_S\overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma')\mathfrak{h}_\Gamma(\gamma''))\\ &:= f_\Gamma(\hat{\varrho}_S^L(\gamma')\varrho_S^L(\gamma')\mathfrak{h}_\Gamma(\gamma')(\hat{\varrho}_S^R(\gamma'^{-1})\varrho_S^R(\gamma'^{-1}))^{-1}\hat{\varrho}_S^L(\gamma'')\varrho_S^L(\gamma'')\mathfrak{h}_\Gamma(\gamma'')(\hat{\varrho}_S^R(\gamma'')\varrho_S^R(\gamma''))^{-1})\\ &= f_\Gamma(\hat{g}_Sg_S\mathfrak{h}_\Gamma(\gamma')g_S^{-1}\hat{g}_S^{-1}\hat{g}_Sg_S\mathfrak{h}_\Gamma(\gamma'')g_S^{-1}\hat{g}_S^{-1})\\ &= f_\Gamma(\hat{g}_Sg_S\mathfrak{h}_\Gamma(\gamma'\circ\gamma'')g_S^{-1}\hat{g}_S^{-1})\end{aligned} \tag{6.61}$$

Summarising, the actions $\overleftarrow{\alpha}^{\mathrm{A},l}$,$\overrightarrow{\alpha}^{\mathrm{A},l}$, $\overleftarrow{\alpha}^{\mathrm{A},r}$ and $\overrightarrow{\alpha}^{\mathrm{A},r}$ depend on the orientation of both paths of the pair $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ w.r.t. the surface orientation of S.

But if two surfaces S,S' intersecting a composed path $\gamma\circ\gamma'$ in the same vertex $v=t(\gamma)$ such that the intersection behavior w.r.t. S and S' of the the paths γ and γ' is different, then there are no well-defined actions $\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))\circ\overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S',\Gamma}(\Gamma'))$, $\overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))\circ\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S',\Gamma}(\Gamma'))$, $\overleftarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))\circ\overrightarrow{\alpha}^{\mathrm{A},r}(\varrho_{S',\Gamma}(\Gamma'))$ or $\overrightarrow{\alpha}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma'))\circ\overleftarrow{\alpha}^{\mathrm{A},r}(\varrho_{S',\Gamma}(\Gamma'))$ derivable.

A summary of the results above is given by the following. If the non-standard identification of the configuration space is used, then the holonomy maps are defined on arbitrary elements of the finite graph groupoid. In particular, a graph Γ', which is a subgraph of Γ and contains a path $\gamma\circ\gamma'$. Then consider a surface S that intersects only the paths γ and γ' of the graph Γ in the vertex $t(\gamma)$. Then different actions of an element $\varrho_{S,\Gamma}(\Gamma')$ on a function in $C_0(\bar{\mathcal{A}}_\Gamma)$ can be considered. There are two different kinds of actions. One referes to a translation of the center of the flux group $\bar{\mathcal{Z}}_{S,\Gamma}$. The other is related to a translation of the fluxes related to admissible maps. Moreover each action depends on the orientation of the paths γ and γ' w.r.t. the surface S.

Problem 6.1.2: There is an ambiugity for actions on graphs consisting of paths that are not composable. Let $\Gamma:=\{\gamma_1,...,\gamma_N\}$ be a disconnected graph such that all paths intersect a surface S in the target vertex of each path and all paths lie below the surface S.

Then there is an ambiguity, since an action is defined by

$$\begin{aligned}(\alpha_R^{\mathrm{A},x}(\varrho_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\varrho_S(\gamma_1^{-1})^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)\varrho_S(\gamma_N^{-1})^{-1})\\ &= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)g_S,...,\mathfrak{h}_\Gamma(\gamma_N)g_S)\end{aligned}$$

where either x is equivalent to l such that

$$(\alpha_R^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\hat{\varrho}_S(\gamma_1^{-1})\varrho_S(\gamma_1^{-1}))^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)(\hat{\varrho}_S(\gamma_N^{-1})\varrho_S(\gamma_N^{-1}))^{-1})$$

or x is equivalent to r such that

$$(\alpha_R^{\mathrm{A},r}(\varrho_{S,\Gamma}(\Gamma)\hat{\varrho}_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\varrho_S(\gamma_1^{-1})\hat{\varrho}_S(\gamma_1^{-1}))^{-1},...,\mathfrak{h}_\Gamma(\gamma_N)(\varrho_S(\gamma_N^{-1})\hat{\varrho}_S(\gamma_1^{-1}))^{-1})$$

whenever $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\varrho_{S,\Gamma} \in G^{\mathrm{A}}_{S,\Gamma}$ is satisfied.

The same is developed for an action $\alpha_L^{\mathrm{A},x}(\varrho_{S,\Gamma}(\Gamma))$. Hence one has to make a choice. At the beginning of this section, the actions $\alpha_L^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))$, $\alpha_R^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))$ and $\alpha_{L,R}^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma))$ are defined in equation (6.6).

▪

Problem 6.1.3: In comparison with the problem 6.1.0.1 stated before, there is a problem if one defines the action of $\bar{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ by

$$\begin{aligned}&(\alpha^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma)\tilde{\varrho}_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &\quad:= f_\Gamma(\varrho_S(\gamma_1)\tilde{\varrho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)(\tilde{\varrho}_S(\gamma_N^{-1})\varrho_S(\gamma_N^{-1}))^{-1})\end{aligned} \tag{6.62}$$

Since for such an action conclude

$$\begin{aligned}&(\alpha^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma)\tilde{\varrho}_{S,\Gamma}(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &= f_\Gamma(\varrho_S(\gamma_1)\tilde{\varrho}_S(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)(\tilde{\varrho}_S(\gamma_N^{-1})\varrho_S(\gamma_N^{-1}))^{-1})\\ &= f_\Gamma(g_S\tilde{g}_S\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)g_S^{-1}\tilde{g}_S^{-1})\\ &\neq (\alpha^{\mathrm{A},l,r}(\varrho^1_{S,\Gamma})(\alpha^{\mathrm{A},l,r}(\tilde{\varrho}^1_{S,\Gamma}f_\Gamma(\mathfrak{h}_\Gamma))))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

Consequently in this framework the actions of $\bar{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ have to be restricted to particular sets of surfaces and subgraphs such that actions of the form (6.62) do not occur. In general there only exists actions of $\bar{G}^{\mathrm{A}}_{\breve{S},\Gamma}$ on $\mathfrak{A}_\Gamma$, which are of the form $\alpha^{\mathrm{A},l}$ or $\alpha^{\mathrm{A},r}$ defined in the definitions 6.1.31, 6.1.32, 6.1.33 and 6.1.34 for arbitrary subgraphs of Γ.

For example, there is a problem if a surface S and a graph $\Gamma' := \{\gamma\circ\gamma',\gamma''\circ\gamma'''\}$ containing two composable pairs (γ,γ') and (γ'',γ'''), which contain the ingoing path γ lying below the surface S, the outgoing path γ' lying above, the ingoing path γ'' lying below and the outgoing path γ''' lying below, are considered. Since, in this case it is true that

$$\begin{aligned}&(\hat{\alpha}^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma')\hat{\varrho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))\\ &:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)(\varrho_S(\gamma^{-1})\hat{\varrho}_S(\gamma^{-1}))^{-1}\varrho_S(\gamma')\hat{\varrho}_S(\gamma')\mathfrak{h}_\Gamma(\gamma'),\\ &\qquad \mathfrak{h}_\Gamma(\gamma'')(\hat{\varrho}_S(\gamma''^{-1})\varrho_S(\gamma''^{-1}))^{-1}\hat{\varrho}_S(\gamma''')\varrho_S(\gamma''')\mathfrak{h}_\Gamma(\gamma'''))\\ &\neq ((\hat{\alpha}^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma))\circ\hat{\alpha}^{\mathrm{A},l,r}(\hat{\varrho}_{S,\Gamma}(\Gamma)))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'),\mathfrak{h}_\Gamma(\gamma'')\mathfrak{h}_\Gamma(\gamma'''))\end{aligned} \tag{6.63}$$

Such surface and graph configurations have to be concerned seperatly in such a way that action of the type (6.63) are excluded. Clearly, all these problems can be solved if maps $\rho_{S,\Gamma} \in \mathcal{Z}^{\mathrm{A}}_{\breve{S},\Gamma}$ for the center $\mathcal{Z}(G)$ of the group G are considered.

▪

The question is if the action $\check{\alpha}^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma'))$ can be reformulated such that the action is automorphic.

Remark 6.1.35. *For some elements* $\varrho_S, \tilde{\varrho}_S, ..., \breve{\varrho}_S \in G^{\mathrm{A}}_{S,\Gamma}$ *set* $\varrho_S(\gamma) = g_S$, $\tilde{\varrho}_S(\gamma) = \tilde{g}_S$ *and so on. Recall that for a connected semisimple Lie group G the following is true. For each* $g \in G$ *there exists a finite number of elements* $g_S, \tilde{g}_S, ...\breve{g}_S, \breve{g}_S$ *such that* $g = g_S\tilde{g}_S...\breve{g}_S\breve{g}_S$ *and* $e_G = \breve{g}_S\breve{g}_S...\tilde{g}_S g_S$. *Hence it is true that*

$$g_S\tilde{g}_S...\breve{g}_S g_S^{-1}\tilde{g}_S^{-1}...\breve{g}_S^{-1} = g_S\tilde{g}_S...\breve{g}_S$$

Furthermore for a surface S and a graph $\Gamma' := \{\gamma \circ \gamma'\}$, *which contains a composable pair* (γ, γ') *such that* γ *is ingoing and lies above and* γ' *is outgoing and lies below the surface S, one can define the following action*

$$\begin{aligned}
&(\breve{\alpha}_1^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma')\hat{\varrho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))\\
&:= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S(\gamma^{-1})\hat{\varrho}_S(\gamma^{-1})(\hat{\varrho}_S(\gamma')\varrho_S(\gamma'))^{-1}\mathfrak{h}_\Gamma(\gamma'))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S(\gamma^{-1})\hat{\varrho}_S(\gamma^{-1})\varrho_S(\gamma')^{-1}\hat{\varrho}_S(\gamma')^{-1}\mathfrak{h}_\Gamma(\gamma'))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)g_S\tilde{g}_S g_S^{-1}\tilde{g}_S^{-1}\mathfrak{h}_\Gamma(\gamma'))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\varrho_S(\gamma')\hat{\varrho}_S(\gamma')\mathfrak{h}_\Gamma(\gamma'))\\
&=: (\breve{\alpha}_1^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma')\hat{\varrho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))
\end{aligned} \tag{6.64}$$

whenever $\varrho_S(\gamma_i^{-1}) = \varrho_S(\gamma_j)$, $\hat{\varrho}_S(\gamma_i^{-1}) = \hat{\varrho}_S(\gamma_j)$ *for all* $(\gamma_i, \gamma_j) \in \mathcal{P}^{(2)}_\Gamma$ *for* $1 \leq i, j \leq |\Gamma|$. *Consequently observe that in this case*

$$\begin{aligned}
&(\breve{\alpha}_1^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma')\hat{\varrho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))\\
&= ((\breve{\alpha}_1^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma)) \circ \breve{\alpha}_1^{\mathrm{A},l,r}(\hat{\varrho}_{S,\Gamma}(\Gamma)))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))
\end{aligned}$$

holds.

But for any finite number of elements $g_S, \tilde{g}_S, ...\breve{g}_S, \breve{g}_S$ *such that* $g = g_S\tilde{g}_S...\breve{g}_S\breve{g}_S$ *it is not necessary that* $e_G = \breve{g}_S\breve{g}_S...\tilde{g}_S g_S$ *yields. Consequently in general*

$$\begin{aligned}
&((\breve{\alpha}_1^{\mathrm{A},l,r}(\varrho_{S,\Gamma}(\Gamma)) \circ \breve{\alpha}_1^{\mathrm{A},l,r}(\hat{\varrho}_{S,\Gamma}(\Gamma)))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma'))\\
&\neq (\breve{\alpha}_1^{\mathrm{A},l}(\varrho_{S,\Gamma}(\Gamma')\hat{\varrho}_{S,\Gamma}(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))
\end{aligned}$$

The set of actions of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$

Assume that, the configuration space $\bar{\mathcal{A}}_\Gamma$ of generalised connections is identified in the natural way with $G^{|\Gamma|}$. Certainly, there are a lot of different actions on $C_0(\bar{\mathcal{A}}_\Gamma)$ corresponding to different surfaces and graph configurations, which are build from left and right actions of the group $\bar{G}_{\breve{S},\Gamma}$ on the C^*-algebra. In general there is an exceptional set of all well-defined point-norm continuous automorphic actions.

Definition 6.1.36. *Denote the set of all point-norm continuous automorphic actions of* $\bar{G}_{\breve{S},\Gamma}$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$ *by* $\mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ *for an arbitrary set* $\breve{S}$ *of surfaces, a graph* Γ *and a finite graph system* $\mathcal{P}_\Gamma$.

Let Φ_M be the multiplication representation of $C_0(\bar{\mathcal{A}}_\Gamma)$ on $\mathcal{H}_\Gamma$. For all automorphic and point-norm continuous actions presented above there are unitary representations U of $\bar{G}_{S,\Gamma}$ or $\bar{G}_{\breve{S},\Gamma}$ on the Hilbert space $\mathcal{H}_\Gamma$, which satisfy for a suitable surface S or surface set $\breve{S}$ and graph configuration one of the following or some equivalent Weyl relations

$$\begin{aligned}
&U_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))\Phi_M(f_\Gamma)U_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))^{-1} = \Phi_M(\alpha^p_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))f_\Gamma)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall p \in \mathbb{N},\\
&U^k_{\overleftarrow{L}}(\rho^k_{S,\Gamma})\Phi_M(f_\Gamma)U^k_{\overleftarrow{L}}(\rho^k_{S,\Gamma})^{-1} = \Phi_M(\alpha^k_{\overleftarrow{L}}(\rho^k_{S,\Gamma})f_\Gamma) \quad \forall k \in \mathbb{N},\\
&U^{\overleftarrow{R}}_k(\rho^k_{S,\Gamma})\Phi_M(f_\Gamma)U^{\overleftarrow{R}}_k(\rho^k_{S,\Gamma})^{-1} = \Phi_M(\alpha^{\overleftarrow{R}}_k(\rho^k_{S,\Gamma})f_\Gamma) \quad \forall k \in \mathbb{N},\\
&U^{\overleftarrow{R}}_{p,k}(\rho^{p,k}_{S,\Gamma})\Phi_M(f_\Gamma)U^{\overleftarrow{R}}_{p,k}(\rho^{p,k}_{S,\Gamma})^{-1} = \Phi_M(\alpha^{\overleftarrow{R}}_{p,k}(\rho^{p,k}_{S,\Gamma})f_\Gamma) \quad \text{ for } p \leq k \leq N-1 \text{ or}\\
&U^k_{\overleftarrow{L}}(\rho^k_{S,\Gamma})U^{\overleftarrow{R}}_k(\rho^k_{S,\Gamma})\Phi_M(f_\Gamma)U^{\overleftarrow{R}}_k(\rho^k_{S,\Gamma})^{-1}U^k_{\overleftarrow{L}}(\rho^k_{S,\Gamma})^{-1} = \Phi_M(\alpha^{\overleftarrow{R},k}_{\overleftarrow{L}}(\rho^k_{S,\Gamma})f_\Gamma) \quad \forall k \in \mathbb{N}
\end{aligned} \tag{6.65}$$

and so on for all $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho^k_{S,\Gamma} \in \bar{G}_{S,\Gamma}$, $\rho^{p,k}_{S,\Gamma} \in \bar{G}_{\breve{S},\Gamma}$ and $\rho_{S_i,\Gamma} \in G_{\breve{S},\Gamma}$ for $i = 1, ..., k$ where $\breve{S} := \{S_i\}_{1\leq i\leq p}$ is suitable.

Observe that, for each unitary U defined above the pair (U, Φ_M) consisting of $U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ and $\Phi_M \in \text{Mor}(C_0(\bar{\mathcal{A}}_\Gamma), \mathcal{L}(\mathcal{H}_\Gamma))$ is a covariant pair of the dynamical C^*-system $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma}, \alpha)$ for an action $\alpha \in \text{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$.

Problem 6.1.4: There are no unitary operators, which satisfy the Weyl relations for the actions

$$\alpha^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}), \quad \alpha^1_{\overrightarrow{L},\overrightarrow{R}}(\rho^1_{S,\Gamma}), ...$$

which are presented in the problem 6.1.0.1 and which are not automorphic actions of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$. This is true, since there are unitary operators such that

$$\begin{aligned}
&U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})(U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))^* \\
&= U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1), ..., \tilde{\rho}_S(\gamma_1)\rho_S(\gamma_1))U^1_{\overrightarrow{L},\overleftarrow{L}}(\tilde{\rho}_S(\gamma_1)^{-1}\rho_S(\gamma_1)^{-1}, ..., \rho_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)^{-1}) \\
&= U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)\tilde{\rho}_S(\gamma_1)^{-1}\rho_S(\gamma_1)^{-1}, ..., \rho_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)\rho_S(\gamma_1)) \\
&= \mathbb{1}
\end{aligned}$$

holds. But $U^1_{\overrightarrow{L},\overleftarrow{L}}(\bar{G}_{\breve{S},\Gamma})$ does not form a group. Consequently it is true that

$$U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma})U^1_{\overrightarrow{L},\overleftarrow{L}}(\tilde{\rho}^1_{S,\Gamma})\Phi_M(f_\Gamma)(U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})U^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))^* \neq \Phi_M(\alpha^1_{\overrightarrow{L},\overleftarrow{L}}(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))$$

yields.

■

Remark 6.1.37. *Recall the definitions of the actions of $\bar{G}^{\text{A}}_{\breve{S},\Gamma}$ on $C_0(\mathcal{A}_\Gamma)$ and recall the problem 6.1.0.2.*

Let $\Gamma := \{\gamma_1, ..., \gamma_N\}$ be a disconnected graph such that all paths intersect a surface S in the target vertex of each path and all paths lie below the surface S. Then after the choice of the definition of an action of $\bar{G}^{\text{A}}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ the objects and the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ form a C^*-dynamical system. Hence there exists for every action a unitary U on the Hilbert space $\mathcal{H}_\Gamma$ such that

$$\begin{aligned}
&(U(\rho^1_{S,\Gamma})U(\rho^1_{S,\Gamma})^*\psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) = (U(\rho^1_{S,\Gamma})U((\rho^1_{S,\Gamma})^{-1})\psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) \\
&= \psi_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)(\rho_S(\gamma_1)\rho_S(\gamma_1)^{-1})^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)(\rho_S(\gamma_N)\rho_S(\gamma_N)^{-1})^{-1}) \\
&= \psi_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)\rho_S(\gamma_1)^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)\rho_S(\gamma_N)\rho_S(\gamma_N)^{-1}) \\
&= \psi_\Gamma(\mathfrak{h}_\Gamma(\Gamma))
\end{aligned} \tag{6.66}$$

for $\psi_\Gamma \in \mathcal{H}_\Gamma$. Notice that, the adjoint unitary $U(\rho^1_{S,\Gamma})^*$ is identified with $U(\rho^1_{S^{-1},\Gamma})$.

Definition 6.1.38. *Let $\breve{S}$ be an arbitrary set of surfaces. The Hilbert space $\mathcal{H}_\Gamma$ is identified with $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.*

*Each unitary $U(\rho_S(\Gamma))$ for $U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ and $\rho_S(\Gamma) \in \bar{G}_{\breve{S},\Gamma}$ is called a **Weyl element**. The set of all linearly independent Weyl elements is denoted by $W(\bar{G}_{\breve{S},\Gamma})$. The vector space of all finite complex linear combinations of Weyl elements, which are unitary operators satisfying the Weyl relations* (6.65) *on their own, is symbolised by $\mathbf{W}(\bar{G}_{\breve{S},\Gamma})$.*

*Each unitary $U(\rho_S(\Gamma))$ for $U \in \text{Rep}(\bar{\mathcal{Z}}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ is called the **commutative Weyl element**. The set of all linearly independent commutative Weyl elements is denoted by $W(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$. The vector space $\mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$ is given by the set of all finite complex linear combinations of commutative Weyl elements such that the linear combinations are unitary operators and satisfy the Weyl relations* (6.65)*.*

In general finite linear combinations of unitary elements (elements such that $UU^* = U^*U = \mathbb{1}$) form a unital *-algebra. Otherwise, for each fixed suitable surface set and $\bar{G}_{\breve{S},\Gamma}$, the set $U(\bar{G}_{\breve{S},\Gamma})$ forms a group with the usual left multiplication operation. The group $U(\bar{G}_{\breve{S},\Gamma})$ is a subgroup of the group $U(\mathcal{H}_\Gamma)$ of unitaries on a Hilbert space $\mathcal{H}_\Gamma$. Notice that, the Weyl algebra associated to surfaces and a graph, which is introduced in the next section, is generated by the constant function $\mathbb{1}_\Gamma$, the elements of the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ and the unitaries associated to surfaces. For example, an element of the Weyl algebra is of the form

$$\sum_{l=1}^{L} \mathbb{1}_\Gamma U(\rho^l_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K}\sum_{i=1}^{M} f^k_\Gamma U(\rho^i_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K}\sum_{i=1}^{M} U(\rho^i_{S,\Gamma}(\Gamma)) f^k_\Gamma U(\rho^i_{S,\Gamma}(\Gamma))^* + \sum_{p=1}^{P} f^p_\Gamma$$

whenever $f^k_\Gamma, f^p_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\rho^l_{S,\Gamma}, \rho^i_{S,\Gamma} \in G_{\breve{S},\Gamma}$. Notice that, for a compact group G the analytic holonomy C^*-algebra is unital.

In this context a reformulation of the Weyl C^*-algebra is given as follows.

Lemma 6.1.39. *Let $\mathcal{H}_\Gamma$ be the Hilbert space $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ with norm $\|.\|_2$.*

With the involution * and the natural product of unitaries the vector space $\mathbf{W}(\bar{G}_{\breve{S},\Gamma})$ is a unital *-algebra, where $\mathcal{W}(\bar{G}_{\breve{S},\Gamma})$ stands for the *-algebra of Weyl elements.

The *-algebra $\mathcal{W}(\bar{G}_{\breve{S},\Gamma})$ of Weyl elements completed w.r.t. the strong operator norm is a C^*-algebra. Denote this algebra by $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$.

Notice that, $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ is a C^*-subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ of bounded operators on the Hilbert space $\mathcal{H}_\Gamma$, which is equal to $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.

Recall that G is assumed to be a locally compact unimodular group and $\bar{\mathcal{A}}_\Gamma$ is identified in the natural way with $G^{|\Gamma|}$. Then remember the definition in lemma 6.1.27 of the action $\beta_{\overrightarrow{L}}^{\overrightarrow{R},1}$, which defines a C^*-dynamical system $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma}, \beta_{\overrightarrow{L}}^{\overrightarrow{R},1})$ for a fixed graph and a suitable surface set $\breve{S}$.

Proposition 6.1.40. *Let $\breve{S}'$ and $\breve{S}$ be two suitable surface sets, let Γ be a graph and let $\mathcal{P}_\Gamma$ be the finite graph system associated to Γ.*

Furthermore let $(U_{\overrightarrow{L}}^{\overrightarrow{R},1}, \Phi_M)$ be a covariant pair of a dynamical C^*-system $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S}',\Gamma}, \beta_{\overrightarrow{L}}^{\overrightarrow{R},1})$ for the surface set $\breve{S}'$ and a finite graph system $\mathcal{P}_\Gamma$ associated to a graph Γ.

Denote a general covariant pair by (U, Φ_M) of a dynamical C^*-system $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma}, \alpha)$ for an action $\alpha \in \mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ for the surface set $\breve{S}$ and a finite graph system $\mathcal{P}_\Gamma$. The set $\mathbb{S}$ of all suitable surface sets for Γ contains all surface sets such that there exists an action in $\mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$.

Then there exists a GNS-triple $(\mathcal{H}_\Gamma, \Phi_M, \Omega_\Gamma)$ where Ω_Γ is the cyclic vector for Φ_M on $\mathcal{H}_\Gamma$. Moreover the associated GNS-state ω^Γ_M on $C_0(\bar{\mathcal{A}}_\Gamma)$ is $\bar{G}_{\breve{S},\Gamma}$- and $\bar{G}_{\breve{S}',\Gamma}$-invariant, i.e.

$$\begin{aligned}\omega^\Gamma_M(\beta_{\overrightarrow{L}}^{\overrightarrow{R},1}(\rho^1_{S',\Gamma})(f_\Gamma)) &= \omega^\Gamma_M(f_\Gamma) := \langle \Omega_\Gamma, \Phi_M(f_\Gamma)\Omega_\Gamma\rangle_\Gamma \\ &= \omega^\Gamma_M(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma))\end{aligned}$$

for $\alpha, \beta_{\overrightarrow{L}}^{\overrightarrow{R},1} \in \mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$, $\rho^1_{S,\Gamma}, \rho_{S,\Gamma}(\Gamma) \in \bar{G}_{\breve{S},\Gamma}$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

The state ω^Γ_M is contained in the state space $\mathcal{S}^{\mathbb{S},\Gamma}(C_0(\bar{\mathcal{A}}_\Gamma))$ of $\bar{G}_{\breve{S},\Gamma}$-invariant states on $C_0(\bar{\mathcal{A}}_\Gamma)$ for each suitable surface set $\breve{S}$ in $\mathbb{S}$.

Moreover the set $\mathcal{M} := \Phi_M(C_0(\bar{\mathcal{A}}_\Gamma)) \cup \{U(\bar{G}_{\breve{S},\Gamma}) : U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathfrak{A}_\Gamma)\}$ is irreducible on $\mathcal{H}_\Gamma$.

Remark that the state ω^Γ_M on $C_0(\bar{\mathcal{A}}_\Gamma)$ is $\bar{G}_{\breve{S},\Gamma}$-invariant for many different suitable surface sets. The surface sets are required to intersect the graph Γ only in vertices and hence such that there exists an action in $\mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$.

Proof : The GNS-triple is constructed on the Hilbert space $\mathcal{H}_\Gamma$, which is given by $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. The unitaries U of the set $\mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ for every suitable surface set $\breve{S}$ in $\mathbb{S}$ and the representation Φ_M are given on the same Hilbert space $\mathcal{H}_\Gamma$. Consequently, a state ω^Γ_M exists.

The crucial property is the irreducibility of the set $\Phi_M(C_0(\bar{\mathcal{A}}_\Gamma)) \cup \{U(\bar{G}_{\breve{S},\Gamma})) : U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))\}$. Notice that, $\mathcal{M}' = \Phi_M(C_0(\bar{\mathcal{A}}_\Gamma))' \cap \{U(\bar{G}_{\breve{S},\Gamma})) : U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))\}'$. First notice that $\Phi_M(C_0(\bar{\mathcal{A}}_\Gamma)) \subset \Phi_M(C_0(\bar{\mathcal{A}}_\Gamma))'$. Then it is true that $U^k_{\overleftarrow{L}}(\bar{G}_{\breve{S},\Gamma}) \subset U^{\overleftarrow{R}}_k(\bar{G}_{\breve{S},\Gamma})'$ and $U^{\overleftarrow{R}}_k(\bar{G}_{\breve{S},\Gamma}) \subset U^k_{\overleftarrow{L}}(\bar{G}_{\breve{S},\Gamma})'$ whenever $U^k_{\overleftarrow{L}}, U^{\overleftarrow{R}}_k \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))$ for $1 \leq k \leq N-1$. Clearly, $U^{\overleftarrow{R}}_k(\bar{G}_{\breve{S},\Gamma}) \nsubseteq U^{\overleftarrow{R}}_k(\bar{G}_{\breve{S},\Gamma})'$. Then $U^k_{\overleftarrow{L}}(\bar{G}_{\breve{S},\Gamma})' \cap U^{\overleftarrow{R}}_k(\bar{G}_{\breve{S},\Gamma})' = \{\lambda \cdot \mathbb{1} : \lambda \in \mathbb{R}\}$. Hence $\mathcal{M}'$ is given by $\{\lambda \cdot \mathbb{1} : \lambda \in \mathbb{R}\}$.

■

The proposition 6.1.40 can be reformulated for the non-standard identification of the configuration space $\bar{\mathcal{A}}_\Gamma$ if $\bar{G}_{\breve{S},\Gamma}$ is replaced by $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$. It is also possible to consider $\bar{G}^{\text{A}}_{\breve{S},\Gamma}$ and the actions $\alpha^{\text{A},l}_L, \alpha^{\text{A},l}_R, \alpha^{\text{A},l}_{L,R}, \overleftarrow{\alpha}^{\text{A},l}, \overrightarrow{\alpha}^{\text{A},l}, \overleftarrow{\alpha}^{\text{A},r}, \overrightarrow{\alpha}^{\text{A},r}$ and derived actions of $\bar{G}^{\text{A}}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$.

Corollary 6.1.41. *Let $\mathcal{P}_\Gamma$ be a identified in the natural way finite graph system associated to the graph Γ. Let $\bar{\mathcal{A}}_\Gamma$ be the set of generalised connections, which is identified in the natural way with G^N.*

There is a unique measure on $\bar{\mathcal{A}}_\Gamma$ given by the Haar measure μ_Γ on the product G^N of a locally compact unimodular group G.

If $\bar{G}_{\breve{S},\Gamma}$ is identified with G^N, then there is a unique state ω^Γ_M on $C_0(\bar{\mathcal{A}}_\Gamma)$, which is G^N-invariant and which is given by

$$\begin{aligned}\omega^\Gamma_M(f_\Gamma) &= \int_{G^N} \mathrm{d}\,\mu_N(\mathfrak{h}_\Gamma(\Gamma)) f_\Gamma(\mathfrak{h}_\Gamma(\Gamma)) = \int_{G^N} \mathrm{d}\,\mu_N(\mathfrak{h}_\Gamma(\Gamma)) f_\Gamma(R(\boldsymbol{g})(\mathfrak{h}_\Gamma(\Gamma)))\\ &= \int_{G^N} \mathrm{d}\,\mu_N(\mathfrak{h}_\Gamma(\Gamma)) f_\Gamma(L(\boldsymbol{g})(\mathfrak{h}_\Gamma(\Gamma)))\end{aligned}$$

for all $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\boldsymbol{g} \in G^N$.

The same result can be obtained for the non-standard identification of the finite graph system and the configuration space.

Recall the holonomy maps $\mathfrak{h}_{\Gamma,\Lambda_1}$ correspond directly to path-connections Λ_1. Let $\Lambda_1, \Lambda_2, \Lambda_3$ be three path-connections. Then three holonomy maps along a path γ are defined by $\mathfrak{h}_{\Gamma,\Lambda_1}(\gamma)$, $\mathfrak{h}_{\Gamma,\Lambda_2}(\gamma)$ and $\mathfrak{h}_{\Gamma,\Lambda_3}(\gamma)$. Notice the set $\bar{\mathcal{A}}_\Gamma$ contains all images of holonomy maps $\mathfrak{h}_{\Gamma,\Lambda_i}$ for $i = 1, 2, 3$ along paths in a graph. The set $\bar{\mathcal{A}}_\Gamma$ can be equipped with a multiplication $\mathfrak{h}_{\Gamma,\Lambda_1}(\Gamma') \cdot \mathfrak{h}_{\Gamma,\Lambda_2}(\Gamma') = \mathfrak{h}_{\Gamma,\Lambda_3}(\Gamma')$ for a fixed subgraph Γ' of Γ, which is inherited from the group G. In particular, there exists an inversion such that

$$\mathfrak{h}_{\Gamma,\Lambda}(\Gamma') \cdot \mathfrak{h}_{\Gamma,\Lambda}(\Gamma')^{-1} = (e_G, ..., e_G)$$

Usually the notion $\mathfrak{h}_{\Gamma,\Lambda_i}$ is not used and is for example replaced by $\mathfrak{h}_\Gamma$, $\hat{\mathfrak{h}}_\Gamma$ and $\tilde{\mathfrak{h}}_\Gamma$. Notice that, the multiplication operation is not related to any groupoid morphism and hence there is no map

$$(\mathfrak{h}_{\Gamma,\Lambda_1}(\gamma), \mathfrak{h}_{\Gamma,\Lambda_2}(\gamma')) \mapsto \mathfrak{h}_{\Gamma,\Lambda_1}(\gamma)\mathfrak{h}_{\Gamma,\Lambda_2}(\gamma') \tag{6.67}$$

where $(\gamma, \gamma') \in \mathcal{P}_\Gamma\Sigma^{(2)}$.

Consequently, the following remark can be done.

Remark 6.1.42. *Let $\mathcal{P}_\Gamma$ be a finite graph system associated to the graph Γ identified with a subset of Γ in the natural way. Let $\bar{\mathcal{A}}_\Gamma$ be the space of generalised connections identified in the natural way with $G^{|\Gamma|}$ and G be locally compact unimodular group.*

Let f be a function in the convolution *-algebra $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$, which is given by the algebra $C_c(\bar{\mathcal{A}}_\Gamma)$ of compactly supported functions on $\bar{\mathcal{A}}_\Gamma$ equipped with the convolution as the multiplication operation. For example, for two continuous compactly supported function f, k on $\bar{\mathcal{A}}_\Gamma$ the convolution product is illustrated by

$$(f * k)(\mathfrak{h}_\Gamma(\gamma), \mathfrak{h}_\Gamma(\gamma')) = \int_{\bar{\mathcal{A}}_\Gamma} \mathrm{d}\,\mu_\Gamma(\hat{\mathfrak{h}}_\Gamma(\Gamma')) f(\mathfrak{h}_\Gamma(\gamma)\hat{\mathfrak{h}}_\Gamma(\gamma)^{-1}, \mathfrak{h}_\Gamma(\gamma')\hat{\mathfrak{h}}_\Gamma(\gamma')^{-1}) k(\hat{\mathfrak{h}}_\Gamma(\gamma), \hat{\mathfrak{h}}_\Gamma(\gamma'))$$

for $\Gamma' := \{\gamma,\gamma'\}$*. The involution is given by*

$$f(\mathfrak{h}_\Gamma(\Gamma'))^* = f(\mathfrak{h}_\Gamma(\gamma),\mathfrak{h}_\Gamma(\gamma'))^* := \overline{f(\mathfrak{h}_\Gamma(\gamma)^{-1},\mathfrak{h}_\Gamma(\gamma')^{-1})}$$

Notice that, $\mathfrak{h}_\Gamma(\Gamma)$ *and* $\mathfrak{h}_\Gamma(\Gamma')$ *are elements of* G^N *and hence the convolution* **-algebra* $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ *is identified with* $\mathcal{C}(G^N)$*.*

Then there exists a state on $C_0(\bar{\mathcal{A}}_\Gamma)$ *given by*

$$\omega^\Gamma_{M,f}(f_\Gamma) = \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f(\mathfrak{h}_\Gamma(\Gamma'))$$

where $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ *and* $f \in \mathcal{C}(\bar{\mathcal{A}}_\Gamma)$*. Derive from*

$$\begin{aligned}\omega^\Gamma_{M,f}(\alpha(\boldsymbol{k})(f_\Gamma)) &= \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(R(\boldsymbol{k}^{-1})(\mathfrak{h}_\Gamma(\Gamma')))f(\mathfrak{h}_\Gamma(\Gamma'))\\ &= \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f(R(\boldsymbol{k})(\mathfrak{h}_\Gamma(\Gamma')))\\ &\stackrel{!}{=} \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f(\mathfrak{h}_\Gamma(\Gamma'))\end{aligned}$$

that $\omega^\Gamma_{M,f}$ *is* G^N*-invariant iff*

$$f(R(\boldsymbol{k})(\mathfrak{h}_\Gamma(\Gamma'))) = f(\mathfrak{h}_\Gamma(\Gamma')) \tag{6.68}$$

for any $\boldsymbol{k} \in G^N$*. If* $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ *is identified with* $\mathcal{C}(G^N)$*, then for* $f_{\breve{S}} \in \mathcal{C}(G^N)$ *and* $f_{\breve{S}}(\boldsymbol{g}\boldsymbol{k}^{-1}) = f_{\breve{S}}(\boldsymbol{g})$ *for all* $\boldsymbol{k} \in G^N$ *there is another state on* $C_0(G^N)$ *defined by*

$$\omega^\Gamma_{M,f_{\breve{S}}}(f_\Gamma) = \int_{G^N} \mathrm{d}\,\mu_N(\boldsymbol{g})f_\Gamma(\boldsymbol{g})f_{\breve{S}}(\boldsymbol{g})$$

whenever $f_\Gamma \in C_0(G^N)$*. But both states will be not finite path-and graph-diffeomorphism invariant. This is shown in problem 6.2.5. Clearly, the states* $\omega^\Gamma_{M,f}$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$ *are not invariant under all actions* $\alpha \in \mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$*.*

Notice the function f *can be also an element of* $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$*, but in this case the state* $\omega^\Gamma_{M,f}$ *is obviously not finite graph-diffeomorphism invariant.*

Proof of corollary 6.1.41:
After the natural identication of $\bar{\mathcal{A}}_\Gamma$ with G^N there is a unique Haar measure μ_Γ on $G^{|\Gamma|}$. The dual of $C_0(\bar{\mathcal{A}}_\Gamma)$ is given by the Banach space of all bounded complex Baire measures on $\bar{\mathcal{A}}_\Gamma$. Furthermore there exists an extension of each Baire measure to a regular Borel measure on $\bar{\mathcal{A}}_\Gamma$. The linear space of regular Borel measures equipped with the convolution operation form a Banach *-algebra $\mathbf{M}(G^N)$. The algebra decomposes into norm closed subspaces consisting of measures absoluely continuous with respect to the Haar measure of G^N, continuous measures singular with respect to the Haar measure and discrete measures (refer to [50, chap.: 19]). The Banach *-algebra generated by Dirac point measures is excluded in the following considerations, a closer look on this structure is presented in section 7.1. The subspace $\mathbf{M}_s(G^N)$ of all continuous measures singular with respect to the Haar measure is not a subalgebra of $\mathbf{M}(G^N)$, in general. Consequently, the space $\mathbf{M}_s(G^N)$ is not considered. Notice the space $\mathbf{M}_a(G^N)$ consisting of measures absoluely continuous with respect to the Haar measure of G^N can be identified with $L^1(G^N,\mu_N)$.

The norm-closed subspace of all regular Borel measures on $\bar{\mathcal{A}}_\Gamma$, which are absolutely continuous to the uniquely defined Haar measure μ_Γ is given by $L^1(\bar{\mathcal{A}}_\Gamma,\mu_\Gamma)$. Hence a state on $C_0(\bar{\mathcal{A}}_\Gamma)$ is given by

$$\omega^\Gamma_{M,f}(f_\Gamma) = \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f(\mathfrak{h}_\Gamma(\Gamma'))$$

for all $f \in L^1(\bar{\mathcal{A}}_\Gamma,\mu_\Gamma)$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$. The Banach *-algebra $L^1(\bar{\mathcal{A}}_\Gamma,\mu_\Gamma)$ is the completition of $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ w.r.t. the $\|.\|_1$-norm.

But there is only one state on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\omega^\Gamma_M(f_\Gamma) = \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))$$

which is invariant under all actions $\alpha \in \operatorname{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ for any arbitrary set $\breve{S}$ of suitable surfaces. The invariance of the state under different actions of $\bar{G}_{\breve{S},\Gamma}$ is derived from the fact that the product Haar measure of the product of the unimodular locally compact group G is left and right invariant.

■

Remark 6.1.43. *Let $\bar{\mathcal{A}}_\Gamma$ be the space of generalised connections identified in the natural way with G^N. Let H be a closed subgroup of the locally compact group G.*

Let f be a function in $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ such that $f(\mathfrak{h}_\Gamma(\Gamma')\boldsymbol{k}^{-1}) = f(\mathfrak{h}_\Gamma(\Gamma'))$ for any $\boldsymbol{k} \in H^N$ and consider the state

$$\omega^\Gamma_{H,f}(f_\Gamma) = \int_{G^N} \mathrm{d}\,\mu_N(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f(\mathfrak{h}_\Gamma(\Gamma'))$$

where $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$. Then this state $\omega^\Gamma_{H,f}$ is H^N-invariant. But this state will be not path- or graph-diffeomorphism invariant in general.

6.2 Dynamical systems of actions of the group of bisections on two C^*-algebras

Actions of the group of bisections of the analytic holonomy algebra for finite graph systems

In this section the new concept for graph changing operations is introduced. On the level of finite path groupoids the bisections of the path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ implement path-diffeomorphisms. A path-diffeomorphism is a pair of maps such that one bijective mapping maps vertices to vertices and the second bijective mapping maps non-trivial paths to non-trivial paths. A fixed set of independent paths is a graph, on the other hand, each path of a graph is an element of a path groupoid. Hence there is a concept of bisections of finite graph systems. This concepts are presented in section 3.3.4.4. The action of a bisection of finite path groupoids changes paths by adding or deleting segments of paths. An action of a bisection of a finite graph system transforms graphs to graphs. Actions of bisections of a finite path groupoid are either right-, left- or inner-translations in the finite path groupoid. Hence actions of bisections of a finite graph system are defined by right-, left- or inner-translations in the finite graph system. For a detail analysis refer to definition 3.3.36 of a right translation in a finite graph system. Recall the lemma 3.3.38, which states that the set $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections form a group w.r.t. a multiplication operation $*_2$ and an inverse $^{-1}$. Furthermore each bisection σ defines a right translation R_σ on a finite graph system $\mathcal{P}_\Gamma$.

In the following investigations the non-standard identification of the configuration space is used, but it is also possible to derive results for the natural identification.

Proposition 6.2.1. *Let $\mathcal{P}_\Gamma$ be a finite graph system associated to a graph Γ and let $C_0(\bar{\mathcal{A}}_\Gamma)$ be the analytic holonomy C^*-algebra associated to Γ.*

There is an action ζ of the group $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections equipped with $*_2$ and an inverse $^{-1}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ defined by

$$(\zeta_\sigma f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma((\mathfrak{h}_\Gamma \circ R_\sigma)(\Gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\Gamma_\sigma))$$

whenever $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and for the subgraphs Γ', Γ_σ of Γ. The inverse action is given by

$$(\zeta^{-1}_\sigma f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma((\mathfrak{h}_\Gamma \circ R_{\sigma^{-1}})(\Gamma')) = f_\Gamma(\mathfrak{h}_\Gamma(\Gamma_{\sigma^{-1}}))$$

whenever $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and for the subgraphs $\Gamma', \Gamma_{\sigma^{-1}}$ of Γ.

This action ζ is point-norm continuous and automorphic.

Hence $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma)$ is a C^*-dynamical system.

Proof : First, it is proved that ζ defines an automorphic action. Moreover for simplicity reasons the arguments are verified for one particular graph. The proof can be simply generalises to arbitraray graphs. Assume that, $\tilde{\Gamma}$

is a graph, $V_{\tilde{\Gamma}} = \{v_1', v_2', v_1, v_2, w_1, w_2\}$ and $\Gamma' := \{\gamma_1\}$ is a subgraph of $\tilde{\Gamma}$. Let σ and σ' be two bisections of $\mathcal{P}_{\tilde{\Gamma}}$ such that $\sigma'(z) = \mathbb{1}_z$ if $z \neq v_1$ and $z \neq w_1$ for $z \in V_{\tilde{\Gamma}}$. Moreover let $v_i' = s(\gamma_i)$, $v_i = t(\gamma_i)$ and $w_i = t(\sigma'(v_i))$ ($v_i = s(\sigma'(v_i))$) for $i = 1, 2$. Then derive

$$\begin{aligned}
&((\zeta_\sigma \circ \zeta_{\sigma'}) f_{\tilde{\Gamma}})(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma)) = f_{\tilde{\Gamma}}(\mathfrak{h}_{\tilde{\Gamma}}((\gamma_1 \circ \sigma'(v_1)) \circ \sigma(w_1)))\\
&= f_{\tilde{\Gamma}}\Big(\mathfrak{h}_{\tilde{\Gamma}}(\gamma_1 \circ (\sigma'(v_1) \circ \sigma(w_1)))\Big) = f_{\tilde{\Gamma}}(\mathfrak{h}_{\tilde{\Gamma}}(\gamma_1 \circ (\sigma *_2 \sigma')(v_1)))\\
&= (\zeta_{\sigma *_2 \sigma'} f_{\tilde{\Gamma}})(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma))
\end{aligned}$$

whenever $f_{\tilde{\Gamma}} \in C_0(\bar{\mathcal{A}}_{\tilde{\Gamma}})$. This shows condition (i) of definition 6.1.4. The conditions condition (ii) and condition (iii) are obvious due to the properties of the C^*-algebra $C_0(\bar{\mathcal{A}}_{\tilde{\Gamma}})$. Clearly this generalises to arbitrary subgraphs Γ of an arbitrary graph $\tilde{\Gamma}$.

The point-norm continuity follows if the map $\zeta : \mathfrak{B}(\mathcal{P}_\Gamma) \ni \sigma \mapsto \zeta_\sigma(f_\Gamma)$ is norm-continuous. Notice that, $\mathfrak{B}(\mathcal{P}_\Gamma)$ is finite, since the number of paths $|\Gamma|$ is finite. Assume that, $\Gamma = \{\gamma\}$ is a subgraph of $\tilde{\Gamma}$. Let $f_{\tilde{\Gamma}} \in C_0(\bar{\mathcal{A}}_{\tilde{\Gamma}})$ then

$$\begin{aligned}
&\lim_{\sigma \to \mathrm{id}} \|\zeta_\sigma(f_{\tilde{\Gamma}}) - f_{\tilde{\Gamma}}\|_{\tilde{\Gamma}} = \lim_{\sigma \to \mathrm{id}} \|f_{\tilde{\Gamma}}(\mathfrak{h}_{\tilde{\Gamma}}(\gamma) \cdot \mathfrak{h}_{\tilde{\Gamma}}(\sigma(v))) - f_{\tilde{\Gamma}}(\mathfrak{h}_{\tilde{\Gamma}}(\gamma))\|_{\tilde{\Gamma}}\\
&= 0
\end{aligned}$$

for a graph $\Gamma = \{\gamma\}$ and $\Gamma_\sigma = \{\gamma \circ \sigma(v)\}$ being subgraphs of $\tilde{\Gamma}$, and if the bisection $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ the equality $\sigma(w) = \mathbb{1}_w$ holds for any $w \in V_{\tilde{\Gamma}} \setminus \{v\}$ where $v = t(\gamma)$. Moreover the map $\mathrm{id}(w) = \mathbb{1}_w$ for all $w \in V_{\tilde{\Gamma}}$, in particular for $v = t(\gamma)$, is the identity bisection. Clearly, one can deduce the properties for arbitrary subgraphs Γ of an arbitrary graph $\tilde{\Gamma}$.

■

Clearly, there are also C^*-dynamical systems $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^L)$ and $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^I)$ for the actions

$$(\zeta_\sigma^L f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma((\mathfrak{h}_\Gamma \circ L_\sigma)(\Gamma'))$$

and

$$(\zeta_\sigma^I f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma((\mathfrak{h}_\Gamma \circ I_\sigma)(\Gamma'))$$

whenever L_σ and I_σ are translations in $\mathcal{P}_\Gamma$. Refer to section 3.3.4.4 for the definition of these objects.

Proposition 6.2.2. *The C^*-dynamical systems $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma)$ and $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^L)$ (or $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma)$ and $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^I)$, or $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^L)$ and $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma^I)$) are exterior equivalent [109, Def.: 2.66].*

Proof : The strictly continuous unitary-valued function $u : \mathfrak{B}(\mathcal{P}_\Gamma) \to C_0(\bar{\mathcal{A}}_\Gamma)$, which satisfies

$$\begin{aligned}
&\zeta_\sigma(f_\Gamma) = u_\sigma \zeta_\sigma^L(f_\Gamma) u_\sigma^*\\
&u_{\sigma * \sigma'} = u_\sigma \zeta_\sigma^L(u_{\sigma'})
\end{aligned}$$

for all $\sigma, \sigma' \in \mathfrak{B}(\mathcal{P}_\Gamma)$ is constructed as follows. Recall the left- and right-translation L_σ and R_σ in $\mathcal{P}_\Gamma$, which is presented in section 3.3.4.4. Then derive that

$$\zeta_\sigma(f_\Gamma) \cdot k_\Gamma = f_\Gamma \circ R_\sigma \cdot k_\Gamma = f_\Gamma \circ L_{\sigma^{-1}} \circ L_\sigma \cdot k_\Gamma \circ R_{\sigma^{-1}} \quad = u_\sigma \zeta_\sigma^L(f_\Gamma) u_\sigma^*$$

whenever $f_\Gamma, k_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $u_\sigma f_\Gamma := f_\Gamma \circ L_{\sigma^{-1}}$, $u_\sigma^* f_\Gamma := f_\Gamma \circ R_{\sigma^{-1}}$ and $\cdot$ is the multiplication in $C_0(\bar{\mathcal{A}}_\Gamma)$, holds. Notice that, $u_\sigma u_\sigma^* = L_{\sigma^{-1}} \circ R_{\sigma^{-1}} = \mathrm{id}$.

■

For every graph-diffeomorphism in $\mathrm{Diff}(\mathcal{P}_\Gamma)$ there exists a bisection $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and either a left-, or right- or inner-translation such that $\Phi_\Gamma = X_\sigma$, where X is equivalent to L, or R or I, and $\varphi_\Gamma = t \circ \sigma$. The set $\mathrm{Diff}(\mathcal{P}_\Gamma)$ does not form a group in general. If one of the actions is fixed, then loosely speaking, the group of graph-diffeomorphisms is the set of graph-diffeomorphism in $\mathrm{Diff}(\mathcal{P}_\Gamma)$, which are defined by a bisection $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and a left- translation such that $\Phi_\Gamma = L_\sigma$ and $\varphi_\Gamma = t \circ \sigma$. Clearly, the left-translation can be replaced by right- or inner-translation.

Note that, the C^*-dynamical systems $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta_\sigma)$ and $(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ are not exterior or equivariantly isomorphic [109, Def.: 2.64].

Assume that, Γ' is a subgraph of Γ. Moroever, let $\sigma \in \mathfrak{B}(\mathcal{P}_{\Gamma'})$. Then notice that there is an injective *-homomorphism $\beta_{\Gamma',\Gamma} : C_0(\bar{\mathcal{A}}_{\Gamma'}) \longrightarrow C_0(\bar{\mathcal{A}}_\Gamma)$ where $\mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma_\sigma} \leq \mathcal{P}_\Gamma$ such that

$$((\beta_{\Gamma',\Gamma} \circ \zeta_\sigma) f_{\Gamma'})(\mathfrak{h}_{\Gamma'}(\Gamma')) = ((\beta_{\Gamma',\Gamma} f_{\Gamma'})(\mathfrak{h}_{\Gamma'}(\Gamma_\sigma)) = f_\Gamma(\mathfrak{h}_\Gamma(\Gamma_\sigma))$$

such that the action α of the flux group $\bar{G}_{S,\Gamma}$ can be defined by

$$(\alpha(\rho_{S,\Gamma}(\Gamma_\sigma)) \circ \beta_{\Gamma',\Gamma} \circ \zeta_\sigma) f_{\Gamma'})(\mathfrak{h}_{\Gamma'}(\Gamma')) = (\alpha(\rho_{S,\Gamma}(\Gamma_\sigma)) f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma_\sigma))$$

whenever $\alpha \in \mathrm{Act}(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$. Hence $\alpha(\rho_{S,\Gamma}(\Gamma_\sigma)) \circ \beta_{\Gamma',\Gamma} \circ \zeta_\sigma : C_0(\bar{\mathcal{A}}_{\Gamma'}) \to C_0(\bar{\mathcal{A}}_\Gamma)$ if Γ' is a subgraph of Γ. Note that, in general there is an action $\tilde{\alpha} \in \mathrm{Act}(\bar{G}_{S,\Gamma'}, C_0(\bar{\mathcal{A}}_{\Gamma'}))$ such that

$$\alpha(\rho_{S,\Gamma}(\Gamma_\sigma)) \circ \beta_{\Gamma',\Gamma} \circ \zeta_\sigma = \beta_{\Gamma',\Gamma} \circ \zeta_\sigma \circ \tilde{\alpha}(\rho_{S,\Gamma'}(\Gamma'))$$

This is studied in the next considerations.

In the following investigation assume that Γ is a subgraph of Γ', $N := |\Gamma|$ and $M := |\Gamma'|$. Then the algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ is a C^*-subalgebra of $C_0(\bar{\mathcal{A}}_{\Gamma'})$. Recall the inductive limit C^*-algebra of the inductive family $\{(C_0(\bar{\mathcal{A}}_\Gamma, \beta_{\Gamma',\Gamma} : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}\}$ of C^*-algebras, which is denoted by $C_0(\bar{\mathcal{A}})$. Now another concept of graph changing actions on $C_0(\bar{\mathcal{A}}_{\Gamma'})$ and hence on $C(\bar{\mathcal{A}})$ is presented.

Example 6.2.1: Assume that, $V_{\Gamma'} \setminus V_\Gamma = \{v\}$. Then for example consider a map $\beta_{\Gamma,\Gamma'}(v) : C_0(\bar{\mathcal{A}}_\Gamma) \longrightarrow C_0(\bar{\mathcal{A}}_{\Gamma'})$ at $v \in V_{\Gamma'} \setminus V_\Gamma$, which is given by

$$(\beta_{\Gamma,\Gamma'}(v) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) := f_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\gamma_1'), \mathfrak{h}_{\Gamma'}(\gamma'), ..., \mathfrak{h}_{\Gamma'}(\gamma_N))$$

if $v = s(\gamma')$, $\Gamma' := \{\gamma_1', \gamma', \gamma_2, ..., \gamma_N\}$, $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_N\}$ and $\gamma_1' \circ \gamma' = \gamma_1$. In this particular example the vertex v is contained in the image of the path γ_1. Notice that, the graph $\{\gamma_1'\}$ is not contained in $\mathcal{P}_\Gamma$.

▪

Definition 6.2.3. *Let $V_{\Gamma'}^v$ be a set of vertices, which contains all target vertices of paths in Γ' such that $\{V_{\Gamma'} \setminus V_\Gamma\} \cup \mathrm{Im}(\Gamma) \neq \varnothing$, i.e.*

$$V_{\Gamma'}^v := \{t(\gamma) \in V_{\Gamma'} \setminus V_\Gamma : \mathrm{Im}(\gamma) \cup \{v\} \neq \varnothing\}.$$

Set $\{v\} := \{V_{\Gamma'} \setminus V_\Gamma\} \cup \mathrm{Im}(\Gamma)$. Denote the set $\{V_{\Gamma'}^v : v \in \{V_{\Gamma'} \setminus V_\Gamma\} \cup \mathrm{Im}(\Gamma)\}$ by $V_{\Gamma'}^{\{v\}}$.

Let $v \in \{V_{\Gamma'} \setminus V_\Gamma\} \cup \mathrm{Im}(\Gamma)$. Then for each path γ in a graph Γ such that $\mathrm{Im}(\gamma) \cup \{v\} \neq \varnothing$, there exists a unique path γ' in Γ' such that $t(\gamma') = t(\gamma)$ and $s(\gamma') = v$. Denote the set of paths in Γ such that $\mathrm{Im}(\gamma) \cup \{v\} \neq \varnothing$ by Γ_v and the set $\{\gamma' \in \mathcal{P}_{\Gamma'}\Sigma : t(\gamma') = t(\gamma) \forall \gamma \in \Gamma_v\}$ by ${\Gamma'}_{v,t}$.

Then there are two bisections $\sigma' : \{v\} \to \mathcal{P}_{\Gamma'}$ and $\sigma'' : V_{\Gamma'}^{\{v\}} \to \mathcal{P}_{\Gamma'}$ such that

$$\begin{aligned}(R_{\sigma'} \circ R_{\sigma''})(\Gamma) = \{&\gamma_1, ..., \gamma_i, \sigma'(v_1), ..., \sigma'(v_k), \gamma_{i+1} \circ (\sigma''(t(\gamma_{i+1})))^{-1}, ..., \gamma_N \circ (\sigma''(t(\gamma_N)))^{-1}\} \\ =: &\Gamma_{ref}\end{aligned}$$

whenever

- *each γ_l for $1 \leq l \leq i$ satisfies $\mathrm{Im}(\gamma) \cup \{v\} = \{\varnothing\}$,*
- *each v_m for $1 \leq m \leq k$ is contained in $\{V_{\Gamma'} \setminus V_\Gamma\} \cup \mathrm{Im}(\Gamma)$ the elements $\sigma'(v_m) = \gamma_m''$ are contained in $\Gamma' \setminus \{\gamma_l\}_{1 \leq l \leq i}$,*
- *each $\sigma''(t(\gamma_w)) = {\gamma'}_w$ for $i + 1 \leq w \leq N$ is contained in ${\Gamma'}_{v,t}$.*

Hence $\Gamma_{\text{ref}} \in \mathcal{P}_{\Gamma'}$ and $R_{\sigma'} \circ R_{\sigma''} = R_{\sigma''} \circ R_{\sigma'}$.

Recall the example 6.2.1. Then notice

$$(\beta_{\Gamma,\Gamma'}(v) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) := f_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\gamma_1 \circ {\gamma'}^{-1}), \mathfrak{h}_{\Gamma'}(\gamma'), ..., \mathfrak{h}_{\Gamma'}(\gamma_N))$$

where $\gamma_1 \circ {\gamma'}_1^{-1} = \gamma'_1$.

Definition 6.2.4. *There is a map $\beta_{\Gamma,\Gamma'}(\{v\}) : C_0(\bar{\mathcal{A}}_\Gamma) \rightarrow C_0(\bar{\mathcal{A}}_{\Gamma'})$, which is called the refinement map, defined by the two bisections $\sigma' : \{v\} \rightarrow \mathcal{P}_{\Gamma'}$ and $\sigma'' : V_{\Gamma'}^{\{v\}} \rightarrow \mathcal{P}_{\Gamma'}$ such that*

$$(\beta_{\Gamma,\Gamma'}(\{v\}) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) := (\zeta_{\sigma'} \circ \zeta_{\sigma''} \circ \beta_{\Gamma,\Gamma'}(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma)) = f_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\Gamma_{ref}))$$

whenever $\beta_{\Gamma,\Gamma'} : C_0(\bar{\mathcal{A}}_\Gamma) \rightarrow C_0(\bar{\mathcal{A}}_{\Gamma'})$ such that $f_\Gamma \mapsto f_{\Gamma'}$ is an injective *-homomorphism and $R_{\sigma'} \circ R_{\sigma''} : \mathcal{P}_{\Gamma'} \rightarrow \mathcal{P}_{\Gamma'}$ and $\Gamma \mapsto (R_{\sigma'} \circ R_{\sigma''})(\Gamma) := \Gamma_{ref}$.

Now recall the actions of the flux group $\bar{G}_{\breve{S},\Gamma}$ for a suitable surface set $\breve{S}$ and a graph Γ. Fix one action and derive the following lemma.

Lemma 6.2.5. *Let $\beta_{\Gamma,\Gamma'}(\{v\}) : C_0(\bar{\mathcal{A}}_\Gamma) \longrightarrow C_0(\bar{\mathcal{A}}_{\Gamma'})$ be the refinement map at $\{v\} = \{V_{\Gamma'} \setminus V_\Gamma\} \cup \text{Im}(\Gamma)$.*

Recall example 6.2.1. Moreover let S be surface such that only the path γ_1 in Γ intersects S in the target vertex and which lies above the surface S. There are no other intersection points of Γ' and S. Hence for example an action of $\bar{G}_{S,\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ presented by

$$(\alpha_{\overrightarrow{R}}(\rho_{S,\Gamma}(\Gamma)) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) = f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)) \tag{6.69}$$

whenever $\rho_{S,\Gamma}(\Gamma) \in \bar{G}_{S,\Gamma}$, then this action and the refinement action commute

$$\alpha_{\overrightarrow{R}}(\rho_{S,\Gamma'}(\Gamma)) \circ \beta_{\Gamma,\Gamma'}(\{v\}) = \beta_{\Gamma,\Gamma'}(\{v\}) \circ \alpha_{\overrightarrow{R}}(\rho_{S,\Gamma}(\Gamma))$$

if $\bar{G}_{S,\Gamma}$ is understand as a subgroup of $\bar{G}_{S,\Gamma'}$.

Proof : This can be easily verified by

$$\begin{aligned}
&(\beta_{\Gamma,\Gamma'}(\{v\})(\alpha_{\overrightarrow{R}}(\rho_{S,\Gamma}(\Gamma) f_\Gamma))(\mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N)) = (\beta_{\Gamma,\Gamma'}(\{v\})(\alpha_{\overrightarrow{R}}(\rho_{S,\Gamma}(\Gamma) f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1)^{-1}, ..., \mathfrak{h}_\Gamma(\gamma_N)) \\
&= f_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\gamma'_1), \mathfrak{h}_{\Gamma'}(\gamma')\rho_S(\gamma')^{-1}, ..., \mathfrak{h}_{\Gamma'}(\gamma_N)) \\
&= (\alpha_{\overrightarrow{R}}(\rho_{S,\Gamma'}(\Gamma')) f_{\Gamma'})(\mathfrak{h}_{\Gamma'}(\gamma'_1), \mathfrak{h}_{\Gamma'}(\gamma'), ..., \mathfrak{h}_{\Gamma'}(\gamma_N))
\end{aligned}$$

since $\rho_S(\gamma_1) = \rho_S(\gamma') \in \bar{\mathbb{G}}_{S,\Gamma'}$.

■

The lemma can be reformulated for arbitrary actions of $\bar{G}_{S,\Gamma}$ associated to suitabel surface sets and a graph Γ on $C_0(\bar{\mathcal{A}}_\Gamma)$ and refinement maps $\beta_{\Gamma,\Gamma'}(\{v\}) : C_0(\bar{\mathcal{A}}_\Gamma) \longrightarrow C_0(\bar{\mathcal{A}}_{\Gamma'})$ at $\{v\} = \{V_{\Gamma'} \setminus V_\Gamma\} \cup \text{Im}(\Gamma)$.

Notice this can be used to find a map between continuous functions on the configuration space, which is identified with G^N in the non-standard way, to continuous functions on configuration space, which is identified in the natural way with G^N.

Groups of surface or surface-orientation-preserving bisections for finite graph systems

Up to now only action of bisections of the holonomies are considered. Therefore, in the next investigations the behavior of an action of bisections on the flux operators are analysed. Clearly, this action is required to preserve the structure of the group-valued quantum flux operators.

Definition 6.2.6. *Let $\breve{S}$ be a set of surfaces, which can be embedded in Σ, and φ be a diffeomorphism, which leave each surface S contained in $\breve{S}$ and a suitable neighborhood around S in Σ invariant. Let M_S be the number of paths of a graph Γ that intersect a surface S in $\breve{S}$.*

Then a path-diffeomorphism $(\varphi_\Gamma, \Phi_\Gamma)$ of $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ such that

- $\Phi_\Gamma(\gamma)\in\mathcal{P}_\Gamma\Sigma$ *for all* $\gamma\in\mathcal{P}_\Gamma\Sigma$, $\varphi_\Gamma(v)\in V_\Gamma$ *for all* $v\in V_\Gamma$ *and* $\varphi|_\Gamma=\varphi_\Gamma$;
- *if* γ *does not intersect* S, *then* $\Phi_\Gamma(\gamma)$ *does not intersect* S *and*
- *the number of generators in* Γ *and the number of all transformed paths* $\{\Phi_\Gamma(\gamma_i)\}_{1\leq i\leq N}$ *that intersect* S *in a source and/or target vertex are constant and given by the constant* M_S

is called ***surface-preserving path-diffeomorphism*** *(for a surface set* $\breve{S}$*).*

Denote the set of surface-preserving path-diffeomorphisms by $\text{Diff}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma)$.

With no doubt the set of surface-preserving graph-diffeomorphisms are defined and are denoted by $\text{Diff}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ associated to Γ.

In particular, a path-diffeomorphism that maps a path γ to $\Phi_\Gamma(\gamma)=\gamma\circ\gamma'$ such that a surface S intersect γ in $t(\gamma)$ is not a surface-preserving path-diffeomorphism. This follows from the fact that the number of generators in Γ of the transformed paths increase by one, since γ' has to be added to the set of transformed paths.

On the other hand, such path-diffeomorphisms can be replaced by the following bisections.

Proposition 6.2.7. *Let* $\breve{S}$ *be a set of surfaces and* Γ *be a graph such that* $\breve{S}\cap\Gamma\subset V_\Gamma$. *Let* φ *be a diffeomorphism in* Σ, *which maps each surface* $S\in\breve{S}$ *to a surface* $S_\sigma\in\breve{S}$.

A ***surface-preserving bisection*** σ ***of a finite path groupoid*** *and a set of surfaces* $\breve{S}$ *is defined by a bisection* $\sigma:V_\Gamma\longrightarrow\mathcal{P}_\Gamma\Sigma$ *in* $\mathcal{P}_\Gamma$ *such that*

- *the map* $\varphi_\Gamma:V_\Gamma\longrightarrow V_\Gamma$, *which is given by* $\varphi_\Gamma=t\circ\sigma$, *is bijective and* $(t\circ\sigma)(v)=v$ *whenever* $v\in S\cap V_\Gamma$ *and each* $S\in\breve{S}$ *yields,*
- *for each path* $\gamma\in\mathcal{P}_\Gamma\Sigma$ *that intersects a surface* S *and, such that* $\gamma\cap S=\{s(\gamma)\}$ *holds, the non-trivial transformed path is presented by* $\gamma\circ\sigma(t(\gamma))$ *for* $S\in\breve{S}$,
- *for each path* $\gamma\in\mathcal{P}_\Gamma\Sigma$ *that intersects a surface* S *and, such that* $\gamma\cap S=\{t(\gamma)\}$ *is satisfied, then* $\sigma(t(\gamma))=\mathbb{1}_{t(\gamma)}$ *and* $\gamma\circ\sigma(t(\gamma))=\gamma$ *yields for a surface* $S\in\breve{S}$,
- *the bisection* $\sigma:V\to\mathcal{P}_\Gamma\Sigma$, *where* $V=V_\Gamma\setminus V_{\breve{S}}$ *and* $V_{\breve{S}}=V_\Gamma\cap\{S_i:S_i\in\breve{S}\}$, *is such that* $(t\circ\sigma)(v)\in V$ *for all* $v\in V$ *holds. Then for each* $\gamma\in\mathcal{P}_\Gamma\Sigma$ *such that* $\gamma\cap S=\{\varnothing\}$, *the transformed path is given by* $\gamma\circ\sigma(t(\gamma))$.

The set of all surface-preserving bisections for a path groupoid form a group equipped with $*$ *and* $^{-1}$ *and is called the* ***group*** $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma)$ ***of surface-preserving bisections of a finite path groupoid****.*

Observe for a certain bisection $\sigma\in\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma_v)$ the right translation $R_{\sigma(v)}(\gamma)$ in $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$ for a $v\in V_\Gamma$ defines a path-diffeomorpism $\Phi_\Gamma(\gamma)$ such that $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma\Sigma_v)$.

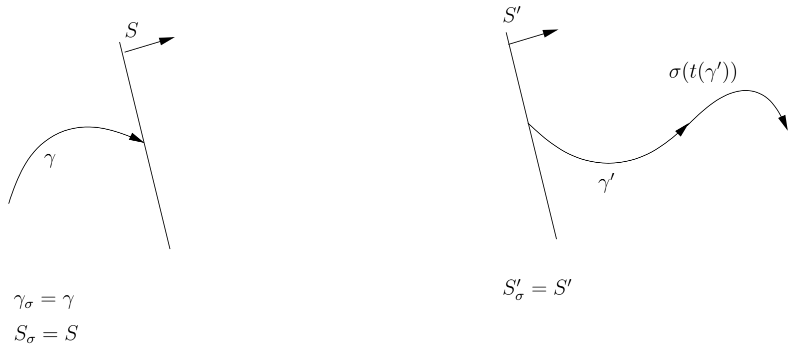
In general it follows that $R_\sigma(\mathbb{1}_v)=\sigma(v)$ for $v\in V$, where $V=V_\Gamma\setminus V_S$ and $V_S=V_\Gamma\cap S$, and $R_\sigma(\mathbb{1}_w)=\mathbb{1}_w$ for $w\in V_S$ for each $S\in\breve{S}$ is satisfied. Since, $\mathbb{1}_v$ and $\mathbb{1}_w$ for $v\in V$ and $w\in V_S$ are elements of $\mathcal{P}_\Gamma\Sigma$.

Proposition 6.2.8. *A* ***surface-preserving bisection*** σ ***of a finite graph system*** *is defined as a bisection* $\sigma:V_\Gamma\longrightarrow\mathcal{P}_\Gamma$ *in* $\mathcal{P}_\Gamma$ *such that there is a surface-preserving bisection* $\tilde{\sigma}$ *of a finite path groupoid* $\mathcal{P}_\Gamma\Sigma\rightrightarrows V_\Gamma$ *and* $\sigma_\Gamma(V)=\{\tilde{\sigma}(v_i):v_i\in V\}$ *whenever* V *is a subset of* V_Γ.

The set surface-preserving bisections for a finite graph system form a group equipped with $*_2$ *and* $^{-1}$ *and is called the* ***group*** $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ ***of surface-preserving bisections of a finite graph system and a surface set*** $\breve{S}$.

A right-translation R_σ in the finite graph system $\mathcal{P}_\Gamma$ is defined for a bisection $\sigma\in\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ in definition 3.3.36.

For example, for a path γ that intersect a surface S in $t(\gamma)$, then $\gamma_\sigma=\gamma\circ\sigma(t(\gamma))=\gamma$. For a path γ' that intersect S' in $s(\gamma')$ the transformed path $\gamma_\sigma=\gamma\circ\sigma(t(\gamma))$. In both cases the surfaces satisfy $S=S_\sigma$ and $S'=S'_\sigma$.



Especially, for a finite surface-preserving graph-diffeomorphism $(\varphi_\Gamma, \Phi_\Gamma)$ this implies that the number of intersections between surfaces in $\breve{S}$ and a set of paths Γ does not change. Certainly, it follows that the surfaces are fixed, whereas the graph Γ is changed to Γ_σ.

Now the question arise, if another dissimilar situation is possible. The idea is to implement bisections of such a way that the orientation of the transformed surface with respect to path obtained by the bisection is preserved, and paths that are ingoing w.r.t. the orientation of the unchanged surface are ingoing w.r.t. the transformed surface. The same is true for paths that are outgoing w.r.t. the orientation of the unchanged surface.

Now consider a diffeomorphism which maps a surface S into another S_σ. Then the modified path-diffeomorphisms are maps such that paths γ, which intersect a surface S and are ingoing w.r.t. a surface S, to paths $\Phi_\Gamma(\gamma)$ that are ingoing w.r.t the transformed surface S_σ. It is possible that the path γ lies above the surface S, whereas $\Phi_\Gamma(\gamma)$ lies below the surface $\varphi(S)$. The new diffeomorphism is required to preserve the orientation properties of the paths and the surfaces such that there is no interchange of the left and right unitary representation of $\bar{G}_{\breve{S},\Gamma}$ on $\mathcal{H}_\Gamma$.

Lemma 6.2.9. *Consider a surface S and a graph Γ consisting of one path γ that intersects S only in the target vertex $v = t(\gamma)$. Then consider a bisection such that $\sigma(t(\gamma)) = \gamma'$ and hence $\varphi_\Gamma(v) = (t \circ \sigma)(v) = t(\gamma')$, $v = S \cap \gamma$ and therefore $\varphi(S) \cap \gamma \circ \gamma' = t(\gamma')$. Assume that, $\sigma(v) \neq \gamma^{-1}$. Then the number of non-trivial intersection points (i.e. a trivial intersection is an intersection of a surface and a trivial loop $\mathbb{1}_v$ in v) does not change.*

In general for each path γ in $\mathcal{P}_\Gamma\Sigma$ such that $t(\gamma) \cap S = t(\gamma)$ a transformed path γ_σ is considered. There is always a path such that $\sigma(t(\gamma')) = {\gamma'}^{-1}$, since σ is bijective map from V_Γ to $\mathcal{P}_\Gamma\Sigma$. Hence the number of non-trivial intersection points of paths in the path groupoid and the surface S is not constant.

Definition 6.2.10. *Let $\breve{S}$ be a set of oriented surfaces, which is embedded in Σ, and φ be a diffeomorphism, which maps each surface S contained in $\breve{S}$ into another surface $S_\sigma \in \breve{S}$.*

Then $(\varphi_\Gamma, \Phi_\Gamma)$ let be a finite path-diffeomorphism on $\mathcal{P}_\Gamma\Sigma \rightrightarrows V_\Gamma$ such that $\Phi_\Gamma(\gamma) \in \mathcal{P}_\Gamma\Sigma$ for all $\gamma \in \mathcal{P}_\Gamma\Sigma$, $\varphi_\Gamma(v) \in V_\Gamma$ for all $v \in V_\Gamma$ and $\varphi|_\Gamma = \varphi_\Gamma$ and if γ does not intersect S, then $\Phi_\Gamma(\gamma)$ does not intersect S_σ, too. Moreover for paths that intersect a surface S and lie above and ingoing (or below and ingoing, or above and outgoing, or below and outgoing) then the transformed path is non-trivial and has the same property w.r.t S_σ.

*A path-diffeomorphism $(\varphi_\Gamma, \Phi_\Gamma)$ with this property is called **surface-orientation-preserving path- diffeomorphism**. Denote the set of surface-orientation-preserving path-diffeomorphisms by $\mathrm{Diff}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma\Sigma)$.*

With no doubt the set of surface-orientation-preserving graph-diffeomorphisms is defined and is denoted by $\mathrm{Diff}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma)$ associated to Γ.

Definition 6.2.11. *Let $\breve{S}$ be a set of surfaces and Γ be a graph such that $\breve{S} \cap \Gamma \subset V_\Gamma$. Let φ be a diffeomorphism in Σ, which maps each surface $S \in \breve{S}$ to a surface $S_\sigma \in \breve{S}$.*

*A map σ is called **surface-orientation-preserving bisection for a finite path groupoid**, if σ is a bisection in $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ such that*

- *the map $\varphi_\Gamma : V_\Gamma \longrightarrow V_\Gamma$, which is given by $\varphi_\Gamma = t \circ \sigma$, is bijective and $\varphi_\Gamma = \varphi|_{V_\Gamma}$, and*
- *for each $\gamma \in \mathcal{P}_\Gamma\Sigma$ that intersects a surface S and, such that $\gamma \cap S = \{t(\gamma)\}$ holds, the non-trivial transformed path is given by $\gamma \circ \sigma(t(\gamma)) =: \gamma_\sigma$ for a surface $S \in \breve{S}$ yields. Moreover if γ lie above (or below) the surface S and γ_σ is non-trivial, then γ_σ lie above (or below) the surface S_σ. Except of a vertex $s(\gamma)$ such that $s(\gamma) \cap S_\sigma = \{s(\gamma)\}$, the vertex $t(\gamma_\sigma)$ is the only intersection vertex of S_σ and γ_σ.*

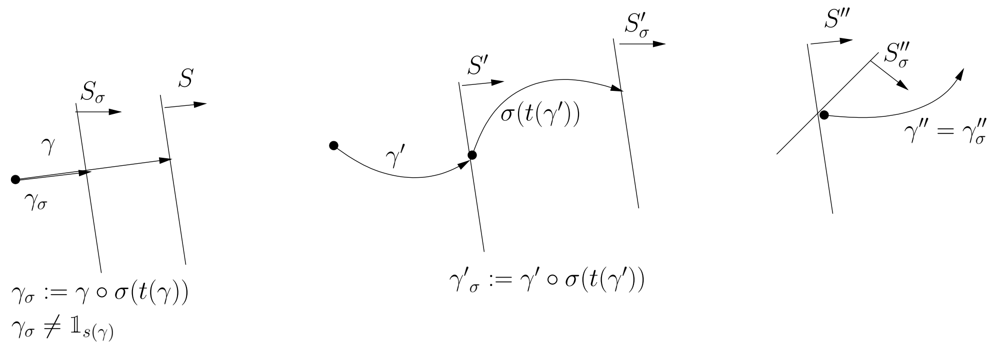
- *For each $\gamma \in \mathcal{P}_\Gamma\Sigma$ that intersects a surface S, and such that $\gamma \cap S = \{s(\gamma)\}$, it is true that $\sigma(s(\gamma)) = \mathbb{1}_{s(\gamma)}$, $(t \circ \sigma)(s(\gamma)) = s(\gamma)$ and hence $\gamma \circ \sigma(s(\gamma)) = \gamma \circ \mathbb{1}_{s(\gamma)} = \gamma$ for a surface $S \in \breve{S}$. Furthermore if γ is located above S, then γ is located above the surface S_σ.*
- *The map $\sigma : V \rightarrow \mathcal{P}_\Gamma\Sigma$, where $V = V_\Gamma \setminus V_{\breve{S}}$ and $V_{\breve{S}} = V_\Gamma \cap \{S_i : S_i \in \breve{S}\}$, is such that $(t \circ \sigma)(v) \in V$ for all $v \in V$ yields. Then for each $\gamma \in \mathcal{P}_\Gamma\Sigma$ such that $\gamma \cap S = \{\varnothing\}$, the transformed path is given by $\gamma \circ \sigma(t(\gamma))$.*

Clearly, this concept can be generalised to surface-orientation-preserving bisections of a finite graph system.

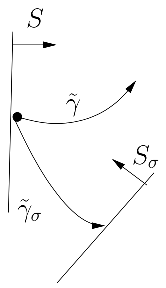
Corollary 6.2.12. *The set $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma\Sigma)$ of all surface-orientation-preserving bisections of a finite path groupoid equipped with multiplication $*$ and inversion $^{-1}$ form a group and it is called the* ***group of surface-orientation-preserving bisections of a finite path groupoid associated to surfaces****.*

The set $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ of all surface-orientation-preserving bisections of a finite graph system equipped with multiplication $*_2$ and inversion $^{-1}$ form a group and is called the ***group $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ of surface-orientation-preserving bisections associated to graphs and surfaces****.*

For example, consider a graph $\Gamma = \{\gamma, \gamma', \gamma''\}$ and the surfaces $\breve{S} := \{S, S', S''\}$, which are presented in the picure below. Then let $(\varphi_\Gamma, \Phi_\Gamma)$ be a surface-orientation-preserving path-diffeomorphism for $\breve{S}$ defined by a bisection σ such that S is mapped to S_σ and so on. Let the path γ intersect the surface S in $t(\gamma)$ such that γ lies below S and γ_σ intersect S_σ in $t(\gamma_\sigma)$ such that γ_σ lies below S_σ. Moreover let γ' intersect S' in $t(\gamma')$ such that γ' lies below S' and ${\gamma'}_\sigma := \gamma' \circ \sigma(t(\gamma'))$ intersect S'_σ in $t({\gamma'}_\sigma)$ and ${\gamma'}_\sigma$ lies below S'_σ. Finally, for a path γ'' intersecting S'' in $s(\gamma'')$ and the path γ'' lies above, then the transformed path γ''_σ is equivalent to γ'' and γ'' is outgoing and above the surface S''_σ. Summarising, there is a map $\Phi_\Gamma(\Gamma) = \{\gamma_\sigma, {\gamma'}_\sigma, \gamma''_\sigma\}$.



There is a problem if the bisection σ maps $s(\tilde{\gamma})$ to a path $\tilde{\gamma}_\sigma$, which is not equivalent to $\tilde{\gamma}$. Since the resulting path $\tilde{\gamma}_\sigma$ is ingoing and lies above the surface S_σ whereas $\tilde{\gamma}$ is outgoing and above S.



The situation can be restricted to the case that the surface set $\breve{S}$ is chosen such that each path γ_i in $\mathcal{P}_\Gamma\Sigma$, which intersects only the surfaces S_j, which is contained in $\breve{S} := \{S_j\}_{1 \leq j \leq K}$, and lie above and ingoing, (or below and ingoing, or above and outgoing, or below and outgoing) w.r.t. S_j and there are no other intersection vertices of this path with any other surface S_k for $k \neq j$. Then it is required that the path $\gamma_i \circ \sigma(t(\gamma_i))$ (or $\sigma(s(\gamma_i))$) lie above and ingoing (or below and ingoing, or above and outgoing, or below and outgoing) w.r.t. each $\varphi(S_j)$. Hence all actions of these bisections preserve the quantm flux operators associated to different surface sets and graphs presented in section3.4 can be treated.

Action of the group of surface-preserving bisections of the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ of Weyl elements

The next question is related to different actions of group of bisections of the algebra of Weyl elements. First consider the action of a surface-preserving group $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ of bisections of a finite graph system on the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$.

Lemma 6.2.13. *The action is trivial, i.e. for* $\Gamma' \in \mathcal{P}_\Gamma$

$$(\zeta_\sigma U)(\rho_{S,\Gamma}(\Gamma')) = U(\rho_{S,\Gamma}(\Gamma'_\sigma)) = U(\rho_{S,\Gamma}(\Gamma'))$$

for $\rho_{S,\Gamma} \in G_{\breve{S},\Gamma}$.

Proof : This is true, since $\rho_S(\gamma) = e_G$ holds if the path $\gamma \in \Gamma'$ does not intersect with a surface S in $\breve{S}$ in a source or target vertex. For a subgraph Γ' of $\Gamma = \{\gamma_1, ..., \gamma_N\}$, where each path γ_i in Γ intersect a surface in $\breve{S}$ in a vertex, the action is given by $(\zeta_\sigma U)(\rho_{S,\Gamma}(\Gamma')) = U(\tilde{R}_\sigma(\rho_{S,\Gamma}(\Gamma')))$ where $\tilde{R}_\sigma(\rho_{S,\Gamma}(\Gamma')) = \rho_{\varphi(S),\Gamma}(R_\sigma(\Gamma')) = \rho_{\varphi(S),\Gamma}(\Gamma'_\sigma)$ for all $S \in \breve{S}$ and $\varphi(S) = S$, $(t \circ \sigma)(v) = v$ and hence $\sigma(v) = \mathbb{1}_v$ for all $v \in V_\Gamma \cap \breve{S}$, then finally deduce $\Gamma'_\sigma = \Gamma'$.

■

Action of the group of surface-orientation-preserving bisections of the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ of Weyl elements

Proposition 6.2.14. *Let $\breve{S}$ be a finite set of surfaces.*

The action of a surface-orientation-preserving group $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ of bisections of a finite graph system on the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ is presented by

$$(\zeta_\sigma U)(\rho_{S,\Gamma}(\Gamma')) = U(\rho_{\varphi(S),\Gamma}(R_\sigma(\Gamma'))) \text{ for all } U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$$

satisfies the following properties.

(i) The action ζ of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ in $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$, which is a C^*-subalgebra of $\mathcal{L}(\mathcal{H}_\Gamma)$ is automorphic,

(ii) The action ζ of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ in $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ is point-norm continuous.

(iii) The automorphic action α the group of bisection $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ is inner such that there exists an unitary representation V of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$, i.e. $V \in \text{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}))$, which satisfy

$$V_\sigma U(\rho_{S,\Gamma}(\Gamma'))V_\sigma^* = \big(\zeta_\sigma(U)\big)(\rho_{S,\Gamma}(\Gamma')) \quad \forall U \in \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \sigma \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma) \tag{6.70}$$

For each $\sigma \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ the unitary V_σ are called the ***unitary bisections of a finite graph system and surfaces in*** *$\breve{S}$.*

For example, for an appropriate surface set $\breve{S}$, a graph $\Gamma = \{\gamma, \gamma_1, ..., \gamma_M\}$ and a subgraph $\Gamma' = \{\gamma\}$ it is true that

$$(\zeta_\sigma U_{\overrightarrow{L}})(\rho_S(\gamma)) = U_{\overrightarrow{L}}(\rho_{\varphi(S)}(\gamma \circ \sigma(v))) \tag{6.71}$$

whenever $v = t(\gamma)$ and $\Gamma'_\sigma := \{\gamma \circ \sigma(v)\}$.

Certainly, there is also an action of the group of bisection $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ given by a left translation or a inner automorphism of the path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ. Each action is inner, and hence for each action there is a unitary representation of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$.

Proof : For the proof of the action being automorphic (property (i)) derive the following. Set $\varphi|_\Gamma = \varphi_\Gamma = t\circ\sigma$ and $\varphi'|_\Gamma = \varphi'_\Gamma = t\circ\sigma'$ for two bisections $\sigma,\sigma'\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$. For simplicity assume that $\Gamma'=\{\gamma\}$ such that $\rho_{S,\Gamma}(\Gamma')=\rho_S(\gamma)$ and $v=t(\gamma)$, $w=t(\sigma'(v))$, then derive

$$\begin{aligned}&((\zeta_\sigma\circ\zeta_{\sigma'})U)(\rho_S(\gamma))\\ &=U\Big(\rho_{(\varphi\circ\varphi')(S)}\big(\big(\gamma\circ\sigma'(v)\big)\circ\sigma(w)\big)\Big)=U\Big(\rho_{(\varphi\circ\varphi')(S)}(\gamma\circ(\sigma'(v)\circ\sigma(w)))\Big)\\ &=(\zeta_{\sigma*\sigma'}U)(\rho_S(\gamma))\end{aligned}$$

which proves condition (i) of definition 6.1.4.

Condition (iii) of definition 6.1.4 can be shown by the observation that

$$(\zeta_\sigma U^*)(\rho_S(\gamma))=U(\rho^{-1}_{\varphi(S)}(\gamma\circ\sigma(v))=(\zeta_\sigma U)^*(\rho_S(\gamma))$$

where $v=t(\gamma)$ and $\varphi(S)^{-1}=\varphi(S^{-1})$.

The action satisfies property (ii) of the proposition, since

$$\begin{aligned}&\lim_{\sigma\to\text{id}}\|\zeta_\sigma(U(\rho_S(\Gamma')))-U(\rho_S(\Gamma'))\|_\Gamma=\lim_{\sigma\to\text{id}}\|U(\rho_{\varphi(S)}({\Gamma'}_\sigma))-U(\rho_S(\Gamma'))\|_\Gamma\\ &=0\end{aligned}$$

for a graph $\Gamma'=\{\gamma\}$ and $\Gamma'_\sigma=\{\gamma\circ\sigma(v)\}$ being subgraphs of Γ, $v=t(\gamma)$, $\sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ and where $\text{id}(v)=v$ for $v=t(\gamma)$ is the identity bisection.

The action is indeed inner (property (iii) of the proposition), since there is a unitary representation $V:\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)\longrightarrow M(\mathsf{W}(\bar{G}_{\breve{S},\Gamma}))$, where $M(\mathsf{W}(\bar{G}_{\breve{S},\Gamma}))$ is the multiplier algebra of $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$, such that

$$(\zeta_\sigma U)(\rho_S(\gamma))=V_\sigma U(\rho_S(\gamma))V^*_\sigma$$

Observe that, $\rho_S(\mathbb{1}_v)=e_G$ for any $v\in V_\Gamma$. Set $(V_\sigma U)(\rho_{S,\Gamma}(\Gamma'))=U(\rho_{S,\Gamma}({\Gamma'}_\sigma))$ and $(V^*_\sigma U)(\rho_{S,\Gamma}(\Gamma'))=U(\rho_{S,\Gamma}({\Gamma'}_{\sigma^{-1}}))$, in example

$$\begin{aligned}&(V_\sigma U)(\rho_S(\gamma))=U(\rho_S(\gamma\circ\sigma(v))\\ &(V^*_\sigma U)(\rho_S(\mathbb{1}_w))=(V_{\sigma^{-1}}U)(\rho_S(\mathbb{1}_w))=U(\rho_{S,\Gamma}(\mathbb{1}_w\circ(\sigma((t\circ\sigma)^{-1}(v)))^{-1}))\\ &V_\sigma V_{\sigma^{-1}}=\text{id}=V_{\sigma^{-1}}V_\sigma\\ &(V_\sigma V_{\sigma^{-1}}U)(\rho_S(\gamma))=U(\rho_S(\gamma\circ(\sigma*\sigma^{-1})(v))=U(\rho_S(\gamma))\\ &(V_{\sigma^{-1}}V_\sigma U)(\rho_S(\mathbb{1}_w))=U(\rho_S(\mathbb{1}_w\circ(\sigma^{-1}*\sigma)(v))=U(\rho_S(\mathbb{1}_w))\end{aligned}$$

whenever $w=(t\circ\sigma)^{-1}(v)$, $v=t(\gamma)$, $\Gamma'=\{\gamma,\mathbb{1}_w,\gamma_1,...,\gamma_M\}$, $\Gamma'_\sigma:=\{\gamma\circ\sigma(v),\mathbb{1}_w,\gamma_1,...,\gamma_M\}$ and ${\Gamma'}_{\sigma^{-1}}:=\{\gamma,\mathbb{1}_w\circ\sigma^{-1}(v),\gamma_1,...,\gamma_M\}$.

To show that the *-representation V of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on the Hilbert space $\mathcal{H}_\Gamma$ is really unitary consider the following example. Let γ_1 and γ_2 be two disjoint paths defining two disjoint graphs, then derive

$$\begin{aligned}&(V_\sigma V^*_\sigma\psi_{\gamma_1})(\mathfrak{h}_{\gamma_1}(\gamma_1))=\psi_{\gamma_1}(\mathfrak{h}_{\gamma_1}(\gamma_1))\\ &(V^*_\sigma V_\sigma\psi_{\gamma_2})(\mathfrak{h}_{\gamma_2}(\gamma_2))=\psi_{\gamma_2}(\mathfrak{h}_{\gamma_2}(\gamma_2))\end{aligned}\tag{6.72}$$

and, consequently,

$$\begin{aligned}&(V_\sigma V^*_\sigma)\Big(\psi_{\gamma_1}(\mathfrak{h}_{\gamma_1}(\gamma_1)),\psi_{\gamma_2}(\mathfrak{h}_{\gamma_2}(\gamma_2))\Big)=\Big(\psi_{\gamma_1}(\mathfrak{h}_{\gamma_1}(\gamma_1)),\psi_{\gamma_2}(\mathfrak{h}_{\gamma_2}(\gamma_2))\Big)\\ &=(V^*_\sigma V_\sigma)\Big(\psi_{\gamma_1}(\mathfrak{h}_{\gamma_1}(\gamma_1)),\psi_{\gamma_2}(\mathfrak{h}_{\gamma_2}(\gamma_2))\Big)\end{aligned}$$

for $v=t(\gamma_2)$, $t(\gamma_1)=(t\circ\sigma)^{-1}(v)$, $\psi_{\gamma_1}(\mathfrak{h}_{\gamma_1}(\gamma_1))\in\mathcal{H}_{\gamma_1}$, $\psi_{\gamma_2}(\mathfrak{h}_{\gamma_2}(\gamma_2))\in\mathcal{H}_{\gamma_2}$ where $\Gamma=\{\gamma_1,\gamma_2\}$ and $\mathcal{H}_\Gamma=\mathcal{H}_{\gamma_1}\otimes\mathcal{H}_{\gamma_2}$.

Then collect the following facts

(i) V_σ is unitary for any $\sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$

(ii) $V_\sigma V_{\sigma'} = V(\sigma * \sigma')$ for all $\sigma, \sigma' \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$

(iii) V_σ is point-norm continuous, since the associated action ζ_σ is.

to conclude that V is a unitary representation of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ in $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$.

■

Dynamical systems of an action of the group of surface-preserving bisections of the C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ of Weyl elements and states on $C_0(\bar{\mathcal{A}}_\Gamma)$

Then the last proposition imply the following.

Proposition 6.2.15. *Let $\breve{S}$ be a set of surfaces.*

The triple $(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \zeta)$ of a surface-orientation-preserving group $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ of bisections, a C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ w.r.t. a set $\breve{S}$ of surfaces and a graph Γ and the action ζ is a C^*-dynamical system.

Lemma 6.2.16. *Let $\Phi : \mathsf{W}(\bar{G}_{\breve{S},\Gamma}) \longrightarrow \mathcal{L}(\mathcal{H}_\Gamma)$ be the natural *-homomorphism.*

Then the set $\{\Phi(W)B : W \in \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), B \in \mathcal{L}(\mathcal{H}_\Gamma)\}$ is dense in $\mathcal{L}(\mathcal{H}_\Gamma)$. Consequently, Φ is a morphism of C^*-algebras $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ and $\mathcal{L}(\mathcal{H}_\Gamma)$.

Hence it is obvious to consider the following covariant pair.

Proposition 6.2.17. *The pair (Φ, V) consisting of a morphism $\Phi \in \text{Mor}(\mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \mathcal{L}(\mathcal{H}_\Gamma))$ and a unitary representation V of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on the Hilbert space $\mathcal{H}_\Gamma$, i.e. $V_\sigma \in \text{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathcal{K}(\mathcal{H}_\Gamma))$ such that*

$$\Psi(\zeta_\sigma W) = V_\sigma \Psi(W) V_\sigma^*$$

whenever $W \in \mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ and $\sigma \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$, is a covariant representation of the C^*-dynamical system $(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \zeta_\sigma)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$.

Proof : Conclude for a subgraph $\Gamma' = \{\gamma\}$ and $\Gamma'_\sigma = \{\gamma \circ \sigma(v)\}$ of Γ, $v = t(\gamma)$, a surface S such that γ is outgoing and lies below and $\psi_\Gamma \in \mathcal{H}_\Gamma$ that

$$\begin{aligned}&(V_\sigma U(\rho_{S,\Gamma}(\Gamma'))V_\sigma^* \psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) = (V_\sigma U(\rho_{S,\Gamma}(\Gamma'))\psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) = (V_\sigma \psi_\Gamma)(\rho_S(\gamma)\mathfrak{h}_\Gamma(\gamma)) \\ &= U(\rho_S(\gamma \circ \sigma(v)))\psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'_\sigma)) = (\zeta_\sigma(U(\rho_{S,\Gamma}(\Gamma')))\psi_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))\end{aligned}$$

holds for $U \in \mathsf{W}(\bar{G}_{\breve{S},\Gamma})$. Hence the proposition is true.

■

On the other hand, covariant pairs can be constructed from the multiplication representation Φ_M of the holonomy algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ for a finite graph system $\mathcal{P}_\Gamma$. The next proposition is valid for both identifications of the configuration space $\bar{\mathcal{A}}_\Gamma$.

Proposition 6.2.18. *Let Φ_M be the multiplication represention of $C_0(\bar{\mathcal{A}}_\Gamma)$ on $\mathcal{H}_\Gamma$.*

Then there is an unitary representation V_σ of $\mathfrak{B}(\mathcal{P}_\Gamma)$ on $\mathcal{H}_\Gamma$ such that

$$V_\sigma \Phi_M(f_\Gamma) V_\sigma^* = \Phi_M(\zeta_\sigma f_\Gamma) \tag{6.73}$$

whenever $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and (V, Φ_M) is a covariant pair of the C^*-dynamical system $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta)$.

Then there is a $\mathfrak{B}(\mathcal{P}_\Gamma)$-invariant state $\omega^\Gamma_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ such that

$$\omega^\Gamma_{\mathfrak{B}}(\zeta_\sigma(f_\Gamma)) = \omega^\Gamma_{\mathfrak{B}}(f_\Gamma) := \langle \Omega^\Gamma_{\mathfrak{B}}, \Phi_M(f_\Gamma)\Omega^\Gamma_{\mathfrak{B}} \rangle$$

where $\Omega^\Gamma_{\mathfrak{B}}$ is a cyclic vector in $\mathcal{H}_\Gamma$ for the GNS-triple $(\mathcal{H}_\Gamma, \Phi_M, \Omega^\Gamma_{\mathfrak{B}})$.

Notice that, the state $\omega^\Gamma_{\mathfrak{B}}$ is also a $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-and $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$-invariant state on $C_0(\bar{\mathcal{A}}_\Gamma)$. Hence

$$V_\sigma\Omega^\Gamma_{\mathfrak{B}} = \Omega^\Gamma_{\mathfrak{B}} \text{ for all } \sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$$

Recall the $\bar{G}_{\breve{S},\Gamma}$-invariant state ω^Γ_M of $C_0(\bar{\mathcal{A}}_\Gamma)$ defined in proposition 6.1.40.

Problem 6.2.5: For a path-diffeomorphism $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma)$ such that for a graph $\Gamma := \{\gamma_1',\gamma_1'',...,\gamma_M',\gamma_M'',\gamma_1''',...,\gamma_M'''\}$ where $|\Gamma| := N = 3M$ and $\Phi_\Gamma(\Gamma') = (\gamma_1'\circ\gamma_1'',...,\gamma_M'\circ\gamma_M'')$ the natural identification can be used, i.e.

$$\Phi_\Gamma(\Gamma') = (\gamma_1',\gamma_1'',...,\gamma_M',\gamma_M'') \tag{6.74}$$

where $\Gamma' := (\gamma_1',...,\gamma_M')$ and $\Gamma'' := (\gamma_1'',...,\gamma_M'')$ are subgraphs of Γ, then

$$\begin{aligned}
&\omega^\Gamma_M(\zeta_{(\varphi_\Gamma,\Phi_\Gamma)}(f_\Gamma)) = \int_{G^{2M}} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))f_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))\\
&= \int_{G^{2M}} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1'),\mathfrak{h}_\Gamma(\gamma_1''),...,\mathfrak{h}_\Gamma(\gamma_M'),\mathfrak{h}_\Gamma(\gamma_M''))\\
&= \int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))\int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma''))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1'),\mathfrak{h}_\Gamma(\gamma_1''),...,\mathfrak{h}_\Gamma(\gamma_M'),\mathfrak{h}_\Gamma(\gamma_M''))\\
&= W\int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1'),...,\mathfrak{h}_\Gamma(\gamma_M'))\\
&\neq \omega^\Gamma_M(f_\Gamma)
\end{aligned}$$

where W is a suitable constant. Clearly, the state ω^Γ_M is not graph-diffeomorphism invariant. Notice that, the map

$$(\mathfrak{h}_\Gamma(\gamma),\mathfrak{h}_\Gamma(\gamma'))\mapsto\mathfrak{h}_\Gamma(\gamma)\mathfrak{h}_\Gamma(\gamma')$$

whenever $(\gamma,\gamma')\in\mathcal{P}_\Gamma\Sigma^{(2)}$ is distinguished from the map defined in (6.67).

On the other hand, if instead of the natural identification (6.74) the non-standard identification is taken into account, then this state is graph-diffeomorphism invariant, since for a graph-diffeomorphism $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma)$ such that for $\Gamma := \{\gamma_1',\gamma_1'',...,\gamma_M',\gamma_M'',\gamma_1''',...,\gamma_M'''\}$ and $N = 3M$,

$$\Phi_\Gamma(\Gamma') = (\gamma_1'\circ\gamma_1'',...,\gamma_M'\circ\gamma_M'')$$

then derive

$$\begin{aligned}
&\omega^\Gamma_M(\zeta_{(\varphi_\Gamma,\Phi_\Gamma)}(f_\Gamma)) = \int_{G^{2M}} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))f_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))\\
&= \int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1')\mathfrak{h}_\Gamma(\gamma_1''),...,\mathfrak{h}_\Gamma(\gamma_M')\mathfrak{h}_\Gamma(\gamma_M''))\\
&= \int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1'),...,\mathfrak{h}_\Gamma(\gamma_M'))\\
&= \omega^\Gamma_M(f_\Gamma)
\end{aligned}$$

Furthermore recall the state $\omega^\Gamma_{M,f}$ defined in remark 6.1.42. Then this state is not path-diffeomorphism invariant, since for a path-diffeomorphism $(\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma)$ such that for $\Gamma := \{\gamma_1',\gamma_1'',...,\gamma_M',\gamma_M'',\gamma_1''',...,\gamma_M'''\}$, $N = 3M$ and

$$\Phi_\Gamma(\Gamma') = (\gamma_1'\circ\gamma_1'',...,\gamma_M'\circ\gamma_M'')$$

and for $f\in\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ such that $f(\mathfrak{h}_\Gamma(\Gamma')\mathbf{k}^{-1}) = f(\mathfrak{h}_\Gamma(\Gamma'))$ for all $\mathbf{k}\in G^N$ holds the state satisfies

$$\begin{aligned}
&\omega^\Gamma_{M,f}(\zeta_{(\varphi_\Gamma,\Phi_\Gamma)}(f_\Gamma))\\
&= \int_{G^N} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma')))f(\mathfrak{h}_\Gamma(\Gamma'))\\
&= \int_{G^M} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma'))f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1')\mathfrak{h}_\Gamma(\gamma_1''),...,\mathfrak{h}_\Gamma(\gamma_M')\mathfrak{h}_\Gamma(\gamma_M''))f(\mathfrak{h}_\Gamma(\gamma_1'),...,\mathfrak{h}_\Gamma(\gamma_M'))\\
&\neq \omega^\Gamma_{M,f}(f_\Gamma)
\end{aligned}$$

Consequently, if the function f satisfies addtionally

$$f(\mathfrak{h}_\Gamma(\Gamma')) = f(\mathfrak{h}_\Gamma(\gamma'_1),...,\mathfrak{h}_\Gamma(\gamma'_M)) = f(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma'))) = f(\mathfrak{h}_\Gamma(\gamma'_1)\mathfrak{h}_\Gamma(\gamma''_1),...,\mathfrak{h}_\Gamma(\gamma'_M)\mathfrak{h}_\Gamma(\gamma''_M)) \qquad (6.75)$$

for all $(\varphi_\Gamma,\Phi_\Gamma) \in \text{Diff}(\mathcal{P}_\Gamma)$ then $\omega^\Gamma_{M,f}$ is $\text{Diff}(\mathcal{P}_\Gamma)$-invariant. But the only function, which satisfies (6.75) for all graph-diffeomorphism for a finite graph groupoid is the constant function.

Restrict the state $\omega^\Gamma_{H,f}$ presented in remark 6.1.43, which is H^N-invariant, to functions in $f \in \mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ such that

$$\begin{aligned}&f(R(\mathbf{k})(\mathfrak{h}_\Gamma(\Gamma'))) = f(\mathfrak{h}_\Gamma(\Gamma')) = f(L(\mathbf{k})(\mathfrak{h}_\Gamma(\Gamma'))) \text{ for all } \mathbf{k}\in H^N \text{ and}\\ &f(\mathfrak{h}_\Gamma(\Gamma')) = f(\mathfrak{h}_\Gamma(\Phi_\Gamma(\Gamma'))) \text{ for all } \Gamma'\in\mathcal{P}_\Gamma \text{ and } (\varphi_\Gamma,\Phi_\Gamma)\in\text{Diff}(\mathcal{P}_\Gamma)\end{aligned}$$

Then the state $\omega^\Gamma_{H,f}$ is H^N- and $\text{Diff}(\mathcal{P}_\Gamma)$-invariant. Observe that, this would give a new state for the holonomy-flux C^*-algebra, but the flux operators would be implemented by maps $H_{\breve{S},\Gamma}$ instead of $G_{\breve{S},\Gamma}$.

▪

Recognize that the elements of $\bar{\mathcal{A}}_\Gamma$ are of the form $\mathfrak{h}_\Gamma(\Gamma')$, where Γ' is a subgraph of Γ. The natural or the non-standard identification is applied to identify $\bar{\mathcal{A}}_\Gamma$ with G^N. Hence there exists, additionally to the actions in $\text{Act}(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$, the actions of $\bar{G}^A_{\breve{S},\Gamma}$ or $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, which are defined in remark 6.1.2.

Furthermore due to the fact that the number of subgraphs of Γ generated by the edges of Γ is finite, there exists a finite set $\mathfrak{B}^\Gamma_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ of certain bisections. Each bisection is a map from the set V_Γ to a distinct subgraph of Γ such that all elements of $\mathcal{P}_\Gamma$ can be construced from this finite set $\mathfrak{B}^\Gamma_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$. Call a set of these certain bisections a **generating system of bisections for a graph** Γ.

Proposition 6.2.19. *Let $\mathfrak{B}^\Gamma_{\breve{S},\text{or}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

Then there is a state $\hat{\omega}^\Gamma_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$\hat{\omega}^\Gamma_{\mathfrak{B}}(f_\Gamma) := \frac{1}{k}\sum_{l=1}^k \omega^\Gamma_M(\zeta_{\sigma_l}(f_\Gamma)) \text{ for } \sigma_l \in \mathfrak{B}^\Gamma_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$$

which is $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$-invariant and where k is the maximal number of subgraphs, which can be generated by all edges and their compositions of the graph Γ.

Consequently, the state $\hat{\omega}^\Gamma_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is $\bar{\mathcal{Z}}_{S,\Gamma}$-, $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$- and $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-invariant, i.o.w. $\hat{\omega}^\Gamma_{\mathfrak{B}}$ is contained in the set $\mathcal{S}^{\mathcal{Z},\text{surf},\text{or}}(C_0(\bar{\mathcal{A}}_\Gamma))$ of all $\bar{\mathcal{Z}}_{S,\Gamma}$-, $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$- and $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-invariant states on $C_0(\bar{\mathcal{A}}_\Gamma)$.

The actions $\alpha \in \text{Act}(\bar{\mathcal{Z}}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ and $\zeta \in \text{Act}(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma))$ commute, i.e.

$$(\alpha(\rho_{S,\Gamma}(\Gamma'))\circ\zeta_\sigma)(f_\Gamma) = (\zeta_\sigma\circ\alpha(\rho_{S,\Gamma}(\Gamma')))(f_\Gamma) \quad \forall f_\Gamma\in\mathfrak{A}_\Gamma \qquad (6.76)$$

The state $\hat{\omega}^\Gamma_{\mathfrak{B}}$ and the actions $\alpha \in \text{Act}(\bar{\mathcal{Z}}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ and $\zeta \in \text{Act}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma))$ satisfy

$$\hat{\omega}^\Gamma_{\mathfrak{B}}\circ\alpha(\rho_{S,\Gamma}(\Gamma'))\circ\zeta_\sigma = \hat{\omega}^\Gamma_{\mathfrak{B}}\circ\zeta_\sigma\circ\alpha(\rho_{S,\Gamma}(\Gamma')) \qquad (6.77)$$

One can also use the flux group $\bar{G}^{\text{A}}_{S,\Gamma}$ constructed from admissible maps instead of $\bar{\mathcal{Z}}_{S,\Gamma}$. In this case, the Weyl algebra is generated by the unitaries corresponding to the covariant representations of the C^*-dynamical systems $(\bar{G}^{\text{A}}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha^{\text{A},l})$ where $\alpha^{\text{A},l}$ are suitable point-norm continuous automorphic actions on $C_0(\bar{\mathcal{A}}_\Gamma)$. This can be done, but there is one more structure, since one has to distinguish all paths intersecting a surface S in ingoing and outgoing, above or below and if paths are composable, then additionally the property of both paths lying both above or below, one above and one below lead to different actions $\alpha^{\text{A},l}$ or $\alpha^{\text{A},r}$. Refer to problem 6.2.0.1 for that issue. Moreover either the group of surface-preserving bisections is concerned or the group of surface-orientation-preserving bisections has to be restricted to those maps that additionally preserve the structure of composable pairs of paths. This is due to the requirement that the actions of a suitable subgroup of the group of surface-orientation-preserving bisections of the Weyl elements associated to admissible maps should preserve supplementary the left or right action of the flux group $\bar{G}^{A}_{S,\Gamma}$ on $\bar{\mathcal{A}}_\Gamma$. Therefore, the set of actions $\mathrm{Act}(\bar{G}^{A}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ are very complicated. Since they depend extremely on the set $\breve{S}$ of surfaces and the graph Γ. Consequently, in the next considerations mostly the actions $\mathrm{Act}(\bar{\mathcal{Z}}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ are used.

Despite the difficulty for action associated to admissible maps, even in the case of actions of the flux group $\bar{G}_{\breve{S},\Gamma}$ on the analytic holonomy C^*-algebra the following problem occurs.

Problem 6.2.6: Let $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ be a subgraph of Γ. Then the following computation for $\rho_{S,\Gamma}(\Gamma') \in \bar{G}_{\breve{S},\Gamma}$

$$\begin{aligned}
&(\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma'))(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma')) = (\zeta_\sigma f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\rho_S(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_S(\gamma_M))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\sigma(t(\gamma_1)))\rho_{S_\sigma,\Gamma}(\gamma_1 \circ \sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\mathfrak{h}_\Gamma(\sigma(t(\gamma_M)))\rho_{S_\sigma,\Gamma}(\gamma_M \circ \sigma(t(\gamma_M))))\\
&\neq f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_\sigma,\Gamma}(\gamma_1)\mathfrak{h}_\Gamma(\sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_{S_\sigma,\Gamma}(\gamma_M)\mathfrak{h}_\Gamma(\sigma(t(\gamma_M))))\\
&= ((\alpha(\rho_{S_\sigma,\Gamma}(\Gamma')) \circ \zeta_\sigma)(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma'))
\end{aligned}$$

where $S_\sigma = S$, $\rho_{S_\sigma,\Gamma}(\gamma_i \circ \sigma(t(\gamma_i))) = \rho_{S_\sigma,\Gamma}(\gamma_i)$ for $i = 1, ..., M$ yields for $\sigma \in \mathfrak{B}_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma)$, $\gamma_i \cap S = \{t(\gamma_i)\}$. Clearly, equality holds for every $\rho_{S,\Gamma}(\Gamma') \in \bar{\mathcal{Z}}_{\breve{S},\Gamma}$. Hence the action of $\bar{G}_{\breve{S},\Gamma}$ and the action of $\mathfrak{B}_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma)$ on the analytic holonomy C^*-algebra do not commute.

In particular, observe that if the natural identification of $\bar{\mathcal{A}}_\Gamma$ is assumed, then there exists no map D such that

$$\begin{aligned}
&(D(\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma'))(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma'))\\
&= (Df_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\mathfrak{h}_\Gamma(\sigma(t(\gamma_1)))\rho_{S_\sigma,\Gamma}(\gamma_1 \circ \sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\mathfrak{h}_\Gamma(\sigma(t(\gamma_M)))\rho_{S_\sigma,\Gamma}(\gamma_M \circ \sigma(t(\gamma_M))))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_\sigma,\Gamma}(\gamma_1), \mathfrak{h}_\Gamma(\sigma(t(\gamma_1)))\rho_{S_\sigma,\Gamma}(\sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_{S_\sigma,\Gamma}(\gamma_M), \mathfrak{h}_\Gamma(\sigma(t(\gamma_M)))\rho_{S_\sigma,\Gamma}(\sigma(t(\gamma_M))))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_\sigma,\Gamma}(\gamma_1), \mathfrak{h}_\Gamma(\sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_{S_\sigma,\Gamma}(\gamma_M), \mathfrak{h}_\Gamma(\sigma(t(\gamma_M))))
\end{aligned}$$

for every element $\rho_{S,\Gamma} \in G_{\breve{S},\Gamma}$, $\rho_{S_\sigma,\Gamma} \in G_{\breve{S},\Gamma}$ and whenever $\sigma(t(\gamma_i))$ and S_σ do not intersect each other for every $i = 1, ..., M$.

But if $\rho_{S,\Gamma} \in \mathcal{Z}_{S,\Gamma}$, then there is a map D such that

$$\begin{aligned}
&(D(\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma')))(f_\Gamma))(\mathfrak{h}_\Gamma(\Gamma'))\\
&= (Df_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_\sigma,\Gamma}(\gamma_1)\mathfrak{h}_\Gamma(\sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_{S_\sigma,\Gamma}(\gamma_M)\mathfrak{h}_\Gamma(\sigma(t(\gamma_M))))\\
&= f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1)\rho_{S_\sigma,\Gamma}(\gamma_1), \mathfrak{h}_\Gamma(\sigma(t(\gamma_1))), ..., \mathfrak{h}_\Gamma(\gamma_M)\rho_{S_\sigma,\Gamma}(\gamma_M), \mathfrak{h}_\Gamma(\sigma(t(\gamma_M))))
\end{aligned}$$

▪

Consequently, the actions of the flux group and the group of bisections can be treated simulaneously only in the case of the commutative flux group $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$.

Proof : First observe that the state $\hat{\omega}^{\Gamma}_{\mathfrak{B}}$ defined in the proposition is well-defined in both cases of a natural or non-standard identication of $\bar{\mathcal{A}}_\Gamma$ and G^N. To conclude that (6.76) yields for an action of $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, investigate the computation

$$\begin{aligned}
\hat{\omega}^{\Gamma}_{\mathfrak{B}}((\zeta_\sigma \circ \alpha(\rho_{S,\Gamma}(\Gamma')))(f_\Gamma)) &= \frac{1}{k}\sum_{l=1}^{k} \omega^{\Gamma}_{M}(\zeta_{\sigma_{l+1}}(\alpha(\rho_{S,\Gamma})(f_\Gamma)))\\
&= \frac{1}{k}\sum_{l=1}^{k} \omega^{\Gamma}_{M}(\alpha(\rho_{S_\sigma,\Gamma}(\Gamma'))(\zeta_{\sigma_{l+1}}(f_\Gamma)))\\
&= \frac{1}{k}\sum_{l=1}^{k} \omega^{\Gamma}_{M}(\zeta_{\sigma_{l+1}}(f_\Gamma))\\
&= \hat{\omega}^{\Gamma}_{\mathfrak{B}}(f_\Gamma) = \hat{\omega}^{\Gamma}_{\mathfrak{B}}((\alpha(\rho_{S_\sigma,\Gamma}(\Gamma')) \circ \zeta_\sigma)(f_\Gamma))
\end{aligned}$$

for $\rho_{S_\sigma,\Gamma}(\Gamma'), \rho_{S,\Gamma}(\Gamma') \in \bar{\mathcal{Z}}_{\breve{S},\Gamma}$, $\sigma \in \mathfrak{B}_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma)$ or $\sigma \in \mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_\Gamma)$.

■

It is possible to construct a state $\breve{\omega}^{\Gamma}_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$, which is $\bar{G}^{A}_{S,\Gamma}$-invariant. But this need more technical details.

An action of the local flux group on the holonomy algebra for finite graph systems

Let $\Gamma' := \{\gamma_1, ..., \gamma_M\}$ be a subgraph of Γ. Recall the maps of the set G^{loc}_Γ presented in definition 3.4.21. For an element $\mathbf{g}_\Gamma \in G^{\text{loc}}_\Gamma$ it is true that, $\mathbf{g}_\Gamma(\Gamma')$ is identified with the element $(g_\Gamma(s(\gamma_1)), ..., g_\Gamma(s(\gamma_M)))$ in $G^{|\Gamma|}$. Notice that, it is not necessary to focus on natural identified graphs in a finite graph system. Then there is an action α_{loc} of $\bar{G}^{\text{loc}}_\Gamma$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ given by

$$(\alpha_{\text{loc}}(\mathbf{g}_\Gamma(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) := f_\Gamma(g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}, ..., g_\Gamma(s(\gamma_M))\mathfrak{h}_\Gamma(\gamma_M)g_\Gamma(t(\gamma_M))^{-1})$$

Consider the example, which is given by a graph $\Gamma := \{\gamma_1, \gamma_2\}$ and asubgraph $\Gamma' := \{\gamma_1 \circ \gamma_2\}$. Then calculate

$$\begin{aligned}(\alpha_{\text{loc}}(\mathbf{g}_\Gamma(\Gamma'))f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma')) &= (D_S\alpha_{\text{loc}}(\mathbf{g}_\Gamma(\Gamma))D_S^{-1}f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1 \circ \gamma_2))\\ &= (D_S\alpha_{\text{loc}}(\mathbf{g}_\Gamma(\Gamma))f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1), \mathfrak{h}_\Gamma(\gamma_2))\\ &= (D_S f_\Gamma)(g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}, g_\Gamma(s(\gamma_2))\mathfrak{h}_\Gamma(\gamma_2)g_\Gamma(t(\gamma_2))^{-1})\\ &= f_\Gamma(g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}g_\Gamma(s(\gamma_2))\mathfrak{h}_\Gamma(\gamma_2)g_\Gamma(t(\gamma_2))^{-1})\\ &= f_\Gamma(g_\Gamma(s(\gamma_1))\mathfrak{h}_\Gamma(\gamma_1 \circ \gamma_2)g_\Gamma(t(\gamma_2))^{-1})\end{aligned}$$

Definition 6.2.20. *The $\bar{G}^{\text{loc}}_\Gamma$-fixed point subalgebra of $C_0(\bar{\mathcal{A}}_\Gamma) =: \mathfrak{A}_\Gamma$ is given by*

$$\mathfrak{A}^{\text{loc}}_\Gamma := \{f_\Gamma \in \mathfrak{A}_\Gamma : \alpha_{\text{loc}}(\boldsymbol{g}_\Gamma(\Gamma'))(f_\Gamma) = f_\Gamma \quad \forall \boldsymbol{g}_\Gamma(\Gamma') \in \bar{G}^{\text{loc}}_\Gamma\}$$

*and this algebra is called the C^***-algebra of gauge invariant holonomies restricted to finite graph systems**.*

Notice that, there is a isomorphism between the C^*-algebras $\mathfrak{A}^{\text{loc}}_\Gamma$ and $C_0(\bar{\mathcal{A}}_\Gamma/\bar{\mathfrak{G}}_\Gamma)$.

6.3 Weyl C^*-algebras associated to surfaces and inductive limits of finite graph systems

Weyl C^*-algebras associated to surfaces and finite graph systems

Definition 6.3.1. *Let Γ be a graph and $\mathcal{P}_\Gamma$ a finite graph system associated to Γ and $\breve{S}$ a surface set. Let $\mathbb{S}$ be the set of all suitable surface sets for Γ.*

The algebra generated by all elements of $C_0(\bar{\mathcal{A}}_\Gamma)$ and $\mathbf{W}(\bar{G}_{\breve{S},\Gamma})$, which satisfy the canonical commutator relations (6.65)*, forms an **abstract Weyl *-algebra** $\mathbb{W}(\breve{S}, \Gamma)$ **for a surface set and a finite graph system** associated to a graph Γ. The algebra generated by all elements of $C_0(\bar{\mathcal{A}}_\Gamma)$ and $\mathbf{W}(\bar{G}_{\mathbb{S},\Gamma})$ forms an **abstract Weyl *-algebra** $\mathbb{W}(\mathbb{S}, \Gamma)$ **for surfaces and a finite graph system** associated to a graph Γ.*

The algebra generated by all elements of $C_0(\bar{\mathcal{A}}_\Gamma)$, $\mathbf{W}(\bar{\mathcal{Z}}_{\mathbb{S},\Gamma})$, $\text{Rep}(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), \mathcal{K}(\mathcal{H}_\Gamma))$ and $\text{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathcal{K}(\mathcal{H}_\Gamma))$ for all $\breve{S} \in \mathbb{S}$, which satisfy the canonical commutator relations (6.65),(6.70) *and* (6.70)*, forms an **abstract commutative Weyl and graph-diffeomorphism *-algebra** $\mathbb{W}_{\text{diff}}(\breve{S}, \Gamma)$ **for surfaces and a finite graph system** associated to a graph Γ.*

Due to the fact that all unitaries U (or V) define a homomorphism of $\bar{G}_{\breve{S},\Gamma}$ (or $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$) into a unitary group of $\mathcal{L}(\mathcal{H}_\Gamma)$, the abstract Weyl *-algebra $\mathbb{W}(\breve{S}, \Gamma)$ can be completed to a C^*-algebra.

Summarising, the Weyl C^*-algebra of Loop Quantum Gravity is generated by continuous functions depending on holonomies along paths and the (strongly) continuous unitary flux operators.

Proposition 6.3.2. *Let $\mathcal{H}_\Gamma$ be the Hilbert space $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ with norm $\|.\|^\Gamma_2$.*

The *-algebra[3] *generated by all elements of $C_0(\bar{\mathcal{A}}_\Gamma)$ and $\mathbf{W}(\bar{G}_{\breve{S},\Gamma})$ for every surface set $\breve{S}$ in $\mathbb{S}$, which satisfy the canonical commutator relations* (6.65)*, completed w.r.t. the $\|.\|^\Gamma_2$-norm is a C^*-algebra. This C^*-algebra is called the **Weyl C^*-algebra for surfaces and a finite graph system**. Denote this C^*-algebra by* $\mathsf{Weyl}(\mathbb{S}, \Gamma)$*.*

[3] modulo the two-sided self-adjoint ideal of the *-algebra defined by $I = \{W : \|W\|_2 = 0\}$

The *-algebra[4] *generated by all elements of* $C_0(\bar{\mathcal{A}}_\Gamma)$ *and* $\mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$ *for every surface set* $\breve{S}$ *in* $\mathbb{S}_{\mathcal{Z}}$*, which satisfy the canonical commutator relations* (6.65)*, completed w.r.t. the* $\|.\|_2^\Gamma$*-norm is a* C^**-algebra. This* C^**-algebra is called the* ***commutative Weyl*** C^****-algebra for surfaces and a finite graph system****. Denote this* C^**-algebra by* $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$.

The *-algebra[5] *generated by all elements of* $C_0(\bar{\mathcal{A}}_\Gamma)$, $\mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$, $\mathrm{Rep}(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma),\mathcal{K}(\mathcal{H}_\Gamma))$ *and* $\mathrm{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma),\mathcal{K}(\mathcal{H}_\Gamma))$ *for every surface set* $\breve{S}$ *in* $\mathbb{S}$*, which satisfy the canonical commutator relations* (6.65),(6.70) *and* (6.73)*, completed w.r.t. the* $\|.\|_2^\Gamma$*-norm is a* C^**-algebra, which is called the* ***commutative Weyl and graph-diffeomorphism*** C^****-algebra for surfaces and a finite graph system****. Denote this* C^**-algebra by* $\mathsf{Weyl}_{\text{diff}}(\mathbb{S},\Gamma)$.

The set $\mathbb{S}$ of surface sets, which is used to define the C^*-algebra $\mathsf{Weyl}(\mathbb{S},\Gamma)$, and the set $\mathbb{S}_{\mathcal{Z}}$, which defines $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$, are distinguished from each other. The set $\mathbb{S}$ contains the set $\mathbb{S}_{\mathcal{Z}}$.

Proposition 6.3.3. *The *-algebra*[6] *generated by all elements of* $C_0(\bar{\mathcal{A}}_\Gamma)$ *and* $\mathbf{W}(G_{\breve{S},\Gamma})$ *for every surface set* $\breve{S}$ *in* $\mathbb{S}$*, which satisfy the canonical commutator relations* (6.65)*, completed w.r.t. the norm*

$$\|W\| := \sup\{\|\pi_r(W)\|_r : \pi_r \text{ a unital }^*\text{-representation of } \mathbb{W}(\breve{S},\Gamma) \text{ on } \mathcal{H}_r\ \forall \breve{S}\in\mathbb{S}\}$$

is a C^**-algebra. This* C^**-algebra is called the* ***universal Weyl*** C^****-algebra for surfaces and a finite graph system*** *and is denoted by* $\mathcal{W}eyl(\mathbb{S},\Gamma)$.

The Weyl algebra for surfaces

Proposition 6.3.4. *Define the action of* $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ *(or* $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$*) on* $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ *by*

$$\zeta_\sigma(U(\rho_{S,\Gamma}(\Gamma'))f_\Gamma) := \big(\zeta_\sigma(U)\big)(\rho_{S,\Gamma}(\Gamma'))\zeta_\sigma(f_\Gamma)$$

whenever $\sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ *and* $U(\rho_{S,\Gamma}(\Gamma')), f_\Gamma\in\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ *for every surface* S *in the set* $\breve{S}$*, which is contained in* $\mathbb{S}_{\mathcal{Z}}$*. This action is automorphic and point-norm continuous.*

Let $\breve{S}$ and $\breve{S}'$ be two disjoint surface sets in $\mathbb{S}_{\mathcal{Z}}$. Then the action of $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ on $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ satisfies

$$\big(\zeta_\sigma(U)\big(\rho_{S,\Gamma}(\Gamma')\big) := \mathbb{1}_\Gamma$$

for $\sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ and $U(\rho_{S,\Gamma}(\Gamma'))\in\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ and $S\in\breve{S}'$.

Proposition 6.3.5. *(i) The state* ω_M^Γ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$*, which is defined in 6.1.40 and which is* $\bar{G}_{\breve{S},\Gamma}$*-invariant for every surface set* $\breve{S}$ *in* $\mathbb{S}$*, extends to a state* $\breve{\omega}_M^\Gamma$ *on* $\mathsf{Weyl}(\mathbb{S},\Gamma)$*. The state* $\breve{\omega}_M^\Gamma$ *is pure and unique.*

(ii) The state $\hat{\omega}_{\mathfrak{B}}^\Gamma$ *on* $C_0(\bar{\mathcal{A}}_\Gamma)$*, which is defined in 6.2.19 and which is* $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$*-,* $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$*- and* $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$*-invariant for every surface set* $\breve{S}$ *in* $\mathbb{S}_{\mathcal{Z}}$*, extends to a state on* $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$*. The state* $\omega_{M,\mathfrak{B}}^\Gamma$ *on* $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ *is* $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$*-,* $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$*- and* $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$*-invariant.*

Definition 6.3.6. *The set of all not necessarily pure states on* $\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ *that are* $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$*-,* $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$*- and* $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$*-invariant are denoted by* $\mathcal{S}^{\text{surf,or}}(\mathsf{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}},\Gamma))$.

Proof of proposition 6.3.5
The part (i) of the proposition follows from the corollary 6.1.41 and the proposition 6.1.40. The part (ii) of the proposition follows from the following derivations. First, fix a surface set $\breve{S}$ in $\mathbb{S}_{\mathcal{Z}}$.
Step 1:
For the covariant pair (Φ_M,V) of $(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma),C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ or $(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma),C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ on $\mathcal{H}_\Gamma$ there exists a invariant state. In proposition 6.2.19 this state is defined and satisfies

$$\hat{\omega}_{\mathfrak{B}}^\Gamma(\zeta_\sigma(f_\Gamma)) = \hat{\omega}_{\mathfrak{B}}^\Gamma(f_\Gamma)$$

[4] modulo the two-sided self-adjoint ideal of the *-algebra defined by $I=\{W : \|W\|_2 = 0\}$
[5] modulo the two-sided self-adjoint ideal of the *-algebra defined by $I=\{W : \|W\|_2 = 0\}$
[6] modulo the two-sided self-adjoint ideal of $\mathbf{Weyl}(\breve{S},\Gamma)$ defined by $I=\{W : \|W\| = 0\}$ for all $\breve{S}\in\mathbb{S}$

for all $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and for arbitrary $\sigma \in \mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ or $\sigma \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$. Recall the state $\omega^\Gamma_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is required to be $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$-invariant, too. Hence the state satisfies

$$\hat{\omega}^\Gamma_{\mathfrak{B}}(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)) = \hat{\omega}^\Gamma_{\mathfrak{B}}(f_\Gamma)$$

for all $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\rho_{S,\Gamma}(\Gamma) \in \bar{G}_{\breve{S},\Gamma}$. While $\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma) \in C_0(\bar{\mathcal{A}}_\Gamma)$ and the actions ζ and α commute, the state $\hat{\omega}^\Gamma_{\mathfrak{B}}$ fulfill

$$\begin{aligned}\hat{\omega}^\Gamma_{\mathfrak{B}}(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma)) &= \hat{\omega}^\Gamma_{\mathfrak{B}}(\zeta_\sigma(\alpha(\rho_{S,\Gamma}(\Gamma))(f_\Gamma))) = \hat{\omega}^\Gamma_{\mathfrak{B}}(\alpha(\rho_{S_\sigma,\Gamma}(\Gamma))(\zeta_\sigma(f_\Gamma)))\\ &= \hat{\omega}^\Gamma_{\mathfrak{B}}(f_\Gamma)\end{aligned}$$

Clearly, there is a morphism $\Phi \in \text{Mor}(C_0(\bar{\mathcal{A}}_\Gamma), \mathsf{Weyl}_{\mathcal{Z}}(\breve{S},\Gamma))$.

Step 2:
On the other hand, there are covariant representations (Ψ, V) of the C^*-dynamical systems $(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \zeta)$ and $(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}), \zeta)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$. There is a $\bar{G}_{\breve{S},\Gamma}$-invariant, $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-invariant and $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$-invariant state $\tilde{\omega}^\Gamma_{M,\mathfrak{B}}$ on $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$.

Step 3:
There are covariant representations (Φ_Γ, V) of the C^*-dynamical systems $(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathsf{Weyl}_{\mathcal{Z}}(\breve{S},\Gamma), \zeta)$ and $(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), \mathsf{Weyl}_{\mathcal{Z}}(\breve{S},\Gamma), \zeta)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$, where $\Phi_\Gamma(W) = \Psi(W)$ for $W = U \in \mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$ or $\Phi_\Gamma(W) = \Phi_M(W)$ for $W = f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$. Consequently there exists a $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$-invariant, $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-invariant and $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$-invariant state $\omega^\Gamma_{M,\mathfrak{B}}$ on $\mathsf{Weyl}_{\mathcal{Z}}(\breve{S},\Gamma)$. This state is an extension of the state $\omega^\Gamma_{\mathfrak{B}}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ by Hahn-Banach theorem.

Then the state restricted to $C_0(\bar{\mathcal{A}}_\Gamma)$ is given by

$$\omega^\Gamma_{M,\mathfrak{B}}(f_\Gamma) = \hat{\omega}^\Gamma_{\mathfrak{B}}(f_\Gamma) \quad \forall f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$$

and restricted to $\mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$

$$\omega^\Gamma_{M,\mathfrak{B}}(W) = \tilde{\omega}^\Gamma_{M,\mathfrak{B}}(\mathbb{1}_\Gamma) \quad \forall W \in \mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$$

such that $\tilde{\omega}^\Gamma_{M,\mathfrak{B}}(W^*W) = 1$ Then

$$\omega^\Gamma_{M,\mathfrak{B}}(Wf_\Gamma) = \omega^\Gamma_{M,\mathfrak{B}}(f_\Gamma) = \omega^\Gamma_{M,\mathfrak{B}}(f_\Gamma W)$$

for all $W \in \mathbf{W}(\bar{\mathcal{Z}}_{\breve{S},\Gamma})$ and $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$.

Finally,

$$\omega^\Gamma_{M,\mathfrak{B}}(V^*_\sigma V_\sigma) = 1 \quad \forall V \in \text{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma), \mathcal{K}(\mathcal{H}_\Gamma)) \text{ or } V \in \text{Rep}(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), \mathcal{K}(\mathcal{H}_\Gamma))$$

Finally, all steps are true for every surface set $\breve{S}$ and hence for all surface sets in $\mathbb{S}_{\mathcal{Z}}$.

■

The inductive limit Γ_∞ of a family of graphs is a graph, which consists of an infinite number of paths. The inductive limit of an inductive family of finite graph systems contains an infinite number of subgraphs of Γ_∞, each of them is a finite set of arbitrary independent paths in Σ.

Let G be a compact group. The analytic holonomy C^*-algebra $C(\bar{\mathcal{A}})$ is given by the inductive limit of the family of C^*-algebras $\{(C(\bar{\mathcal{A}}_{\Gamma_i}), \beta_{\Gamma_i,\Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i,j \in \mathbb{N}\}$ for an inductive family of finite graph systems $\mathcal{P}_{\Gamma_i}$. The maps $\beta_\Gamma : C(\bar{\mathcal{A}}_\Gamma) \longrightarrow C(\bar{\mathcal{A}})$ are injective unit-preserving *-homomorphism satisfying consistency conditions

$$\beta_{\Gamma,\Gamma''} = \beta_{\Gamma,\Gamma'} \circ \beta_{\Gamma',\Gamma''}$$

whenever $\mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma''}$. Notice that, the maps $\beta_{\Gamma,\Gamma''} : C(\bar{\mathcal{A}}_\Gamma) \longrightarrow C(\bar{\mathcal{A}}_{\Gamma''})$ are isometries:

$$\|f_{\Gamma_i}\| = \|\beta_{\Gamma_i,\Gamma_j} f_{\Gamma_i}\| = \|f\|$$

whenever $f \in C(\bar{\mathcal{A}})$ and $f_{\Gamma_i} \in C(\bar{\mathcal{A}}_{\Gamma_i})$ for all i.

Now, in the next investigations the focus lies on actions of fluxes and diffeomorphisms on the inductive limit algebra $C(\bar{\mathcal{A}})$ and the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$. Simply speaking a graph-diffeomorphism for a limit graph system $\mathcal{P}_{\Gamma_\infty}$ maps graphs to graphs in $\mathcal{P}_{\Gamma_\infty}$ such that the number of edges of the graphs is preserved.

Proposition 6.3.7. *Let Γ_∞ be the inductive limit of an inductive family $\{\Gamma_i\}$ of graphs. Then $\mathcal{P}_{\Gamma_\infty}$ denotes the inductive limit of an inductive family of finite graph systems $\mathcal{P}_{\Gamma_i}$. Moreover let G^∞ be the projective limit of the family of groups $\{G^{N_i}\}$ if $G^{N_i} = G\times ...\times G$ and G is a compact group.*

*Let $(\varphi,\Phi)\in \mathrm{Diff}(\mathcal{P})$ be a path-diffeomorphism of a path groupoid $\mathcal{P}\rightrightarrows\Sigma$ such that $\varphi:\Sigma\longrightarrow\Sigma$ and $\Phi:\mathcal{P}\longrightarrow\mathcal{P}$. Then a **graph-diffeomorphism for a limit graph system** $\mathcal{P}_{\Gamma_\infty}$ is given by the pair $(\varphi_\Sigma,\Phi_\infty)$ of maps such that $\varphi_\Sigma:\Sigma\longrightarrow\Sigma$, $\Phi_\infty:\mathcal{P}_{\Gamma_\infty}\longrightarrow\mathcal{P}_{\Gamma_\infty}$ and*

$$\Phi_\infty(\Gamma) = (\Phi(\gamma_1),...,\Phi(\gamma_N)) = \Gamma_\Phi$$

for $\Gamma:=\{\gamma_1,...,\gamma_N\}$ and Γ_Φ being two subgraphs of $\mathcal{P}_{\Gamma_\infty}$. The set of such graph-diffeomorphism for a limit graph system $\mathcal{P}_{\Gamma_\infty}$ is denoted by $\mathrm{Diff}(\mathcal{P}_{\Gamma_\infty})$.

*Then there is an **action of graph-diffeomorphisms for a limit graph system** $\mathcal{P}_{\Gamma_\infty}$ on the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}})$ defined by*

$$(\theta_{(\varphi_\Sigma,\Phi_\infty)}f)(\mathfrak{h}(\Gamma)) := f(\mathfrak{h}(\Phi_\infty(\Gamma))) = (\beta_{\Phi_\infty(\Gamma)}f_{\Phi_\infty(\Gamma)})(\mathfrak{h}_{\Phi_\infty(\Gamma)}(\Phi_\infty(\Gamma))) = f(\mathfrak{h}(\Gamma_\Phi)) \tag{6.78}$$

whenever $\Gamma,\Gamma_\Phi\in\mathcal{P}_{\Gamma_\infty}$, $(\varphi_\Sigma,\Phi_\infty)\in\mathrm{Diff}(\mathcal{P}_{\Gamma_\infty})$ and for

$$f(\mathfrak{h}(\Gamma)) = (f_\Gamma\circ\pi_\Gamma)(\mathfrak{h}(\Gamma)) = (\beta_\Gamma f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) \tag{6.79}$$

where $\pi_\Gamma:\bar{\mathcal{A}}\longrightarrow\bar{\mathcal{A}}_\Gamma$ is a surjective projection, $\beta_\Gamma: C(\bar{\mathcal{A}}_\Gamma)\longrightarrow C(\bar{\mathcal{A}})$ are injective unit-preserving *-homomorphism satisfying consistency conditions.

On the other hand, the group of bisections $\mathfrak{B}(\mathcal{P})$ is defined to be the set of all smooth maps σ from Σ to the path groupoid $\mathcal{P}\overset{s}{\underset{t}{\rightrightarrows}}\Sigma$ such that $s\circ\sigma=\mathrm{id}_\Sigma$ and $t\circ\sigma:\Sigma\longrightarrow\Sigma$ is a diffeomorphism. Therefore, due to the group morphism $\mathfrak{B}(\mathcal{P})\ni\sigma\mapsto t\circ\sigma\in\mathrm{Diff}(\Sigma)$ there exists also an action of the bisections of the C^*-algebra $C(\bar{\mathcal{A}})$. Recognize that it is possible to rewrite

$$\Phi_\infty(\Gamma) =: \Gamma_{\sigma_\Sigma} \text{ for a bisection } \sigma_\Sigma\in\mathfrak{B}(\mathcal{P}_{\Gamma_\infty}) \text{ on the limit graph system } \mathcal{P}_{\Gamma_\infty}$$

Recall the action ζ of the group $\mathfrak{B}(\mathcal{P}_{\tilde\Gamma})$ of bisections for a finite graph system $\mathcal{P}_{\tilde\Gamma}$ on $C(\bar{\mathcal{A}}_{\tilde\Gamma})$, which is given in proposition 6.2.1 by

$$(\zeta_\sigma f_{\tilde\Gamma})(\mathfrak{h}_{\tilde\Gamma}(\Gamma)) = (f_{\tilde\Gamma}\circ R_\sigma)(\mathfrak{h}_{\tilde\Gamma}(\Gamma)) = f_{\tilde\Gamma}(\mathfrak{h}_{\tilde\Gamma}(\Gamma_\sigma))$$

whenever $\sigma\in\mathfrak{B}(\mathcal{P}_{\tilde\Gamma})$, $f_{\tilde\Gamma}\in C(\bar{\mathcal{A}}_{\tilde\Gamma})$ and $\mathcal{P}_{\Gamma_\sigma}\leq\mathcal{P}_{\tilde\Gamma}$.

Definition 6.3.8. *There is an **action of the group of global bisections** $\mathfrak{B}(\mathcal{P}_{\Gamma_\infty})$ on the algebra $C(\bar{\mathcal{A}})$ given by*

$$\begin{aligned}(\zeta_{\sigma_\Sigma}f)(\mathfrak{h}(\Gamma)) &:= f(R_{\sigma_\Sigma}\mathfrak{h}(\Gamma)) = f(\mathfrak{h}(\Gamma_{\sigma_\Sigma}))\\ &= \Big((\beta_{\tilde\Gamma}\circ\beta_{\Gamma_{\sigma_\Sigma},\tilde\Gamma})f_{\Gamma_{\sigma_\Sigma}}\Big)(\mathfrak{h}_{\Gamma_{\sigma_\Sigma}}(\Gamma_{\sigma_\Sigma}))\\ &= (\beta_{\tilde\Gamma}f_{\tilde\Gamma})(\mathfrak{h}_{\tilde\Gamma}(\Gamma_\sigma)) = (\beta_{\tilde\Gamma}(\zeta_\sigma f_{\tilde\Gamma}))(\mathfrak{h}_{\tilde\Gamma}(\Gamma))\\ &= (\beta_{\tilde\Gamma}f_{\tilde\Gamma})(R_\sigma(\mathfrak{h}_{\tilde\Gamma}(\Gamma)))\end{aligned}$$

whenever $\mathcal{P}_\Gamma\leq\mathcal{P}_{\Gamma_{\sigma_\Sigma}}\leq\mathcal{P}_{\tilde\Gamma}$, for a function $f\in C(\bar{\mathcal{A}})$, where $\beta_{\tilde\Gamma}:C(\bar{\mathcal{A}}_{\tilde\Gamma})\longrightarrow C(\bar{\mathcal{A}})$, $\beta_{\Gamma_{\sigma_\Sigma},\tilde\Gamma}:C(\bar{\mathcal{A}}_{\Gamma_{\sigma_\Sigma}})\longrightarrow C(\bar{\mathcal{A}}_{\tilde\Gamma})$ are unit-preserving injective *-homomorphisms satisfying consistency conditions and for a global bisections $\sigma_\Sigma\in\mathfrak{B}(\mathcal{P}_{\Gamma_\infty})$ such that for a bisection $\sigma\in\mathfrak{B}(\mathcal{P}_{\tilde\Gamma})$ on a finite graph system $\mathcal{P}_{\tilde\Gamma}$ it is true that $\sigma_\Sigma(V_{\tilde\Gamma})=\sigma(V_{\tilde\Gamma})$.

The limit Hilbert space $\mathcal{H}_\infty$ with norm $\|.\|_\infty$ is constructed from the inductive family of Hilbert spaces $\mathcal{H}_\Gamma$.

But these actions related to the limit graph system $\mathcal{P}_{\Gamma_\infty}$ are not norm-point continuous. This can be proved by the following argument. Since from $\mathfrak{h}_\Gamma$ is not a continuous groupoid morphism between $\mathcal{P}\rightrightarrows\Sigma$ to G over $\{e_G\}$, it follows that

$$\begin{aligned}&\lim_{\sigma_\Sigma(\Sigma)\rightarrow\mathrm{id}(\Sigma)}\|\zeta_{\sigma_\Sigma}(f)-f\|_\infty = \lim_{\sigma_\Sigma(\Sigma)\rightarrow\mathrm{id}(\Sigma)}\|\beta_{\tilde\Gamma}(\zeta_\sigma f_{\tilde\Gamma})-f\|_\infty\\ &= \lim_{\sigma_\Sigma(\Sigma)\rightarrow\mathrm{id}(\Sigma)}\|(\beta_{\tilde\Gamma}f_{\tilde\Gamma})(\mathfrak{h}_{\tilde\Gamma}(\Gamma)\mathfrak{h}_{\tilde\Gamma}(\sigma(V^t)),\mathfrak{h}_{\tilde\Gamma}(\sigma(V)))-(\beta_{\tilde\Gamma}f_{\tilde\Gamma})(\mathfrak{h}_{\tilde\Gamma}(\Gamma))\|_\infty\neq 0\end{aligned}$$

for a function $f \in C(\bar{\mathcal{A}})$, subgraph $\Gamma := \{\gamma_1, ..., \gamma_N\}$ of $\tilde{\Gamma}$, $V_\Gamma := V^t \cup V$ is a subset of $V_{\tilde{\Gamma}}$ where $V^t := \{t(\gamma_1), ..., t(\gamma_N)\}$ and $N = |\Gamma|$, for a global bisection $\sigma_\Sigma \in \mathfrak{B}(\mathcal{P}_{\Gamma_\infty})$ such that $\sigma_\Sigma(V_\Gamma) = (\tilde{\sigma}_\Sigma(v_1), ..., \tilde{\sigma}_\Sigma(v_{2N}))$ where $\tilde{\sigma}_\Sigma \in \mathfrak{B}(\mathcal{P})$ and there is a bisection $\sigma \in \mathfrak{B}(\mathcal{P}_{\tilde{\Gamma}})$ such that $\sigma(V_{\tilde{\Gamma}}) = \sigma_\Sigma(V_{\tilde{\Gamma}})$. Since there is an group morphism between $\mathfrak{B}(\mathcal{P})$ and the group of diffeomorphisms $\text{Diff}(\Sigma)$ on the spatial manifold Σ, the diffeomorphism cannot be implemented as strongly or weakly continuous representations on the limit Hilbert space $\mathcal{H}_\infty$. Nevertheless the action ζ of $\mathfrak{B}(\mathcal{P}_{\Gamma_\infty})$ on $C(\bar{\mathcal{A}})$ is automorphic. Denote the set of automorphic actions of a group $\mathfrak{B}(\mathcal{P}_{\Gamma_\infty})$ on the commutative C^*-algebra $C(\bar{\mathcal{A}})$ by $\text{Act}_0(\mathfrak{B}(\mathcal{P}_{\Gamma_\infty}), C(\bar{\mathcal{A}}))$.

Despite the discontinuity of the action, on the inductive limit of C^*-algebras $C(\bar{\mathcal{A}})$ there are injective *-homomorphisms $\beta_{\Gamma,\Gamma_\sigma}$ such that

$$f(\mathfrak{h}(\Gamma')) = \Big((\beta_{\tilde{\Gamma}} \circ \beta_{\Gamma_\sigma,\tilde{\Gamma}}) f_{\Gamma_\sigma}\Big)(\mathfrak{h}_{\Gamma_\sigma}(\Gamma'))$$

for any graphs such that $\mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma_\sigma} \leq \mathcal{P}_{\tilde{\Gamma}}$.

In LQG literature, the set of surfaces is not restricted, the set $\breve{S}$ is an infinite set of surfaces and the inductive limit of a family of graph systems is constructed from a limit of a family of graphs.

In this work the infinite set of surfaces is decomposed into several finite sets. To implement an action of $\bar{G}_{\breve{S},\Gamma}$ the inductive limit structure of graphs has to preserve the particular sort of the action for a fixed suitable surface set $\breve{S}$.

Proposition 6.3.9. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that the set $\breve{S}$ of surfaces has the same surface intersection property for each graph Γ_i of the family. Let $\breve{S}$ be a suitable surface set with same right surface intersection property for each graph Γ_i of the family. Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems.*

Then there is an action of $\bar{G}_{\breve{S}}$ on $C(\bar{\mathcal{A}})$ given by

$$\begin{aligned}(\alpha(\rho_{S,\Gamma_\infty}(\Gamma))f)(\mathfrak{h}(\Gamma)) &:= f(L(\rho_{S',\Gamma_\infty}(\Gamma))(\mathfrak{h}(\Gamma)))\\ &= (\beta_{\tilde{\Gamma}} \circ \alpha(\rho_{S,\tilde{\Gamma}}(\Gamma)) f_{\tilde{\Gamma}})(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma)) = (\beta_{\tilde{\Gamma}} f_{\tilde{\Gamma}})(L(\rho_{S,\tilde{\Gamma}}(\Gamma))(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma)))\end{aligned}$$

for $\mathcal{P}^{\mathfrak{o}}_\Gamma \leq \mathcal{P}^{\mathfrak{o}}_{\tilde{\Gamma}} \leq \mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$, injective unit-preserving *-homomorphism $\beta_{\tilde{\Gamma}} : C(\bar{\mathcal{A}}_{\tilde{\Gamma}}) \longrightarrow C(\bar{\mathcal{A}})$ satisfying consistency conditions, elements $\rho_{S,\Gamma_\infty}(\Gamma) \in \bar{G}_{\breve{S}}$ and there are element $\rho_{S,\tilde{\Gamma}}(\Gamma) \in \bar{G}_{\breve{S},\tilde{\Gamma}}$ such that $\rho_{S,\tilde{\Gamma}}(\Gamma) = \rho_{S,\Gamma_\infty}(\Gamma)$ for all $\Gamma \in \mathcal{P}_{\tilde{\Gamma}}$ and every $S \in \breve{S}$.

There is another action of $\bar{G}_{\breve{S},\Gamma_\infty}$ on $C(\bar{\mathcal{A}})$ defined by

$$\begin{aligned}(\alpha(\rho_{S',\Gamma_\infty}(\Gamma))f)(\mathfrak{h}(\Gamma)) &:= f(R(\rho_{S',\Gamma_\infty}(\Gamma))(\mathfrak{h}(\Gamma)))\\ &= (\beta_{\tilde{\Gamma}} \circ \alpha(\rho_{S',\tilde{\Gamma}}(\Gamma)) f_{\tilde{\Gamma}})(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma)) = (\beta_{\tilde{\Gamma}} f_{\tilde{\Gamma}})(R(\rho_{S',\tilde{\Gamma}}(\Gamma))(\mathfrak{h}_{\tilde{\Gamma}}(\Gamma)))\end{aligned}$$

for $\mathcal{P}^{\mathfrak{o}}_\Gamma \leq \mathcal{P}^{\mathfrak{o}}_{\tilde{\Gamma}} \leq \mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$, injective unit-preserving *-homomorphism $\beta_{\tilde{\Gamma}} : C(\bar{\mathcal{A}}_{\tilde{\Gamma}}) \longrightarrow C(\bar{\mathcal{A}})$ satisfying consistency conditions, elements $\rho_{S',\Gamma_\infty}(\Gamma) \in \bar{G}_{\breve{S},\Gamma_\infty}$ and there are elements $\rho_{S',\tilde{\Gamma}}(\Gamma) \in \bar{G}_{\breve{S},\tilde{\Gamma}}$ such that $\rho_{S',\tilde{\Gamma}}(\Gamma) = \rho_{S',\Gamma_\infty}(\Gamma)$ for all $\Gamma \in \mathcal{P}_{\tilde{\Gamma}}$ and every surface $S' \in \breve{S}$.

These actions of the flux group $\bar{G}_{\breve{S}}$ for the surface set $\breve{S}$ and $\bar{G}_{\breve{S},\Gamma_\infty}$ for surfaces in $\breve{S}$ on $C(\bar{\mathcal{A}})$ are automorphic and point-norm continuous.

Proof : The point-norm continuity follows from the observation

$$\begin{aligned}&\lim_{\rho_{S,\Gamma_\infty}(\Gamma)\rightarrow \text{id}_{S,\Gamma_\infty}(\Gamma)} \big\|\alpha(\rho_{S,\Gamma_\infty}(\Gamma)(f) - f\big\|_{\sup}\\ &= \lim_{\rho_{S,\Gamma_\infty}(\Gamma)\rightarrow \text{id}_{S,\Gamma_\infty}(\Gamma)} \Big\| \Big(\beta_{\tilde{\Gamma}}(\alpha(\rho_{S,\tilde{\Gamma}}(\Gamma)) f_{\tilde{\Gamma}})\Big) - \beta_{\tilde{\Gamma}}(f_{\tilde{\Gamma}})\Big\|_{\sup}\\ &= 0\end{aligned}$$

whenever $\mathcal{P}_\Gamma \leq \mathcal{P}_{\tilde{\Gamma}}$, $\rho_{S,\Gamma_\infty}(\Gamma) \in \bar{G}_{\breve{S},\Gamma_\infty}$ and $\text{id}_{S,\Gamma_\infty}(\Gamma) \in \bar{G}_{\breve{S},\Gamma_\infty}$, which is defined by $\text{id}_{S,\Gamma_\infty}(\Gamma) = (\text{id}_S(\gamma_1), ..., \text{id}_S(\gamma_N)) = (e_G, .., e_G)$ for a graph $\Gamma := \{\gamma_1, ..., \gamma_N\}$.

∎

Definition 6.3.10. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that the set $\breve{S}$ of surfaces has the surface intersection property for each graph Γ_i of the family. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.*

Let $(\varphi,\Phi)\in\operatorname{Diff}(\mathcal{P})$ be a path-diffeomorphism of a path groupoid $\mathcal{P}\rightrightarrows\Sigma$ such that

- *$\varphi:\Sigma\longrightarrow\Sigma$, which leave each surface in $\breve{S}$ and a suitable neighborhood of each surface in $\breve{S}$ invariant and $\Phi:\mathcal{P}\longrightarrow\mathcal{P}$;*
- *if a path γ in $\mathcal{P}$ does not intersect all surfaces, then $\Phi(\gamma)$ does not intersect all surfaces and*
- *the number of all generators $\{\gamma_j\}$ of Γ_i and the number of all transformed paths $\{\Phi(\gamma_j)\}$ that intersect each surface in $\breve{S}$ in their target vertices are constant and equal*

*is called a **surface-preserving path-diffeomorphism for a path groupoid** $\mathcal{P}\rightrightarrows\Sigma$ and a surface set $\breve{S}$.*

*Then a **surface-preserving graph-diffeomorphism for a limit graph system** $\mathcal{P}_{\Gamma_\infty}$ is given by the pair $(\varphi_\Sigma,\Phi_\infty)$ of maps such that*

- *$\varphi_\Sigma:\Sigma\longrightarrow\Sigma$, $\Phi_\infty:\mathcal{P}_{\Gamma_\infty}\longrightarrow\mathcal{P}_{\Gamma_\infty}$ and*

  $$\Phi_\infty(\Gamma)=(\Phi(\gamma_1),...,\Phi(\gamma_N))=\Gamma_\Phi$$

  for $\Gamma:=\{\gamma_1,...,\gamma_N\}$ and Γ_Φ being two subgraphs of $\mathcal{P}_{\Gamma_\infty}$ and
- *(φ_Σ,Φ) is a surface-preserving path-diffeomorphism for a path groupoid $\mathcal{P}\rightrightarrows\Sigma$ and a surface set $\breve{S}$.*

The set of surface-preserving graph-diffeomorphism for a limit graph system $\mathcal{P}_{\Gamma_\infty}$ is denoted by $\operatorname{Diff}_{\text{surf}}(\mathcal{P}_{\Gamma_\infty})$.

With no doubt, the group $\mathfrak{B}_{\text{surf}}(\mathcal{P}_{\Gamma_\infty})$ of surface-preserving bisections of a limit graph system $\mathcal{P}_{\Gamma_\infty}$ can be defined.

Definition 6.3.11. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that the set $\breve{S}$ of surfaces has the simple surface intersection property for each graph Γ_i of the family. Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems.*

Let $(\varphi,\Phi)\in\operatorname{Diff}(\mathcal{P})$ be a path-diffeomorphism of a path groupoid $\mathcal{P}\rightrightarrows\Sigma$ such that

- *$\varphi:\Sigma\longrightarrow\Sigma$ such that each surface S in $\breve{S}$ is mapped to another surface S_σ in $\breve{S}$ and $\Phi:\mathcal{P}\longrightarrow\mathcal{P}$;*
- *if a path γ in $\mathcal{P}$ does not intersect a surface in $\breve{S}$, then $\Phi(\gamma)$ does not intersect a surface in $\breve{S}$ and*
- *if a path intersects a surface S, lies below and is outgoing (or above and outgoing, below and ingoing, above and ingoing) and the transformed path $\Phi(\gamma)$ is non-trivial, then $\Phi(\gamma)$ intersects the transformed surface S_σ, lies below and is outgoing (or above and outgoing, below and ingoing, above and ingoing), too,*

*is called a **surface-orientation-preserving path-diffeomorphism for a path groupoid** $\mathcal{P}\rightrightarrows\Sigma$ and a surface set $\breve{S}$.*

*Then a **surface-orientation-preserving graph-diffeomorphism for a limit orientation preserved graph system** $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is given by the pair $(\varphi_\Sigma,\Phi_\infty)$ of maps such that*

- *$\varphi_\Sigma:\Sigma\longrightarrow\Sigma$, $\Phi_\infty:\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}\longrightarrow\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ and*

  $$\Phi_\infty(\Gamma)=(\Phi(\gamma_1),...,\Phi(\gamma_N))=\Gamma_\Phi$$

  for $\Gamma:=\{\gamma_1,...,\gamma_N\}$ and Γ_Φ being two subgraphs of $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ and
- *(φ_Σ,Φ) is a surface-orientation-preserving path-diffeomorphism for a path groupoid $\mathcal{P}\rightrightarrows\Sigma$ and a surface set $\breve{S}$.*

The set of surface-preserving graph-diffeomorphism for a orientation preserved limit graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ *is denoted by* $\text{Diff}_{\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty})$.

With no doubt, the group $\mathfrak{B}_{\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty})$ of surface-orientation-preserving bisections of a limit orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ can be defined.

Apart from the problems of defining a point-norm continuous action of the group of global bisections of the inductive limit holonomy C^*-algebra, the Weyl algebra can be realized as inductive limit C^*-algebra, too.

Definition 6.3.12. *Let* $\breve{S}$ *be a finite set of surfaces in* Σ. *Moreover let* Γ_∞ *be the inductive limit of a family of graphs* $\{\Gamma_i\}$ *such that each graph* Γ_i *of the family has the surface intersection property for the set* $\breve{S}$ *of surfaces. Then* $\mathcal{P}_{\Gamma_\infty}$ *is the inductive limit of an inductive family* $\{\mathcal{P}_{\Gamma_i}\}$ *of finite graph systems.*

The ***Weyl*** C^****-algebra*** $\mathfrak{Weyl}(\breve{S})$ ***for a surface set*** *is generated by the inductive limit* $C(\bar{\mathcal{A}})$ *of the family of* C^**-algebras* $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_j}, i,j\in\mathbb{N}\}$ *and the Weyl elements* $\mathsf{W}(\bar{G}_{\breve{S}})$, *which satisfy the canonical commutator relations* (6.65), *completed w.r.t. the* $\|.\|_\infty$*-norm defined by the Hilbert space* $\mathcal{H}_\infty$, *which is given as the limit of the Hilbert spaces* $\mathcal{H}_\Gamma$.

Definition 6.3.13. *Let* Γ_∞ *be the inductive limit of a family of graphs* $\{\Gamma_i\}$ *and let* $\mathcal{P}_{\Gamma_\infty}$ *be the inductive limit of an inductive family* $\{\mathcal{P}_{\Gamma_i}\}$ *of finite graph systems associated to the family of graphs* $\{\Gamma_i\}$. *Let* $\mathbb{S}$ *and* $\mathbb{S}_{\mathcal{Z}}$ *be two sets of suitable surfaces for each graph* Γ_i *of the family of graphs* $\{\Gamma_i\}$.

Moreover $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ *is the inductive limit of an inductive family* $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ *of finite orientation preserved graph systems.*

The ***Weyl*** C^****-algebra*** $\mathfrak{Weyl}(\mathbb{S})$ ***for surfaces*** *is generated by the inductive limit* $C(\bar{\mathcal{A}})$ *of the family of* C^**-algebras* $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_j}, i,j\in\mathbb{N}\}$ *and the Weyl elements* $\mathsf{W}(\bar{G}_{\breve{S}})$ *for each surface set* $\breve{S}$ *in* $\mathbb{S}$, *which satisfy the canonical commutator relations* (6.65), *completed w.r.t. the* $\|.\|_\infty$*-norm defined by the Hilbert space* $\mathcal{H}_\infty$, *which is given as the limit of the Hilbert spaces* $\mathcal{H}_\Gamma$.

The ***commutative Weyl*** C^****-algebra*** $\mathfrak{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}})$ ***for surfaces*** *is generated by the inductive limit of the family of* C^**-algebras* $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_j}, i,j\in\mathbb{N}\}$ *and the Weyl elements* $\mathsf{W}(\bar{\mathcal{Z}}_{\breve{S}})$ *for each surface set* $\breve{S}$ *in* $\mathbb{S}_{\mathcal{Z}}$, *which satisfy the canonical commutator relations* (6.65), *completed w.r.t. the* $\|.\|_\infty$*-norm defined by the Hilbert space* $\mathcal{H}_\infty$.

The ***commutative Weyl and graph-diffeomorphism*** C^****-algebra*** $\mathfrak{Weyl}_{\text{diff}}(\mathbb{S}_{\mathcal{Z}})$ ***for surfaces*** *is generated by the inductive limit* $C(\bar{\mathcal{A}})$ *of the family of* C^**-algebras* $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\leq\mathcal{P}^{\mathfrak{o}}_{\Gamma_j}, i,j\in\mathbb{N}\}$, *the Weyl elements* $\mathsf{W}(\bar{\mathcal{Z}}_{\breve{S}})$, $\text{Rep}(\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}),\mathcal{K}(\mathcal{H}_\infty))$ *and* $\text{Rep}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}),\mathcal{K}(\mathcal{H}_\infty))$ *for each surface set* $\breve{S}$ *in* $\mathbb{S}_{\mathcal{Z}}$, *which satisfy the canonical commutator relations* (6.65),(6.70) *and* (6.73), *completed w.r.t. the* $\|.\|_\infty$*-norm defined by the Hilbert space* $\mathcal{H}_\infty$, *which is given as the limit of the Hilbert spaces* $\mathcal{H}_\Gamma$.

6.4 Flux and graph-diffeomorphism group-invariant states of the Weyl C^*-algebra for surfaces

Consider the inductive limit algebra $C(\bar{\mathcal{A}})$ of the family of C^*-algebras
$\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_j}, i,j\in\mathbb{N}\}$, where $C(\bar{\mathcal{A}}_\Gamma)$ is isomorphic to $C(G^N)$ by the natural or non-standard identification.

Proposition 6.4.1. *Let* $\breve{S}$ *be a finite set of surfaces in* Σ. *Moreover let* Γ_∞ *be the inductive limit of a family of graphs* $\{\Gamma_i\}$ *such that each graph* Γ_i *of the family has the surface intersection property for the set* $\breve{S}$ *of surfaces. Then* $\mathcal{P}_{\Gamma_\infty}$ *is the inductive limit of an inductive family* $\{\mathcal{P}_{\Gamma_i}\}$ *of finite graph systems. Let* $\bar{\mathcal{A}}_{\Gamma_i}$ *be identified naturally with* $G^{|\Gamma_i|}$.

The limit $\hat{\omega}_{\mathfrak{B}_\Sigma}$ *on* $C(\bar{\mathcal{A}})$ *is defined by*

$$\hat{\omega}_{\mathfrak{B}_\Sigma}(f):=\lim_{\Gamma_i\rightarrow\Gamma_\infty}\frac{1}{k_{\Gamma_i}}\sum_{l=1}^{k_{\Gamma_i}}\omega^{\Gamma_i}_M(\zeta_{\sigma_l}(f_{\Gamma_i}))\text{ for }\sigma_l\in\mathfrak{B}^{\Gamma_i}_{\breve{S},\text{surf}}(\mathcal{P}_{\Gamma_i})$$

and which is $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_{\Gamma_\infty})$*-invariant, and where* k_{Γ_i} *is the maximal number of subgraphs in* $\mathcal{P}_{\Gamma_i}$, *which can be generated by all edges and their compositions of the graph* Γ_∞. *The limit* $\hat{\omega}_{\mathfrak{B}_\Sigma}$ *does not converge in weak***-topology.*

Proof : There are two disjoint families of graph $\{\Gamma_i'\}$ and $\{\Gamma_i\}$ such that the union converge to Γ_∞ and such that for a suitable constants $k_{\Gamma_i'}$ it is true that

$$\lim_{\Gamma_i\longrightarrow\Gamma_\infty}\Big|\hat{\omega}_{\mathfrak{B}_\Sigma}(f)-\frac{1}{k_{\Gamma_i}}\sum_{l=1}^{k_{\Gamma_i}}\omega_M^{\Gamma_i}(\zeta_{\sigma_l}(f_\Gamma))\Big| > \lim_{\Gamma_i'\longrightarrow\Gamma_\infty}\Big|\frac{1}{k_{\Gamma_i'}}\sum_{l=1}^{k_{\Gamma_i'}}\omega_M^{\Gamma_i'}(\zeta_{\sigma_l}(f_{\Gamma_i'}))\Big| > 0$$

■

If there would be a asymptotic condition for the state such that for the limit to the infinite graph the state does not depend on the action of ζ anymore, then the state defined above would be weakly converging.

Consequently if the natural identication of $\bar{\mathcal{A}}_\Gamma$ with G^N is used, then there is no state which is $\bar{\mathcal{Z}}_{S,\Gamma_\infty}$-, $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_\infty})$- and $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_{\Gamma_\infty})$-invariant on $C(\bar{\mathcal{A}})$.

Let each finite graph system $\mathcal{P}_{\Gamma_i}$ be identified naturally or in the non-standard way, then the following observations can be made.

Proposition 6.4.2. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph Γ_i of the family has the surface intersection property for the set $\breve{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.*

There is a $\bar{G}_{\breve{S}}$-invariant state ω_M on $C(\bar{\mathcal{A}})$ presented by

$$\omega_M(f)=\int_{G^{N_i}} f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i'))\,\mathrm{d}\,\mu_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i'))$$

for all $f\in C(\bar{\mathcal{A}})$, $\mathcal{P}_{\Gamma_i'}\leq\mathcal{P}_{\Gamma_i}$ and which satisfies

$$\omega_M=\omega_M^{\Gamma_i}\circ\beta_{\Gamma_i}$$

for $\beta_{\Gamma_i}: C(\bar{\mathcal{A}}_{\Gamma_i})\rightarrow C(\bar{\mathcal{A}})$ is an injective *-homomorphism.

Moreover there is a state on $C(\bar{\mathcal{A}})$ given by

$$\omega_{\mathfrak{B}}(f)=(\omega_{\mathfrak{B}}^{\Gamma_i}\circ\beta_{\Gamma_i})(f):=\frac{1}{k_{\Gamma_i}}\sum_{l=1}^{k_{\Gamma_i}}\beta^*_{\Gamma_i}\omega_M^{\Gamma_i}(\zeta_{\sigma_l}(f_{\Gamma_i}))$$

for $\sigma_l\in\mathfrak{B}^{\Gamma_i}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i})$ and $f_{\Gamma_i}\in C(\bar{\mathcal{A}}_{\Gamma_i})$, which is invariant under the automorphic actions of the groups $\text{Diff}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i})$, $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i})$ for a fixed graph Γ_i and $\bar{\mathcal{Z}}_{\breve{S}}$ for a suitable set $\breve{S}$ of surfaces in $\mathbb{S}_{\mathcal{Z}}$.

In other words,

$$\begin{aligned}&\beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i}(\theta_{(\varphi_{\Gamma_i},\Phi_{\Gamma_i})}f_{\Gamma_i})=\beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i}(f_{\Gamma_i}),\quad \beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i}(\zeta_\sigma f_{\Gamma_i})=\beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i}(f_{\Gamma_i}),\\ &\omega_{\mathfrak{B}}(\alpha(\rho_{S,\Gamma_\infty}(\Gamma_i'))(f))=\omega_{\mathfrak{B}}(f)\end{aligned}$$

for all $f_{\Gamma_i}\in C(\bar{\mathcal{A}}_{\Gamma_i})$, $f\in C(\bar{\mathcal{A}})$, $(\varphi_{\Gamma_i},\Phi_{\Gamma_i})\in\text{Diff}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i})$, $\theta\in\text{Act}_0(\text{Diff}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i}),C(\bar{\mathcal{A}}))$, $\sigma\in\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i})$, $\zeta\in\text{Act}_0(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i}),C(\bar{\mathcal{A}}))$, $\rho_{S,\Gamma_\infty}(\Gamma_i')\in\bar{\mathcal{Z}}_{\breve{S}}$, $\alpha\in\text{Act}(\bar{\mathcal{Z}}_{\breve{S}},C(\bar{\mathcal{A}}))$ for any surface set $\breve{S}$ in $\mathbb{S}_{\mathcal{Z}}$.

Furthermore the state $\omega_{\mathfrak{B}}$ and the actions $\alpha\in\text{Act}(\bar{\mathcal{Z}}_{\breve{S}},C(\bar{\mathcal{A}}))$ and $\zeta\in\text{Act}(\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_{\Gamma_i}),C(\bar{\mathcal{A}}))$ satisfy

$$(\beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i})\circ\alpha(\rho_{S,\Gamma_i}(\Gamma'))\circ\zeta_\sigma=(\beta^*_{\Gamma_i}\omega_{\mathfrak{B}}^{\Gamma_i})\circ\zeta_\sigma\circ\alpha(\rho_{S,\Gamma_i}(\Gamma'))\tag{6.80}$$

Note that, the state $\omega_{\mathfrak{B}}$ is even invariant under a graph-diffeomorphism $(\varphi_{\Gamma_i},\Phi_{\Gamma_i})$ such that there exists a diffeomorphism $\varphi:\Sigma\rightarrow\Sigma$ that maps surfaces into surfaces in $\breve{S}$ and $\varphi(v)=\varphi_{\Gamma_i}(v)$ for all $v\in V_{\Gamma_i}$. In the following only graph-diffeomophisms in $\text{Diff}(\mathcal{P}_\Gamma)$ and consequently also the induced bisections of $\mathfrak{B}(\mathcal{P}_\Gamma)$, which satisfy this requirement are considered. Therefore, the restricted sets are denoted by $\text{Diff}_{\breve{S}}(\mathcal{P}_\Gamma)$ and $\mathfrak{B}_{\breve{S}}(\mathcal{P}_\Gamma)$.

Proof : Recall the inductive limit $C(\bar{\mathcal{A}})$ of the C^*-algebras $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}):\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_j},i,j\in\mathbb{N}\}$ where $\beta_{\Gamma_i,\Gamma_j}$ is an injective *-homomorphism satisfying $\beta_{\Gamma_i,\Gamma_j}=\beta_{\Gamma_i,\Gamma_k}\circ\beta_{\Gamma_k,\Gamma_j}$ whenever $\mathcal{P}_{\Gamma_i}\leq\mathcal{P}_{\Gamma_k}\leq\mathcal{P}_{\Gamma_j}$ for all $i,k,j\in\mathbb{N}$.

There is a state ω_M^Γ on $C(\bar{\mathcal{A}}_\Gamma)$, which is $\bar{G}_{\breve{S},\Gamma}$-invariant and a state $\omega_{\mathfrak{B}}^\Gamma$ which is $\bar{\mathcal{Z}}_{\breve{S},\Gamma}$- and $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$-invariant due to proposition 6.1.9 and 6.2.19. The inductive limit of C^*-algebras induces a projective limit of states of the C^*-algebras. Hence there are conjugate maps $\beta^*_{\Gamma,\Gamma'}: C(\bar{\mathcal{A}}_\Gamma) \longrightarrow C(\bar{\mathcal{A}}_{\Gamma'})$ such that $\omega_M^{\Gamma'} = \beta^*_{\Gamma,\Gamma'}\omega_M^\Gamma$ and $\omega_M^{\Gamma'} = \beta^*_{\Gamma,\Gamma'}\omega_{\mathfrak{B}}^\Gamma$. Denote the projective limit state of $\{(\omega_M^{\Gamma'}, \beta^*_{\Gamma,\Gamma'}) : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}\}$ by ω_M on $C(\bar{\mathcal{A}})$, respectively, $\{(\omega_{\mathfrak{B}}^{\Gamma'}, \beta^*_{\Gamma,\Gamma'}) : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}\}$ by $\omega_{\mathfrak{B}}$ on $C(\bar{\mathcal{A}})$.

Then the state on $C(\bar{\mathcal{A}})$ satisfies

$$\begin{aligned}\omega_M(f) &= \beta^*_\Gamma(\omega_{\mathfrak{B}}^\Gamma(f_\Gamma)) = (\beta^*_{\Gamma'} \circ \beta^*_{\Gamma,\Gamma'})(\omega_M^\Gamma(f_\Gamma)) = \beta^*_{\Gamma'}(\omega_M^{\Gamma'}(f_{\Gamma'}))\\ &= \int_{G^N} f_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\,\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\\ &= \int_{G^{N'}} f_{\Gamma'}(L(\rho_{S,\Gamma'}(\Gamma'))(\mathfrak{h}_{\Gamma'}(\Gamma')))\,\mathrm{d}\,\mu_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\Gamma'))\\ &= \int_{G^{N''}} f_{\Gamma''}(R(\rho_{S',\Gamma''}(\Gamma''))(\mathfrak{h}_{\Gamma''}(\Gamma'')))\,\mathrm{d}\,\mu_{\Gamma''}(\mathfrak{h}_{\Gamma''}(\Gamma''))\end{aligned}$$

for suitable surface S and S', graphs Γ, Γ' and Γ'', maps $\rho_{S,\Gamma'} \in G_{\breve{S},\Gamma'}$ and $\rho_{S',\Gamma''} \in G_{\breve{S}',\Gamma''}$.

■

Notice that, the state $\omega_{\mathfrak{B}}$ is only invariant under the group $\bar{\mathcal{Z}}_{\breve{S}}$. This follows from the fact that the action ζ for $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}_\Gamma)$ and the action α for $\bar{G}_{\breve{S}}$ do not commute.

Corollary 6.4.3. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems. Let $\bar{\mathcal{A}}_\Gamma$ be identified in the non-standard identification with $G^{|\Gamma|}$.*

Then the state ω_M defined by

$$\omega_M(f) := \int_{\bar{\mathcal{A}}} f(\mathfrak{h}(\Gamma_i'))\,\mathrm{d}\,\mu_{\Gamma_\infty}(\mathfrak{h}(\Gamma_i')) = \int_{G^{N_i}} f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i'))\,\mathrm{d}\,\mu_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i'))$$

for all $f \in C(\bar{\mathcal{A}})$, $\mathcal{P}_{\Gamma_i'} \leq \mathcal{P}_{\Gamma_i}$ and which satisfies

$$\omega_M \circ \beta_{\Gamma_i} = \omega_M^{\Gamma_i}$$

is the unique state on $C(\bar{\mathcal{A}})$, which is invariant under the automorphic actions of the groups $\mathrm{Diff}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$, $\mathfrak{B}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$ for each graph Γ_i and $\bar{\mathcal{Z}}_{\breve{S}}$.

Proof : Recall the corollary 6.1.41, the state ω_M^Γ on $C(\bar{\mathcal{A}}_\Gamma)$. In the remark 6.1.43 the state $\omega_{\mathcal{Z},f}^\Gamma$ for a suitable function $f \in \mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ is defined. Assume that, f satisfies $f(\mathfrak{h}_\Gamma(\Gamma')\rho_{S,\Gamma}(\Gamma')^{-1}) = f(\mathfrak{h}_\Gamma(\Gamma')) = f(\rho_{S,\Gamma}(\Gamma')\mathfrak{h}_\Gamma(\Gamma'))$ for all $\rho_{S,\Gamma} \in \mathcal{Z}_{\breve{S},\Gamma}$. Then problem 6.4.0.5 indicates that additionally f is assumed to satisfy

$$f(\mathfrak{h}_{\Gamma_i}(\Gamma')) = f(\mathfrak{h}_{\Gamma_i}(\Phi_{\Gamma_i}(\Gamma')))$$

for all $(\varphi_{\Gamma_i}, \Phi_{\Gamma_i}) \in \mathrm{Diff}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$. Since the state $\omega_M = \omega_M^{\Gamma_i} \circ \beta_{\Gamma_i}$ is required to be invariant under all $\mathrm{Diff}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$ for all i. Thus, the state $\omega_{N,f}^{\Gamma_i}$ is required to be $\mathrm{Diff}_{\breve{S}}(\mathcal{P}_{\Gamma_j})$-invariant for all $\mathcal{P}_{\Gamma_j} \leq \mathcal{P}_{\Gamma_i}$ and $1 \leq j \leq i \leq \infty$. But only the constant function satisfies this requirement. Hence ω_M is the only graph-diffeomorphism and $\bar{\mathcal{Z}}_{\breve{S}}$-invariant state for an arbitrary set $\breve{S}$ of surfaces.

■

The same result can be obtained for the non-standard identification of the finite graph system and the configuration space.

Let each finite graph system $\mathcal{P}_{\Gamma_i}$ be identified naturally or in the non-standard way.

Proposition 6.4.4. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph Γ_i of the family has the surface intersection property for the set $\breve{S}$ of surfaces[7]. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.*

There is a GNS-representation $(\mathcal{H}_\infty,\Phi,\Omega_M)$ of $\mathfrak{W}eyl(\breve{S})$ on $\mathcal{H}_\infty$ such that the corresponding pure and unique state $\bar{\omega}_M$ on $\mathfrak{W}eyl(\breve{S})$ is given by

$$\begin{aligned}\bar{\omega}_M(f) &= (\omega_M^\Gamma\circ\beta_\Gamma)(f) = \langle\Omega_M,\Phi_M(f)\Omega_M\rangle\\ &= \int_{G^{N_i}} f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i))\,\mathrm{d}\,\mu_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i))\\ \bar{\omega}_M(U^*(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma))) &= \langle\Omega_M,\Phi(U^*(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma)))\Omega_M\rangle = \bar{\omega}_M(\mathbb{1})\end{aligned}$$

where $\mathbb{1}$ is the identity on $\mathcal{H}_\infty$ and for $f\in C(\bar{\mathcal{A}})$ and $U(\rho_{S,\Gamma}(\Gamma))\in\mathsf{W}(\bar{G}_{\breve{S}})$, where $\Phi\in\mathrm{Mor}(\mathfrak{W}eyl(\breve{S}),\mathcal{L}(\mathcal{H}_\infty))$ and $\Phi_M:=\Phi\big|_{C(\bar{\mathcal{A}})}$. The state $\bar{\omega}_M$ is invariant under the automorphic actions of the flux group $\bar{G}_{\breve{S}}$.

Proof. First use the proposition 6.3.5 for finite graph systems and the uniqueness of the construction of the limit of states on the inductive limit of C^*-algebras. Equivalently, one show that the representation $\Phi_M\in\mathrm{Rep}(C(\bar{\mathcal{A}}),\mathcal{L}(\mathcal{H}_\infty))$ extends uniquely to a representation $\Phi\in\mathrm{Mor}(\mathfrak{W}\mathrm{eyl}(\breve{S}),\mathcal{L}(\mathcal{H}_\infty))$. □

Furthermore this proposition extends to sets of surface sets.

Proposition 6.4.5. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ and let $\mathcal{P}_{\Gamma_\infty}$ be the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems associated to the family of graphs $\{\Gamma_i\}$. Let $\mathbb{S}$ be a set of suitable surfaces for each graph Γ_i of the family of graphs $\{\Gamma_i\}$.*

There is a GNS-representation $(\mathcal{H}_\infty,\Phi,\Omega_M)$ of $\mathfrak{W}eyl(\mathbb{S})$ on $\mathcal{H}_\infty$ such that the corresponding pure and unique state $\bar{\omega}_M$ on $\mathfrak{W}eyl(\mathbb{S})$ is given by

$$\begin{aligned}\bar{\omega}_M(f) &= (\omega_M^\Gamma\circ\beta_\Gamma)(f) = \langle\Omega_M,\Phi_M(f)\Omega_M\rangle\\ &= \int_{G^{N_i}} f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i))\,\mathrm{d}\,\mu_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\Gamma_i))\\ \bar{\omega}_M(U^*(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma))) &= \langle\Omega_M,\Phi(U^*(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma)))\Omega_M\rangle = \bar{\omega}_M(\mathbb{1})\end{aligned}$$

where $\mathbb{1}$ is the identity on $\mathcal{H}_\infty$ and for $f\in C(\bar{\mathcal{A}})$ and $U(\rho_{S,\Gamma}(\Gamma))\in\mathsf{W}(\bar{G}_{\breve{S}})$ for every suitable surface set $\breve{S}$ in $\mathbb{S}$, where $\Phi\in\mathrm{Mor}(\mathfrak{W}eyl(\mathbb{S}),\mathcal{L}(\mathcal{H}_\infty))$ and $\Phi_M:=\Phi\big|_{C(\bar{\mathcal{A}})}$ and which is invariant under the automorphic actions of the flux group $\bar{G}_{\breve{S}}$ for each suitable surface set $\breve{S}$ in $\mathbb{S}$.

Now, the focus lies on graph-diffeomorphism invariant states on a Weyl C^*-algebra. Then the following theorem can be stated.

Theorem 6.4.6. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph Γ_i of the family has the surface intersection property for the set $\breve{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems and each $\mathcal{P}_{\Gamma_i}$ is identified in the non-standard way.*

The state $\bar{\omega}_M$ on $\mathfrak{W}eyl_{\mathcal{Z}}(\breve{S})$ given in proposition 6.4.4 is the unique state, which is invariant under the automorphic actions of the groups $\mathrm{Diff}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$, $\mathfrak{B}_{\breve{S}}(\mathcal{P}_{\Gamma_i})$ for each graph Γ_i and $\bar{\mathcal{Z}}_{\breve{S}}$ and pure.

Furthermore the state $\bar{\omega}_M$ and the actions $\alpha\in\mathrm{Act}(\bar{\mathcal{Z}}_{\breve{S}},\mathfrak{W}eyl_{\mathcal{Z}}(\breve{S}))$ and $\zeta\in\mathrm{Act}(\mathfrak{B}_{\breve{S}}(\mathcal{P}_{\Gamma_i}),\mathfrak{W}eyl_{\mathcal{Z}}(\breve{S}))$ satisfy

$$(\beta^*_{\Gamma_i}\bar{\omega}^{\Gamma_i}_M)\circ\alpha(\rho_{S,\Gamma_i}(\Gamma'))\circ\zeta_\sigma = (\beta^*_{\Gamma_i}\bar{\omega}^{\Gamma_i}_M)\circ\zeta_\sigma\circ\alpha(\rho_{S,\Gamma_i}(\Gamma')) \qquad (6.81)$$

Notice that, this theorem generalises for a suitable set $\mathbb{S}_{\mathcal{Z}}$ of surface sets.

Proof : This follows from corollary 6.4.3 and proposition 6.4.4.

■

[7] This condition is necessary, since otherwise $\bar{G}_{\breve{S},\Gamma_i}$ doesn't form a group.

For the natural or non-standard and natural identification the next theorem follows.

Theorem 6.4.7. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.*

There is a GNS- representation $(\mathcal{H}_\infty, \Phi, \Omega_{M,\mathfrak{B}})$ of $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ on $\mathcal{H}_\infty$ such that the corresponding state $\omega_{M,\mathfrak{B}}$ on $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$, which is given by

$$\begin{aligned}
\omega_{M,\mathfrak{B}}(f_{\Gamma_i}) &:= \frac{1}{k_{\Gamma_i}} \sum_{l=1}^{k_{\Gamma_i}} \omega_M(\zeta_{\sigma_l}(f_{\Gamma_i})) \\
&= \langle \Omega_{M,\mathfrak{B}}, \Phi_M(f_{\Gamma_i})\Omega_{M,\mathfrak{B}}\rangle \\
\omega_{M,\mathfrak{B}}(f) &= \omega_M(f) \\
\omega_{M,\mathfrak{B}}(U^*(\rho_{S,\Gamma}(\Gamma'))U(\rho_{S,\Gamma}(\Gamma'))) &= \langle \Omega_{M,\mathfrak{B}}, \Phi(U^*(\rho_{S,\Gamma}(\Gamma'))U(\rho_{S,\Gamma}(\Gamma')))\Omega_{M,\mathfrak{B}}\rangle = \omega_{M,\mathfrak{B}}(\mathbb{1})
\end{aligned}$$

where $\mathbb{1}$ is the identity on $\mathcal{H}_\infty$ and for $\sigma_l \in \mathfrak{B}^{\Gamma_i}_{\breve{S},\mathrm{surf}}(\mathcal{P}_{\Gamma_i})$, $f_{\Gamma_i} \in C(\bar{\mathcal{A}}_{\Gamma_i})$, $f \in C(\bar{\mathcal{A}})$, $U(\rho_{S,\Gamma}(\Gamma')) \in \mathsf{W}(\bar{\mathcal{Z}}_{S,\Gamma_\infty})$, where $\Phi \in \mathrm{Mor}(\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}), \mathcal{L}(\mathcal{H}_\infty))$ and $\Phi_M := \Phi\big|_{C(\bar{\mathcal{A}})}$.

The state $\omega_{M,\mathfrak{B}}$ on $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ is $\mathrm{Diff}_{\breve{S},\mathrm{surf}}(\mathcal{P}_{\Gamma_i})$-, $\mathrm{Diff}_{\breve{S},\mathrm{or}}(\mathcal{P}_{\Gamma_i})$-, $\mathfrak{B}_{\breve{S},\mathrm{surf}}(\mathcal{P}_{\Gamma_i})$-, $\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_{\Gamma_i})$- for a fixed graph Γ_i and $\bar{\mathcal{Z}}_{\breve{S}}$-invariant.

In other words,

$$\begin{aligned}
&\omega_{M,\mathfrak{B}}(\theta_{(\varphi_{\Gamma_i},\Phi_{\Gamma_i})}(W)) = \omega_{M,\mathfrak{B}}(W), \quad \omega_{M,\mathfrak{B}}(\zeta_\sigma(W)) = \omega_{M,\mathfrak{B}}(W), \\
&\omega_{M,\mathfrak{B}}(\alpha(\rho_{S,\Gamma_\infty}(\Gamma'))(W)) = \omega_{M,\mathfrak{B}}(W)
\end{aligned}$$

for all $W \in \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ and $\theta \in \mathrm{Act}_0(\mathrm{Diff}(\mathcal{P}_{\Gamma_i}), \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$,
$\zeta \in \mathrm{Act}_0(\mathfrak{B}_{\breve{S},\mathrm{surf}}(\mathcal{P}_{\Gamma_i}), \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$ or $\zeta \in \mathrm{Act}_0(\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_{\Gamma_i}), \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$ for each fixed graph Γ_i,
$\alpha \in \mathrm{Act}(\bar{\mathcal{Z}}_{S,\Gamma_\infty}, \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$ for any surface $S \in \breve{S}$.

Finally, the state $\omega_{M,\mathfrak{B}}$ and the actions $\alpha \in \mathrm{Act}(\bar{\mathcal{Z}}_{\breve{S}}, \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$ and $\zeta \in \mathrm{Act}(\mathfrak{B}_{\breve{S},\mathrm{or}}(\mathcal{P}_{\Gamma_i}), \mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))$ satisfy

$$(\beta^*_{\Gamma_i}\omega^{\Gamma_i}_{M,\mathfrak{B}}) \circ \alpha(\rho_{S,\Gamma_i}(\Gamma')) \circ \zeta_\sigma = (\beta^*_{\Gamma_i}\omega^{\Gamma_i}_{M,\mathfrak{B}}) \circ \zeta_\sigma \circ \alpha(\rho_{S,\Gamma_i}(\Gamma')) \tag{6.82}$$

This theorem can be generalised to $\mathfrak{Weyl}_{\mathcal{Z}}(\mathbb{S}_{\mathcal{Z}})$, since with no doubt this theorem is true for all surface sets in $\mathbb{S}_{\mathcal{Z}}$.

6.5 The holonomy-flux von Neumann algebra and the Weyl C^*-algebra for surfaces

The holonomy-flux von Neumann algebra and KMS-states

Let G be a compact connected Lie group. Identify $\bar{\mathcal{A}}_\Gamma$ with G^N in the non-standard way. Recall the representation Φ_Γ associated to the state $\omega^\Gamma_{M,\mathfrak{B}}$ of the commutative Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\breve{S}, \Gamma)$ for a surface set and a finite graph system generated by Γ on the Hilbert space $\mathcal{H}_\Gamma$ presented in proposition 6.3.5. Moreover let Φ be the representation associated to the state $\omega_{M,\mathfrak{B}}$ of the Weyl C^*-algebra $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ on $\mathcal{H}_\infty$.

Definition 6.5.1. *Let $\breve{S}$ be a finite set of surfaces in Σ. Moreover let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.*

Let $(\Phi_M, \mathcal{H}_\Gamma, \Omega^\Gamma_{M,\mathfrak{B}})$ be the GNS-triple associated to the state $\omega^\Gamma_{M,\mathfrak{B}}$ of the commutative Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\breve{S}, \Gamma)$. Furthermore let $(\Phi_M, \mathcal{H}_\infty, \Omega_{M,\mathfrak{B}})$ be the GNS-triple associated to the state $\omega_{M,\mathfrak{B}}$ of the commutative Weyl C^*-algebra $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$.

*Then the von Neumann algebra $\Phi_M(\mathsf{Weyl}_{\mathcal{Z}}(\breve{S}, \Gamma))''$ is called the **holonomy-flux von Neumann algebra for a graph and a surface set** and is denoted by $\mathfrak{M}(\breve{S}, \Gamma)$.*

*The von Neumann algebra $\Phi_M(\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S}))''$ is called the the **holonomy-flux von Neumann algebra for a surface set** and is denoted by $\mathfrak{M}(\breve{S})$.*

Notice that, the quantum flux operators generate a von Neumann algebra on their own, since $\{U(\rho_{S,\Gamma}(\Gamma)) : \rho_{S,\Gamma}(\Gamma) \in \bar{G}_{\breve{S},\Gamma}\}''$ is a von Neumann algebra.

The next step is to show that the von Neumann algebra with respect to $(\Phi_\Gamma, \mathcal{H}_\Gamma, \Omega_{M,\mathfrak{B}})$ does not have a standard form[8].

Proposition 6.5.2. *Let $(\Phi_\Gamma, \mathcal{H}_\Gamma, \Omega^\Gamma_{M,\mathfrak{B}})$ be the GNS-triple associated to the state $\omega^\Gamma_{M,\mathfrak{B}}$ of the commutative Weyl C^*-algebra $\mathsf{Weyl}_{\mathcal{Z}}(\breve{S}, \Gamma)$. Furthermore let $(\Phi, \mathcal{H}_\infty, \Omega_{M,\mathfrak{B}})$ be the GNS-triple associated to the state $\omega_{M,\mathfrak{B}}$ of the commutative Weyl C^*-algebra $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$.*

The cyclic vector $\Omega^\Gamma_{M,\mathfrak{B}}$ is not separating for $\mathfrak{M}(\breve{S}, \Gamma)$ and the cyclic vector $\Omega_{M,\mathfrak{B}}$ is not separating for $\mathfrak{M}(\breve{S})$.

Proof : For any $U(\rho_{S,\Gamma}(\Gamma)) \in \mathfrak{M}(\breve{S}, \Gamma)$ it follows that

$$U(\rho_{S,\Gamma}(\Gamma))\Omega^\Gamma_{M,\mathfrak{B}} = \Omega^\Gamma_{M,\mathfrak{B}} \neq 0$$

for all $U(\rho_{S,\Gamma}(\Gamma)) \in \mathfrak{M}(\breve{S}, \Gamma)$. Moreover for $f_\Gamma \in \mathfrak{M}(\breve{S}, \Gamma)$ and

$$f_\Gamma \Omega^\Gamma_{M,\mathfrak{B}} = 0$$

it does not follow that $f_\Gamma = 0$. Then for

$$(U(\rho_{S,\Gamma}(\Gamma)) - U(\rho_{\tilde{S},\Gamma}(\Gamma)))\Omega^\Gamma_{M,\mathfrak{B}} = 0$$

for $(U(\rho_{S,\Gamma}(\Gamma)) - U(\rho_{\tilde{S},\Gamma}(\Gamma))) \in \mathfrak{M}(\breve{S}, \Gamma)$ it does not follows that $(U(\rho_{S,\Gamma}(\Gamma)) - U(\rho_{\tilde{S},\Gamma}(\Gamma))) = 0$ for all surfaces in $\breve{S}$. Consequently $\Omega^\Gamma_{M,\mathfrak{B}}$ is not separating for $\mathfrak{M}(\breve{S}, \Gamma)$.

In analogy it can be shown that $\Omega_{M,\mathfrak{B}}$ is not separating for $\mathfrak{M}(\breve{S})$.

■

It is possible to construct a von Neumann algebra for the holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$, which is indeed of standard form for the state $\omega_{\mathfrak{B}}$. Then one can easily verify that the operator I defined given in definition 6.1.19 in section 6.1 by the map $I : C(\bar{\mathcal{A}}_\Gamma) \longrightarrow C(\bar{\mathcal{A}}_\Gamma)$, $I(f_\Gamma) = \breve{f}_\Gamma$ is the associated modular conjugation operator. Denote the holonomy von Neumann algebra by $\mathfrak{M}_\Gamma$.

Let $\breve{S}$ has the surface intersection property for a graph Γ. Then a condidate for the modular operator $\triangle_\Gamma$ associated to the holonomy von Neumann algebra $\mathfrak{M}_\Gamma$ for a graph Γ is given by $U(\exp(E_S(\Gamma)))$ for a fixed $U \in \text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$, a surface S in $\breve{S}$ and $\exp(E_S(\Gamma)) \in \bar{G}_{\breve{S},\Gamma}$ such that

$$\triangle^{it}_\Gamma \mathfrak{M}_\Gamma \triangle^{-it}_\Gamma = \mathfrak{M}_\Gamma \tag{6.83}$$

for all $t \in \mathbb{R}$.

Definition 6.5.3. *A state ω of a von Neuman algebra $\mathfrak{M}$ is said to be **normal** iff for every increasing bounded net of positive elements M_λ converging to M in $\mathfrak{M}$ it follows that $\omega(M_\lambda)$ converges to $\omega(M)$.*

The difficulty is to find a faithful, normal state $\omega_{\mathfrak{M}}$ on the holonomy von Neumann algebra $\mathfrak{M}_\Gamma$ for a graph Γ such that there exists a *-automorphism

$$\alpha^{\omega_{\mathfrak{M}}}_t(f_\Gamma) = \pi^{-1}_{\omega_{\mathfrak{M}}}(\triangle^{it}_\Gamma \pi_{\omega_{\mathfrak{M}}}(f_\Gamma)\triangle^{-it}_\Gamma) \tag{6.84}$$

which defines a σ-weakly continuous one-parameter group. A faithful, normal state $\omega_{\mathfrak{M}}$ with the property above on the holonomy von Neumann algebra $\mathfrak{M}_\Gamma$ is normal iff the state is for example of the form

$$\omega_{\mathfrak{M}}(A) = \frac{\text{tr}(\triangle^{it} A)}{\text{tr}(\triangle^{it})} \tag{6.85}$$

Indeed such a state $\omega_{\mathfrak{M}}$ satisfies the KMS-condition for $\beta > 0$

$$\omega_{\mathfrak{M}}(A_1 \alpha^{\omega_{\mathfrak{M}}}_{i\beta}(A_2)) = \omega_{\mathfrak{M}}(A_2 A_1) \text{ for } A_1, A_2 \in \mathfrak{M}_\Gamma \tag{6.86}$$

The state ω^Γ_M or $\omega^\Gamma_{\mathfrak{B}}$ on $\mathfrak{M}_\Gamma$ are not normal.

[8]A von Neumann algebra $\pi(\mathfrak{A})''$ is in standard form if the GNS-triple $(\mathcal{H}, \pi, \Omega)$ contains a cyclic vector Ω, which is separating for $\pi(\mathfrak{A})''$

Definition 6.5.4. *A triple $(\mathfrak{M},\mathbb{R},\alpha)$ is called a W^*-dynamical system if $\mathfrak{M}$ is a von Neumann algebra and α is a σ-weakly continuous one-parameter group of automorphisms of $\mathfrak{M}$.*

Definition 6.5.5. *Let $(\mathfrak{M},\mathbb{R},\alpha)$ be a W^*-dynamical system. The state ω over $\mathfrak{M}$ is defined to be a KMS-state at value $\beta\in\mathbb{R}$ if ω is normal and*

$$\omega(A\alpha_{i\beta}(B)) = \omega(BA) \tag{6.87}$$

for all A,B contained in a σ-weakly dense, α-invariant *-subalgebra[9] of the entire analytic elements $\mathfrak{M}_\alpha$ for α.

Consequently modular theory and KMS-states for the holonomy-flux von Neumann algebra $\mathfrak{M}(\breve{S})$ for a surface set $\breve{S}$ or for the holonomy von Neumann algebra is not easy to consider. Notice that, normal states of the form (6.85) are naturally given for matrix algebras. Therefore, the group C^*-algebra $C^*(G)$ over a compact Lie group G or the C^*-completion of the *- algebra $AP(G)$ of almost periodic functions on G, which are presented in section 8.1, have a natural normal state ω defined by the trace and a modular operator given by an exponentiated flux operator. In the case of the group C^*-algebra $C^*(G)$, this state is a KMS-state for the von Neumann algebra $\pi_\omega(C^*(G))''$.

One can define KMS-states also for C^*-algebra dynamical systems $(\mathfrak{A},\mathbb{R},\alpha)$ where $\mathfrak{A}$ is a C^*-algebra and an action $\alpha\in\mathrm{Act}(\mathbb{R},\mathfrak{A})$.

KMS states on the Weyl C^*-algebra for surfaces

Definition 6.5.6. *For a given state ω of a C^*-algebra $\mathfrak{A}$, the representation $t\mapsto\alpha_t$ of the additive group $\mathbb{R}$ of real numbers in the automorphism group of $\mathfrak{A}$ is a homomorphism with the property that $\omega(A\alpha_t(B))$ is a continuous function of t, is called an **evolution**.*

Let S be a surface such that S has the surface intersection property for a graph Γ and for the finite graph system $\mathcal{P}_\Gamma$. Then for the Weyl C^*-algebra $\mathsf{Weyl}(\breve{S},\Gamma)$ for an arbitrary surface set $\breve{S}$ and a finite graph system generated by a graph Γ on the Hilbert space $\mathcal{H}_\Gamma$, there is an automorphism α_t for $t\in\mathbb{R}$ given by

$$\begin{aligned}\alpha_t(f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) &:= f_\Gamma(L(\exp(itE_S^+(\Gamma)E_S(\Gamma))(\mathfrak{h}_\Gamma(\Gamma)))\\ \alpha_t(U)(\rho_S(\Gamma)) &= U(L(\exp(itE_S^+(\Gamma)E_S(\Gamma)))(\rho_S(\Gamma)))\end{aligned} \tag{6.88}$$

for every $E_S(\Gamma)\in\bar{\mathfrak{g}}_{S,\Gamma}$. One searches for a state ω^Γ such that $t\mapsto\alpha_t$ is an evolution. Consequently the following maps are required to be continuous

$$\begin{aligned}&t\mapsto\omega^\Gamma(k_\Gamma\alpha_t(f_\Gamma))\\ &t\mapsto\omega^\Gamma(U(\rho_S(\Gamma))\alpha_t(f_\Gamma))\\ &t\mapsto\omega^\Gamma(U(\hat\rho_S(\Gamma))\alpha_t(U(\rho_S(\Gamma))))\\ &t\mapsto\omega^\Gamma(U(\rho_S(\Gamma))\alpha_t(f_\Gamma)))\end{aligned}$$

Let ω be a state of $\mathfrak{A}$ and let $t\mapsto\alpha_t$ define an evolution. Then the functions $t\mapsto\omega(A\alpha_t(B))\in\mathbb{C}$ and $t\mapsto\omega(\alpha_t(B)A)\in\mathbb{C}$ are bounded functions on $\mathbb{R}$. Consequently these functions can be considered as elements of the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$. Refer to Reed Simon [78]. Then the KMS-condition can be formulated as follows.

Definition 6.5.7. *[21, Def 5.3.1,Prop 5.3.12] A state ω_β of the C^*-algebra $\mathfrak{A}$ is called a **KMS-state** with respect to an evolution $t\mapsto\alpha_t$ of $\mathfrak{A}$ at inverse temperature $0<\beta<\infty$ if the KMS-condition*

$$\omega_\beta(A\alpha_{i\beta}(B)) = \omega_\beta(BA)$$

for all $A\in\mathfrak{A}$ and for all B contained in a norm-dense, α-invariant *-subalgebra[10] $\mathfrak{A}_D^\alpha$ of entire analytic elements for α holds.

[9] Refer to [22, Def.2.5.20].

[10] The set of analytic vectors of α_t consists of those $A\in\mathfrak{A}$ such that the map $t\mapsto\omega(\alpha_t(A))$ extends to an analytic function on the strip $0<|\Im(z)|<\beta$ for all $\omega\in\mathcal{S}(\mathfrak{A})$. Refer to [22, Def.2.5.20].

In the if $t \mapsto \alpha_t(A)$ is norm-continuous in t for each $A \in \mathfrak{A}$, then the set The construction of the subalgebra $\mathfrak{A}_D^\alpha$ associated to a general C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \alpha)$ can be found in the article [52] of Kastler, Pool and Poulsen.

A KMS-state is invariant under the automorphism α_t. Given a KMS-state ω_β then there exists

(i) a representation π_β of $\mathfrak{A}$ with cyclic vector Ω_β on a Hilbert space $\mathcal{H}_\beta$,

(ii) Ω_β is separating for $\pi_\beta(\mathfrak{A})''$,

(iii) an antiunitary operator J such that

$$J\pi_\beta(\mathfrak{A})''J = \pi_\beta(\mathfrak{A})'$$

(iv) a unitary strongly continuous group $t \mapsto U_\beta(t)$ such that

$$U_\beta(t)\Omega_\beta = \Omega_\beta$$

which implements $t \mapsto \alpha_t$,

(v) the KMS-condition extend to the weak-closure $\pi_\beta(\mathfrak{A})''$ of $\pi_\beta(\mathfrak{A})$ w.r.t. the mapping $N \mapsto U_\beta(t)NU_\beta(-t)$ for all $N \in \pi_\beta(\mathfrak{A})''$,

(vi) a generator H of U_β and $T = \exp(-\frac{\beta}{2}H)$ such that

$$N\Omega_\beta = JTN\Omega_\beta$$

for all $N \in \pi_\beta(\mathfrak{A})''$.

The states $\bar{\omega}_M^\Gamma$ on $\mathsf{Weyl}(\breve{S}, \Gamma)$ and $\bar{\omega}_M$ on $\mathsf{Weyl}(\breve{S})$ are invariant under the action α_t. But both states do not satisfy the KMS-condition. Hence one has to search for other automorphism on $\mathsf{Weyl}(\breve{S}, \Gamma)$, or other states on $\mathsf{Weyl}(\breve{S}, \Gamma)$ or both. The next theorem shows that for the automorphism α defined above in (6.88) there is no KMS-state for the Weyl C^*-algebra $\mathsf{Weyl}(\breve{S}, \Gamma)$.

Theorem 6.5.8. *The C^*-dynamical system $(\mathsf{Weyl}(\breve{S}, \Gamma), \mathbb{R}, \alpha)$ does not admit any KMS-state.*

This can be shown by the theorem of Woronowicz in [111]. For the proof, the following mathematical objects have to be considered.

Definition 6.5.9. *The opposite algebra $\mathsf{Weyl}(\breve{S}, \Gamma)^{\mathrm{opp}}$ is defined by the set $\mathsf{Weyl}(\breve{S}, \Gamma)$ with a multiplication operation m*

$$\bar{m}(W_1, W_2) = W_2 W_1$$

for all elements $W_1, W_2 \in \mathsf{Weyl}(\breve{S}, \Gamma)^{\mathrm{opp}}$.

There is an antilinear multiplicative *-invariant isometry $j : \mathsf{Weyl}(\breve{S}, \Gamma) \to \mathsf{Weyl}(\breve{S}, \Gamma)^{\mathrm{opp}}$ and the image $j(W)$ is denoted by $\bar{W}$.

Then define the smallest closed left ideal L in the maximal C^*-tensor product C^*-algebra $\mathsf{Weyl}(\breve{S}, \Gamma)^{\mathrm{opp}} \otimes \mathsf{Weyl}(\breve{S}, \Gamma)$ containing all elements of the form

$$\bar{W} \otimes \mathbb{1} - \bar{\mathbb{1}} \otimes \alpha_{i/2}(W^*)$$

for all $W \in \mathsf{Weyl}(\breve{S}, \Gamma)_D^\alpha$.

Proof of the theorem 6.5.8: First observe that ω_M^Γ is the state on $\mathsf{Weyl}(\breve{S}, \Gamma)$, which is presented in proposition 6.3.5. Assume that, there is a state $\tilde{\omega}_M^\Gamma$ on $\mathsf{Weyl}(\breve{S}, \Gamma)^{\mathrm{opp}} \otimes \mathsf{Weyl}(\breve{S}, \Gamma)$ such that $\tilde{\omega}_M^\Gamma(\bar{\mathbb{1}} \otimes W) = \omega_M^\Gamma(W)$ for all $W \in \mathsf{Weyl}(\breve{S}, \Gamma)$. Now, calculate

$$\begin{aligned}
&\tilde{\omega}_M^\Gamma((\bar{W}_2 \otimes W_3)\left(\bar{W}_1 \otimes \mathbb{1} - \bar{\mathbb{1}} \otimes \alpha_{i/2}(W_1^*)\right))\\
&= \langle \Omega, (\bar{W}_2 \otimes W_3)\left(\bar{W}_1 \otimes \mathbb{1}\right)\Omega\rangle - \langle \Omega, (\bar{W}_2 \otimes W_3)\left(\bar{\mathbb{1}} \otimes \alpha_{i/2}(W_1^*)\right)\Omega\rangle\\
&= \langle \Omega, \bar{W}_1\bar{W}_2 \otimes W_3\Omega\rangle - \langle \Omega, \bar{W}_2 \otimes W_3 U\Big(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big)\Big)W_1^*)\Omega\rangle
\end{aligned}$$

for all $W_1, W_2, W_3 \in \mathsf{Weyl}(\breve{S},\Gamma)^\alpha_D$. Set $W_i := U(\rho^i_S(\Gamma))$ for $i = 2,3$, $W_1 = f \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $W_1^* = \breve{f} = J(f)$. Then recall that $\bar{m}(\bar{U}(\rho^i_S(\Gamma)), \bar{W}_1) = j(m(W_1, U(\rho^i_S(\Gamma))))$ for every $\bar{W}_i, \bar{W}_1 \in \mathsf{Weyl}(\breve{S},\Gamma)^{\text{opp}}$ and $i = 2,3$. Then derive

$$\begin{aligned}
&\tilde{\omega}^\Gamma_M((\bar{W}_2 \otimes W_3)\left(\bar{W}_1 \otimes \mathbb{1} - \bar{\mathbb{1}} \otimes \alpha_{i/2}(W_1^*)\right))\\
&= \langle \Omega, \overline{f}\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\Omega\rangle\\
&\quad - \langle \Omega, \bar{U}(\rho^2_S(\Gamma)) \otimes W_3 U\Big(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big)\Big)J(f)\Omega\rangle\\
&= \langle \Omega, \overline{f}\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\Omega\rangle\\
&\quad - \langle \Omega, \bar{U}(\rho^2_S(\Gamma)) \otimes m\left(W_3, U\Big(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big)\Big)J(f)\right)\Omega\rangle\\
&= \langle \Omega, \overline{f}\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\Omega\rangle\\
&\quad - \langle \Omega, \bar{m}\left(\bar{U}(\rho^2_S(\Gamma)), \bar{U}\Big(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big)\Big)\overline{J(f)}\right) \otimes W_3\Omega\rangle\\
&= \langle \Omega, \overline{f}\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\Omega\rangle\\
&\quad - \langle \Omega, \overline{J(f)}\bar{U}(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big))\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\Omega\rangle
\end{aligned}$$

Set $\Omega := \tilde{\Omega}^\Gamma_M \times \Omega^\Gamma_M$. Moreover remember $U(\rho^i_S))\Omega^\Gamma_M = \Omega^\Gamma_M$.

$$\begin{aligned}
&\tilde{\omega}^\Gamma_M((\bar{W}_2 \otimes W_3)\left(\bar{W}_1 \otimes \mathbb{1} - \bar{\mathbb{1}} \otimes \alpha_{i/2}(W_1^*)\right))\\
&= \langle \tilde{\Omega}^\Gamma_M \times \Omega^\Gamma_M, \overline{f}\bar{U}(\rho^2_S(\Gamma)) \otimes W_3\tilde{\Omega}^\Gamma_M \times \Omega^\Gamma_M\rangle\\
&\quad - \langle \tilde{\Omega}^\Gamma_M \times \Omega^\Gamma_M, \bar{U}(\rho^2_S(\Gamma)) \otimes W_3 U\Big(\exp\Big(-\frac{i}{2}E_S(\Gamma)^+E_S(\Gamma)\Big)\Big)J(f)\tilde{\Omega}^\Gamma_M \times \Omega^\Gamma_M\rangle\\
&= \langle \Omega, \overline{f} \otimes \mathbb{1}\Omega\rangle - \langle \Omega, \overline{J(f)} \otimes \mathbb{1}\Omega\rangle\\
&= 0
\end{aligned}$$

Finally, deduce that for all linear combinations of element of $C_0(\bar{\mathcal{A}}_\Gamma)$ and $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ it is true that

$$\tilde{\omega}^\Gamma_M((\bar{W}_2 \otimes W_3)(fW)) = 0$$

for all $fW \in L$.

Since the set of all linear combinations of elements of the form $(\bar{W}_2 \otimes W_3)\left(\overline{fW} \otimes \mathbb{1} - \bar{\mathbb{1}} \otimes \alpha_{t/2}((fW)^*)\right)$ is dense in $\mathsf{Weyl}(\breve{S},\Gamma)^{\text{opp}} \otimes \mathsf{Weyl}(\breve{S},\Gamma)$, it follows that $\tilde{\omega}^\Gamma_M = 0$. Hence by Woronowicz's theorem [111, Theorem 3] the theorem follows.

■

Theorem 6.5.10. *The C^*-dynamical system $(C(\bar{\mathcal{A}}_\Gamma), \mathbb{R}, \alpha)$ does not admit any KMS-state.*

Proof : This follows easily from same arguments used in the proof of theorem 6.5.8.

■

Time avarages for the Weyl C^*-algebra for surfaces

The concept of time avarages are studied in mathematics under the term ergodicity or noncommutative mean theory. Refer to Radin [75]. Let $\mathbb{R} \ni t \mapsto \alpha_t \in \mathfrak{Aut}(\mathfrak{A})$ be an evolution of a C^*-algebra $\mathfrak{A}$ with respect to a state ω. Denote the set of states, which are invariant under the automorphism α_t, i.e. $\omega \circ \alpha_t = \omega$ for all $t \in \mathbb{R}$, by $\mathcal{S}^{\mathbb{R}}(\mathfrak{A})$.

Let $W(\mathcal{S}(\mathfrak{A}))$ be the set of all states such that the function $t \mapsto \omega(\alpha_t(A))$ is a weakly almost periodic function[11] on $\mathbb{R}$ for all $A \in \mathfrak{A}$.

[11] The set of weakly almost periodic functions on $\mathbb{R}$ is a subset of the C^*-algebra $C_b(\mathbb{R})$ of complex bounded continuous functions on $\mathbb{R}$, which consists of those functions f such that $\{R_s f : s \in \mathbb{R}\}$ is conditionally weakly compact. $(R_s f)(t) = f(t+s)$.

Proposition 6.5.11. *[75, Prop. 3]Let $\mathfrak{A}$ be a C^*-algebra and let $\omega \in W(\mathcal{S}(\mathfrak{A}))$.*

Then there exists a unique state $\tilde{\omega}$ such that

$$\begin{aligned}\tilde{\omega}(A) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \omega(\alpha_t(A))\,\mathrm{d}\,t\\ \tilde{\omega}\circ\alpha_t &= \tilde{\omega}\end{aligned} \tag{6.89}$$

The state $\tilde{\omega}$ is called a ***time avarage****.*

If ω is a vector density matrix and a KMS-state on $\mathfrak{A}$ then it follows that the time avarage of this state on a matrix C^*-algebra $\mathfrak{A}$ is uniquely defined and it is a density matrix state.

The proposition can be reformulated in terms of a set $\mathcal{S}^{\mathbb{R}}_{AP}(\mathfrak{A})$ of states such that $t \mapsto \omega(\alpha_t(A))$ is an almost periodic functions on $\mathbb{R}$ and a orthogonal measure μ over $\mathcal{S}^{\mathbb{R}}_{AP}(\mathfrak{A})$. Refer to the results in [22, prop.4.3.30] presented by Bratteli and Robinson.

Assume that, all finite graph systems are identified in the non-standard way and forget about the quantum diffeomorphism constraint for the next arguments. Then recall the state $\bar{\omega}^{\Gamma}_M$ on $\mathsf{Weyl}(\breve{S},\Gamma)$ and $\bar{\omega}_M$ on $\mathsf{Weyl}(\breve{S})$ defined in 6.4.6, which are time-independent for the automorphism α defined by equation (6.88). Fix an element $W \in \mathsf{Weyl}(\breve{S},\Gamma)$ and hence the map $t \mapsto \bar{\omega}^{\Gamma}_M(\alpha_t(W))$ is a weakly almost periodic function on $\mathbb{R}$. Then the state (6.89) is unique. But the exponentiated quantum Hamiltonian constraint is much more complicated than $\exp(E^+_S(\Gamma)E_S(\Gamma))$. Furthermore the full quantum constraint set is even more complicated and a lot of problems occur, which have been presented in chapter 2.

Chapter 7

The holonomy-flux cross-product C^*-algebra

In this section a new C^*-algebra in the framework of Loop Quantum Gravity is presented. The algebra is basically a specific cross-product algebra, which is well-known in mathematics (for example refer to Williams [109], Blackadar [17]). Moreover a lot of arguments given in this chapter can be found in Hewitt and Ross [50] or Pedersen [74]. The constuction of the new algebra is developed in several steps.

First of all, the considerations focus on a flux group associated to a surface set only. For generality, it is assumed that G is a unimodular locally compact group. The flux group $\bar{G}_{\breve{S},\Gamma}$ for a suitable surface set $\breve{S}$ and a finite graph system associated to a graph Γ is identified with $G^{|\Gamma|}$.

If only the flux group associated to a surface set and a graph is concerned, then the following algebras are derived. In general, convolution *-algebras are the starting point of the construction of cross-products. Hence there exists a flux convolution *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ associated to a surface set, which is in general a non-commutative *-algebra. Moreover the **flux group C^*-algebra** $C^*(\bar{G}_{\breve{S},\Gamma})$ and the **flux transformation group C^*-algebra** $C^*(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$ **associated to a surface set $\breve{S}$ and a graph** Γ is constructed in section 7.1.

Secondly, consider the configuration space $\bar{\mathcal{A}}_\Gamma$, which is naturally identified with $G^{|\Gamma|}$. Then the convolution holonomy algebra $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ and the holonomy C^*-algebra $C^*(\bar{\mathcal{A}}_\Gamma)$ associated to a graph Γ is derivable. The construction of cross-products for the holonomy variable is related to the observation, which has been noticed by Ashtekar and Lewandowski [9]. The difference between their approach and the construction in this dissertation is the following. In the context of heat kernels the Lewandowski and Ashtekar [9, section 6.2] have presented an object, which can be understood as a generalised heat kernel representation π_I^H of the **non-commutative holonomy C^*-algebra** $C^*(\bar{\mathcal{A}}_\Gamma)$ **associated to a graph** Γ on the Hilbert space $\mathcal{H}_\Gamma := L^2(\bar{\mathcal{A}}_\Gamma, \mathrm{d}\,\mu_\Gamma)$. This object is given by

$$\begin{aligned}\pi_I^H(\rho_{t,\Gamma})\psi_\Gamma &= \int_{\bar{\mathcal{A}}_\Gamma} \mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma)\rho_{t,\Gamma}(\hat{\mathfrak{h}}_\Gamma^{-1}\mathfrak{h}_\Gamma)\psi_\Gamma(\mathfrak{h}_\Gamma)\\ &= \rho_{t,\Gamma} * \psi_\Gamma\end{aligned} \tag{7.1}$$

where $\hat{\mathfrak{h}}_\Gamma$ and $\mathfrak{h}_\Gamma$ are two different holonomies along paths of a graph Γ, and for $\rho_{t,\Gamma} \in C^*(\bar{\mathcal{A}}_\Gamma)$ and $\psi_\Gamma \in \mathcal{H}_\Gamma$. Notice that, $C^*(\bar{\mathcal{A}}_\Gamma)$ is generated by matrix elements of the fundamental representation of $\bar{\mathcal{A}}_\Gamma$, which is identified with $G^{|\Gamma|}$. The inductive limit of the inductive family $\{C^*(\bar{\mathcal{A}}_\Gamma)\}$ of C^*-algebras is called the **non-commutative holonomy C^*-algebra**. Furthermore it is possible to explore the inductive limit C^*-algebra of the inductive family $\{C^*(\bar{\mathcal{A}}_\Gamma, \bar{\mathcal{A}}_\Gamma)\}$ of transformation group algebras, which is called the **heat-kernel-holonomy C^*-algebra**. The last algebras presented in this paragraph are given in section 7.1.

In the context of holonomies and fluxes a similar algebra, which is generated by the flux group associated to a surface set and a graph, and the analytic holonomy C^*-algebra associated to a graph, is developed. This algebra is called the **holonomy-flux cross-product C^*-algebra associated to a graph and a fixed suitable surface set** $\breve{S}$. Let G be compact, then the inductive limit C^*-algebra of the inductive family of holonomy-flux cross-product C^*-algebras associated to graphs and the suitable surface set, is called the **holonomy-flux cross-product C^*-algebra associated to the surface set** $\breve{S}$. This new algebra is investigated in section 7.2. It is shown that, the state associated to a representation of the holonomy-flux cross-product C^*-algebra associated to a surface set on a particular Hilbert space, is not invariant under surface-orientation-preserving graph-diffeomorphism and, hence, under general graph-diffeomorphisms.

Furthermore only for special surface sets and graphs the configuration space and the flux group is identifiable with $G^{|\Gamma|}$. Only in this situation the space of generalised connections and the flux group are not distinguishable from each other. Then the cross-product C^*-algebra is constructed such that the generalised Stone-von Neumann theorem can be applied. This theorem imply the uniqueness of an irreducible representations of the holonomy-flux cross-product C^*-algebra associated to a fixed surface set and a fixed graph on a certain Hilbert space.

Finally the theory of cross-products allows to construct some other new C^*-algebras. For example, in the case of the group of suitable bisections the **holonomy-flux-graph-diffeomorphism cross-product C^*-algebra associated to the surface set $\breve{S}$** is discovered in section 7.3. Moreover the group and transformation group C^*-algebra in the context of Loop Quantum Cosmology is investigated in section 7.4.

7.1 The flux and flux transformation group, n.c. and heat-kernel-holonomy C^*-algebra

The flux group C^*-algebra associated to graphs and a surface set

Let $\mathcal{C}(G)$ be the convolution *-algebra of continuous functions $C_c(G)$ on a locally compact unimodular group G equipped with the convolution product, an inversion and supremum norm.

Recall that a surface S has the same surface intersection property for a graph Γ, if each path of Γ intersect the surface S exactly once in a source (or target) vertex of the path and the path is outgoing and lies below (or ingoing and below, ingoing and above or outgoing and above).

Corollary 7.1.1. *Let S be a surface with same surface intersection property for a finite graph system associated to a graph Γ. Let G be a unimodular locally compact group and let $\bar{G}_{S,\Gamma}$ be the flux group.*

Then the ***convolution flux *-algebra $\mathcal{C}(\bar{G}_{S,\Gamma})$ associated to a surface*** *and a graph Γ is defined by the following product*

$$\begin{aligned}&(f_1 * f_2)(\rho_{S,\Gamma}(\Gamma)) = (f_1 * f_2)(\rho_S(\gamma_1), ..., \rho_S(\gamma_N))\\
&= \int_{G_{S,\Gamma}} f_1(\rho_S(\gamma_1)\hat{\rho}_S(\gamma_1)^{-1}, ..., \rho_S(\gamma_N)\hat{\rho}_S(\gamma_N)^{-1}) f_2(\hat{\rho}_S(\gamma_1), ..., \hat{\rho}_S(\gamma_N))\\
&\qquad \mathrm{d}\,\mu_{S,\Gamma}(\hat{\rho}_S(\gamma_1), ..., \hat{\rho}_S(\gamma_N))\\
&= \int_{G_{S,\Gamma}} f_1(\rho_{S,\Gamma}(\Gamma)\hat{\rho}_{S,\Gamma}(\Gamma)^{-1}) f_2(\hat{\rho}_{S,\Gamma}(\Gamma))\,\mathrm{d}\,\mu_{S,\Gamma}(\hat{\rho}_S(\Gamma))\end{aligned}$$

for $\Gamma = \{\gamma_1, ..., \gamma_N\}$ and where $\rho_{S,\Gamma}, \hat{\rho}_{S,\Gamma} \in G_{S,\Gamma}$ and $\rho_S, \hat{\rho}_S \in \mathbb{G}_{S,\Gamma}$ which reduces to

$$(f_1 * f_2)(\rho_{S,\Gamma}(\Gamma')) = (f_1 * f_2)(\rho_S(\gamma_i)) = \int_{G_{S,\Gamma}} f_1(\rho_S(\gamma_i)\hat{\rho}_S(\gamma_i)^{-1}) f_2(\hat{\rho}_S(\gamma_i))\,\mathrm{d}\,\mu_{S,\Gamma}(\hat{\rho}_S(\gamma_i))$$

for any $i = 1, ..., N$ and $\Gamma' = \{\gamma_i\} \in \mathcal{P}_\Gamma$, the involution

$$f_\Gamma^*(\rho_{S,\Gamma}(\Gamma')) = \overline{f_\Gamma(\rho_{S,\Gamma}(\Gamma')^{-1})}$$

for any $i = 1, ..., N$ and equipped with the supremum norm.

Remark that, if all paths γ_i are ingoing and above (respectively outgoing and below), $\Gamma' := \{\gamma_i\}$, then the product reads

$$(f_1 * f_2)(\rho_{S,\Gamma}(\Gamma')) = \int_G f_1(g_S\hat{g}_S^{-1}) f_2(\hat{g}_S)\,\mathrm{d}\,\mu(\hat{g}_S) \tag{7.2}$$

otherwise

$$(f_1 * f_2)(\rho_{S,\Gamma}(\Gamma')) = \int_G f_1(g_S^{-1}\hat{g}_S) f_2(\hat{g}_S)\,\mathrm{d}\,\mu(\hat{g}_S) \tag{7.3}$$

This implies that, only for one surface the structure is identified with $\mathcal{C}(G)$. The convolution algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ is defined similarly to the one defined in corollary 7.1.1 for a surface set $\breve{S}$ with same surface intersection property for a finite graph system associated to a graph Γ.

Recall that, a set $\breve{S}$ of N surfaces has the simple surface intersection property for a graph Γ with N independent edges, if it contains only surfaces, for which each path γ_i of a graph Γ intersect only one surface S_i only once in the target vertex of the path γ_i, the path γ_i lies above and there are no other intersection points of each path γ_i and each surface in $\breve{S}$. Then the convolution algebra is defined as follows.

Corollary 7.1.2. *Let $\breve{S} := \{S_i\}_{1\leq i\leq N}$ be a set of surfaces with simple surface intersection property for a finite graph system associated to a graph Γ.*

Then the ***convolution flux *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ associated to a surface set*** *and a graph Γ is defined by the following product*

$$\begin{aligned}&(f_1 * f_2)(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N))\\ &= \int_G f_1(\rho_{S_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)\hat{\rho}_{S_N}(\gamma_N)^{-1}) f_2(\hat{\rho}_{S_1}(\gamma_1), ..., \hat{\rho}_{S_N}(\gamma_N))\,\mathrm{d}\,\mu(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\end{aligned}$$

where $\rho_{\breve{S},\Gamma} \in G_{\breve{S},\Gamma}$, $\rho_{S_i} \in \mathbb{G}_{\breve{S},\Gamma}$ for $i = 1, ..., N$, $\rho_{\breve{S},\Gamma}(\Gamma) := (\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N))$, the involution is defined by

$$f^*_\Gamma(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N)) = \overline{f_\Gamma(\rho_{S_1}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)^{-1})}$$

and equipped with the supremum norm.

Clearly $\bar{G}_{\breve{S},\Gamma}$ is identified with G^N for N being the number of independent paths in Γ such that each of the path γ_i intersects a surface S_i.

The convolution algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ can be also studied for other situations, as far as the surface set $\breve{S}$ has one of the surface intersection properties, which have been given in section 3.4.

The dual space $C_0(\bar{G}_{\breve{S},\Gamma})^*$ is identified by the Riesz-Markov theorem with the Banach space of bounded complex Baire measures on $\bar{G}_{\breve{S},\Gamma}$. Moreover each Baire measure has a unique extension to a regular Borel measure on $\bar{G}_{\breve{S},\Gamma}$. The Banach space of all regular Borel measures is denoted by $\mathbf{M}(\bar{G}_{\breve{S},\Gamma})$. There is a convolution multiplication

$$\begin{aligned}&\int_{\bar{G}_{\breve{S},\Gamma}} f(\rho_{\breve{S},\Gamma}(\Gamma'))\,\mathrm{d}(\mu * \nu)(\rho_{\breve{S},\Gamma}(\Gamma'))\\ &= \int_{\bar{G}_{\breve{S},\Gamma}}\int_{\bar{G}_{\breve{S},\Gamma}} f(\rho_{\breve{S},\Gamma}(\Gamma')\hat{\rho}_{\breve{S},\Gamma}(\Gamma'))\,\mathrm{d}\,\mu(\rho_{\breve{S},\Gamma}(\Gamma'))\,\mathrm{d}\,\nu(\hat{\rho}_{\breve{S},\Gamma}(\Gamma'))\end{aligned} \tag{7.4}$$

where $\rho_{\breve{S},\Gamma} \in G_{\breve{S},\Gamma}$, $\rho_{S_i} \in \mathbb{G}_{\breve{S},\Gamma}$ for $i = 1, ..., N$, $\rho_{\breve{S},\Gamma}(\Gamma') := (\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_M))$, $\Gamma' := \{\gamma_1, ..., \gamma_M\}$, $\mu, \nu \in \mathbf{M}(\bar{G}_{\breve{S},\Gamma})$ and $f \in C_0(\bar{G}_{\breve{S},\Gamma})$ and an inversion

$$\int_{\bar{G}_{\breve{S},\Gamma}} f(\rho_{\breve{S},\Gamma}(\Gamma'))\,\mathrm{d}\,\mu^*(\rho_{\breve{S},\Gamma}(\Gamma')) = \overline{\int_{\bar{G}_{\breve{S},\Gamma}} \bar{f}(\rho_{\breve{S},\Gamma}(\Gamma')^{-1})\,\mathrm{d}\,\mu(\rho_{\breve{S},\Gamma}(\Gamma'))} \tag{7.5}$$

which transfer $\mathbf{M}(\bar{G}_{\breve{S},\Gamma})$ to a Banach *-algebra. Then restrict $\mathbf{M}(\bar{G}_{\breve{S},\Gamma})$ to the norm closed subspace consisting of measures absolutely continuous w.r.t. the Haar measure $\mu_{\breve{S},\Gamma}$, which is identified with $L^1(\bar{G}_{\breve{S},\Gamma})$ by $\mathrm{d}\,\mu(\rho_{\breve{S},\Gamma}(\Gamma')) = f_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma'))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma'))$ for $f_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma})$.

Corollary 7.1.3. *Let $\breve{S} := \{S_i\}_{1\leq i\leq N}$ be a set of surfaces with same surface intersection property for a finite graph system associated to a graph Γ.*

The Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma})$ is the continuous extension of $\mathcal{C}(\bar{G}_{\breve{S},\Gamma})$ in the L^1-norm.

There is a non-degenerate *-representation π_0 of $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ on the Hilbert space $\mathcal{H}_\Gamma = L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$, which is of the form

$$\pi_0(f_\Gamma) := \int_{\bar{G}_{\breve{S},\Gamma}} f_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) \tag{7.6}$$

for $f_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ (defined in the sense of a Bochner integral).

Notice that, the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ has an approximate unit. Then for a *-representation π_0 of $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ on $\mathcal{H}_\Gamma$ exists a GNS-triple $(\mathcal{H}_\Gamma,\pi_0,\Omega_0)$ and an associated state ω_0 [89, section 8.6]. Furthermore there is a left regular unitary representation $U_{\overleftarrow{L}}^N$ of $\bar{G}_{\breve{S},\Gamma}$ on $\mathcal{H}_\Gamma$ presented in lemma 6.1.16. Then observe that, for $f_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ and $\rho^N_{\breve{S},\Gamma},\hat{\rho}^N_{\breve{S},\Gamma} \in \bar{G}_{\breve{S},\Gamma}$ the unitary $U_{\overleftarrow{L}}^N$ satisfies

$$\begin{aligned}
&U_{\overleftarrow{L}}^N(\hat{\rho}^N_{\breve{S},\Gamma})\pi_0(f_\Gamma)\Omega_0\\
&= \int_{\bar{G}_{\breve{S},\Gamma}} d\mu_{\breve{S},\Gamma}(\rho^N_{\breve{S},\Gamma}(\Gamma))U_{\overleftarrow{L}}^N(\hat{\rho}^N_{\breve{S},\Gamma})f_\Gamma(\rho^N_{\breve{S},\Gamma})\Omega_0\\
&= \int_{\bar{G}_{\breve{S},\Gamma}} d\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))f_\Gamma(\hat{\rho}_{S_1}(\gamma_1)\rho_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N)\rho_{S_N}(\gamma_N))\Omega_0\\
&= \pi_0(\alpha_{\overleftarrow{L}}^N(\rho^N_{\breve{S},\Gamma})f_\Gamma)\Omega_0
\end{aligned} \tag{7.7}$$

where $\rho^N_{\breve{S},\Gamma}(\Gamma) := (\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))$, Ω_0 is a cyclic vector. This imply

$$\omega_0(\alpha_{\overleftarrow{L}}^N(\hat{\rho}^N_{\breve{S},\Gamma})f_\Gamma) = \omega_0(f_\Gamma) \tag{7.8}$$

and hence that, the state ω_0 on the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ associated to the representation π_0 is $\bar{G}_{\breve{S},\Gamma}$-invariant.

The same is true, if all paths in Γ intersect in vertices of the set V_Γ with a surface S such that all paths are outgoing and lie below the surface S and $U_{\overleftarrow{L}}^1(\hat{\rho}^1_{S,\Gamma})$ is analysed. Clearly this can be also studied for other situations presented in the section 6.3. But for readers only interested in the construction of the holonomy-flux cross-product C^*-algebra, need for a first reading only the definition of the action $\alpha_{\overleftarrow{L}}^N$, which is presented in lemma 6.1.16.

Notice that, the Banach *-algebra $L^1(\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma})$ generated by all Dirac point measures $\{\delta(\rho_{S,\Gamma}(\Gamma')) : \rho_{S,\Gamma}(\Gamma') \in \bar{G}_{\breve{S},\Gamma}\}$ such that

$$\begin{aligned}
&\delta(\rho_{S,\Gamma}(\Gamma')) * \delta(\hat{\rho}_{S,\Gamma}(\Gamma')) = \delta(\rho_{S,\Gamma}(\Gamma')\hat{\rho}_{S,\Gamma}(\Gamma'))\\
&\delta^*(\rho_{S,\Gamma}(\Gamma')) = \delta(\rho_{S,\Gamma}(\Gamma')^{-1})
\end{aligned}$$

Moreover recognize that

$$\begin{aligned}
&(\delta(\rho_{S,\Gamma}(\Gamma')) * f_\Gamma)(\hat{\rho}_{S,\Gamma}) = f_\Gamma(\rho_{S,\Gamma}(\Gamma')^{-1}\rho_{S,\Gamma}(\Gamma'))\\
&(f_\Gamma * \delta(\rho_{S,\Gamma}(\Gamma')))(\hat{\rho}_{S,\Gamma}) = f_\Gamma(\rho_{S,\Gamma}(\Gamma')\rho^{-1}_{S,\Gamma}(\Gamma'))
\end{aligned}$$

for all $f_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ and $\rho_{S,\Gamma} \in \bar{G}_{\breve{S},\Gamma}$.

Observe that for $A = \sum_{i=1}^n a_i\delta(\rho_{S_i,\Gamma}(\Gamma'))) \in L^1(\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma})$ and $\breve{S} := \{S_i\}_{1\leq i\leq N}$, there is a state $\hat{\omega}_0$ on $L^1(\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma})$ such that

$$\begin{aligned}
\hat{\omega}_0(A^*A) &= \sum_{n,m}\overline{a_n}a_m\hat{\omega}_0(\delta^*(\rho_{S_n,\Gamma}(\Gamma'))\delta(\rho_{S_m,\Gamma}(\Gamma')))\\
&= \sum_{n,m}\overline{a_n}a_m\hat{\omega}_0(\delta(\rho_{S_n,\Gamma}(\Gamma')^{-1}\rho_{S_m,\Gamma}(\Gamma')))
\end{aligned} \tag{7.9}$$

Moreover for an action α of $\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma}$ on $L^1(\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma})$ the action is automorphic and point-norm continuous. The state is defined by

$$\hat{\omega}_0(\delta(\rho_{S,\Gamma}(\Gamma'))) := \begin{cases} 1 & \text{for } \rho_{S,\Gamma}(\Gamma') = e_G\\ 0 & \text{for } \rho_{S,\Gamma}(\Gamma') \neq e_G\end{cases}$$

Derive

$$\begin{aligned}\hat{\omega}_0(\alpha(\tilde{\rho}_{S,\Gamma})(\delta(\rho_{S_n,\Gamma}(\Gamma')))) &= \hat{\omega}_0(\delta(\tilde{\rho}_{S_n,\Gamma}(\Gamma')\rho_{S_n,\Gamma}(\Gamma')\tilde{\rho}_{S_n,\Gamma}(\Gamma')^{-1}))\\ &= \hat{\omega}_0(\delta(\rho_{S_n,\Gamma}(\Gamma')))\end{aligned} \tag{7.10}$$

The state $\hat{\omega}_0$ is $\bar{G}^{\mathrm{d}}_{\breve{S},\Gamma}$-invariant.

Definition 7.1.4. *Let the surface S has the same surface intersection property for a graph Γ, let $\breve{S}$ be a set of surfaces $S_1,...,S_N$ having the same surface intersection property for a graph Γ.*

The ***generalised group-valued quantum flux operator for a surface*** *S is given by the following non-degenerate representation $\pi_{S,\Gamma}$ of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ on the Hilbert space $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$, which satisfy $\|\pi_{S,\Gamma}(f_\Gamma)\|_2 \leq \|f_\Gamma\|_1$ and is defined as a $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$-valued Bochner integral*

$$\pi_{S,\Gamma}(f_\Gamma)\psi_\Gamma \quad := \int_{\bar{G}_{S,\Gamma}} \mathrm{d}\,\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma)) f_\Gamma(\rho_{S,\Gamma}(\Gamma)) U(\rho_{S,\Gamma}(\Gamma))\psi_\Gamma \qquad \text{for } f_\Gamma \in L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$$

and a weakly continuous unitary representation U of $\bar{G}_{S,\Gamma}$ acting on a vector ψ_Γ in $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$.

The ***generalised group-valued quantum flux operator for a set of surfaces*** *$\breve{S}$ is given by the following non-degenerate representation $\pi_{\breve{S},\Gamma}$ of $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ on the Hilbert space $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$, which satisfy $\|\pi_{\breve{S},\Gamma}(f_\Gamma)\|_2 \leq \|f_\Gamma\|_1$ and is defined as a $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$-valued Bochner integral*

$$\begin{aligned}&\pi_{\breve{S},\Gamma}(f_\Gamma)\psi_\Gamma\\ &:= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &\qquad f_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) U_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\end{aligned}$$

for $f_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ and a weakly continuous unitary representation U of $G_{\breve{S},\Gamma}$ acting on a vector ψ_Γ in $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$.

It is easy to show that, for example the representation associated to a left regular representation $U^N_{\overleftarrow{L}}$ of $\bar{G}_{\breve{S},\Gamma}$ on $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ fulfill

$$\begin{aligned}&\Phi_M(f_\Gamma)\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),..,\hat{\rho}_{S_N}(\gamma_N))\\ &\quad= \int_{\bar{G}_{\breve{S},\Gamma}} d\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) f_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &\qquad\qquad U^N_{\overleftarrow{L}}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),..,\hat{\rho}_{S_N}(\gamma_N))\\ &\quad= \int_{\bar{G}_{\breve{S},\Gamma}} d\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) f_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &\qquad\qquad \psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),..,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))\\ &\quad= f_\Gamma * \psi_\Gamma \text{ for } \psi_\Gamma \in L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})\end{aligned} \tag{7.11}$$

It is a *-representation on the Hilbert space $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$, since it is true that

$$\begin{aligned}&\Phi_M(f^1_\Gamma * f^2_\Gamma)\psi_\Gamma = \Phi_M(f^1_\Gamma)\Phi_M(f^2_\Gamma)\psi_\Gamma\\ &\Phi_M(\lambda_1 f^1_\Gamma + \lambda_2 f^2_\Gamma)\psi_\Gamma = \lambda_1\Phi_M(f^1_\Gamma)\psi_\Gamma + \lambda_2\Phi_M(f^2_\Gamma)\psi_\Gamma\\ &\Phi_M(f^*_\Gamma)\psi_\Gamma = \Phi_M(f_\Gamma)^*\psi_\Gamma\end{aligned} \tag{7.12}$$

for $f_\Gamma, f^1_\Gamma, f^2_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ and $\lambda_1,\lambda_2 \in \mathbb{C}$.

The representation associated to the right regular representation $U^N_{\overleftarrow{R}}$ of $\bar{G}_{\breve{S},\Gamma}$ on $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ is equivalent to

$$\begin{aligned}&\Phi_M(f_\Gamma)\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &:= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\\ &\qquad f_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma)) U^N_{\overleftarrow{R}}(\rho^N_{\breve{S},\Gamma})\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &= \psi_\Gamma * f_\Gamma\end{aligned} \tag{7.13}$$

Clearly there is a representation of $L^1(\bar{G}_{S,\Gamma},\mu_{\breve{S},\Gamma})$, which correponds to the situation of all paths intersecting with one surface S and such that all paths are outgoing and lie below the surface S, on the Hilbert space $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$. The representation is illustrated by

$$\begin{aligned}
&\pi_{\overleftarrow{L},N}(f_\Gamma)\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),..,\hat{\rho}_{S_N}(\gamma_N))\\
&\quad=\int_G d\mu(\rho_S(\gamma_1),...,\rho_S(\gamma_N))f_\Gamma(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\\
&\qquad\qquad U^N_{\overleftarrow{L}}(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),..,\hat{\rho}_{S_N}(\gamma_N))\\
&\quad=\int_G d\mu(\rho_S(\gamma_1),...,\rho_S(\gamma_N))f_\Gamma(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\\
&\qquad\qquad \psi_\Gamma(\rho_S(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),..,\rho_S(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\\
&\quad=\int_G d\mu(g_S)f_\Gamma(g_S)\psi_\Gamma(g_S\hat{g}_{S_1},..,g_S\hat{g}_{S_N})\\
&\quad=f_\Gamma*\psi_\Gamma\text{ for }\psi_\Gamma\in\mathcal{H}_\Gamma
\end{aligned}\tag{7.14}$$

for any $i=1,...,N$ and where all surfaces S_i are elements of the surface set $\breve{S}$.

On the other hand, a representation of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ on the Hilbert space $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ is defined by

$$\begin{aligned}
&\pi_{\overleftarrow{L},1}(f_\Gamma)\psi_\Gamma(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))\\
&\quad=\int_G d\mu_{S,\Gamma}(\rho_S(\gamma_1),...,\rho_S(\gamma_N))f_\Gamma(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\\
&\qquad U^1_{\overleftarrow{L}}(\rho_S(\gamma_1),...,\rho_S(\gamma_N))\psi_\Gamma(\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))\\
&\quad=\int_G d\mu_{S,\Gamma}(\rho_S(\gamma_i))f_\Gamma(\rho_S(\gamma_i))\psi_\Gamma(\rho_S(\gamma_i)\hat{\rho}_S(\gamma_i))\\
&\quad=\int_G d\mu_{S,\Gamma}(g_S)f_\Gamma(g_S)\psi_\Gamma(g_S\hat{g}_S)\\
&\quad=f_\Gamma*\psi_\Gamma\text{ for }\psi_\Gamma\in L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})
\end{aligned}\tag{7.15}$$

Moreover a general representation $\pi_{\breve{S},\Gamma}$ is a faithful regular [1] *-representation of $\mathcal{C}^*_r(\bar{G}_{\breve{S},\Gamma})$ in $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$. It is a faithful representation, since from $f_\Gamma*\psi_\Gamma=0$ it is deducible that $f_\Gamma=0$. The left and the right regular representations $U^1_{\overleftarrow{L}}$ and $U^1_{\overleftarrow{R}}$ are unitarily equivalent, hence, the generalised representations $\pi_{\overleftarrow{L},1}$ and $\pi_{\overleftarrow{R},1}$ are unitarily equivalent, too.

Definition 7.1.5. *Let S be a surface and $\breve{S}$ be a set of surfaces such that S and $\breve{S}$ have the same surface intersection property for a graph Γ.*

The ***reduced flux group C^*-algebra $\mathcal{C}^*_r(\bar{G}_{S,\Gamma})$ for a surface S or $\mathcal{C}^*_r(\bar{G}_{\breve{S},\Gamma})$ for a set $\breve{S}$ of surfaces*** *is defined as the closure of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ or, respectively, $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ in the norm $\|f_\Gamma\|_r:=\|\pi_{S,\Gamma}(f_\Gamma)\|_2$ or $\|f_\Gamma\|_r:=\|\pi_{\breve{S},\Gamma}(f_\Gamma)\|_2$.*

In fact, all continuous unitary representations U of the flux group $\bar{G}_{S,\Gamma}$ on $L^2(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ give a non-degenerate representation $\pi_{S,\Gamma}$ of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$. Each representation is given by

$$\pi_{S,\Gamma}(f_\Gamma):=\int_{\bar{G}_{S,\Gamma}}\mathrm{d}\,\mu_{S,\Gamma}(\rho_{S,\Gamma}(\Gamma))f_\Gamma(\rho_{S,\Gamma}(\Gamma))U(\rho_{S,\Gamma}(\Gamma))\tag{7.16}$$

Definition 7.1.6. *Let S be a surface and $\breve{S}$ be a set of surfaces such that S and $\breve{S}$ have the same surface intersection property for a graph Γ.*

The ***flux group C^*-algebras $C^*(\bar{G}_{S,\Gamma})$ for a surface S or $C^*(\bar{G}_{\breve{S},\Gamma})$ for a set $\breve{S}$ of surfaces*** *is the closure of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$ or $L^1(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$ in the norm $\|f_\Gamma\|:=\sup_\pi\|\pi(f_\Gamma)\|_2$ where the supremum is taken over all*

[1] A representation $(\pi,\mathcal{H})$ of a C^*-algebra $\mathfrak{A}$ of the form (7.1.4) is called **regular** iff the unitary representation U of a locally compact group G is weak operator continuous on $\mathcal{H}$.

non-degenerate L^1-norm decreasing[2] *-representations of $L^1(\bar{G}_{S,\Gamma},\mu_{S,\Gamma})$, or respectively, all representations π of the form (7.16), where U is a continuous unitary representation (one representative of each equivalence class) of the flux group $\bar{G}_{S,\Gamma}$ on a Hilbert space.

Remark 7.1.7. *In the case of a (second countable) compact group G the structures above are well known. Let $\hat{G}$ be the unitary dual consisting of all unitary equivalence classes of irreducible, continuous and unitary and, therefore, finite-dimensional representations π_{s,γ_i} of G w.r.t. a graph $\Gamma := \{\gamma_i\}$ on a finite dimensional Hibert space $\mathcal{H}_{s,\gamma_i}$. Notice that, every element of $\hat{G}$ is one-dimensional iff G is commutative. The dual $\hat{G}$ is discrete and countable. The set $\hat{G}$ is finite iff G is finite. The finite-dimensional representation U_{s,γ_i} is equivalent to the left-regular representation $U_L : G \to U(L^2(G))$.*

There exists an isomorphisms betweeen Hilbert spaces such that

$$\mathcal{H}_\Gamma := L^2_{\gamma_i}(G) \simeq L^2_{\gamma_i}(\hat{G}) := \hat{\mathcal{H}}_\Gamma = \bigoplus_{s\in\hat{G}} M_{d_{s,\gamma_i}}(\mathbb{C})$$

where d_{s,γ_i} is the dimension of s in $\hat{G}$, given by the unitary Plancherel transform $\mathcal{F} : L^2_{\gamma_i}(G) \to L^2_{\gamma_i}(\hat{G})$ with

$$\hat{\psi}_{\gamma_i}(s) := (\mathcal{F}\psi_{\gamma_i})(s) = \sqrt{d_{s,\gamma_i}} \int_G \mathrm{d}\,\mu(\rho_{S_i}(\gamma_i)) U_{s,\gamma_i}(\rho_{S_i}(\gamma_i))\psi_{\gamma_i}(\rho_{S_i}(\gamma_i)) \tag{7.17}$$

where $\rho_{S_i}(\gamma_i) \in \bar{\mathbb{G}}_{\breve{S},\Gamma}$ is identified with G if $\breve{S} := \{S_i\}_{1\leq i\leq N}$ has the same intersection property for Γ. The inverse transform is given by

$$\mathcal{F}^{-1}\hat{\psi}_{\gamma_i}(s) := \sum_{s\in\hat{G}} \sqrt{\dim \pi_{s,\gamma_i}}\, \mathrm{tr}(\hat{\psi}_{\gamma_i}(s) U_{s,\gamma_i}(\rho_{S_i}(\gamma_i))^*)$$

Clearly, if $\psi_\Gamma \in L^2_{\gamma_i}(G)$ and $\hat{\psi}_\Gamma \in L^2_{\gamma_i}(\hat{G})$ it is true that

$$\int_G |\psi_\Gamma(\rho_{S_i}(\gamma_i))|^2 \,\mathrm{d}\,\mu(\rho_{S_i}(\gamma_i)) = \sum_{s\in\hat{G}} (\dim \pi_\sigma)\, \mathrm{tr}(\hat{\psi}_\Gamma(s)\hat{\psi}_\Gamma(s)^*) \tag{7.18}$$

holds. Let Γ be equivalent to $\{\gamma\}$ and S has the same intersection property for Γ. The representation $\pi_{S,\Gamma}$ of the C^*-algebra $C^*_r(\bar{G}_{S,\Gamma})$ on the Hilbert space $\mathcal{H}_\Gamma := L^2(\bar{G}_{S,\gamma},\mu_{S,\gamma})$ is given for a path γ that intersect S such that the path is outgoing and lie below by

$$\pi_{S,\Gamma}(f_\Gamma)\psi_\Gamma := \int_G \mathrm{d}\,\mu_{S,\gamma}(\rho_S(\gamma)) f_\Gamma(\rho_S(\gamma)) U_{s,\Gamma}(\rho_S(\gamma))\psi_\Gamma(\rho_S(\gamma)) \tag{7.19}$$

for $\psi_\Gamma \in \mathcal{H}_\Gamma$. Notice that, for an abelian (locally) compact flux group $\bar{G}_{S,\Gamma}$ there is an isomorphism $\mathcal{F} : C^*_r(\bar{G}_{S,\Gamma}) \to C_0(\widehat{\bar{G}_{S,\Gamma}})$ given by

$$\mathcal{F}(f_\Gamma)(s) := \int_G \mathrm{d}\,\mu_{S,\gamma}(\rho_S(\gamma)) f_\Gamma(\rho_S(\gamma)) U_{s,\Gamma}(\rho_S(\gamma))$$

which is called the generalised Fourier transform. The set of characters is denoted by $\widehat{\bar{G}_{S,\Gamma}}$.

Example 7.1.1: For an abelian locally compact group G the group algebra $C^*(G)$ coincide with $C^*_r(G)$. This is true since for $s \in \hat{G}$ the representation π_s of G on $L^2(G)$ coincide with $\hat{f}(s) \in \mathbb{C}$ and, consequently, the norm $\|.\|_r$ and $\|.\|$ are the same.

Moreover since $\mathbb{R}$ and $\hat{\mathbb{R}}$ are equal, there are the following isomorphisms

$$C_0(\mathbb{R}) \simeq C^*(\mathbb{R}) \simeq C^*_r(\mathbb{R})$$

Notice that, this statement generalises for an abelian locally compact group G. There is an isomorphism $C^*(G)$ and $C(\hat{G})$.

[2] A norm $\|.\|$ of $\mathfrak{A}$ is called L^1-norm decreasing if $\|\pi_I(f)\|_2 \leq \|f\|$ for all $f \in \mathfrak{A}$.

For a general locally compact group $\bar{G}_{\breve{S},\Gamma}$, it is true that

$$C_r^*(\bar{G}_{\breve{S},\Gamma}) := \pi_{S,\Gamma}(C^*(\bar{G}_{\breve{S},\Gamma})) \simeq C^*(\bar{G}_{\breve{S},\Gamma}) \setminus \ker(\pi_{S,\Gamma})$$

Therefore, a Lie group is called amenable, when $C^*(\bar{G}_{\breve{S},\Gamma})$ coincide with $C_r^*(\bar{G}_{\breve{S},\Gamma})$ and hence, iff $\pi_{S,\Gamma}$ is faithful. Since for locally compact groups, the representation $\pi_{S,\Gamma}$ is always faithful, these groups are always amenable.

▪

Proposition 7.1.8. *Let S be a surface with the same surface intersection property for a graph Γ.*

For a compact Lie group G the flux group C^*-algebras for surface S and a graph $\Gamma := \{\gamma\}$ is given by

$$C_r^*(\bar{G}_{\breve{S},\Gamma}) \simeq C^*(\bar{G}_{\breve{S},\Gamma}) \simeq \bigoplus_{\pi_{s,\Gamma}\in\hat{G}} M_{d_{s,\Gamma}}(\mathbb{C}) =: M_\Gamma$$

or

$$C_r^*(\bar{G}_{S,\Gamma}) \simeq C^*(\bar{G}_{S,\Gamma}) \simeq \bigoplus_{\pi_{s,\Gamma}\in\hat{\bar{G}}_{S,\Gamma}} M_{d_{s,\Gamma}}(\mathbb{C})$$

and, hence, $\bar{G}_{\breve{S},\Gamma}$ is amenable.

Proof : This is due to the remark 7.1.7.

■

The flux transformation group C^*-algebra associated to graphs and a surface set

In the general theory, for arbitrary locally compact groups the transform $\pi_{\overleftarrow{L},1}(f_\Gamma)$ given by the left regular representation $\pi_{\overleftarrow{L},1}$ of $\bar{G}_{S,\Gamma}$, which is defined by $\pi_{\overleftarrow{L},1}(f_\Gamma)\psi_\Gamma := f_\Gamma * \psi_\Gamma$ for $f_\Gamma \in L^1(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$ on the Hilbert space $L^2(\bar{G}_{S,\Gamma}, \mu_{S,\Gamma})$, is a compact operator.

Notice that the set of functions $\mathcal{C}(\bar{G}_{S,\Gamma}, \bar{G}_{S,\Gamma})$ for a locally compact group, which is a linear subspace of $\mathcal{C}(\bar{G}_{S,\Gamma}, C_0(\bar{G}_{S,\Gamma}))$.

Theorem 7.1.9. *[109, Theorem 4.24]* ***(Generalised Stone- von Neumann theorem):***

Let $\breve{S}$ be be a set of surfaces with the simple surface intersection property for a graph Γ.

Let G be a locally compact unimodular group, $\bar{G}_{\breve{S},\Gamma}$ be the flux group and let U be a continuous, irreducible and unitary representation of $\bar{G}_{\breve{S},\Gamma}$ on $L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}) =: \mathcal{H}_\Gamma$. Hence $U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$.

Let $C_0(\bar{G}_{\breve{S},\Gamma})$ be the C^*-algebra of continuous functions vanishing at infinity on $\bar{G}_{\breve{S},\Gamma}$ with a pointwise multiplication and sup*-norm and let Φ_M is the multiplication representation of $C_0(\bar{G}_{\breve{S},\Gamma})$ on $\mathcal{H}_\Gamma$. Therefore, $\Phi_M \in \mathrm{Mor}(C_0(\bar{G}_{\breve{S},\Gamma}), \mathcal{L}(\mathcal{H}_\Gamma))$.*

Then the linear map $\pi_I : \mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma})) \to \mathcal{L}(L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}))$ of the form

$$\begin{aligned}&(\pi_I(F_\Gamma)\psi_\Gamma)(\hat{\rho}_{S_1}(\gamma_1), ..., \hat{\rho}_{S_N}(\gamma_N))\\ &:= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\Phi_M(F_\Gamma(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N); \hat{\rho}_{S_1}(\gamma_1), ..., \hat{\rho}_{S_N}(\gamma_N)))\\ &\qquad U(\rho_{\breve{S},\Gamma}(\Gamma))\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1), ..., \hat{\rho}_{S_N}(\gamma_N))\end{aligned} \tag{7.20}$$

is a faithful and irreducible representation of the convolution *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ of continuous functions $\bar{G}_{\breve{S},\Gamma} \to C_0(\bar{G}_{\breve{S},\Gamma})$ with compact support acting on the Hilbert space $L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma})$. The convolution *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ is equipped with a norm $\|.\|_1$ such that its completion is given by the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$. Consequently $\pi_I \in \mathrm{Rep}(L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma})), \mathcal{L}(\mathcal{H}_\Gamma))$.

Set $\|F_\Gamma\|_u := \sup_{\pi_I} \|\pi_I(F_\Gamma)\|$, where the supremum is taken over all non-degenerate L^1-norm decreasing[3] **-representations π_I of the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$, or respectively, over all representations π_I of the form (7.20) where $(\Phi_M, U_{\overleftarrow{L}}^N)$ is a covariant Hilbert space representation of the C^*-dynamical system $(C_0(\bar{G}_{\breve{S},\Gamma}), \alpha_{\overleftarrow{L}}^N, \bar{G}_{\breve{S},\Gamma})$.*

Then the range of the closure of $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ w.r.t. the norm $\|.\|_u$ is called the ***flux transformation group C^*-algebra $C^*(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$ for a set $\breve{S}$ of surfaces and a graph Γ.***

Moreover $C^*(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$ is isomorphic to the C^*-algebra $\mathcal{K}(L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}))$ of compact operators. The C^*-algebras $C^*(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$ and $\mathcal{K}(L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}))$ are Morita equivalent C^*-algebras.

Proof : *Step A.: Existence of the flux transformation group algebra for a graph*
The convolution *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ is given by the convolution product

$$\begin{aligned}&(F^1_\Gamma * F^2_\Gamma)(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N),\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu((\gamma_1),...,\rho_{S_N}(\gamma_N)) F^1_\Gamma(\tilde{\rho}_{S_1}(\gamma_1),...,\tilde{\rho}_{S_N}(\gamma_N),\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &\qquad F^2_\Gamma(\tilde{\rho}_{S_1}(\gamma_1)^{-1}\rho_{S_1}(\gamma_1),...,\tilde{\rho}_{S_N}(\gamma_N)^{-1}\rho_{S_N}(\gamma_N),\tilde{\rho}_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\tilde{\rho}_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))\end{aligned}$$

and involution

$$F^*_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N),\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N)) = \overline{F(\rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1},\hat{\rho}_{S_1}(\gamma_1)^{-1},...,\hat{\rho}_{S_N}(\gamma_N)^{-1})}$$

Equipp the convolution *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ with the $\|.\|_1$-norm, which is defined by

$$\begin{aligned}&\|F_\Gamma\|_1\\ &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu((\gamma_1),...,\rho_{S_N}(\gamma_N)) \sup_{\substack{(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ \in \bar{G}_{\breve{S},\Gamma}}} |F_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N),\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))|\end{aligned}$$

and complete the algebra to the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$.

Set $\mathcal{H}_\Gamma := L^2(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$. Assume that the surface set has the simple surface property for a graph Γ and all paths lie below and are outgoing. Let $U_{\overleftarrow{L}}^N \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$, $F_\Gamma \in \mathcal{C}(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$. Then the map

$$\begin{aligned}&\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &:= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1))...\,\mathrm{d}\,\mu(\rho_{S_N}(\gamma_N))\\ &\qquad F_\Gamma(\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N);\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N)) U^N_{\overleftarrow{L}}(\rho^N_{\breve{S},\Gamma})\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\ &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1))...\,\mathrm{d}\,\mu(\rho_{S_N}(\gamma_N))\\ &\qquad F_\Gamma(\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N);\hat{\rho}_S(\gamma_1),...,\hat{\rho}_S(\gamma_N))\psi_\Gamma(\rho_{S_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\end{aligned}$$

defines a *-homomorphism $\pi_I : \mathcal{C}(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma}) \to \mathcal{L}(\mathcal{H}_\Gamma)$, which is extended to a *-homomorphism from $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ to $\mathcal{L}(\mathcal{H}_\Gamma)$ and, therefore, define a *-representation of $\mathcal{C}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$. Furthermore it extends to a *-representation of $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{G}_{\breve{S},\Gamma}))$ on $\mathcal{H}_\Gamma$. The representation is faithful, since from $\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\psi_\Gamma = F_\Gamma * \psi_\Gamma = 0$ it follows that $F_\Gamma = 0$. Clearly, this investigation carry over for arbitrary surface sets, which have the simple surface intersection property for Γ.

Step B.: Isomorphism between $C^*(\bar{G}_{\breve{S},\Gamma}, \bar{G}_{\breve{S},\Gamma})$ and $\mathcal{K}(L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}))$
Secondly, $\pi_I(F_\Gamma)$ is Hilbert-Schmidt if $\|\pi_I(F_\Gamma)\|^2_2 < \infty$, which is verified by the following computation

$$\begin{aligned}\|\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\|^2_2 &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu(\rho_{\breve{S},\Gamma}(\Gamma)) \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\\ &\qquad |F_\Gamma(\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N);\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))|^2\end{aligned}$$

[3] A norm $\|.\|$ of $\mathfrak{A}$ is called L^1-norm decreasing if $\|\pi_I(f)\| \leq \|f\|_1$ for all $f \in \mathfrak{A}$.

which is finite for every $F_\Gamma\in C^*(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$. Consequently $\pi^N_{I,\overleftarrow{L}}(\mathcal{C}(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma}))$ is a subset of the Hilbert Schmidt class $\mathcal{K}_{HS}(L^2(\bar G_{\breve S,\Gamma}))$, which is a dense subspace (w.r.t. in the usual operator norm) of the C^*-algebra of compact operators $\mathcal{K}(L^2(\bar G_{\breve S,\Gamma}))$. Hence the closure of $\pi^N_{I,\overleftarrow{L}}(\mathcal{C}(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma}))$ is equivalent to $\mathcal{K}(L^2(\bar G_{\breve S,\Gamma}))$ in the operator norm and equality of the C^*-algebra $\pi^N_{I,\overleftarrow{L}}(\mathcal{C}(\bar G_{\breve S,\Gamma},C_0(\bar G_{\breve S,\Gamma}))$ and $\mathcal{K}(L^2(\bar G_{\breve S,\Gamma}))$ is due to the fact that $\pi^N_{I,\overleftarrow{L}}$ is faithful.

Step C.: All non-degenerate representations of $L^1(\bar G_{\breve S,\Gamma},C_0(\bar G_{\breve S,\Gamma}))$ are unitarily equivalent to $\pi^N_{I,\overleftarrow{L}}$
To show that there is an isomorphism between the categories of representations of $C^*(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$ and $\mathcal{K}(L^2(\bar G_{\breve S,\Gamma}))$, which is isomorphic to the representations of $\mathbb{C}$, on a Hilbert space. This is equivalent to the property of $C^*(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$ and $\mathcal{K}(L^2(\bar G_{\breve S,\Gamma}))$ being Morita equivalent C^*-algebras.

Step 1.: two pre-C^*-algebras $\mathfrak{A}_\Gamma$,$\mathfrak{B}$ and a full pre-Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$
Assume that the surface set has the simple surface property for a graph Γ and all paths lie below and are outgoing. Let $U^N_{\overleftarrow{L}}\in\operatorname{Rep}(\bar G_{\breve S,\Gamma},\mathcal{K}(\mathcal{H}_\Gamma))$.

Consider the pre-C^*-algebras $\mathfrak{A}_\Gamma=\mathcal{C}(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$ and $\mathfrak{B}=\mathbb{C}$. Moreover let $\mathcal{E}_\Gamma=C_c(\bar G_{\breve S,\Gamma})$ be a full pre-Hilbert $\mathbb{C}$-module, which is defined by the $\mathbb{C}$-action π_R on $C_c(\bar G_{\breve S,\Gamma})$, i.o.w. $\pi_R(\lambda)\psi_\Gamma=\psi_\Gamma\lambda$, and the inner product

$$\langle\psi_\Gamma,\phi_\Gamma\rangle_{\mathbb{C}}:=\langle\psi_\Gamma,\phi_\Gamma\rangle_2$$

Step 2.: full right Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$
The completition of $\mathcal{E}_\Gamma$ is a Hilbert $\mathbb{C}$-module.

Step 3.: left-action of $\mathfrak{A}_\Gamma$ on $\mathcal{E}_\Gamma$ s.t. $\mathcal{E}_\Gamma$ is a full left pre-Hilbert $\mathfrak{A}_\Gamma$-module
The left-action of $\mathfrak{A}_\Gamma$ on $\mathcal{E}_\Gamma$ is defined by $F_\Gamma\psi_\Gamma:=\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\psi_\Gamma$ and, therefore,

$$\begin{aligned}&(\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\psi_\Gamma)(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\\&=\int_{\bar G_{\breve S,\Gamma}}\mathrm{d}\,\mu_{\breve S,\Gamma}(\rho_{\breve S,\Gamma}(\Gamma))F_\Gamma(\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N);\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))\\&\quad\psi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\end{aligned}$$

and for $F^*_\Gamma\psi_\Gamma:=\pi^N_{I,\overleftarrow{L}}(F^*_\Gamma)\psi_\Gamma$

$$\begin{aligned}&(\pi^N_{I,\overleftarrow{L}}(F^*_\Gamma)\psi_\Gamma)(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\&\quad=\int_{\bar G_{\breve S,\Gamma}}\mathrm{d}\,\mu_{\breve S,\Gamma}(\rho_{\breve S,\Gamma}(\Gamma))\psi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\\&\qquad F^*_\Gamma(\hat\rho_{S_1}(\gamma_1)\rho_{S_1}(\gamma_1)^{-1},....,\hat\rho_{S_1}(\gamma_N)\rho_{S_N}(\gamma_N)^{-1};\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N))\\&\quad=\int_{\bar G_{\breve S,\Gamma}}\mathrm{d}\,\mu_{\breve S,\Gamma}(\rho_{\breve S,\Gamma}(\Gamma))\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))\\&\qquad\overline{F_\Gamma(\hat\rho_{S_1}(\gamma_1)^{-1},....,\hat\rho_{S_1}(\gamma_N)^{-1};\rho_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N))}\end{aligned}$$

and there is a $\mathcal{C}(\bar G_{\breve S,\Gamma}\times\bar G_{\breve S,\Gamma})$-valued inner product on $C_c(\bar G_{\breve S,\Gamma})$ given by

$$\begin{aligned}&\langle\phi_\Gamma,\varphi_\Gamma\rangle_{\mathcal{C}(\bar G_{\breve S,\Gamma}\times\bar G_{\breve S,\Gamma})}\\&:=\phi_\Gamma(\rho_{\breve S,\Gamma}(\Gamma))\overline{\varphi_\Gamma(\hat\rho_{S_1}(\gamma_1)^{-1}\rho_{S_1}(\gamma_1),....,\hat\rho_{S_N}(\gamma_N)^{-1}\rho_{S_N}(\gamma_N))}\end{aligned}$$

Notice, $\mathcal{C}(\bar G_{\breve S,\Gamma}\times\bar G_{\breve S,\Gamma})\subset\mathcal{C}(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$. Consequently $C_c(\bar G_{\breve S,\Gamma})$ is a full pre-Hilbert $\mathfrak{A}_\Gamma$-module.

Step 4.: full left Hilbert $\mathfrak{A}$-module $\mathcal{E}_\Gamma$
The completition of $\mathcal{E}_\Gamma$ is a Hilbert $C^*(\bar G_{\breve S,\Gamma},\bar G_{\breve S,\Gamma})$-module.

Step 4.: $\mathfrak{A}_\Gamma$-$\mathfrak{B}$-imprimitivity bimodule $\mathcal{E}_\Gamma$
Step 4.1:

Derive

$$\langle\psi_\Gamma, F_\Gamma\phi_\Gamma\rangle_{\mathbb{C}} = \int_G \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma})\overline{\psi_\Gamma(\rho_{\breve{S},\Gamma})}\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\phi_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma))$$

$$\begin{aligned}\langle\psi_\Gamma, F_\Gamma\phi_\Gamma\rangle_{\mathbb{C}} = &\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\overline{\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))}\\ &F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma);\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\phi_\Gamma(\rho_{S_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\\ = &\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\overline{\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))}\\ &F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma);\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))\phi_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma))\\ = &\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\overline{\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))}\\ &\overline{\left(\overline{F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma);\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))}\right)}\phi_\Gamma(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\end{aligned}$$

$$\begin{aligned}\langle\psi_\Gamma, F_\Gamma\phi_\Gamma\rangle_{\mathbb{C}} = &\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\\ &\overline{\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))}\\ &\overline{F^*_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma);\hat{\rho}_{S_1}(\gamma_1)\rho_{S_1}(\gamma_1)^{-1},....,\hat{\rho}_{S_N}(\gamma_N)\rho_{S_N}(\gamma_N)^{-1})}\\ &\phi_\Gamma(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\\ = &\langle F^*_\Gamma\psi_\Gamma,\phi_\Gamma\rangle_{\mathbb{C}} = \langle\pi_I(F^*_\Gamma)\psi_\Gamma,\phi_\Gamma\rangle_{\mathbb{C}}\end{aligned}$$

for $F_\Gamma \in \mathcal{C}(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})$, $\psi_\Gamma,\phi_\Gamma \in C_c(\bar{G}_{\breve{S},\Gamma})$ and

$$\langle\lambda\psi_\Gamma,\phi_\Gamma\rangle_{C(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})} = \langle\psi_\Gamma,\lambda^*\phi_\Gamma\rangle_{C(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})} = \langle\psi_\Gamma,\overline{\lambda}\phi_\Gamma\rangle_{C(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})}$$

for $\lambda \in \mathbb{C}$ and $\psi_\Gamma,\phi_\Gamma \in C_c(\bar{G}_{\breve{S},\Gamma})$.

Step 4.2:

$$\phi_\Gamma\langle\psi_\Gamma,\varphi_\Gamma\rangle_{\mathbb{C}} = \pi_R(\langle\psi_\Gamma,\varphi_\Gamma\rangle_{\mathbb{C}})\phi_\Gamma = \langle\phi_\Gamma,\psi_\Gamma\rangle_{\mathcal{C}(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})}\varphi_\Gamma = \pi_I(\langle\phi_\Gamma,\psi_\Gamma\rangle_{\mathcal{C}(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})})\varphi_\Gamma$$

for $\phi_\Gamma,\psi_\Gamma,\varphi_\Gamma \in \mathcal{E}$.

Step 5.: Morita equivalence
Hence conclude that the C^*-algebras $C^*(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})$ and $\mathbb{C}$ are Morita equivalent. Moreover for two Morita equivalent C^*-algebras there is a bijective correspondence between the non-degenerate representations of those two C^*-algebras. Consequently all irreducible representations of the *-algebra $\mathcal{C}(\bar{G}_{\breve{S},\Gamma},C_0(\bar{G}_{\breve{S},\Gamma}))$ are unitarily equivalent to $\pi^N_{I,\overleftarrow{L}}$. Clearly, for different unitarily inequivalent irreducible representations of $\bar{G}_{\breve{S},\Gamma}$, there are different inequivalent irreducible representations of $\mathcal{C}(\bar{G}_{\breve{S},\Gamma},C_0(\bar{G}_{\breve{S},\Gamma}))$, which corresponds, therefore, to possible superselections of the system. Remark that every non-degenerate representation of the compact operators $\mathcal{K}(\mathcal{H}_\Gamma)$ is equivalent to a direct sum of copies of the identity representation. Hence it follows that every non-degenerate representation of $C^*(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma})$ is equivalent to a direct sum of copies of $\pi^N_{I,\overleftarrow{L}} := \Phi_M \ltimes U^N_{\overleftarrow{L}}$, where Φ_M is the multiplication representation of $C_0(\bar{G}_{\breve{S},\Gamma})$ on $\mathcal{H}_\Gamma$.

■

Proposition 7.1.10. *The states on $\mathcal{K}(L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma}))$ are normalised density matrices given by*

$$\omega(A) = \mathrm{tr}(DA) \quad \forall A \in \mathcal{K}(L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma}))$$

whenever D is a trace-class opertor on $L^2(\bar{G}_{\breve{S},\Gamma},\mu_{\breve{S},\Gamma})$.

To summarise the Generalised Stone- von Neumann theorem 7.1.9 states that there is a bijective correspondence strongly continuous unitary representations of a group $\bar{G}_{\breve{S},\Gamma}$ on the C^*-algebra $\mathcal{L}(\mathcal{H}_\Gamma)$ and elements of $\mathrm{Mor}(C^*(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma}),\mathcal{K}(\mathcal{H}_\Gamma))$. This correspondence preserves direct sums and irreducibility.

Furthermore all unitary representations of $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ for surface sets, which have the simple surface property for Γ are naturally elements of the multiplier algebra $M(C^*(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma}))$. Or equivalently, all unitary representations of $\bar{G}_{S,\Gamma}$ for a surface S having the same surface intersection property for Γ are naturally elements of the multiplier algebra $M(C^*(\bar{G}_{S,\Gamma},\bar{G}_{S,\Gamma}))$. Clearly the closed linear span $\{U(\rho_{S,\Gamma}(\Gamma)) : \rho_{S,\Gamma}(\Gamma)\in\bar{G}_{\breve{S},\Gamma}\}$ of all unitary representations of $\bar{G}_{\breve{S},\Gamma}$ on the C^*-algebra $C(\bar{G}_{\breve{S},\Gamma})$ form a C^*-subalgebra of $M(C^*(\bar{G}_{\breve{S},\Gamma},\bar{G}_{\breve{S},\Gamma}))$.

In the next investigations the question is what happen if different surface sets are used for the construction of the flux transformation group C^*-algebra. In particular, is there a generalised von Neumann theorem available?

For a simplification the following identifications are used. The flux group $\bar{G}_{\breve{S},\Gamma}$ is identified with G^N. Then the following coset space (or space of orbits) G^N/G of a group G^N and a closed subgroup G is defined by the sets

$$\begin{aligned}G^N/G &:= \{(\rho_{S_1}(\gamma_1)\rho_S(\gamma_1),...,\rho_{S_N}(\gamma_N)\rho_S(\gamma_N)): \quad \rho_S\in\mathbb{G}_{S,\Gamma},\rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\\ &\qquad \rho_S(\gamma_i)=\rho_S(\gamma_j)=g_S\in G; 1\leq i,j\leq N\}\\ G^N\setminus G &:= \{(\rho_S(\gamma_1)\rho_{S_1}(\gamma_1),...,\rho_S(\gamma_N)\rho_{S_N}(\gamma_N)) \quad \rho_S\in\mathbb{G}_{S,\Gamma},\rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\\ &\qquad \rho_S(\gamma_i)=\rho_S(\gamma_j)=g_S\in G; 1\leq i,j\leq N\}\end{aligned}$$

whenever $\breve{S}$ is a surface set with simple surface intersection property for Γ and S has the same surface intersection property for Γ.

The space G^N/G^2 is identified with $G^2/G^2\times G^{N-2}$, which is given by

$$\begin{aligned}G^2/G^2\times G^{N-2} &:= \{(\rho_{S_1}(\gamma_1)\rho_{\bar{S}_1}(\gamma_1),\rho_{S_2}(\gamma_2)\rho_{\bar{S}_2}(\gamma_2),\rho_{S_3}(\gamma_3)...,\rho_{S_N}(\gamma_N)):\\ &\qquad \rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\rho_{\bar{S}_l}\in\mathbb{G}_{\breve{S},\Gamma},\forall l=1,2; i=1,...,N \text{ and } (\rho_{\bar{S}_1}(\gamma_1),\rho_{\bar{S}_2}(\gamma_2))\in G^2\}\\ &= G^{N-2}\end{aligned}$$

whenever $\breve{S}$ is a surface set with simple surface intersection property for Γ and $\breve{S}:=\{\bar{S}_1,\bar{S}_2\}$ has the simple surface intersection property for $\{\gamma_1,\gamma_2\}$. The space $G^2/G\times G^{N-2}$ is derivable as

$$\begin{aligned}G^2/G\times G^{N-2} &:= \{(\rho_{S_1}(\gamma_1)\rho_S(\gamma_1),\rho_{S_2}(\gamma_2)\rho_S(\gamma_2),\rho_{S_3}(\gamma_3)...,\rho_{S_N}(\gamma_N)):\\ &\qquad \rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\forall i=1,...,N;\rho_S\in\mathbb{G}_{S,\Gamma}\text{ and }\rho_S(\gamma_k)\in G,\forall k=1,2\}\end{aligned}$$

whenever $\breve{S}$ is a surface set with simple surface intersection property for Γ.

Or more general define

$$\begin{aligned}&G^N/G^{N-M} = G^{N-M}/G^{N-M}\times G^M\\ &:= \{(\rho_{S_1}(\gamma_1)\rho_{\bar{S}_1}(\gamma_1),...,\rho_{S_{N-M}}(\gamma_{N-M})\rho_{\bar{S}_{N-M}}(\gamma_{N-M}),\rho_{S_{N-M+1}}(\gamma_{N-M+1}),...,\rho_{S_N}(\gamma_N)):\\ &\qquad \rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\rho_{\bar{S}_i}\in\mathbb{G}_{\breve{S},\Gamma}\text{ and }(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{N-M}}(\gamma_{N-M}))\in G^{N-M}\}\\ &\text{or}\\ &G^{N-M}/G\times G^M\\ &:= \{(\rho_{S_1}(\gamma_1)\rho_S(\gamma_1),...,\rho_{S_{N-M}}(\gamma_{N-M})\rho_S(\gamma_{N-M}),\rho_{S_{N-M+1}}(\gamma_{N-M+1}),...,\rho_{S_N}(\gamma_N)):\\ &\qquad \rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\rho_S\in\mathbb{G}_{S,\Gamma},(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{N-M}}(\gamma_{N-M}))\in G^{N-M}\text{ and}\\ &\qquad\quad \rho_S(\gamma_i)=\rho_S(\gamma_j)\in G\quad i,j=1,...,N\}\end{aligned}$$

for suitable surface sets $\breve{S}$ and $\breve{S}$ and a surface S. Hence the coset G^N/G^{N-1} of a group G^N and a closed subgroup G^{N-1} is the set

$$\begin{aligned}&G^N/G^{N-1} = G^{N-1}/G^{N-1}\times G\\ &:= \{(\rho_{S_1}(\gamma_1)\rho_{\bar{S}_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar{S}_{N-1}}(\gamma_{N-1}),\rho_{S_N}(\gamma_N)):\\ &\qquad \rho_{S_i}\in\mathbb{G}_{\breve{S},\Gamma},\rho_{\bar{S}_i}\in\mathbb{G}_{\breve{S},\Gamma}\text{ and }(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1}))\in G^{N-1}\}\end{aligned}$$

for suitable surface sets $\breve{S}$ and $\breve{S}$. For suitable surface sets $\breve{S}$, $\breve{S}$ and a graph Γ the following theorem holds.

Theorem 7.1.11. *It is true that:*

(i) the algebras $C_0(G^N/G^{N-1})\rtimes G^N$ and $C^*(G^{N-1})$ are Morita equivalent C^*-algebras (for $N>1$).

(ii) The algebras $C_0(G^N/G^{N-M})\rtimes G^N$ and $C^*(G^{N-M})$ are Morita equivalent C^*-algebras (for $N>1$ and $1\leq M<N$)

Proof : In the following the case (ii) is considered.

Step 1.: two pre-C^*-algebras $\mathfrak{A}_\Gamma$,$\mathfrak{B}_\Gamma$ and a full pre-Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$
Set N be equivalent to $|\Gamma|$ for a graph Γ. Let $\mathfrak{A}_\Gamma=\mathcal{C}(G^N,G^N/G^{N-1})$ be the dense subalgebra of $C^*(G^N,G^N/G^{N-1})$ such that $\mathcal{C}(G^N,G^N/G^{N-1})$ is a pre-C^*-algebra. Similarly, let $\mathfrak{B}_\Gamma=\mathcal{C}(G^{N-1})$ be a dense subalgebra of $C^*(G^{N-1})$ such that $\mathcal{C}(G^{N-1})$ is a pre-C^*-algebra. Identify G^N/G^{N-1} with G.

The full pre-Hilbert $\mathcal{C}(G^{N-1})$-module is given by $C_c(G^N)$ and the right action $\psi_\Gamma f_\Gamma:=\pi_R(f_\Gamma)\psi_\Gamma$ which is of the form

$$\begin{aligned}\pi_R(f_\Gamma)\psi_\Gamma :=& \int_{G^{N-1}}\mathrm{d}\,\mu(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))\\ &\psi_\Gamma(\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1}),\hat\rho_{S_N}(\gamma_N))\\ &f_\Gamma(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))\\ \pi_R(f^*_\Gamma)\psi_\Gamma :=& \int_{G^{N-1}}\mathrm{d}\,\mu(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))f^*_\Gamma(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))\\ &\psi_\Gamma(\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1)^{-1},...,\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1})^{-1},\hat\rho_{S_N}(\gamma_N))\end{aligned}$$

for $\psi_\Gamma\in C_c(G^N)$ and $f_\Gamma\in\mathcal{C}(G^{N-1})$. The $\mathcal{C}(G^{N-1})$-valued product on $C_c(G^N)$ is given by

$$\begin{aligned}\langle\psi_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})} :=& \int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1,\Gamma},...,\hat\rho_{S_N,\Gamma})\overline{\psi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))}\\ &\phi_\Gamma(\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1}),\hat\rho_{S_N}(\gamma_N))\end{aligned}$$

Step 2.: full right Hilbert $\mathfrak{B}$-module $\mathcal{E}_\Gamma$
The completition of $C_c(G^N)$ is a Hilbert $\mathcal{C}(G^{N-1})$-module.

Step 3.: left-action π_L of $\mathfrak{A}_\Gamma$ on $\mathcal{E}_\Gamma$ s.t. $\mathcal{E}_\Gamma$ is a full left pre-Hilbert $\mathfrak{A}_\Gamma$-module
Then there is a pre-Hilbert $\mathcal{C}(G^N,G)$-module is given by $C_c(G^N)$ and the left action $F_\Gamma\psi_\Gamma:=\pi_L(F_\Gamma)\psi_\Gamma$ which is of the form

$$\begin{aligned}\pi_L(F_\Gamma)\psi_\Gamma :=& \int_G\mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &F_\Gamma(\rho_S(\gamma_N);\hat\rho_{S_1}(\gamma_1)\rho_{S_1}(\gamma_1)^{-1},...,\hat\rho_{S_N}(\gamma_N)\rho_{S_N}(\gamma_N)^{-1})\\ &\psi_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ =& \int_G\mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &F_\Gamma(\rho_S(\gamma_N);\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))\end{aligned}$$

where $\rho_S(\gamma_i)=\rho_S(\gamma_j)$ for $i,j=1,...,N$ and

$$\begin{aligned}\pi_L(F^*_\Gamma)\psi_\Gamma :=& \int_G\mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &F^*_\Gamma(\rho_S(\gamma_N);\rho_{S_1}(\gamma_1)\hat\rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)\hat\rho_{S_N}(\gamma_N)^{-1})\\ &\psi_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ =& \int_G\mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &F^*_\Gamma(\rho_S(\gamma_N);\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\ &\psi_\Gamma(\rho_{S_1}(\gamma_1)\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)\hat\rho_{S_N}(\gamma_N))\end{aligned}$$

for $F_\Gamma \in \mathcal{C}(G^N,G)$ and $\psi_\Gamma \in C_c(G^N)$. The $\mathcal{C}(G^N,G)$-valued inner product on $C_c(G^N)$ is equal to

$$\begin{aligned}
&\langle \psi_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^N,G)}\\
&:= \int_{G^N} \mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))\\
&\qquad\overline{\phi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\rho_S(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))}
\end{aligned}$$

for $\psi_\Gamma,\phi_\Gamma \in C_c(G^N)$.

Step 4.: $\mathfrak{A}_\Gamma$-$\mathfrak{B}_\Gamma$-imprimitivity bimodule $\mathcal{E}_\Gamma$
Step 4.1:

$$\begin{aligned}
&\langle \psi_\Gamma f_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^N,G)} = \langle \pi_R(f_\Gamma)\psi_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^N,G)}\\
&= \int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\int_G \mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\psi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1})^{-1}\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1}),\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))\\
&\quad f_\Gamma(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))\overline{\phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\rho_S(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))}\\
&= \int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\int_G \mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\psi_\Gamma(\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1}),\rho_{S_N}(\gamma_N))\\
&\quad\overline{\overline{f_\Gamma(\rho_{\bar S_1}(\gamma_1),...,\rho_{\bar S_{N-1}}(\gamma_{N-1}))}\phi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\rho_S(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))}\\
&= \int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\int_G \mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\psi_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1}),\rho_{S_N}(\gamma_N))\\
&\quad\overline{f^*_\Gamma(\rho_{\bar S_1}(\gamma_1)^{-1},...,\rho_{\bar S_{N-1}}(\gamma_{N-1})^{-1})}\\
&\quad\overline{\phi_\Gamma(\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1)^{-1},...,\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1})^{-1},\rho_S(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))}\\
&= \langle\psi_\Gamma,\pi_R(f^*_\Gamma)\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})}\\
&= \langle\psi_\Gamma,\phi_\Gamma f^*_\Gamma\rangle_{\mathcal{C}(G^{N-1})}
\end{aligned}$$

and

$$\begin{aligned}
&\langle\psi_\Gamma,F_\Gamma\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})} = \langle\psi_\Gamma,\pi_L(F_\Gamma)\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})}\\
&= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\overline{\psi_\Gamma(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))}F_\Gamma(\rho_S(\gamma_N);\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1})^{-1}\hat\rho_{S_{N-1}}(\gamma_{N-1})\rho_{\bar S_{N-1}}(\gamma_{N-1}),\rho_{S_N}(\gamma_N)^{-1}\hat\rho_{S_N}(\gamma_N))
\end{aligned}$$

$$\begin{aligned}
\langle\psi_\Gamma,F_\Gamma\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})} &= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\hat\rho_{S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N))\int_{G^N}\mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\quad\overline{\psi_\Gamma(\rho_{S_1}(\gamma_1)\hat\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)\hat\rho_{S_N}(\gamma_N))F^*_\Gamma(\rho_S(\gamma_N);\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))}\\
&\quad\phi(\hat\rho_{S_1}(\gamma_1)\rho_{\bar S_1}(\gamma_1),...,\hat\rho_{S_N}(\gamma_N)\rho_{\bar S_N}(\gamma_N))\\
&= \langle\pi_L(F^*_\Gamma)\psi_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})}\\
&= \langle F^*_\Gamma\psi_\Gamma,\phi_\Gamma\rangle_{\mathcal{C}(G^{N-1})}
\end{aligned}$$

Step 4.2:

The following is true

$$\begin{aligned}
&\phi_\Gamma\langle\psi_\Gamma,\varphi_\Gamma\rangle_{\mathcal{C}(G^{N-1})} = \pi_R(\langle\psi_\Gamma,\varphi_\Gamma\rangle_{\mathcal{C}(G^{N-1})})\phi_\Gamma\\
&= \int_{G^{N-1}} \mathrm{d}\,\mu(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})) \int_{G^N} \mathrm{d}\,\mu(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\hat{\rho}_{S_1}(\gamma_1)\rho_{\bar{S}_1}(\gamma_1),...,\hat{\rho}_{S_{N-1}}(\gamma_{N-1})\rho_{\bar{S}_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\overline{\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))}\\
&\qquad \varphi_\Gamma(\hat{\rho}_{S_1}(\gamma_1)\rho_{\bar{S}_1}(\gamma_1),...,\hat{\rho}_{S_{N-1}}(\gamma_{N-1})\rho_{\bar{S}_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\\
&= \int_{G^{N-1}} \mathrm{d}\,\mu(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})) \int_{G^N} \mathrm{d}\,\mu(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\rho_{\bar{S}_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\overline{\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))}\\
&\qquad \varphi_\Gamma(\rho_{\bar{S}_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\\
&\langle\phi_\Gamma,\psi_\Gamma\rangle_{\mathcal{C}(G^N,G)}\varphi_\Gamma = \pi_L(\langle\phi_\Gamma,\psi_\Gamma\rangle_{\mathcal{C}(G^N,G)})\varphi_\Gamma\\
&= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))\overline{\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\rho_S(\gamma_N)^{-1}\hat{\rho}_{S_N}(\gamma_N))}\\
&\qquad \varphi_\Gamma(\rho_{S_1}^{-1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}^{-1}(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\\
&= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\rho_S(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\overline{\psi_\Gamma(\hat{\rho}_{S_1}(\gamma_1),...,\hat{\rho}_{S_N}(\gamma_N))}\\
&\qquad \varphi_\Gamma(\rho_{S_1}^{-1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}^{-1}(\gamma_N)\rho_S(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))
\end{aligned}$$

for $\phi_\Gamma,\psi_\Gamma,\varphi_\Gamma\in C_c(G^N)$. Then

$$\phi_\Gamma\langle\psi_\Gamma,\varphi_\Gamma\rangle_{\mathcal{C}(G^{N-1})} = \langle\phi_\Gamma,\psi_\Gamma\rangle_{\mathcal{C}(G^N,G)}\varphi_\Gamma$$

since the properties of the surfaces and paths force the identity

$$\begin{aligned}
&\int_{G^{N-1}} \mathrm{d}\,\mu(\rho_{\bar{S}_1}(\gamma_1),...,\rho_{\bar{S}_{n-1}}(\gamma_{N-1}))\\
&\qquad \phi_\Gamma(\rho_{\bar{S}_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \varphi_\Gamma(\rho_{\bar{S}_1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{\bar{S}_{N-1}}(\gamma_{N-1})\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\\
&= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)^{-1}\rho_S(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \varphi_\Gamma(\rho_{S_1}^{-1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_N}^{-1}(\gamma_N)\rho_S(\gamma_N)\hat{\rho}_{S_N}(\gamma_N))\\
&= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_N)) \int_{G^N} \mathrm{d}\,\mu(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\\
&\qquad \phi_\Gamma(\rho_{S_1}(\gamma_1)^{-1}\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_{N-1}}(\gamma_{N-1})^{-1}\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))\\
&\qquad \varphi_\Gamma(\rho_{S_1}^{-1}(\gamma_1)\hat{\rho}_{S_1}(\gamma_1),...,\rho_{S_{N-1}}^{-1}(\gamma_N)\hat{\rho}_{S_{N-1}}(\gamma_{N-1}),\hat{\rho}_{S_N}(\gamma_N))
\end{aligned}$$

The case (i) is derivable if one sets $\mathfrak{A}_\Gamma = \mathcal{C}(G^N, G^M)$ be the dense subalgebra of $C^*(G^N, G^M)$ such that $\mathcal{C}(G^N, G^M)$ is a pre-C^*-algebra. Similarly, let $\mathfrak{B}_\Gamma = \mathcal{C}(G^{N-M})$ be a dense subalgebra of $C^*(G^{N-M})$ such that $\mathcal{C}(G^{N-M})$ is a pre-C^*-algebra. Then $C_c(G^N)$ is a full left Hilbert $\mathcal{C}(G^N, G^M)$-module or full right Hilbert $\mathcal{C}(G^{N-M})$-module. Moreover $C_c(G^N)$ is a $\mathfrak{A}_\Gamma$-$\mathfrak{B}_\Gamma$-imprimitivity bimodule.

■

The non-commutative holonomy and the heat-kernel-holonomy C^*-algebra for graphs and a surface set

If the configuration set $\bar{\mathcal{A}}_\Gamma$ is naturally identified with $G^{|\Gamma|}$, then the convolution algebra $\mathcal{C}(\bar{\mathcal{A}}_\Gamma)$ is considered. This algebra has been introduced in remark 6.1.42. Observe that, the convolution product is for example for a graph

$\Gamma := \{\gamma'\}$ defined by

$$(f_\Gamma * k_\Gamma)(\mathfrak{h}_\Gamma(\gamma')) = \int_{\bar{\mathcal{A}}_\Gamma} \mathrm{d}\,\mu_\Gamma(\mathfrak{g}_\Gamma(\gamma')) f_\Gamma(\mathfrak{g}_\Gamma(\gamma')) k_\Gamma(\mathfrak{g}_\Gamma(\gamma')^{-1}\mathfrak{h}_\Gamma(\gamma'))$$

The **non-commutative holonomy C^*-algebra for a graph** is given by the object $C^*_r(\bar{\mathcal{A}}_\Gamma)$ and reduces in the case of a compact Lie group G to the following object.

Remark 7.1.12. *In the case of a compact group G the holonomy algebra $C^*(\bar{\mathcal{A}}_\Gamma)$ for a graph Γ is equivalent to the C^*-algebra of the matrices.*

The new algebra of matrices associated to a graph is given by the infinite matrix algebra

$$M_\Gamma := \bigotimes_{\gamma_i\in\Gamma} \bigoplus_{\pi_{s,\gamma_i}\in\hat{G}} M_{d_{s,\gamma_i}}(\mathbb{C}),$$

where $\hat{G}$ is the dual of G, π_{s,γ_i} is a representation of G associated to a path γ_i and d_{s,γ_i} is the dimension of the representation π_{s,γ_i}. The inductive limit of a increasing family of matrix algebras M_{Γ_i} associated to graphs can be considered.

For an inductive family $\{\Gamma_i\}$ of graphs, there is an injective *-homomorphism $\hat{\beta}_{\Gamma,\Gamma'} : C^*(\bar{\mathcal{A}}_\Gamma) \to C^*(\bar{\mathcal{A}}_{\Gamma'})$ for all $\mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}$. This *-homomorphism is for example given for a subgraph $\Gamma := \{\gamma\}$ of $\Gamma' := \{\gamma\circ\gamma'\}$ by

$$(\hat{\beta}_{\Gamma,\Gamma'}(f_\Gamma))(\mathfrak{h}_\Gamma(\gamma)) := f_{\Gamma'}(\mathfrak{h}_{\Gamma'}(\gamma\circ\gamma'))$$

Consequently there exists an inductive family of C^*-algebras $\{(C^*(\bar{\mathcal{A}}_\Gamma), \hat{\beta}_{\Gamma,\Gamma'}) : \mathcal{P}_\Gamma \leq \mathcal{P}_{\Gamma'}\}$.

An increasing family of finite matrix algebras are used to define UHF (uniformly hyperfinite) algebras, which are often used in quantum statistical mechanical systems. Furthermore some of these algebras can lead to KMS-states, which are fruitful states such that the dynamics of Loop Quantum Gravity can be implemented. This issue is analysed further in section 8.4.

The following theorem can be proven by the theorem of Woronowicz similarly to the procedure done in the proof of theorem 6.5.8. The action α in equation (6.88) is now defined for functions in $C^*(\bar{\mathcal{A}}_\Gamma)$.

Proposition 7.1.13. *The C^*-dynamical system $(C^*(\bar{\mathcal{A}}_\Gamma), \mathbb{R}, \alpha)$ does admit a KMS-state.*

There is a relation of the non-commutative holonomy C^*-algebra for a graph to the holonomy algebra studied in [2]. Clearly a similar algebra for the flux group and the flux transformation group C^*-algebra is constructed for $\bar{\mathcal{A}}_\Gamma$. The holonomy transformation group C^*-algebra $C^*(\bar{\mathcal{A}}_\Gamma, \bar{\mathcal{A}}_\Gamma)$ is called the **heat-kernel-holonomy C^*-algebra**.

7.2 The holonomy-flux cross-product C^*-algebra for surface sets

After the considerations of algebras generated by either quantum configuration or quantum momentum variables, algebras generated by both quantum variables simultaneously is studied in this section.

There is not a particular holonomy-flux cross-product C^*-algebra generated by all group-valued quantum flux operators and certain functions depending on holonomies along paths. But there exists a bunch of holonomy-flux cross-product C^*-algebra associated to a finite graph system and many different surface sets. These algebras are developed in section 7.2.1. The existence of this variety is the consequence of the following facts.

The group-valued quantum flux operators associated to certain surfaces and a graph Γ form the flux group associated to a surface set $\breve{S}$ and a graph Γ. These elements are implemented as point-norm continuous and automorphic actions on the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ restricted to the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$. For a short notation the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ is abreviated by the term *analytic holonomy C^*-algebra associated to the graph* Γ. It is assumed that the configuration space is naturally identified with $G^{|\Gamma|}$. Then the elements of the flux group are represented as unitary operators on the Hilbert space $\mathcal{H}_\Gamma$, which is given by $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.

For each automorphic action defined in section 6.1 a holonomy-flux cross-product C^*-algebra is constructed. Precisely an automorphic action α of the flux group $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ defines a *holonomy-flux cross-product C^*-algebra associated to a graph* Γ *and a surface set* $\breve{S}$. This C^*-algebra is denoted by $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{G}_{\breve{S},\Gamma}$.

There are many different possible actions of flux groups depending on a surface or a surface set. For example, in Lemma 6.1.11 there is the point-norm continuous automorphic action α_L^1 of the flux group $\bar{G}_{S,\Gamma}$ associated to one suitable surface S on the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$. Moreover in Lemma 6.1.16 the action $\alpha_{\overleftarrow{L}}^N$ is defined for a flux group associated to a set $\breve{S}$ of surfaces, which has the simple surface intersection property for a finite orientation preserved graph system associated to the graph Γ. In the following these two actions are often used.

Finally there is an algebra, which unifies all cross-product algebras associated to a graph and different sets of surfaces. This algebra is given by the *multiplier algebra of the cross-product algebra* $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^N} \bar{G}_{\breve{S},\Gamma}$. In theorem 7.2.12 it is proven that this algebra contains every element of the cross-product C^*-algebra associated to a graph and any surface set and every Weyl element, which is obtained by the unitary representation of flux groups associated to a graph and any surface set.

The inductive limit of the inductive families of holonomy-flux cross-product C^*-algebras is studied in section 7.2.2. There the inductive limit C^*-algebra is derived from the inductive limit of C^*-algebras restricted to finite orientation preserved graph systems. This algebra is called the *holonomy-flux cross-product C^*-algebra* (of a special surface configuration $\breve{S}$).

7.2.1 The holonomy-flux cross-product C^*-algebra for a finite graph system and a surface set

To start with the development of such a cross-product algebra generated by holonomies and quantum fluxes a particular Banach *-algebra has to be given.

Definition 7.2.1. *Let $\breve{S}$ be a set of surfaces with same surface intersection property for a finite orientation preserved graph system associated to a graph Γ with N independent edges. Furthermore let $(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ be a C^*-dynamical system defined by a point-norm continuous automorphic flux action α of $\bar{G}_{\breve{S},\Gamma}$ presented in subsection 6.3.*

The space $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha)$ consists of all measurable functions $F_\Gamma : \bar{G}_{\breve{S},\Gamma} \rightarrow C_0(\bar{\mathcal{A}}_\Gamma)$ for which

$$\|F_\Gamma\|_1 := \int_{\bar{G}_{\breve{S},\Gamma}} d\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N)) \|F_\Gamma(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N))\|_2 < \infty$$

whenever $\rho_S \in G_{\breve{S},\Gamma}$ yields.

Proposition 7.2.2. *Let $\breve{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph Γ. Furthermore let $(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^N)$ be a C^*-dynamical system where $\alpha_{\overleftarrow{L}}^N \in \operatorname{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$.*

Then the operations

$$\begin{aligned}
&(F_\Gamma * \hat{F}_\Gamma)(\tilde{\rho}_{S_1}(\gamma_1), ..., \tilde{\rho}_{S_N}(\gamma_N)) \\
&= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma)) \\
&\qquad F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma)) \left(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{S},\Gamma}^N)(\hat{F}_\Gamma)\right) (\rho_{S_1}(\gamma_1)^{-1}\tilde{\rho}_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N)^{-1}\tilde{\rho}_{S_N}(\gamma_N))
\end{aligned}$$

where $\rho_{\breve{S},\Gamma}(\Gamma) = (\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N)) =: \rho_{\breve{S},\Gamma}^N$, $\rho_{S_i}, \tilde{\rho}_{S_i} \in G_{\breve{S},\Gamma}$ and

$$F_\Gamma^*(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N)) = \left(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{S},\Gamma}^N)(F_\Gamma^+)\right) (\rho_{S_1}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)^{-1})$$

where the involution $^+$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is given by

$$F_\Gamma^+(\rho_{S_1}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)^{-1}) := \overline{F_\Gamma(\rho_{S_1}(\gamma_1)^{-1}, ..., \rho_{S_N}(\gamma_N)^{-1})}$$

turn $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^N)$ into a Banach *-algebra.

In particular, let S be a surface having the same surface intersection property for a finite orientation preserved graph system associated to a graph Γ. Then the action $\alpha_{\overleftarrow{L}}^{1}$ is defined in (6.14) for a graph Γ and the convolution product reads

$$\begin{aligned}&(F_\Gamma * \hat{F}_\Gamma)(\tilde{\rho}_S(\gamma_i))\\ &= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_i))F_\Gamma(\rho_S(\gamma_i))\left(\alpha_{\overleftarrow{L}}^{1}(\rho^1_{S,\Gamma})(\hat{F}_\Gamma)\right)(\rho_S^{-1}(\gamma_i)\tilde{\rho}_S(\gamma_i))\\ &= \int_G \mathrm{d}\,\mu(\rho_S(\gamma_i))F_\Gamma(\rho_S(\gamma_i);\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &\qquad \hat{F}_\Gamma(\rho_S^{-1}(\gamma_i)\tilde{\rho}_S(\gamma_i);\rho_S(\gamma_i)\mathfrak{h}_\Gamma(\gamma_1),....,\rho_S(\gamma_i)\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}\tag{7.21}$$

for any $i = 1,..,N$. Since $\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in G$ for all $i,j = 1,...,N$. Clearly, this convolution equipped with an appropriate involution and norm form a *-Banach algebra $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{1})$.

Clearly the *-Banach algebras $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{M})$ for $1 \leq M \leq N$ exists.

Indeed, there are a lot of different Banach *-algebras depending on the choice of the set of surfaces $\breve{S}$. Let $\breve{S}$ has the same surface intersection property for a graph Γ such that each path γ_i, that intersect the surface S_i, lie above and ingoing w.r.t. the surface orientation of S_i. There are no other intersection points of each path γ_i with any other surface S_j where $i \neq j$. Then for the map $F_\Gamma : \bar{G}_{\breve{S},\Gamma} \rightarrow C_0(\bar{\mathcal{A}}_\Gamma)$ write for the image of this function $F_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N)) = F_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N);\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))$ and derive

$$\begin{aligned}&(F_\Gamma * \hat{F}_\Gamma)(\tilde{\rho}_{S_1}(\gamma_1),...,\tilde{\rho}_{S_N}(\gamma_N))\\ &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))\\ &\qquad F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma))\left(\alpha_N^{\overleftarrow{R}}(\rho^N_{\breve{S},\Gamma})(\hat{F}_\Gamma)\right)\left(\rho_{S_1}(\gamma_1)^{-1}\tilde{\rho}_{S_1}(\gamma_1),....,\rho_{S_N}(\gamma_N)^{-1}\tilde{\rho}_{S_N}(\gamma_N)\right)\end{aligned}$$

Hence for a redefined convolution product and involution the *-Banach algebras $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_M^{\overleftarrow{R}})$ for $1 \leq M \leq N$ can be studied. Furthermore it is also possible to construct the *-Banach algebras $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{\overleftarrow{R},M})$ for $1 \leq M \leq N$ and other algebras of that form for a modified convolution product, which is given in general by

$$\begin{aligned}&(F_\Gamma * \hat{F}_\Gamma)(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))\\ &= \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\breve{S},\Gamma}(\Gamma))F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma))\left(\alpha(\rho_{\breve{S},\Gamma}(\Gamma))\hat{F}_\Gamma\right)(L(\rho_{\breve{S},\Gamma}(\Gamma)^{-1})(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))\end{aligned}$$

where $\rho_{\breve{S},\Gamma}(\Gamma) = (\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))$, $\rho_{\breve{S},\Gamma}, \tilde{\rho}_{\breve{S},\Gamma} \in G_{\breve{S},\Gamma}$ and a modified involution

$$F^*_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma)) = \alpha(\rho_{\breve{S},\Gamma}(\Gamma))\left(F^+_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma)^{-1})\right)$$

whenever $\alpha \in \mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$. Hence for all well-defined C^*-dynamical system $(\bar{G}_{\breve{S},\Gamma}, \mathfrak{A}_\Gamma, \alpha)$ there exists a general Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, \mathfrak{A}_\Gamma, \alpha)$.

Theorem 7.2.3. *Let $\breve{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph Γ. Furthermore let $(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{N})$ be a C^*-dynamical system where $\alpha_{\overleftarrow{L}}^{N} \in \mathrm{Act}(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$.*

There is a bijective correspondence between non-degenerate L^1-norm decreasing *- representations π of the Banach *-algebra $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{N})$ and covariant representations $(\Phi_M, U_{\overleftarrow{L}}^{N})$ of the C^*-dynamical system $(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{N})$ in $\mathcal{L}(\mathcal{H}_\Gamma)$.

This correspondence is given in one direction by the fact that the representation $\pi^N_{I,\overleftarrow{L}}$ of $L^1(\bar{G}_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{N})$ is defined by a covariant pair $(\Phi_M, U_{\overleftarrow{L}}^{N})$ via

$$\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\psi_\Gamma := \int_{\bar{G}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho^N_{\breve{S},\Gamma})\Phi_M(F_\Gamma(\rho^N_{\breve{S},\Gamma}))U_{\overleftarrow{L}}^{N}(\rho^N_{\breve{S},\Gamma})\psi_\Gamma$$

where $\rho^N_{\breve S,\Gamma} \in \bar G_{\breve S,\Gamma}$, $F_\Gamma \in L^1(\bar G_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^N)$ *and* $\psi_\Gamma \in \mathcal{H}_\Gamma$.

The other direction is given by the definition of the covariant pair $(\Phi_M, U^N_{\overleftarrow{L}})$ *through the maps*
$F_\Gamma : \rho^N_{\breve S,\Gamma} \mapsto F_\Gamma(\rho^N_{\breve S,\Gamma})$ *and*
$\alpha^N_{\overleftarrow{L}}(\rho^N_{\breve S,\Gamma})(F_\Gamma) : \tilde\rho^N_{\breve S,\Gamma} \mapsto \left(\alpha^N_{\overleftarrow{L}}(\tilde\rho^N_{\breve S})(F_\Gamma)\right)(L((\tilde\rho^N_{\breve S,\Gamma})^{-1})(\rho^N_{\breve S,\Gamma}))$ *such that*

$$\begin{aligned}U^N_{\overleftarrow{L}}(\rho^N_{\breve S,\Gamma})\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\Omega &:= \pi^N_{I,\overleftarrow{L}}\left(\alpha^N_{\overleftarrow{L}}(\rho^N_{\breve S,\Gamma})(F_\Gamma)\right)\Omega\\ \Phi_M(f_\Gamma)\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\Omega &:= \pi^N_{I,\overleftarrow{L}}(f_\Gamma F_\Gamma)\Omega\end{aligned}$$

where Ω *is a cyclic vector for* $\pi^N_{I,\overleftarrow{L}}(\mathcal{C}(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma)))$, $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho^N_{\breve S,\Gamma}, \tilde\rho^N_{\breve S,\Gamma} \in \bar G_{\breve S,\Gamma}$ *and* $F_\Gamma \in L^1(\bar G_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha^N_{\overleftarrow{L}})$.

This bijection preserves unitary equivalence, direct sums and irreducibility.

The ***reduced holonomy-flux group C^*-algebra*** $C^*_r(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ ***associated to a graph*** Γ ***and a set*** $\breve S$ ***of surfaces*** *is defined as the norm-closure of* $L^1(\bar G_{S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha^N_{\overleftarrow{L}})$ *with respect to the norm* $\|F_\Gamma\| := \|\pi^N_{I,\overleftarrow{L}}(F_\Gamma)\|_2$.

With no doubt there are a big bunch of reduced holonomy-flux group C^*-algebra $C^*_r(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ for different graph systems and different sets of surfaces.

Definition 7.2.4. *Let* $\breve S$ *be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph* Γ.

Then the ***Weyl-integrated holonomy-flux representation*** *w.r.t. a finite orientation preserved graph system associated to a graph* Γ *and a set* $\breve S$ *of surfaces is given by*

$$\pi^{I,\Gamma}_{E(\breve S)}(F_\Gamma)\psi_\Gamma = \int_{\bar G_{\breve S,\Gamma}} \mathrm{d}\,\mu_{\breve S,\Gamma}(\rho_{\breve S,\Gamma}(\Gamma))\Phi_M\left(F_\Gamma(\rho_{\breve S,\Gamma}(\Gamma))\right)U(\rho_{\breve S,\Gamma}(\Gamma))\psi_\Gamma$$

for $F_\Gamma \in C^*(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$, $\rho_{\breve S,\Gamma}(\Gamma) \in \bar G_{\breve S,\Gamma}$, $U \in \mathrm{Rep}(\bar G_{\breve S,\Gamma}, \mathcal{K}(L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)))$ *and* $\psi_\Gamma \in L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. *The Weyl-integrated holonomy-flux representation* $\pi^{I,\Gamma}_{E(\breve S)}$ *is a* **-representation of the* C^**-algebra* $C^*(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ *with a norm inherited from the representations* $\pi^{I,\Gamma}_{E(\breve S)}$ *on* $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. *The representation* $\pi^{I,\Gamma}_{E(\breve S)}$ *is also denoted by* $\Phi_M \rtimes U$.

Proposition 7.2.5. *Let* $\breve S$ *be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph* Γ. *Furthermore let* $(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha^N_{\overleftarrow{L}})$ *is a* C^**-dynamical system.*

Define for each $F_\Gamma \in \mathcal{C}(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ *the norm*

$$\|F_\Gamma\|_u := \sup\left\{\|(\Phi_M \rtimes U^N_{\overleftarrow{L}})(F_\Gamma)\|\right\}$$

where the supremum is taken over all covariant Hilbert space representations $(\Phi_M, U^N_{\overleftarrow{L}})$ *of the* C^**-dynamical system* $(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}})$.

Then $\|.\|_u$ *is a norm on* $\mathcal{C}(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$, *which is called the universal norm. The universal norm is dominated by the* $\|.\|_1$*-norm, and the completition of* $\mathcal{C}(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ *with respect to* $\|.\|_u$ *is a* C^**-algebra called the* ***holonomy-flux cross-product C^*-algebra*** *of* $C_0(\bar{\mathcal{A}}_\Gamma)$ *by* $\bar G_{\breve S,\Gamma}$ *for a finite orientation preserved graph system associated to a graph* Γ *and a set* $\breve S$ *of surfaces and is denoted by* $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}} \bar G_{\breve S,\Gamma}$.

Notice that for a surface S having the same surface intersection property for a finite orientation preserved graph system associated to Γ, the $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$-norm of an element $F_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha^1_{\overleftarrow{L}}} \bar G_{S,\Gamma}$ is given by

$$\begin{aligned}\|\pi^{I,\Gamma}_{E(S)}(F_\Gamma)\psi_\Gamma\|_2 = &\int_{\bar{\mathcal{A}}_\Gamma}\int_{\bar G_{S,\Gamma}} \mathrm{d}\,\mu_{S,\Gamma}(\rho_S(\gamma_i))\,\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\\ &|f_\Gamma(\rho_{S,\Gamma}(\gamma_i);\mathfrak{h}_\Gamma(\Gamma))\psi_\Gamma(L(\rho_S(\gamma_i))(\mathfrak{h}_\Gamma(\gamma_1)),...,L(\rho_S(\gamma_i))(\mathfrak{h}_\Gamma(\gamma_N)))|^2\end{aligned} \tag{7.22}$$

whenever $\rho_S(\gamma_i) = \rho_S(\gamma_j) = g_S \in \bar G_{S,\Gamma}$ for $i \neq j$ and $1 \leq i,j \leq N$.

The general holonomy-flux cross-product algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar G_{\breve S,\Gamma}$ for an action $\alpha \in \mathrm{Act}(\bar G_{\breve S,\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ is in the case of a locally compact group G a non-commutative and non-unital C^*-algebra.

Refer to the definitions of resticted graph-diffeomorphisms presented in definition 6.2.10.

Proposition 7.2.6. *The state $\omega^\Gamma_{E(\breve{S})}$ on $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}}\bar{\mathcal{Z}}_{\breve{S},\Gamma}$ associated to the GNS-representation $(\mathcal{H}_\Gamma,\pi^{I,\Gamma}_{E(\breve{S})},\Omega^{I,\Gamma}_{E(\breve{S})})$ is not surface-orientation preserving graph-diffeomorphism invariant, but it is a surface preserving graph- diffeomorphism invariant state.*

Notice that,

$$\zeta_\sigma\circ\alpha(\rho_{S,\Gamma}(\Gamma))\neq\alpha(\rho_{S,\Gamma}(\Gamma_\sigma))\circ\zeta_\sigma$$

for every $\sigma\in\mathfrak{B}(\mathcal{P}^{\mathfrak{o}}_\Gamma)$ and $\rho_{S,\Gamma}\in G_{\breve{S},\Gamma}$ holds. Therefore, it is necessary to restrict the holonomy-flux cross-product C^*-algebra to $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}}\bar{\mathcal{Z}}_{\breve{S},\Gamma}$.

Proof : Let $(\varphi_\Gamma,\Phi_\Gamma)$ be a graph-diffeomorphism on $\mathcal{P}_\Gamma$ over V_Γ, which is surface-orientation preserving. Then investigate the following computation

$$\begin{aligned}
&\omega^\Gamma_E(\theta_{(\varphi_\Gamma,\Phi_\Gamma)}(F_\Gamma))\\
&=\int_{\bar{\mathcal{A}}_\Gamma}\int_{\bar{G}_{\breve{S},\Gamma}}\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Phi_\Gamma(\gamma_1)),...,\mathfrak{h}_\Gamma(\Phi_\Gamma(\gamma_N)))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\varphi_\Gamma(S_1)}(\Phi_\Gamma(\gamma_1)),...,\rho_{\varphi_\Gamma(S_N)}(\Phi_\Gamma(\gamma_N)))\\
&\qquad|F_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma_\sigma);\rho_{\breve{S},\Gamma}(\Gamma_\sigma)^{-1}\mathfrak{h}_\Gamma(\Gamma_\sigma))|^2\\
&=\int_{\bar{\mathcal{A}}_\Gamma}\int_{\bar{G}_{\breve{S},\Gamma}}\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{\tilde{S}_1}(\gamma_1),...,\rho_{\tilde{S}_N}(\gamma_N))|F_\Gamma(\rho_{\tilde{S}_1}(\gamma_1),...,\rho_{\tilde{S}_N}(\gamma_N))|^2\\
&\neq\int_{\bar{\mathcal{A}}_\Gamma}\int_{\bar{G}_{\breve{S},\Gamma}}\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))|F_\Gamma(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))|^2\\
&=\omega^\Gamma_E(F_\Gamma)
\end{aligned}$$

where $\varphi_\Gamma(S_i)=\tilde{S}_i$, $\tilde{S}_i\in\breve{S}$ for all $1\leq i\leq N$ and $\Gamma_\sigma=(\Phi_\Gamma(\gamma_1),...,\Phi_\Gamma(\gamma_N))$. Clearly, for $\varphi_\Gamma(S_i)=S_i$ the invariance property is easy to deduce.

■

The different possibilities of orientation of surfaces and the graphs allow to define a bulk of automorphic actions and C^*-dynamical systems for the holonomy algebra $C_0(\bar{\mathcal{A}}_\Gamma)$. Therefore speak about different surface configurations with respect to graphs and define many different holonomy-flux cross-product C^*-algebras. For example, there are the following holonomy-flux cross-product C^*-algebras constructable $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^M_{\overleftarrow{L}}}\bar{G}_{\breve{S}_2,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^M_{\overleftarrow{L}}}\bar{G}_{\breve{S}_3,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R}}_M}\bar{G}_{\breve{S}_5,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R}}_M}\bar{G}_{\breve{S}_6,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R},M}_{\overleftarrow{L}}}\bar{G}_{\breve{S}_7,\Gamma}$ and $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R},M}_{\overleftarrow{L}}}\bar{G}_{\breve{S}_8,\Gamma}$ for $1\leq M\leq N$ for a set $\{\breve{S}_i\}$ of suitable surface sets. If the tensor C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)\otimes C_0(\bar{\mathcal{A}}_{\Gamma'})$ is used, then the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^N_{\overleftarrow{R}}}\bar{G}_{\breve{S},\Gamma}\otimes C_0(\bar{\mathcal{A}}_{\Gamma'})\rtimes_{\alpha^{N'}_{\overleftarrow{L}}}\bar{G}_{\breve{S},\Gamma'}$ with respect to the minimal C^*-norm is constructed.

Observe that the generalised Stone - von Neumann theorem 7.1.9 stated in [109, Theorem 4.24] is not achievable, since the objects $\bar{G}_{\breve{S},\Gamma}$ and $\bar{\mathcal{A}}_\Gamma$ are not identified in general. It is necessary to distinguish between the two objects, since the holonomies are independent whereas the fluxes are dependent on a surface or surface set. Nevertheless, if it is assumed that $\bar{G}_{\breve{S},\Gamma}$ is identified with G^M and $\bar{\mathcal{A}}_\Gamma$ is identified with G^N, then the holonomy-flux cross-product algebra is identified with $C_0(G^N)\rtimes_{\alpha^M_{\overleftarrow{L}}}G^M$. But the generalised Stone - von Neumann theorem is only available for M equal to N. This is the result of theorem 7.1.9 and theorem 7.1.11. Hence only in the configuration $M=N$ the C^*-algebra $C_0(G^N)\rtimes_{\alpha^M_{\overleftarrow{L}}}G^N$ is isomorphic to $\mathcal{K}(L^2(G^N,\mu_N))$. Notice that the state ω^Γ_E is now given by

$$\omega^\Gamma_E(F_\Gamma)=\int_{G^N}\int_{G^N}\mathrm{d}\,\mu_N(\mathbf{g})\,\mathrm{d}\,\mu_N(\mathbf{h})|F_\Gamma(\mathbf{g},\mathbf{h})|^2$$

for $F_\Gamma\in C_0(G^N)\rtimes_{\alpha^M_{\overleftarrow{L}}}G^N$ and does not depend on the surfaces anymore. If $\bar{G}_{\breve{S},\Gamma}$ for example is identified with G^{N-1}, then a problem occurs. The Morita equivalent C^*-algebra to $C_0(G^N)\rtimes_{\alpha^M_{\overleftarrow{L}}}G^M$ where $M<N$ is not of the form $C^*(G^K)$ for a suitable K where $1\leq K\leq N$. The author does not know any Morita equivalent C^*-algebra to the C^*-algebra $C_0(G^N)\rtimes_{\alpha^M_{\overleftarrow{L}}}G^M$ where $M<N$.

In the book of Pedersen [74, section 7.7] a generalisation of regular representations of cross-products has been presented. These results are adapted to the case of a set $\breve{S}$ of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to Γ.

Set $\mathcal{H}^\Gamma_{E(\breve{S})} := L^2(\bar{G}_{\breve{S},\Gamma},\mathcal{H}_\Gamma)$, where this Hilbert space is identified with $L^2(\bar{G}_{\breve{S},\Gamma})\otimes\mathcal{H}_\Gamma$. In the following investigation the element F_Γ is understood as an element of $\mathcal{K}(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma))$.

First, observe $\Psi_\Gamma(\rho_{\breve{S},\Gamma}(\Gamma))$ is an element of $\mathcal{H}^\Gamma_{E(\breve{S})}$, if there is a map

$$\rho_{\breve{S},\Gamma}(\Gamma)\mapsto\Psi^\Gamma_{E(\breve{S})}(\mathfrak{h}_\Gamma(\Gamma),\rho_{\breve{S},\Gamma}(\Gamma))$$

such that $\Psi^\Gamma_{E(\breve{S})}(\rho_{\breve{S},\Gamma}(\Gamma))\in\mathcal{H}_\Gamma$.

Then recall the C^*-algebra dynamical system $(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha_{\overleftarrow{L}}^N)$ and the covariant pair $(\Phi_M,U_{\overleftarrow{L}}^N)$ of this C^*-dynamical system. There is a morphism $\Phi^M_{\overleftarrow{L}}$ of the C^*-algebras, which maps from $C_0(\bar{\mathcal{A}}_\Gamma)$ to $\mathcal{K}\big(\mathcal{H}^\Gamma_{E(\breve{S})}\big)$, and a representation $U^N_{\overleftarrow{L}}$ of the group $\bar{G}_{\breve{S},\Gamma}$ on the C^*-algebra $\mathcal{K}\big(\mathcal{H}^\Gamma_{E(\breve{S})}\big)$. Both objects are defined by

$$\left(\Phi^M_{\overleftarrow{L}}(f_\Gamma)\Psi^\Gamma_{E(\breve{S})}\right)(\rho_{\breve{S},\Gamma}(\Gamma)) \;=\Phi_M\left(\alpha^N_{\overleftarrow{L}}(\rho_{\breve{S},\Gamma}(\Gamma))(f_\Gamma)\right)\Psi^\Gamma_{E(\breve{S})}(\rho_{\breve{S},\Gamma}(\Gamma))$$

for $\Psi^\Gamma_{E(\breve{S})}\in\mathcal{H}^\Gamma_{E(\breve{S})}$, $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$ and

$$(U^N_{\overleftarrow{L}}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{S})})(\rho_{\breve{S},\Gamma}(\Gamma)) := \Psi^\Gamma_{E(\breve{S})}(L(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))(\rho_{\breve{S},\Gamma}(\Gamma)))$$

for $U^N_{\overleftarrow{L}}\in\operatorname{Rep}(\bar{G}_{\breve{S},\Gamma},\mathcal{K}(\mathcal{H}^\Gamma_{E(\breve{S})}))$, $\rho_{\breve{S},\Gamma},\hat{\rho}_{\breve{S},\Gamma},\tilde{\rho}_{\breve{S},\Gamma}\in G_{\breve{S},\Gamma}$. Then $(\Phi^M_{\overleftarrow{L}},U^N_{\overleftarrow{L}})$ defines a covariant representation of $(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha^N_{\overleftarrow{L}})$ in $\mathcal{K}\big(\mathcal{H}^\Gamma_{E(\breve{S})}\big)$

Definition 7.2.7. *Let $\breve{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph Γ.*

The ***left regular representation of the holonomy-flux cross-product C^*-algebra*** *$C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha^N_{\overleftarrow{L}}}\bar{G}_{\breve{S},\Gamma}$ induced by $(\Phi_M,\mathcal{H}_\Gamma)$ is the representation $\pi^{\Gamma,\breve{S}}_{\overleftarrow{L}}$ on $L^2(\bar{G}_{\breve{S},\Gamma},\mathcal{H}_\Gamma))$, which is expressed by*

$$\begin{aligned}((\pi^{\Gamma,\breve{S}}_{\overleftarrow{L}}(F_\Gamma))\Psi^\Gamma_{E(\breve{S})})(\rho_{\breve{S},\Gamma}(\Gamma)) &= \left(((\Phi^M_{\overleftarrow{L}}\rtimes U^N_{\overleftarrow{L}})(F_\Gamma))\Psi^\Gamma_{E(\breve{S})}\right)(\rho_{\breve{S},\Gamma}(\Gamma))\\
:= \int_{\bar{G}_{\breve{S},\Gamma}}\Phi_M\left(\alpha^N_{\overleftarrow{L}}(\rho_{\breve{S},\Gamma}(\Gamma))(F_\Gamma(\hat{\rho}_{\breve{S},\Gamma}(\Gamma)))\right)&U^N_{\overleftarrow{L}}(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{S})}(\rho_{\breve{S},\Gamma}(\Gamma))\,\mathrm{d}\,\mu_{\breve{S},\Gamma}(\hat{\rho}_{\breve{S}}(\Gamma))\end{aligned}$$

for $F_\Gamma(\hat{\rho}_{\breve{S},\Gamma}(\Gamma))\in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho_{\breve{S},\Gamma},\tilde{\rho}_{\breve{S},\Gamma},\hat{\rho}_{\breve{S},\Gamma}\in G_{\breve{S},\Gamma}$ and $\Psi^\Gamma_{E(\breve{S})}\in\mathcal{H}^\Gamma_{E(\breve{S})}$. The representation $\pi^{\Gamma,\breve{S}}_{\overleftarrow{L}}$ is also denoted by $\Phi^M_{E(\breve{S})}\rtimes U^N_{\overleftarrow{L}}$.

Then recall a general C^*-algebra dynamical system $(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha)$. There is a morphism $\Phi^M_{E(\breve{S})}$ from the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ to $\mathcal{K}\big(\mathcal{H}^\Gamma_{E(\breve{S})}\big)$ and a representation U of the group $\bar{G}_{\breve{S},\Gamma}$ on the C^*-algebra $\mathcal{K}\big(\mathcal{H}^\Gamma_{E(\breve{S})}\big)$. They are defined by

$$\left(\Phi^M_{E(\breve{S})}(f_\Gamma)\Psi^\Gamma_{E(\breve{S})}\right)(\rho_{\breve{S},\Gamma}(\Gamma)) := \Phi_M\left(\alpha(\rho_{\breve{S},\Gamma}(\Gamma))(f_\Gamma)\right)\Psi^\Gamma_{E(\breve{S})}(\rho_{\breve{S},\Gamma}(\Gamma))$$

for $\Psi^\Gamma_{E(\breve{S})}\in\mathcal{H}^\Gamma_{E(\breve{S})}$, $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\tilde{\rho}_{\breve{S},\Gamma},\rho_{\breve{S},\Gamma}\in G_{\breve{S},\Gamma}$. Consequently a general regular representation of the holonomy-flux cross-product is given by

$$\pi^\Gamma_{E(\breve{S})}(f_\Gamma)\Psi^\Gamma_{E(\breve{S})} = (\Phi^M_{E(\breve{S})}\rtimes U)(f_\Gamma)\Psi^\Gamma_{E(\breve{S})} \tag{7.23}$$

whenever $U\in\operatorname{Rep}(\bar{G}_{\breve{S},\Gamma},\mathcal{K}(\mathcal{H}^\Gamma_{E(\breve{S})}))$ and $f_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)$.

Definition 7.2.8. *Let $\breve{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph Γ.*

The ***multiplier algebra of the holonomy-flux cross-product C^*-algebra*** $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}$ *is given by all linear operators*

$$M: C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}\longrightarrow C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}$$

such that for any $\hat{F}_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}$ there exists a $\tilde{F}_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}$ such that for all $F_\Gamma\in C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}$ it is true that

$$\begin{aligned}\hat{F}_\Gamma^* M(F_\Gamma)&=\Big\langle \hat{F}_\Gamma, M(F_\Gamma)\Big\rangle_{C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}}=\Big\langle \tilde{F}_\Gamma, F_\Gamma\Big\rangle_{C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{S},\Gamma}}\\&=\tilde{F}_\Gamma^* F_\Gamma\end{aligned}$$

In particular, the multiplier algebra of the reduced holonomy-flux group C^*-algebra $\mathcal{C}_r^*(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma))$ consists of such linear maps M such that for any $\hat{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma))\in C_0(\bar{\mathcal{A}}_\Gamma)$ there exists a $\tilde{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma))\in C_0(\bar{\mathcal{A}}_\Gamma)$ such that for all $F_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma))\in C_0(\bar{\mathcal{A}}_\Gamma)$ it is true that

$$\begin{aligned}&\Big\langle (\pi_{\overleftarrow{L}}^{\Gamma,\breve{S}}(\hat{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)))\Psi^\Gamma_{E(\breve{S})},\pi_{\overleftarrow{L}}^{\Gamma,\breve{S}}(M(F_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma))))\Phi^\Gamma_{E(\breve{S})}\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{S})}}\\&=\Big\langle \pi_{\overleftarrow{L}}^{\Gamma,\breve{S}}(\tilde{F}_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)))\Psi^\Gamma_{E(\breve{S})},\pi_{\overleftarrow{L}}^{\Gamma,\breve{S}}(F_\Gamma(\hat{\rho}_{S,\Gamma}(\Gamma)))\Phi^\Gamma_{E(\breve{S})}\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{S})}}\end{aligned}\tag{7.24}$$

whenever $\Psi^\Gamma_{E(\breve{S})},\Phi^\Gamma_{E(\breve{S})}\in\mathcal{H}^\Gamma_{E(\breve{S})}$.

Example 7.2.1: In definition 6.1.19 the following map I has been introduced. The map $I: C_0(\bar{\mathcal{A}}_\Gamma)\rightarrow C_0(\bar{\mathcal{A}}_{\Gamma^{-1}})$ is given by

$$I: f_\Gamma\mapsto f_{\Gamma^{-1}},\text{ where }(I\circ f_\Gamma)(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)):=f_{\Gamma^{-1}}(\mathfrak{h}_{\Gamma^{-1}}(\gamma_1)^{-1},...,\mathfrak{h}_{\Gamma^{-1}}(\gamma_N)^{-1})$$

such that $I^2=\text{id}$, where id is the identical automorphism on $C_0(\bar{\mathcal{A}}_\Gamma)$.

Consider a suitable set $\bar{S}$ of surfaces that is contained in the set $\breve{S}$ and let $M\leq N$. Note that if $M<N$, then there is a set of paths $\Gamma'':=\Gamma\setminus\Gamma'$ such that each path of this set does not intersect a surface in $\bar{S}$. Each path in Γ' intersects only one surface in $\bar{S}$ at the source vertes of this path. Then $\bar{G}_{\bar{S},\Gamma'\leq\Gamma}$ is a subgroup of $\bar{G}_{\bar{S},\Gamma}$ and is embedded by $\bar{G}_{\bar{S},\Gamma}:=\bar{G}_{\bar{S},\Gamma'}\times\{e_G\}\times....\times\{e_G\}$ in $\bar{G}_{\breve{S},\Gamma}$. Denote the set of surfaces, which has the simple surface intersection property for the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma'}$, which is contained in $\breve{S}$ and which is not contained in $\bar{S}$, by $\breve{R}$. Note that $\bar{G}_{\breve{R},\Gamma''\leq\Gamma}$ is a subgroup of $\bar{G}_{\breve{R},\Gamma}$ and is embedded by $\bar{G}_{\breve{R},\Gamma}:=\bar{G}_{\breve{R},\Gamma''}\times\{e_G\}\times....\times\{e_G\}$ in $\bar{G}_{\breve{S},\Gamma}$. Let $\bar{R}$ be a set of surfaces, which has the same surface intersection property for a path γ' in a graph, which is contained in the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma'}$.

Situation 1:
Then there is a C^*-dynamical system in $\mathcal{K}(\mathcal{H}^{\Gamma^{-1}}_{\mathcal{E}(\bar{S})})$, which is given by $(\bar{G}_{\bar{S},\Gamma^{-1}},C_0(\bar{\mathcal{A}}_{\Gamma^{-1}}),\alpha_{\overleftarrow{R}}^M)$. Let $(\Phi_M,U_{\overleftarrow{R}}^M)$ be a covariant pair associated to the C^*-dynamical system.

Then observe that $\alpha_{\overleftarrow{R}}^M=I\circ\alpha_{\overleftarrow{L}}^M\circ I^{-1}$ and $U_{\overleftarrow{R}}^M=I\circ U_{\overleftarrow{L}}^M\circ I^{-1}$ hold. Then $(\bar{G}_{\bar{S},\Gamma},C_0(\bar{\mathcal{A}}_{\Gamma^{-1}}),I\circ\alpha_{\overleftarrow{L}}^M\circ I^{-1})$ is a C^*-dynamical system in $\mathcal{K}(\mathcal{H}^{\Gamma^{-1}}_{\mathcal{E}(\bar{S})})$. Respectively, $(\bar{G}_{\bar{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha_{\overleftarrow{L}}^M)$ is a C^*-dynamical system in $\mathcal{K}(\mathcal{H}^\Gamma_{\mathcal{E}(\bar{S})})$.

Note that if $\bar{S}$ is equal to $\breve{S}$, then $\bar{S}$ has the simple surface intersection property for the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_{\Gamma^{-1}}$ and $M=N$.Then $(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_{\Gamma^{-1}}),I\circ\alpha_{\overleftarrow{L}}^N\circ I^{-1})$ and $(\bar{G}_{\breve{S},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha_{\overleftarrow{L}}^N)$ are two C^*-dynamical systems in $\mathcal{K}(\mathcal{H}^\Gamma_{\mathcal{E}(\bar{S})})$.

Situation 2:
Furthermore there is a C^*-dynamical system in $\mathcal{K}(\mathcal{H}^\Gamma_{\mathcal{E}(\breve{R})})$ given by $(\bar{G}_{\breve{R},\Gamma},C_0(\bar{\mathcal{A}}_\Gamma),\alpha_{\overleftarrow{L}}^K)$ for K suitable.

Situation 3:
There is a C^*-dynamical system in $\mathcal{K}(\mathcal{H}^{\gamma'}_{\mathcal{E}(\bar{R})})$ given by $(\bar{G}_{\bar{R},\gamma'}, C_0(\bar{\mathcal{A}}_{\gamma'}), \alpha_{\overleftarrow{L}}^{1})$.

Situation 4:
Finally there is C^*-dynamical system in $\mathcal{K}(\mathcal{H}^{\Gamma}_{\mathcal{E}(\breve{S})})$ given by $(\bar{G}_{\bar{S},\Gamma'^{-1}} \times \bar{G}_{\breve{R},\Gamma''}, C_0(\bar{\mathcal{A}}_\Gamma), (I^{-1}\circ \alpha_{\overleftarrow{R}}^{M}\circ I)\circ\alpha_{\overleftarrow{L}}^{K})$. Note that $(I^{-1}\circ \alpha_{\overleftarrow{R}}^{M}\circ I)\circ\alpha_{\overleftarrow{L}}^{K} = \alpha_{\overleftarrow{L}}^{K}\circ(I^{-1}\circ \alpha_{\overleftarrow{R}}^{M}\circ I)$. Reformulate $(\bar{G}_{\bar{S},\Gamma'} \times \bar{G}_{\breve{R},\Gamma''}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{M}\circ\alpha_{\overleftarrow{L}}^{K})$.

For each C^*-dynamical system given above there is a cross-product C^*-algebra.

∎

In the following proposition the *situation 1* is studied.

Proposition 7.2.9. *Let $\breve{T} := \{T_1,...,T_N\}$ be a set of surfaces with simple surface intersection property for the orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$. Let $\breve{S} := \{S_1,...,S_M\}$ be a set of surfaces that is contained in $\breve{T}$ an such that $M \leq N$.*

The unitaries $U_{\overleftarrow{R}}^{M}(\rho_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))$, whenever $\rho_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}) \in \bar{G}_{\breve{S},\Gamma^{-1}}$, are elements of the multiplier algebra of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{T},\Gamma}$. Moreover the elements of the holonomy-flux cross-product algebra $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{R}}^{M}\circ I} \bar{G}_{\breve{S},\Gamma^{-1}}$ are multipliers of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

Proof. Choose the two surface sets $\breve{S}$ and $\breve{T}$ and a graph Γ such that $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma^{-1}}, I^{-1}\circ\alpha_M^{\overleftarrow{R}}\circ I)$ and $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{T},\Gamma^{-1}}, I^{-1}\circ\alpha_N^{\overleftarrow{R}}\circ I)$ are two C^*-dynamical systems. Then notice that

$$\begin{aligned}&(F_\Gamma * \hat{F}_\Gamma)(\tilde{\rho}_{T_1}(\gamma_1^{-1}),...,\tilde{\rho}_{T_N}(\gamma_N^{-1}))\\ &= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\rho_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\\ &\qquad F_\Gamma(\rho_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\Big((I^{-1}\circ\alpha_M^{\overleftarrow{R}}(\rho^M_{\breve{S},\Gamma^{-1}})\circ I)(\hat{F}_\Gamma)\Big)\left(\rho_{S_1}(\gamma_1^{-1})^{-1}\tilde{\rho}_{T_1}(\gamma_1^{-1}),....,\rho_{S_N}(\gamma_N^{-1})^{-1}\tilde{\rho}_{T_N}(\gamma_N^{-1})\right)\end{aligned}$$

whenever $F_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma^{-1}}, C_0(\bar{\mathcal{A}}_\Gamma), I^{-1}\circ\alpha_M^{\overleftarrow{R}}\circ I)$ and $\hat{F}_\Gamma \in L^1(\bar{G}_{\breve{S},\Gamma^{-1}}, C_0(\bar{\mathcal{A}}_\Gamma), I^{-1}\circ\alpha_N^{\overleftarrow{R}}\circ I)$, holds. Furthermore recognize that

$$F^*_\Gamma(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1})) = (I^{-1}\circ\alpha_M^{\overleftarrow{R}}(\tilde{\rho}^M_{\breve{S},\Gamma^{-1}})\circ I)\left(F^+_\Gamma(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1})^{-1})\right)$$

is true.

Notice that

$$\begin{aligned}\alpha_N^{\overleftarrow{R}}(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})(f_{\Gamma^{-1}})(\mathfrak{h}_{\Gamma^{-1}}(\Gamma^{-1})) &= (I\circ\alpha_{\overleftarrow{L}}^{N}(\tilde{\rho}^N_{\breve{S},\Gamma})\circ I^{-1})(f_{\Gamma^{-1}})(\mathfrak{h}_{\Gamma^{-1}}(\Gamma^{-1}))\\ &= (I\circ f_\Gamma)(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\mathfrak{h}_\Gamma(\Gamma))\\ &= f_{\Gamma^{-1}}(\mathfrak{h}_{\Gamma^{-1}}(\Gamma^{-1})\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\end{aligned}$$

and

$$\int_{\bar{\mathcal{A}}_\Gamma}\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma))(I^{-1}\circ\alpha_N^{\overleftarrow{R}}(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})\circ I)(f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) = \int_{\bar{\mathcal{A}}_\Gamma}\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\Gamma))\alpha_N^{\overleftarrow{L}}(\tilde{\rho}^N_{\breve{S},\Gamma})(f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma))$$

whenever $f_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$ and $\tilde{\rho}^N_{\breve{S},\Gamma} \in \bar{G}_{\breve{T},\Gamma}$.

Clearly, there is a representation $\pi^M_{I,\overleftarrow{R}}$ of $L^1(\bar{G}_{\breve{S},\Gamma^{-1}}, C_0(\bar{\mathcal{A}}_\Gamma), I^{-1}\circ\alpha_M^{\overleftarrow{R}}\circ I)$ on $\mathcal{H}_\Gamma$, which is given by

$$\begin{aligned}\pi^M_{I,\overleftarrow{R}}(F_\Gamma)\psi_\Gamma &:= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\rho^M_{\breve{S},\Gamma^{-1}})\Phi_M(F_\Gamma(\rho^M_{\breve{S},\Gamma^{-1}}))(I^{-1}\circ U^M_{\overleftarrow{R}}(\rho^M_{\breve{S},\Gamma^{-1}})\circ I)\psi_\Gamma\\ &= \int_{\bar{G}_{\breve{S},\Gamma}}\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho^M_{\breve{S},\Gamma})\Phi_M(F_\Gamma(\rho^M_{\breve{S},\Gamma}))\big(U^M_{\overleftarrow{L}}(\rho^M_{\breve{S},\Gamma})\big)\psi_\Gamma\end{aligned}$$

where $\rho^M_{\breve{S},\Gamma^{-1}} \in \bar{G}_{\breve{S},\Gamma^{-1}}$, $F_\Gamma \in L^1(\bar{G}_{S,\Gamma^{-1}}, C_0(\bar{\mathcal{A}}_\Gamma), I^{-1}\circ\alpha_{\overleftarrow{M}}^{\overleftarrow{R}} \circ I)$ and $\psi_\Gamma \in \mathcal{H}_\Gamma$. Then derive that there is an isomorphism $\mathcal{I}$ from $L^1(\bar{G}_{\breve{S},\Gamma^{-1}}, C_0(\bar{\mathcal{A}}_\Gamma), I^{-1}\circ\alpha_{\overleftarrow{M}}^{\overleftarrow{R}}\circ I)$ to $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \alpha_{\overleftarrow{L}}^{M})$.

Then the Hilbert space $L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma})$ is embedded into $L^2(\bar{G}_{\breve{T},\Gamma}, \mu_{\breve{T},\Gamma})$. The left regular representation of $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{M}}^{\overleftarrow{R}}\circ I} \bar{G}_{\breve{S},\Gamma^{-1}}$ on $L^2(\bar{G}_{\breve{T},\Gamma}, \mu_{\breve{T},\Gamma})\otimes\mathcal{H}_\Gamma$ is given by

$$\begin{aligned}
&((\pi_{\overleftarrow{R}}^{\Gamma^{-1},\breve{S}}(F_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)) = \Big(((\Phi_{\overleftarrow{R}}^M \rtimes (I^{-1}\circ U_{\overleftarrow{R}}^M)\circ I)(F_\Gamma))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma))\\
&:= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\hat{\rho}_{\breve{S}}(\Gamma^{-1}))\\
&\qquad \Phi_M\Big((I^{-1}\circ\alpha_{\overleftarrow{R}}^M(\rho_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I)(F_\Gamma(\hat{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1})))\Big)(I^{-1}\circ U_{\overleftarrow{R}}^M(\hat{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I)\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))
\end{aligned}$$

for $F_\Gamma(\hat{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho_{\breve{T},\Gamma}\in G_{\breve{T},\Gamma}$, $\tilde{\rho}_{\breve{S},\Gamma^{-1}}, \hat{\rho}_{\breve{S},\Gamma^{-1}} \in G_{\breve{S},\Gamma^{-1}}$ and $\Psi^\Gamma_{E(\breve{S})}\in\mathcal{H}^\Gamma_{E(\breve{S})}$.

Set $U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)) := \hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))$ and $(I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I)\hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)) := \tilde{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))$. Then the unitaries $I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})\circ I$, whenever $\tilde{\rho}^N_{\breve{S},\Gamma^{-1}}\in\bar{G}_{\breve{S},\Gamma^{-1}}$, are multipliers. This is verified by the following computation:

$$\begin{aligned}
&\Big\langle(\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)), ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(M_U(\hat{F}_\Gamma)))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&:= \Big\langle(\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(\hat{F}_\Gamma)\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)), ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}((I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})\circ I)(\hat{F}_\Gamma)))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \Big\langle\Big(((\Phi_{\overleftarrow{L}}^M\rtimes U_{\overleftarrow{L}}^N)(\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma)), \Big(((\Phi_{\overleftarrow{L}}^M\rtimes U_{\overleftarrow{L}}^N)((I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})\circ I)(\hat{F}_\Gamma)))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\qquad\Big\langle\Phi_M\Big(\big(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))(\hat{F}_\Gamma)\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)),\\
&\qquad\quad \Phi_M\Big(\big((\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))\circ I^{-1}\circ\alpha_{\overleftarrow{R}}^N(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I)\hat{F}_\Gamma\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\\
&\qquad\qquad (I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})\circ I)\hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\qquad\Big\langle\Phi_M\Big(\big(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))(\hat{F}_\Gamma)\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)(I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})^*\circ I)\tilde{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)),\\
&\qquad\Phi_M\Big(\big((I^{-1}\circ\alpha_{\overleftarrow{R}}^N(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))\hat{F}_\Gamma\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\tilde{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \Big\langle\Big(((\Phi_{\overleftarrow{L}}^M\rtimes U_{\overleftarrow{L}}^N)((I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})^*\circ I)\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma)), \Big(((\Phi_{\overleftarrow{L}}^M\rtimes U_{\overleftarrow{L}}^N)(\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \Big\langle(\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(\tilde{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)), ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}
\end{aligned}$$

whenever $(I^{-1}\circ U_{\overleftarrow{R}}^N(\tilde{\rho}^N_{\breve{S},\Gamma^{-1}})^*\circ I)\hat{F}_\Gamma := \tilde{F}_\Gamma$, holds.

Finally each element of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{I^{-1}\circ\alpha_{\overleftarrow{M}}^{\overleftarrow{R}}\circ I}\bar{G}_{\breve{S},\Gamma^{-1}}$ define a linear map M from $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar{G}_{\breve{T},\Gamma}$

to $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^N} \bar{G}_{\breve{T},\Gamma}$ by

$$\begin{aligned}
&((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(M(\hat{F}_\Gamma)))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma))\\
&:= ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(F_\Gamma * \hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)) = \Big(((\Phi_{\overleftarrow{L}}^M \rtimes U_{\overleftarrow{L}}^N)(F_\Gamma * \hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})}\Big)(\rho_{\breve{T},\Gamma}(\Gamma))\\
&= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\qquad\Phi_M\Big(F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))((I^{-1}\circ\alpha_M^{\overleftarrow{R}}(\tilde{\rho}^M_{\breve{S},\Gamma^{-1}})\circ I\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))(\hat{F}_\Gamma))(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\\
&\qquad U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\\
&= \int_{\bar{G}_{\breve{S},\Gamma}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\qquad\Phi_M\Big(F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))((\alpha_{\overleftarrow{L}}^N(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1})\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))(\hat{F}_\Gamma))(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\\
&\qquad U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))
\end{aligned}$$

for $F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)), \hat{F}_\Gamma(\hat{\rho}_{\breve{T},\Gamma}(\Gamma)) \in C_0(\bar{\mathcal{A}}_\Gamma)$, $\rho_{\breve{S},\Gamma} \in G_{\breve{S},\Gamma}$, $\tilde{\rho}_{\breve{T},\Gamma}, \hat{\rho}_{\breve{T},\Gamma} \in G_{\breve{T},\Gamma}$, $\tilde{\rho}_{\breve{T},\Gamma}(\Gamma) := (\tilde{\rho}^M_{\breve{T},\Gamma}, e_G, ..., e_G) \in \bar{G}_{\breve{T},\Gamma}$ and $\Psi^\Gamma_{E(\breve{T})} \in \mathcal{H}^\Gamma_{E(\breve{T})}$. Clearly, the set $\breve{S}$ is replaced by a set $\breve{R}^{-1}$, which is contained in $\breve{T}$, then $\tilde{\rho}_{\breve{R}^{-1},\Gamma}(\Gamma)^{-1} = \tilde{\rho}_{\breve{R},\Gamma}(\Gamma) \in \bar{G}_{\breve{R},\Gamma}$ and $\alpha_{\overleftarrow{L}}^N(\tilde{\rho}_{\breve{R}^{-1},\Gamma}(\Gamma)^{-1}) = \alpha_{\overleftarrow{L}}^N(\tilde{\rho}_{\breve{R},\Gamma}(\Gamma)) \in \mathfrak{Aut}(C_0(\bar{\mathcal{A}}_\Gamma))$ yield.

Set $(I^{-1}\circ U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\circ I)\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)) := \hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))$. Then M is a multiplier since the following derivation:

$$\begin{aligned}
&\Big\langle ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(\hat{F}_\Gamma))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma)), ((\pi_{\overleftarrow{L}}^{\Gamma,\breve{T}}(M(\hat{F}_\Gamma)))\Psi^\Gamma_{E(\breve{T})})(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\quad\langle\Phi_M\big(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))(\hat{F}_\Gamma)\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)),\\
&\qquad\Phi_M\Big(F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))((I^{-1}\circ\alpha_M^{\overleftarrow{R}}(\tilde{\rho}^M_{\breve{S},\Gamma^{-1}})\circ I\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))(\hat{F}_\Gamma))(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\\
&\qquad U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \int_{\bar{G}_{\breve{S},\Gamma^{-1}}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma^{-1}}(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\quad\Big\langle\Phi_M\big(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))(\hat{F}_\Gamma)\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)),\\
&\qquad\Phi_M\Big(F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma^{-1}}(\Gamma^{-1}))((I^{-1}\circ\alpha_M^{\overleftarrow{R}}(\tilde{\rho}^M_{\breve{S},\Gamma^{-1}})\circ I\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))(\hat{F}_\Gamma))(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\\
&\qquad\hat{\Psi}^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}\\
&= \int_{\bar{G}_{\breve{S},\Gamma}}\int_{\bar{G}_{\breve{T},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))\,\mathrm{d}\,\mu_{\breve{T},\Gamma}(\hat{\rho}_{\breve{T}}(\Gamma))\\
&\quad\Big\langle\Phi_M\Big(\big(\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma))(\hat{F}_\Gamma)\big)(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma)),\\
&\qquad\Phi_M\Big(F_\Gamma(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma))((\alpha_M^{\overleftarrow{R}}(\tilde{\rho}^M_{\breve{S},\Gamma})\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve{T},\Gamma}(\Gamma)))(\hat{F}_\Gamma))(\tilde{\rho}_{\breve{S},\Gamma}(\Gamma)^{-1}\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Big)\,U_{\overleftarrow{L}}^N(\hat{\rho}_{\breve{T},\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve{T})}(\rho_{\breve{T},\Gamma}(\Gamma))\Big\rangle_{\mathcal{H}^\Gamma_{E(\breve{T})}}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\bar G_{\breve S,\Gamma^{-1}}}\int_{\bar G_{\breve T,\Gamma}} \mathrm{d}\,\mu_{\breve S,\Gamma^{-1}}(\tilde\rho_{\breve S,\Gamma^{-1}}(\Gamma^{-1}))\,\mathrm{d}\,\mu_{\breve T,\Gamma}(\hat\rho_{\breve T}(\Gamma))\\
&\quad\Big\langle \Phi_M\Big((I^{-1}\circ\alpha^{\overleftarrow{R}}_M(\tilde\rho^M_{\breve S,\Gamma^{-1}})\circ I)\left(F^+_\Gamma(\tilde\rho_{\breve S,\Gamma^{-1}}(\Gamma^{-1})^{-1})\right)\\
&\qquad((I^{-1}\circ\alpha^{\overleftarrow{R}}_M(\tilde\rho^M_{\breve S,\Gamma^{-1}})\circ I\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve T,\Gamma}(\Gamma)))(\hat F_\Gamma))(\tilde\rho_{\breve S,\Gamma}(\Gamma)\hat\rho_{\breve T,\Gamma}(\Gamma))\Big)\hat\Psi^\Gamma_{E(\breve T)}(\rho_{\breve T,\Gamma}(\Gamma)),\\
&\qquad\quad\Phi_M\left(\alpha_{\overleftarrow{L}}^N(\rho_{\breve T,\Gamma}(\Gamma))(\hat F_\Gamma)(\hat\rho_{\breve T,\Gamma}(\Gamma))\right)\hat\Psi^\Gamma_{E(\breve T)}(\rho_{\breve T,\Gamma}(\Gamma))\Big\rangle_{\mathcal H^\Gamma_{E(\breve T)}}\\
&= \langle((\pi^{\Gamma,\breve T}_{\overleftarrow{L}}(F^*_\Gamma * \hat F_\Gamma))\Psi^\Gamma_{E(\breve T)})(\rho_{\breve T,\Gamma}(\Gamma)),\pi^{\Gamma,\breve T}_{\overleftarrow{L}}(\hat F_\Gamma)\Psi^\Gamma_{E(\breve T)})(\rho_{\breve T,\Gamma}(\Gamma))\rangle_{\mathcal H^\Gamma_{E(\breve T)}}
\end{aligned}
$$

holds.

□

Notice that, the same arguments can be used for a surface set $\breve T := \{T_1,...,T_N\}$, which has the simple surface intersection property for the orientation preserved graph system $\mathcal P^{\mathfrak o}_\Gamma$ and $\breve R^{-1} := \{R_1^{-1},...,R_N^{-1}\}$ be a set of surfaces that has the simple surface intersection property for the orientation preserved graph system $\mathcal P^{\mathfrak o}_{\Gamma^{-1}}$. Indeed it can be shown that for all situations of example 7.2.1 except *situation 2* similar results can be obtained. The *situation 2* is not needed in the next theorem and hence is briefly discussed in the following remark.

Remark 7.2.10. *In situation 2 the sets $\breve R$ and $\bar S$ are disjoint. Let $\breve T_2$ and $\breve T_3$ be two disjoint surface sets such that the holonomy-flux cross-product algebras are given by $C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha^{\overleftarrow{L}}_N}\bar G_{\breve T_2,\Gamma}$ and $C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R}}_N}\bar G_{\breve T_3,\Gamma}$.*

Then the elements of these algebras are represented on two different Hilbert spaces $\mathcal H^\Gamma_{E(\breve T_2)} := L^2(\bar G_{\breve T_2,\Gamma},\mu_{\breve T_2,\Gamma})\otimes\mathcal H_\Gamma$ and $\mathcal H^\Gamma_{E(\breve T_3)} := L^2(\bar G_{\breve T_3,\Gamma},\mu_{\breve T_3,\Gamma})\otimes\mathcal H_\Gamma$. Set $\mathcal H_{E(\breve T_i)} := L^2(\bar G_{\breve T_i,\Gamma},\mu_{\breve T_i,\Gamma})$ for $i=2,3$. Hence there are two representations $\pi_{E(\breve T_2)}$ and $\pi_{E(\breve T_3)}$ such that $\pi_{E(\breve T_2)}\otimes\pi_{E(\breve T_3)}$ is a representation on $\mathcal H_{E(\breve T_2)}\otimes\mathcal H_{E(\breve T_3)}$.

The holonomy-flux cross-product C^*-algebra $C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha^{\overleftarrow{R}}_N}\bar G_{\breve T_3,\Gamma}$ is represented on $\mathcal H^\Gamma_{E(\breve T_3)}$ by

$$
\begin{aligned}
&(\pi^{\Gamma,\breve T_3}_{\overleftarrow{R}}(F_\Gamma)\Psi^\Gamma_{E(\breve T_3)})(\rho_{\breve T_3,\Gamma}(\Gamma)) = ((\Phi_M(F_\Gamma)\rtimes U^N_{\overleftarrow{R}}(\hat\rho^N_{\breve T_3,\Gamma}))\Psi^\Gamma_{E(\breve T_3)})(\rho_{\breve T_3,\Gamma}(\Gamma))\\
&= \int_{\bar G_{\breve T_3,\Gamma}}\left(\Phi_M\left(\alpha^{\overleftarrow{R}}_N(\rho^N_{\breve T_3,\Gamma})(F_\Gamma(\hat\rho_{\breve T_3,\Gamma}(\Gamma)))\right)\Psi^\Gamma_{E(\breve T_3)}\right)(L(\hat\rho_{\breve T_3,\Gamma}(\Gamma)^{-1})(\rho_{\breve T_3,\Gamma}(\Gamma)))\,\mathrm{d}\,\mu_{\breve T_3,\Gamma}(\hat\rho_{\breve T_3,\Gamma}(\Gamma))
\end{aligned}
$$

for $F_\Gamma(\hat\rho_{\breve T_3,\Gamma}(\Gamma))\in C_0(\bar{\mathcal A}_\Gamma)$, $\rho_{\breve T_3,\Gamma}(\Gamma),\hat\rho_{\breve T_3,\Gamma}(\Gamma)\in\bar G_{\breve T_3,\Gamma}$ and $\Psi^\Gamma_{E(\breve T_3)}\in\mathcal H^\Gamma_{E(\breve T_3)}$.
Similarly the elements of $C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha^{\overleftarrow{L}}_N}\bar G_{\breve T_2,\Gamma}$ is represented on $\mathcal H^\Gamma_{E(\breve T_2)}$.

Now the multiplier algebra of the cross product C^*-algebra $C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar G_{\breve T_2,\Gamma}$ is studied. First of all unitary elements, i.e. $U^N_{\overleftarrow{R}}(\rho_{\breve T_3,\Gamma}(\Gamma))$ for $\rho_{\breve T_3,\Gamma}(\Gamma)\in\bar G_{\breve T_3,\Gamma}$, are elements of the multiplier algebra. This is verified by the identification of M with the map

$$\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(F_\Gamma)\mapsto\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(U^N_{\overleftarrow{R}}(\rho_{\breve T_3,\Gamma}(\Gamma))F_\Gamma)\in C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar G_{\breve T_2,\Gamma}$$

whenever $\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(F_\Gamma)\in C_0(\bar{\mathcal A}_\Gamma)\rtimes_{\alpha_{\overleftarrow{L}}^N}\bar G_{\breve T_2,\Gamma}$ and $f_\Gamma\in C_0(\bar{\mathcal A}_\Gamma)$ and the computation

$$
\begin{aligned}
&\Big\langle(\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(\hat F_\Gamma)\Psi^\Gamma_{E(\breve T_2)},\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(M(F_\Gamma))\Phi^\Gamma_{E(\breve T_2)}\Big\rangle_{\mathcal H_{E(\breve T_2)}\otimes\mathcal H_\Gamma}\\
&= \Big\langle\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(\hat F_\Gamma)\Psi^\Gamma_{E(\breve T_2)},\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(U^N_{\overleftarrow{R}}(\rho_{\breve T_3,\Gamma}(\Gamma))F_\Gamma)\Phi^\Gamma_{E(\breve T_2),}\Big\rangle_{\mathcal H_{E(\breve T_2)}\otimes\mathcal H_\Gamma}\\
&= \int_{\bar G_{\breve T_2,\Gamma}}\mathrm{d}\,\mu_{\breve T_2,\Gamma}(\hat\rho_{\breve T_2}(\Gamma))\Big\langle\Phi_M\left(\alpha_{\overleftarrow{L}}^N(\rho_{\breve T_2,\Gamma}(\Gamma))(\hat F_\Gamma(\hat\rho_{\breve T_2,\Gamma}(\Gamma)))\right)U^N_{\overleftarrow{L}}(\hat\rho_{\breve T_2,\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve T_2)}(\rho_{\breve T_2,\Gamma}(\Gamma)),\\
&\qquad\Phi_M\left((\alpha^{\overleftarrow{R}}_N(\rho_{\breve T_3,\Gamma}(\Gamma))\circ\alpha_{\overleftarrow{L}}^N(\rho_{\breve T_2,\Gamma}(\Gamma)))(F_\Gamma(\hat\rho_{\breve T_2,\Gamma}(\Gamma)))\right)U^N_{\overleftarrow{R}}(\rho_{\breve T_3,\Gamma}(\Gamma))U^N_{\overleftarrow{L}}(\hat\rho_{\breve T_2,\Gamma}(\Gamma))\Psi^\Gamma_{E(\breve T_2)}(\rho_{\breve T_2,\Gamma}(\Gamma))\Big\rangle_{\mathcal H^\Gamma_{E(\breve T_2)}}\\
&= \Big\langle\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(U^N_{\overleftarrow{R}}(\rho_{\breve T_3,\Gamma}(\Gamma))^*\hat F_\Gamma)\Psi^\Gamma_{E(\breve T_2)},\pi^{\Gamma,\breve T_2}_{\overleftarrow{L}}(F_\Gamma)\Phi^\Gamma_{E(\breve T_2),}\Big\rangle_{\mathcal H_{E(\breve T_2)}\otimes\mathcal H_\Gamma}
\end{aligned}
$$

Then one has to show that the holonomy-flux cross-product C^*- algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{N}}^{R}} \bar{G}_{\breve{T}_3,\Gamma}$ is a subset of the multiplier algebra $M(C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{T}_2,\Gamma})$. The multiplier M is assumed to be the map

$$C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{T}_2,\Gamma} \ni F_\Gamma \mapsto \hat{F}_\Gamma * F_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{T}_2,\Gamma}$$

for a $\hat{F}_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{N}}^{R}} \bar{G}_{\breve{T}_3,\Gamma}$. But since $L(\rho_{\breve{T}_3,\Gamma}(\Gamma)^{-1})(\tilde{\rho}_{\breve{T}_2,\Gamma}(\Gamma))$ is not well-defined, the convolution

$$(\hat{F}_\Gamma * F_\Gamma)(\tilde{\rho}_{\breve{T}_2,\Gamma}(\Gamma)) = \int_{\bar{G}_{\breve{T}_3,\Gamma}} \mathrm{d}\,\mu_{\breve{T}_3,\Gamma}(\rho_{\breve{T}_3,\Gamma}(\Gamma))\hat{F}_\Gamma(\rho_{\breve{T}_3,\Gamma}(\Gamma))\big(\alpha_{N}^{\overleftarrow{R}}(\rho_{\breve{T}_3,\Gamma}(\Gamma))F_\Gamma\big)(L(\rho_{\breve{T}_3,\Gamma}(\Gamma)^{-1})(\tilde{\rho}_{\breve{T}_2,\Gamma}(\Gamma)))$$

is not well-defined, too. Consequently it has to be assumed that either $\bar{G}_{\breve{T}_3,\Gamma}$ is embedded into $\bar{G}_{\breve{T}_2,\Gamma}$ as a subgroup or the other way arround. Clearly the situation 4 is of this form.

Remark 7.2.11. *Let $\breve{S}$ contain only the surface S and let $\bar{S}$ be a surface set with same surface intersection property for a path γ. Then $U_{\overleftarrow{R}}^{1}(\rho_{\bar{S},\gamma}(\gamma))$ is contained in the multiplier algebra of $C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{1}} \bar{G}_{S,\gamma}$. This follows by showing that the map*

$$C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{1}} \bar{G}_{S,\gamma} \ni \pi_{\overleftarrow{L}}^{\gamma,S}(F_\gamma) \mapsto U_{\overleftarrow{R}}^{1}(\rho_{\bar{S},\gamma}(\gamma))\pi_{\overleftarrow{L}}^{\gamma,S}(F_\gamma) \in C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{1}} \bar{G}_{S,\gamma}$$

defines a multiplier map. Furthermore show that,

$$C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{1}} \bar{G}_{S,\gamma} \ni \pi_{\overleftarrow{L}}^{\gamma,S}(F_\gamma) \mapsto \pi_{\overleftarrow{L}}^{\gamma,S}(\hat{F}_\Gamma * F_\gamma) \in C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{1}} \bar{G}_{S,\gamma}$$

defines a multiplier map for each function $\hat{F}_\Gamma \in C_0(\bar{\mathcal{A}}_\gamma) \rtimes_{\alpha_{1}^{\overleftarrow{R}}} \bar{G}_{\bar{S},\gamma}$.

Theorem 7.2.12. *Let $\breve{S}$ be a set of surfaces with simple surface intersection property for a finite orientation preserved graph system associated to a graph Γ. Let $\{\breve{S}_i\}$ be a set of sets of surface such that each surface set $\breve{S}_i$ is suitable for a finite (orientation preserved) graph system associated to a graph Γ.*

Then the following statements are true:

(i) The algebra $C_0(\bar{\mathcal{A}}_\Gamma)$, the group $\bar{G}_{\breve{S}_i,\Gamma}$ and the group $\bar{G}_{\breve{S},\Gamma}$ are not contained in $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

(ii) The analytic holonomy algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ and the unitaries $U_{\overleftarrow{R}}^{M}(\rho_{\breve{S}_2,\Gamma}(\Gamma))$, whenever $\rho_{\breve{S}_i,\Gamma}(\Gamma) \in \bar{G}_{\breve{S}_2,\Gamma}$ where $1 \leq M \leq N$, are elements of the multiplier algebra of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

(iii) The unitaries $U_{\overleftarrow{L}}^{M}(\rho_{\breve{S}_1,\Gamma}(\Gamma))$, $U_{\overleftarrow{L}}^{\overleftarrow{R},M}(\rho_{\breve{S}_3,\Gamma}(\Gamma))$, $U_{\overrightarrow{L}}^{\overrightarrow{R},M}(\rho_{\breve{S}_4,\Gamma}(\Gamma))$, $U_{\overrightarrow{L}}^{M}(\rho_{\breve{S}_5,\Gamma}(\Gamma))$ and so on, whenever $\rho_{\breve{S}_i,\Gamma}(\Gamma) \in \bar{G}_{\breve{S}_i,\Gamma}$ where $1 \leq M \leq N$ and all i,
are elements of the multiplier algebra of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

(iv) The elements of the holonomy-flux cross-product algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{R}}^{M}} \bar{G}_{\breve{S}_2,\Gamma}$ are multipliers of the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

(v) Moreover all elements of $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{M}^{\overleftarrow{L}}} \bar{G}_{\breve{S}_1,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{M}^{\overleftarrow{R}}} \bar{G}_{\breve{S}_6,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{M}} \bar{G}_{\breve{S}_7,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overrightarrow{L}}^{M}} \bar{G}_{\breve{S}_5,\Gamma}$, $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{\overleftarrow{R},M}} \bar{G}_{\breve{S}_3,\Gamma}$ and $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overrightarrow{L}}^{\overrightarrow{R},M}} \bar{G}_{\breve{S}_4,\Gamma}$ for $1 \leq M \leq N$ are contained in the multiplier algebra the C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^{N}} \bar{G}_{\breve{S},\Gamma}$.

Proof : The proof is similar to proposition 7.2.9 and remarks 7.2.10 and 7.2.11.

■

In chapter 3.4 the Lie algebra-valued quantum flux operators $E_S(\Gamma)$ for different surfaces S are considered. Similarly, they are not contained in $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_L} \bar{G}_{\breve{S},\Gamma}$ or $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_R} \bar{G}_{\breve{S},\Gamma}$, but they are affiliated in the sense of Woronowicz [115].

Remark 7.2.13. *If the action of the flux group $\bar{G}_{\breve{S},\Gamma}$ on $C_0(\bar{\mathcal{A}}_\Gamma)$ is assumed to be the identity, then $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_{\mathrm{id}} \bar{G}_{\breve{S},\Gamma}$ is equivalent to $C_0(\bar{\mathcal{A}}_\Gamma) \otimes_{max} C^*(\bar{G}_{\breve{S},\Gamma})$ where $\otimes_{max}$ denotes the maximal C^*-tensor product.*

7.2.2 The holonomy-flux cross-product C^*-algebra for surfaces

Let G be a compact group and $F_\Gamma \in C^*(\bar{G}_{\breve{S},\Gamma}, C(\bar{\mathcal{A}}_\Gamma))$. Recall the Weyl-integrated holonomy-flux representation $\pi^{I,\Gamma}_{E(\breve{S})}(F_\Gamma) = (\Phi_M \rtimes U_{\overleftarrow{L}}^N)(F_\Gamma)$ of the C^*-algebra $C(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^N} \bar{G}_{\breve{S},\Gamma}$ presented in (7.2.4) and consider a *-homomorphisms $\beta_{\Gamma,\Gamma'}$ from $C(\bar{\mathcal{A}}_\Gamma) \rtimes_{\alpha_{\overleftarrow{L}}^N} \bar{G}_{\breve{S},\Gamma}$ to $C(\bar{\mathcal{A}}_{\Gamma'}) \rtimes_{\alpha_{\overleftarrow{L}}^N} \bar{G}_{\breve{S},\Gamma'}$ which satisfies

$$\beta_{\Gamma,\Gamma'}((\Phi_M \rtimes U_{\overleftarrow{L}}^N)(F_\Gamma(\mathfrak{h}_\Gamma, \rho_{\breve{S},\Gamma}(\Gamma)))) = (\Phi_M \rtimes U_{\overleftarrow{L}}^N)(F_{\Gamma'}(\mathfrak{h}_{\Gamma'}, \rho_{\breve{S},\Gamma}(\Gamma'))) \tag{7.25}$$

Definition 7.2.14. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph Γ_i of the family has the same intersection surface property for the set $\breve{S}$ (or the set $\breve{S}$) of surfaces. Set $|\Gamma_i| = N_i$. Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems.*

The ***holonomy-flux cross-product C^*-algebra*** $\mathfrak{A} \rtimes_{\alpha_{\overleftarrow{L}}} \bar{G}_{\breve{S}}$ *(of a special surface configuration $\breve{S}$) is an inductive limit C^*-algebra* $\varinjlim_{\mathcal{P}_{\Gamma_i}\in\mathcal{P}} C(\bar{\mathcal{A}}_{\Gamma_i}) \rtimes_{\alpha_{\overleftarrow{L}}^{N_i}} \bar{G}_{S,\Gamma_i}$ *of the inductive system of C^*-algebras*

$$\{(C(\bar{\mathcal{A}}_{\Gamma_i}) \rtimes_{\alpha_L} \bar{G}_{S,\Gamma_i}, \beta_{\Gamma_i,\Gamma_j}) \quad \beta_{\Gamma_i,\Gamma_j} : \ ^*\text{- homomorphisms } \ s.t.\ \beta_{\Gamma_i,\Gamma_j} = \beta_{\Gamma_i,\Gamma_k} \circ \beta_{\Gamma_k,\Gamma_j}\}$$

completed in the norm (where elements of norm 0 are devided out)

$$\|F\| := \inf_{\mathcal{P}_{\Gamma_j} \supseteq \mathcal{P}_{\Gamma_i}} \|\beta_{\Gamma_i,\Gamma_j}(F_{\Gamma_i})\|_{\Gamma_j} \text{ for } F_{\Gamma_i} \in \mathfrak{A}_{\Gamma_i} \rtimes_{\alpha_{\overleftarrow{L}}^{N_i}} \bar{G}_{\breve{S},\Gamma_i} \tag{7.26}$$

with $\|F_{\Gamma_i}\|_{\Gamma_i} := \sup_{\pi_E} \|\pi_E(F_{\Gamma_i})\|_2$ where the supremum is taken over all non-degenerate L^1-norm decreasing $*$-representations of $L^1(\bar{G}_{\breve{S},\Gamma_i}, C(\bar{\mathcal{A}}_{\Gamma_i}))$.

Proposition 7.2.15. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph Γ_i of the family has the same intersection surface property for the set $\breve{S}$ (or the set $\breve{S}$) of surfaces and such that there is only a finite number of intersections of $\breve{S}$ and all graphs in Γ_∞. Set $|\Gamma_i| = N_i$. Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems. Denote the center of the inductive limit group $\bar{G}_{\breve{S}}$ by $\bar{\mathcal{Z}}_{\breve{S}}$.*

The state $\omega_{E(\breve{S})}$ on $\mathfrak{A} \rtimes_{\alpha_{\overleftarrow{L}}} \bar{\mathcal{Z}}_{\breve{S}}$ associated to the GNS-representation $(\mathcal{H}_\Gamma, \pi^I_{E(\breve{S})}, \Omega^I_{E(\breve{S})})$ is not surface-orientation preserving graph-diffeomorphism invariant, but it is a surface preserving graph-diffeomorphism invariant state.

Proof : This can be deduced from proposition 7.2.6.

■

Theorem 7.2.16. *The* ***multiplier algebra*** $M(\mathfrak{A} \rtimes_{\alpha_{\overleftarrow{L}}} \bar{G}_{\breve{S}})$ ***of the holonomy-flux cross-product C^*-algebra*** $\mathfrak{A} \rtimes_{\alpha_{\overleftarrow{L}}} \bar{G}_{\breve{S}}$ *contains all elements of the holonomy-flux cross-product C^*-algebra of any suitable surface set $\breve{S}$ in $\mathbb{S}$.*

Proof : This can be derived from theorem 7.2.12.

■

7.3 The holonomy-flux-graph-diffeomorphism cross-product C^*-algebra

In this section the holonomy-flux cross-product C^*-algebra is enlarged further such that the new C^*-algebra contains in a suitable sense the finite graph-diffeomorphisms. Hence this algebra contains some constraints of the theory of quantum gravity. This is one further step to the aim of this dissertation. Notice that, the construction in this section is restricted to surface preserving graph-diffeomorphisms, but the development is generalised to surface-orientation preserving graph-diffeomorphisms. The latter are necessary for the interplay with the quantum flux operators.

Recall the C^*-dynamical system $(\mathfrak{B}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta)$ defined in proposition 6.2.1. Similarly to the construction of the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma), \zeta)$ in subsection 7.2.1 the Banach *-algebra $l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma), \zeta)$ is developed in the next paragraph.

Recall the generating system $\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ of bisections for a graph Γ, which is presented in section 6.2. The function $F_{\Gamma,\mathfrak{B}}$ is contained in $l^1(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ if $F_{\Gamma,\mathfrak{B}}$ satisfies

$$\|F_{\Gamma,\mathfrak{B}}\|_1 := \sum_{l=1,\dots,k_\Gamma} \|F_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'_{\sigma_l}))\|_2 < \infty$$

Then the product of two elements $F_{\Gamma,\mathfrak{B}}, K_{\Gamma,\mathfrak{B}} \in l^1(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ is defined by

$$(F_{\Gamma,\mathfrak{B}} * K_{\Gamma,\mathfrak{B}})(\mathfrak{h}_\Gamma(\Gamma'_\sigma)) = \sum_{\substack{\tilde{\sigma},\breve{\sigma}\in\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)\\ \tilde{\sigma}*_2\breve{\sigma}=\sigma}} F_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'_{\tilde{\sigma}}))K_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'_{\breve{\sigma}}))$$

and the involution is

$$F_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma({\Gamma'}_\sigma)) := \overline{F_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma({\Gamma'}_{\sigma^{-1}}))}$$

There is a *-representation $\pi^{\Gamma}_{I,\mathfrak{B}}$ of $l^1(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ on $l^2(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ given by

$$\pi^{\Gamma}_{I,\mathfrak{B}}(F_{\Gamma,\mathfrak{B}}) = \sum_{\sigma\in\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)} F_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'_\sigma))U(\mathfrak{h}_\Gamma(\Gamma'_\sigma))$$

where $U(\mathfrak{h}_\Gamma(\Gamma'_\sigma)) = \delta_\sigma$ and $\delta_\sigma(\mathfrak{h}_\Gamma(\Gamma'_{\breve{\sigma}})) := \delta(\mathfrak{h}_\Gamma(\Gamma'_{\sigma*\breve{\sigma}}))$.

Lemma 7.3.1. *Let $\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

The integrated *-representation $\pi^{\Gamma}_{I,\mathfrak{B}}$ of $l^1(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma),\zeta)$ is non-degenerate.

Proof : This follows from the fact that $\pi^{\Gamma}_{I,\mathfrak{B}}(\mathcal{F}_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'))\delta_{\text{id}}(\mathfrak{h}_\Gamma(\Gamma')) = \mathcal{F}_{\Gamma,\mathfrak{B}}(\mathfrak{h}_\Gamma(\Gamma'))$.

■

Since the group $\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ is finite-dimensional and discrete, the reduced holonomy-graph-diffeomorphism group C^*-algebra coincide with the holonomy-graph-diffeomorphism cross-product C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\zeta \mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$.

But this algebra do not contain any flux variables. Hence recall that, in proposition 6.2.15 it has been shown that, the triple $(\mathfrak{B}(\mathcal{P}_\Gamma\Sigma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}),\zeta)$ of a surface preserving group $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ of bisections, a C^*-algebra $\mathsf{W}(\bar{G}_{\breve{S},\Gamma})$ w.r.t. a suitable set $\breve{S}$ of surfaces and a graph Γ is a C^*-dynamical system in $\mathcal{L}(\mathcal{H}_\Gamma)$.

The pair (Φ, V), which consists of a morphism $\Phi \in \text{Mor}(\mathsf{W}(\bar{G}_{\breve{S},\Gamma}),\mathcal{L}(\mathcal{H}_\Gamma))$ and a unitary representation V of $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ on $\mathcal{L}(\mathcal{H}_\Gamma)$, i.e. $V \in \text{Rep}(\mathfrak{B}(\mathcal{P}_\Gamma\Sigma),\mathcal{K}(\mathcal{H}_\Gamma))$ such that

$$\Phi(\zeta_\sigma(W)) = V(\sigma)\Phi(W)V^*(\sigma)$$

is a covariant representation of $(\mathfrak{B}(\mathcal{P}_\Gamma\Sigma), \mathsf{W}(\bar{G}_{\breve{S},\Gamma}),\zeta)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$.

Lemma 7.3.2. *Let $\breve{S}$ be a set of surfaces with same surface intersection property for Γ. Furthermore let $\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

Then the triple $(\mathfrak{B}^{\Gamma}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma},\zeta)$ is a C^*-dynamical system in $\mathcal{L}(\mathcal{H})$.

Proof : Set $\Gamma = \{\gamma_1, ..., \gamma_N\}$, $\Gamma_\sigma = \{\gamma_1 \circ \sigma(v_1), ..., \gamma_N \circ \sigma(v_N)\}$.

Let $F_\Gamma : C_c(\bar{\mathcal{Z}}_{\breve{S},\Gamma}) \to C_0(\bar{\mathcal{A}}_\Gamma)$ and denote the image of $F_\Gamma(\rho_{S_1}(\gamma_1), ..., \rho_{S_1}(\gamma_1))$ by $F_\Gamma(\rho_{S_1}(\gamma_1), ..., \rho_{S_N}(\gamma_N); \mathfrak{h}_\Gamma(\gamma_1), ..., \mathfrak{h}_\Gamma(\gamma_N))$. Notice that

$$\begin{aligned}&(\zeta_\sigma F_\Gamma)(\rho_{S_1}(\gamma_1), ..., \rho_{S_1}(\gamma_1))\\ &= F_{\Gamma_\sigma}(\rho_{S_1}(\gamma_1 \circ \sigma(v_1)), ..., \rho_{S_N}(\gamma_N \circ \sigma(v_N)); \mathfrak{h}_{\Gamma_\sigma}(\gamma_1 \circ \sigma(v_1)), ..., \mathfrak{h}_{\Gamma_\sigma}(\gamma_N \circ \sigma(v_N)))\end{aligned}$$

holds. Clearly this defines a point-norm continuous automorphic action.

■

Proposition 7.3.3. *Let $\breve{S}$ be a set of surfaces with same surface intersection property for Γ. Furthermore let $\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

The pair $(\pi^{I,\Gamma}_{E(\breve{S})}, V)$ is a covariant pair of the C^*-dynamical system $(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma), C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma},\zeta)$ in $\mathcal{L}(\mathcal{H})$.

Proof : Take the $\pi^{I,\Gamma}_{E(\breve{S})}$ *-representation of $C_c(\bar{\mathcal{Z}}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ on $\mathcal{H}_\Gamma$ and V a regular representation of $\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma\Sigma)$ on $\mathcal{H}_\Gamma$ to observe that

$$\begin{aligned}
&\pi^{I,\Gamma}_{E(\breve{S})}(\zeta_\sigma(F_\Gamma))\Omega^I_{E(\breve{S})}\\
&= \int_{\bar{\mathcal{Z}}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))(\zeta_\sigma F_\Gamma)(\rho_{S_1}(\gamma_1),...,\rho_{S_N}(\gamma_N))\Omega^I_{E(\breve{S})}\\
&= \int_{\bar{\mathcal{Z}}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1\circ\sigma(v_1)),...,\rho_{S_N}(\gamma_N\circ\sigma(v_N)))\\
&\qquad F_\Gamma(\rho_{S_1}(\gamma_1\circ\sigma(v_1)),...,\rho_{S_N}(\gamma_N\circ\sigma(v_N)))\Omega^I_{E(\breve{S})}\\
&= \int_{\bar{\mathcal{Z}}_{\breve{S},\Gamma}} \mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S_1}(\gamma_1\circ\sigma(v_1)),...,\rho_{S_N}(\gamma_N\circ\sigma(v_N)))\\
&\qquad F_\Gamma(\rho_{S_1}(\gamma_1\circ\sigma(v_1)),...,\rho_{S_N}(\gamma_N\circ\sigma(v_N)))V^*_\sigma\Omega^I_{E(\breve{S})}\\
&= V_\sigma\pi^{I,\Gamma}_{E(\breve{S})}(F_\Gamma)V^*_\sigma\Omega^I_{E(\breve{S})}
\end{aligned}$$

yields if $v_i = t(\gamma_i)$ for $i = 1,...,N$. Consequently $(\pi^{I,\Gamma}_{E(\breve{S})}, V)$ is a covariant representation.

■

In proposition 7.2.15 it is shown that the state $\omega_{E(\breve{S})}$ of $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$ is graph-diffeomorphism invariant in general. There is a finite surface-orientation preserving graph-diffeomorphism and, hence, $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$-invariant state of $C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$ on $\mathcal{H}_\Gamma$ given by

$$\begin{aligned}
\omega^\Gamma_{E(\breve{S})}(\zeta_\sigma(\mathcal{F}_{\Gamma,\breve{S}})) &= \langle\Omega^\Gamma_{E(\breve{S})}, V_\sigma\pi^{I,\Gamma}_{E(\breve{S})}(\mathcal{F}_{\Gamma,\breve{S}})V^*_\sigma\Omega^\Gamma_{E(\breve{S})}\rangle\\
&= \omega^\Gamma_{E(\breve{S})}(\mathcal{F}_{\Gamma,\breve{S}})
\end{aligned}$$

for $\sigma\in\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ and where $\mathcal{F}_{\Gamma,\breve{S}}\in C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$.

Proposition 7.3.4. *Let $\breve{S}$ be a set of surfaces with same surface intersection property for Γ. Furthermore let $\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

The space $l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma},\zeta)$ is defined by all functions $\mathcal{F}_{\Gamma,\breve{S}} : \mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)\rightarrow C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$ for which

$$\|\mathcal{F}_{\Gamma,\breve{S}}\|_1 = \sum_{\sigma\in\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)} \|\mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))\|_2 < \infty$$

is true.

The convolution *-algebra $l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma)\rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma},\zeta)$ is presented by the multiplication

$$\begin{aligned}
&(\mathcal{G}_{\Gamma,\breve{S}} * \mathcal{F}_{\Gamma,\breve{S}})(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))\\
&= \sum_{\tilde{\sigma},\breve{\sigma}\in\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)} \mathcal{G}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))\zeta_\sigma\Big(\mathcal{F}_{\Gamma,\breve{S}}((\sigma^{-1}*\sigma')(t(\gamma_1)),...,(\sigma^{-1}*\sigma')(t(\gamma_N)))\Big)
\end{aligned}$$

and the involution

$$\mathcal{F}^*_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N))) = \zeta_\sigma(\mathcal{F}_{\Gamma,\breve{S}}(\sigma^{-1}(t(\gamma_1)),...,\sigma^{-1}(t(\gamma_N)))^*)$$

where the involution * *of* $l^1(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ *is inherited from the involution* * *of* $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$

$$\mathcal{F}^*_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N))) = \alpha(\rho_{\breve{S}}(\Gamma))\Big(\mathcal{F}^+_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1})\Big)$$

and

$$\begin{aligned}&\mathcal{F}^+_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1})\\ &= \overline{\mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1})}\end{aligned}$$

where the map

$$(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N))) \mapsto \mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))$$

define an element in $l^1(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$*, the map*

$$(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1}) \mapsto \mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1})$$

defines an element in $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}$ *and finally the map*

$$\begin{aligned}&(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1}; \mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\\ &\quad\mapsto \mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)); \rho_{S_1}(\gamma_1)^{-1},...,\rho_{S_N}(\gamma_N)^{-1}; \mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))\end{aligned}$$

define an element in $C_0(\bar{\mathcal{A}}_\Gamma)$.

The space $l^1(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ *is a well-defined Banach* **-algebra.*

Definition 7.3.5. *Let* $\breve{S}$ *be a set of surfaces with same surface intersection property for* Γ*. Furthermore let* $\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ *be a subset of* $\mathfrak{B}(\mathcal{P}_\Gamma)$ *that forms a generating system of bisections for the graph* Γ*.*

Let $(\pi^{I,\Gamma}_{E(\breve{S})}, V)$ *be a covariant representation of* $(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ *in* $\mathcal{L}(\mathcal{H}_\Gamma)$*.*

Define the ***integrated holonomy-flux-graph-diffeomorphism representation of*** $l^1(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ *by*

$$\begin{aligned}\pi_{I,\mathfrak{B}}(\mathcal{F}_{\Gamma,\breve{S}}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))) &= \sum_{\sigma\in\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma\Sigma)} \pi^{I,\Gamma}_{E(\breve{S})}\big(\mathcal{F}_{\Gamma,\breve{S},\sigma}(\sigma(t(\gamma_1)),...,\sigma(t(\gamma_N)))\big)V_\sigma\\ = \sum_{\substack{\delta_i\in\mathcal{P}_\Gamma\Sigma^{t(\gamma_i)}\\ i=1,..,N}} \pi^{I,\Gamma}_{E(\breve{S})}\big(\mathcal{F}_{\Gamma,\breve{S}}(\delta_1,...,\delta_N)\big)V(\delta_1,...,\delta_N)\end{aligned}$$

such that the sum is over all paths δ_i*, which start at* $t(\gamma_i)$ *and* $\delta_i \in \mathcal{P}_\Gamma\Sigma$*.*

Definition 7.3.6. *Let* $\breve{S}$ *be a set of surfaces with same surface intersection property for* Γ*. Furthermore let* $\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ *be a subset of* $\mathfrak{B}(\mathcal{P}_\Gamma)$ *that forms a generating system of bisections for the graph* Γ*.*

The ***reduced holonomy-flux-graph-diffeomorphism group*** C^****-algebra*** $C^*_r(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma})$ ***of a graph*** Γ ***and a set of surfaces*** $\breve{S}$ *is defined as the closure of* $l^1(\mathfrak{B}^\Gamma_{\breve{S},\mathrm{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ *in the norm* $\|\mathcal{F}_{\Gamma,\breve{S}}\| := \|\pi_{I,\mathfrak{B}}(\mathcal{F}_{\Gamma,\breve{S}})\|_2$.

Proposition 7.3.7. *Let $\breve{S}$ be a set of surfaces with same surface intersection property for Γ. Furthermore let $\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma) := \{\sigma_l \in \mathfrak{B}(\mathcal{P}_\Gamma)\}_{1\leq l\leq k}$ be a subset of $\mathfrak{B}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph Γ.*

Suppose that $(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ in $\mathcal{L}(\mathcal{H})$ is a C^*-dynamical system and that for each $F_\Gamma \in l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ define

$$\|\mathcal{F}_{\Gamma,\breve{S}}\| := \sup\Big\{\|(\pi \rtimes V)(\mathcal{F}_{\Gamma,\breve{S}})\| :(\pi, V) \text{ is a covariant representation of } (\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)\Big\}$$

Then $\|.\|$ is a norm on $l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ called the universal norm. The universal norm is dominated by the $\|.\|_1$-norm, and the completition of $l^1(\mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma), C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}, \zeta)$ with respect to $\|.\|$ is a C^*-algebra. This C^*-algebra is called the ***holonomy-flux-graph-diffeomorphism cross-product C^*-algebra*** *$\Big(C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S},\Gamma}\Big) \rtimes_\zeta \mathfrak{B}^\Gamma_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma)$ associated to a graph Γ and a set $\breve{S}$ of surfaces.*

In proposition 6.2.2 in section 6.2, it has been argued that there are several C^*-dynamical systems available for the analytic holonomy C^*-algebra and the group of bisections. This can be used to define a bunch of holonomy-flux-graph-diffeomorphism cross-product C-algebras, which can be constructed from C^*-dynamical systems. These cross-product C^*-algebra are exterior equivalent, too. Clearly there is a multiplier algebra of the holonomy-flux-graph-diffeomorphism cross-product algebra associated to a graph and a set of surfaces is derivable. The author of this dissertation suggests that it can be proven that the different holonomy-flux-graph-diffeomorphism cross-product C-algebras are contained in this multiplier algebra by using similiar arguments used in the proof of theorem 7.2.12. The construction of the inductive limit C^*-algebra of a family of C^*-algebras defined above is not mathematically understood very well until now. The detailed study of these objects is a further project.

7.4 The group and the transformation group C^*-algebra in Loop Quantum Cosmology

In this section a simple example of the holonomy-flux cross-product C^*-algebra construction is presented in the context of Loop Quantum Cosmology. This is a further development of the ideas presented in chapter 4. The algebras presented below a not used in the framework of LQC until now. A further project is to explore the new algebras in this context and to redefine the LQC-Hamilton constraint in this new operator algebraic framework.

In a general context the Weyl algebras are constructed from unitary elements that satisfy canonical commutation relations. For example, consider the untaries $w(x,p) := u_x v_p$ for $x, p \in \mathbb{R}_\text{d}$ on the Hilbert space $l^2(\mathbb{R}^2_\text{d})$ and the commutator relation

$$w(x_1, p_1)w(x_1, p_1) = \sigma(x_1, x_2; p_1, p_2)w(x_1 + x_2, p_1 + p_2) \tag{7.27}$$

where $\sigma(x_1, x_2; p_1, p_2) := \exp(-{}^{i}\!/\!{}_{2}(x_1p_2 - x_2p_1))$ such that the map $(p_1, p_2) \mapsto \sigma(x_1, x_2; p_1, p_2)$ is a continuous two-cocycle $\sigma \in Z^2(\mathbb{R}^2_\text{d} \times \mathbb{R}^2_\text{d}, C(\mathbb{R}^2_\text{d}, \mathbb{T}))$[4].

Then the **twisted convolution *-algebra** $\mathcal{C}_\sigma(\mathbb{R}^2_\text{d})$ is defined by the convolution

$$(f * g)(z_1, z_2) := \sum_{y_1,y_2\in\mathbb{R}_d} f(y_1, y_2)g(z_1 - y_1, z_2 - y_2)\sigma(y_1, y_2; z_1 - y_1, z_2 - y_2)$$

and the involution

$$f^*(y_1, y_2) := \overline{f(-y_1, -y_2)}$$

for $f, g \in C_0(\mathbb{R}^2_\text{d})$. The convolution and involution operation are continuous in the $l^1_\sigma(\mathbb{R}^2_\text{d})$-norm defined by

$$\|f\|_1 := \sum_{y_1,y_2\in\mathbb{R}_d} |f(y_1, y_2)|$$

[4] $C(\mathbb{R}^2_\text{d}, \mathbb{T})$ is the multiplier algebra of $C_0(\mathbb{R}^2_\text{d})$

since $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ and $\|f^*\|_1 = \|f\|_1$. Furthermore there is an natural action of the multiplier algebra $C(\mathbb{R}^2_{\mathrm{d}}, \mathbb{T})$ on $C_0(\mathbb{R}^2_d)$, which is used in the construction of the twisted transformation group algebra. The group algebra is constructed by using the generalized momentum representation of the twisted convolution algebra $\mathcal{C}_\sigma(\mathbb{R}^2_{\mathrm{d}})$. This is studied in the next paragraphs.

Let α be an automorphism of $C_0(\mathbb{R}^2_{\mathrm{d}})$ which is given by

$$(\alpha_{p_1,p_2} f)(y_1, y_2) = f(y_1 - p_1, y_2 - p_2) \tag{7.28}$$

Then $(C_0(\mathbb{R}^2_{\mathrm{d}}), \alpha, \mathbb{R}^2_{\mathrm{d}})$ is a C^*-dynamical system.

The continuous unitary representation $\hat{\pi}$ of $\mathbb{R}^2_{\mathrm{d}}$ on the Hilbert space $l^2(\mathbb{R}^2_{\mathrm{d}})$ defined by $\hat{\pi}(y)\chi_z := v_y \chi_z = \chi_{z-y}$ is called the left-regular representation of $\mathbb{R}^2_{\mathrm{d}}$ on $l^2(\mathbb{R}^2_{\mathrm{d}})$. For a function $f \in l^1(\mathbb{R}^2_{\mathrm{d}})$ set

$$\hat{\pi}_\sigma(f) := \sum_{y_1,y_2 \in \mathbb{R}_d} f(y_1, y_2)\sigma(y_1, y_2)\hat{\pi}(y_1 + y_2) \tag{7.29}$$

whenever $\sigma \in Z^2(\mathbb{R}^2_{\mathrm{d}}, C(\mathbb{R}^2_{\mathrm{d}}, \mathbb{T}))$. Then for an element χ in the Hilbert space $l^2(\mathbb{R}^2_{\mathrm{d}})$ it is true that

$$\hat{\pi}_\sigma(f)\chi(z_1, z_2) = (f * \chi)(z_1, z_2)$$

It is easy to verify that $\hat{\pi}_\sigma$ defines a faithful representation and it is called **twisted generalized momentum representation of** $l^1_\sigma(\mathbb{R}^2_{\mathrm{d}})$ on $l^2(\mathbb{R}^2_{\mathrm{d}})$. The closure of $\mathcal{C}_\sigma(\mathbb{R}^2_{\mathrm{d}})$ w.r.t. $\|\hat{\pi}_\sigma(.)\|_2$ is called **twisted reduced group algebra** $C^*_{r,\sigma}(\mathbb{R}^2_{\mathrm{d}})$.

The closure of $\mathcal{C}_\sigma(\mathbb{R}^2_{\mathrm{d}})$ w.r.t. the norm

$$\|f\| := \sup\{\|\hat{\Pi}^\sigma(f)\|_2 : \hat{\Pi}^\sigma \in \mathrm{Rep}_\sigma(l^1_\sigma(\mathbb{R}^2_{\mathrm{d}}))\}$$

where $\mathrm{Rep}_\sigma(l^1_\sigma(\mathbb{R}^2_{\mathrm{d}}))$ denotes the set of all non-degenerate l^1-norm decreasing[5] *-representations of the Banach *-algebra $l^1_\sigma(\mathbb{R}^2_{\mathrm{d}})$, is called **twisted group algebra** $C^*_\sigma(\mathbb{R}^2_{\mathrm{d}})$. Another equivalent formulation of $C^*_\sigma(\mathbb{R}^2_{\mathrm{d}})$ is given by the closure w.r.t. the norm $\|f\| := \sup\{\hat{\pi}_\sigma(f) : \hat{\pi} \in \mathrm{Rep}(\mathbb{R}^2_{\mathrm{d}})\}$ where $\mathrm{Rep}(\mathbb{R}^2_{\mathrm{d}})$ denotes the set of all non-degenerate weakly continuous irreducible unitary representations of the group $\mathbb{R}^2_d$ on a Hilbert space and $\hat{\pi}_\sigma$ is of the form (7.29).

Furthermore for a non-degenerate irreducible weakly continuous representations $\hat{\pi}$ of $\mathbb{R}^2_{\mathrm{d}}$ on $l^2(\mathbb{R}^2_{\mathrm{d}})$ (the characters of $\mathbb{R}^2_{\mathrm{d}}$) define

$$\hat{\pi}_\sigma(f)\hat{\chi} = \sum_{p_1,p_2 \in \mathbb{R}_{\mathrm{d}}} f(p_1, p_2)\sigma(p_1, p_2)\hat{\pi}(p_1 + p_2)\chi =: \hat{f}_\sigma \hat{\chi}$$

where $\sigma \in Z^2(\mathbb{R}^2_{\mathrm{d}}, C(\mathbb{R}^2_{\mathrm{d}}, \mathbb{T}))$ and $\chi \in l^2(\mathbb{R}^2_{\mathrm{d}})$, $\hat{\chi} \in l^2(\hat{\mathbb{R}}^2_{\mathrm{d}})$, the **generalized Fourier transformation** $\hat{\pi}_\sigma(f)$: $C^*_\sigma(\mathbb{R}^2_{\mathrm{d}}) \to C(\hat{\mathbb{R}}^2_{\mathrm{d}})$, where $f \mapsto \hat{f}_\sigma$. Let p_1, p_2 be zero, then $\sigma = \mathbb{1}$ for all $x_1, x_2 \in \mathbb{R}^2_{\mathrm{d}}$ and the generalized Fourier transformation is an *-isomorphism between the group algebra $\mathcal{C}^*(\mathbb{R}^2_{\mathrm{d}})$ and $C(\hat{\mathbb{R}}^2_{\mathrm{d}})$. The C^*-algebra $C(\hat{\mathbb{R}}^2_d)$ is called the **algebra of LQC-configuration variables**.

Equivalently, if $\hat{\Pi}$ is a representation of the Pontryagin dual $\hat{\mathbb{R}}_d$ on $l^2(\mathbb{R}^2_{\mathrm{d}})$. Then the canonical commutator relations read

$$\hat{\Pi}(x_1)\hat{\pi}(p_1)\hat{\Pi}(x_2)\hat{\pi}(p_2) = \hat{\sigma}(x_1, x_2; p_1, p_2)\hat{\Pi}(x_1 + x_2)\hat{\pi}(p_1 + p_2)$$

whenever $\hat{\sigma} \in Z^2(\hat{\mathbb{R}}^2_{\mathrm{d}}, C(\mathbb{R}^2_{\mathrm{d}}, \mathbb{T}))$, for $x_1, x_2 \in \hat{\mathbb{R}}_d$ and $p_1, p_2 \in \mathbb{R}_{\mathrm{d}}$ holds. Then for $\hat{f} \in$ the inverse generalised Fourier transform $\hat{\pi}_{\hat{\sigma}} : C(\hat{\mathbb{R}}^2_{\mathrm{d}}) \to C^*_\sigma(\mathbb{R}^2_{\mathrm{d}})$ is given by

$$\hat{\pi}_{\hat{\sigma}}(\hat{f})\chi := \int_{\hat{\mathbb{R}}_{\mathrm{d}}} \mathrm{d}\,\mu_{\hat{\mathbb{R}}_{\mathrm{d}}}(x_1, x_2)\hat{f}(x_1, x_2)\hat{\sigma}(x_1, x_2)\hat{\Pi}(x_1 + x_2)\hat{\chi} =: f_{\hat{\sigma}}\chi$$

Notice that the Hilbert space $l^2(\hat{\mathbb{R}}_{\mathrm{d}})$ is non-separable and, hence, the unitary representations of $\mathbb{R}$ are not weakly operator continuous (refer to [77]). Recall the usual Weyl algebra in Quantum mechanics (refer to the section 1.3.1.2) and consider the state

$$\omega_0(v_p) = \begin{cases} 0 & \text{if } p \neq 0 \\ 1 & \text{if } p = 0 \end{cases}$$

[5] That means $\|\hat{\Pi}^\sigma(\mathfrak{p})\| \leq \|\mathfrak{p}\|_1$ for all $\mathfrak{p} \in l^1_\sigma(\mathbb{R}^2_{\mathrm{d}})$.

Then the GNS-representation associated to ω_0 is called the polymer representation.

Now the construction of twisted transformation group algebras is straight forward. Recall the action α given by (7.28) of $\mathbb{R}^2_{\text{d}}$ on $C_0(\mathbb{R}^2_{\text{d}})$. Then the convolution product of two functions $\mathfrak{p}_1, \mathfrak{p}_2$ in $l^1(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ is given by

$$(\mathfrak{p}_1 * \mathfrak{p}_2)(y_1, y_2) = \sum_{x_1, x_2 \in \mathbb{R}_{\text{d}}} \mathfrak{p}_1(x_1, x_2)(\alpha_{x_1,x_2}(\mathfrak{p}_2))(x_1^{-1}y_1, x_2^{-1}y_2)\sigma(x_1^{-1}y_1, x_2^{-1}y_2)$$

and involution on $l^1(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$

$$\mathfrak{p}^*(x_1, x_2) = \overline{\sigma(x_1^{-1}, x_2^{-1})}(\alpha_{x_1,x_2}(\mathfrak{p}^+))(x_1^{-1}, x_2^{-1})$$

whenever $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2 \in l^1(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ and $\sigma \in Z^2(\mathbb{R}^2_{\text{d}}, C(\mathbb{R}^2_{\text{d}}, \mathbb{T}))$. Note that $^+$ denotes the involution of $C_0(\mathbb{R}^2_{\text{d}})$. The norm of elements in $l^1(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ is given by

$$\|\mathfrak{p}\|_1 := \sum_{p_1, p_2 \in \mathbb{R}_{\text{d}}} |\mathfrak{p}(p_1, p_2)|$$

Consider the multiplication representation Φ_M of $C_0(\mathbb{R}^2_{\text{d}})$ on $l^2(\mathbb{R}^2_{\text{d}})$ defined by $\Phi_M(f)\chi = f \cdot \chi$. Finally define the intergated representation of $l^1(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ on $l^2(\mathbb{R}^2_{\text{d}})$ by

$$\hat{\Pi}^\sigma_I(\mathfrak{p}) := \sum_{p_1, p_2 \in \mathbb{R}_{\text{d}}} \Phi_M(\mathfrak{p}(p_1, p_2))\sigma(p_1, p_2)\hat{\pi}(p_1 + p_2)$$

and call $\hat{\Pi}^\sigma_I$ the **Weyl-integrated representation of** $l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ acting on $l^2(\mathbb{R}^2_{\text{d}})$. Moreover $(\Phi_M, \hat{\Pi}^\sigma_I)$ is called a covariant pair. Notice that

$$\hat{\Pi}^\sigma_I(\mathfrak{p}v_{p_3}) := \sum_{p_1, p_2 \in \mathbb{R}_{\text{d}}} \Phi_M(\mathfrak{p}(p_1, p_2))\sigma(p_1, p_2)\hat{\pi}(p_1 + p_2 + p_3)$$

Moreover let $\hat{\omega}^\sigma_I$ be a state associated to the GNS-triple $(l^2(\mathbb{R}_d), \hat{\Pi}^\sigma_I, \Omega_I)$ given by

$$\hat{\omega}^\sigma_I(\mathfrak{p}) = \langle \Omega_I, \Pi^\sigma_I(\mathfrak{p})\Omega_I \rangle$$

for all $\mathfrak{p} \in l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$.

The closure of $l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ w.r.t. the norm

$$\|\mathfrak{p}\| := \sup\{\|\hat{\Pi}^\sigma_I(\mathfrak{p})\|_2 : \hat{\Pi}^\sigma_I \in \text{Rep}_\sigma(l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}})))\}$$

where $\text{Rep}_\sigma(l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}})))$ denotes th set of all non-degenerate l^1-norm decreasing[6] *-representations of the Banach algebra $l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$ is called **twisted transformation group algebra** $C^*_\sigma(\mathbb{R}^2_{\text{d}}, \mathbb{R}^2_{\text{d}})$ **of quantum LQC-variables**.

Consequently the quantum operators $\hat{p}$, $\hat{V}$ introduced in the framework of Loop Quantum Cosmology in for example [5] are not elements of $C^*_\sigma(\mathbb{R}_{\text{d}})$. In [41] Fredenhagen and Reszewski have shown that, states on the Weyl algebra of Quantum mechanics can be approximated by states associated to the polymer representation. This due to Fell's theorem, which states that the states associated to every faithful representation of a C^*-algebra form a weakly dense subset of the full state space. If the Weyl algebra is changed, for example by using the discretised real line, then new discretised operators $\hat{p}_{\text{d}}$, $\hat{V}_{\text{d}}$ similarly to the ones defined in LQC can be studied. In a further work one can search for an inductive family of modified Weyl C^*-algebras such that the limit is the Weyl C^*-algebra of quantum mechanics. A possibility for this can be constructed by using a countable dense subset Λ of $\mathbb{R}$, which is invariant under translation by $\mathbb{Z}$. Then an inductive family of C^*-algebras $\{C^*_\sigma(\Lambda)\}$ can be studied. Notice, that the quantum LQC- Hamiltonian constraint presented in [13], [65] and [60] are much more simpler than the LQG-Hamilton constraint. Hence it is easy to find a Weyl algebra such that the constraint set is a subalgebra of the algebra of quantum operators in the LQC framework.

[6] That means $\|\hat{\Pi}^\sigma_I(f)\|_2 \leq \|f\|_1$ for all $f \in l^1_\sigma(\mathbb{R}^2_{\text{d}}, C_0(\mathbb{R}^2_{\text{d}}))$.

Chapter 8

Analytic holonomy and holonomy-flux cross-product *-algebras

8.1 Some analytic holonomy *-algebras

In this section a short overview about some different analytic holonomy *-algebras is presented. In LQG literature the different possibilities are not analysed in detail so far. In Loop Quantum Gravity the holonomy algebras for the analytic category are usually constructed from the matrix elements $T_{\gamma,\pi_s,m,n} := \pi_s(\mathfrak{h}(\gamma))^m_n$ of the holonomy map $\mathfrak{h}$ of a path groupoid $\mathcal{P} \rightrightarrows \Sigma$ along a path γ, where γ runs over all paths in $\mathcal{P}$, π_s runs over all (equivalence classes of) irreducible representations of a compact group G, and m and n runs over all the corresponding matrix indices. In particular, for example an inverse loop transforms on elements of $L^1(G,\mu)$ has been studied by Thiemann in [99]. In this section different algebras constructed from the matrix elements $T_{\gamma,\pi_s,m,n}$ are presented. The underlying mathematical theory can be found in the books of Bump [27] and Hewitt and Ross [50].

For simplicity fix for a moment a path γ. Then the holonomy map $\mathfrak{h}$ along that path γ is rewritten by $\mathfrak{h}(\gamma) =: h_\gamma$ and for another holonomy $\tilde{\mathfrak{h}}$ one writes $\tilde{\mathfrak{h}}(\gamma) := g_\gamma$. Furthermore instead of paths graphs are used usually. Therefore one consideres a graph Γ with $|\Gamma|$ edges and $|\Gamma|$-tuples $(T_{\gamma_1,\pi_s^1,m_1,n_1},...,T_{\gamma_k,\pi_s^k,m_k,n_k})$ where $k := |\Gamma|$.

According to the book of Bump [27, §4.1.] the Peter-Weyl theorem states that the set of matrix coefficients of a compact group G is dense in the space $C(G)$ of continuous complex functions on G, equipped with the supremum norm. Consequently the set of matrix coefficients of a compact group G is dense in the Hilbert space $L^2(G,\mu)$. Following the theory of representation theory of compact groups, which is developed in the book of Dixmier [33] or Hewitt and Ross [50], the set of matrix coefficients of a compact group G is equipped with an involution and a multiplication such that this set is a Banach *-algebra. Then a similar density property can be derived.

Consider the algebra of continuous functions $C(G^{|\Gamma|})$ with convolution product, an involution * and supremum norm and denote this *-algebra by $\mathcal{C}(G^{|\Gamma|})$. The completion of $\mathcal{C}(G^{|\Gamma|})$ w.r.t. the supremum norm is given by the Banach *-algebra $L^\infty(G^{|\Gamma|})$.

The set of all functions f_Γ in $\mathcal{C}(G^{|\Gamma|})$ such that the linear span of the set of all left translates $\{f_\Gamma(g_{\gamma_1}h_{\gamma_1},...,g_{\gamma_N}h_{\gamma_N}) : g_{\gamma_i} \in G; i = 1,...,N\}$ is finite-dimensional is called the **linear space of almost periodic functions of a product of compact groups** $G^{|\Gamma|}$ and is denoted by $AP(G^{|\Gamma|})$. Set $N := |\Gamma|$. The functions of $AP(G^N)$ are also called representative functions by Schmüdgen. Equivalently, one can define $AP(G^N)$ to be the linear span of the set of all right translates $\{f_\Gamma(h_{\gamma_1}g_{\gamma_1},...,h_{\gamma_N}g_{\gamma_N}) : g_{\gamma_i} \in G; i = 1,...,N\}$ or the set of all left and right translates $\{f_\Gamma(g_{\gamma_1}h_{\gamma_1}k_{\gamma_1},...,g_{\gamma_N}h_{\gamma_N}k_{\gamma_N}) : g_{\gamma_i}, k_{\gamma_i} \in G; i = 1,...,N\}$, which are finite-dimensional.

Then $AP(G^{|\Gamma|})$ is a linear subspace of $\mathcal{C}(G^{|\Gamma|})$. In general, for every element f_Γ of $AP(G^{|\Gamma|})$ there exists a finite number of functions $g_{\Gamma,1},...,g_{\Gamma,k}, k_{\Gamma,1},...,k_{\Gamma,k}$ in $L^\infty(G^{|\Gamma|})$ such that

$$f_\Gamma(g_{\gamma_1}h_{\gamma_1},...,g_{\gamma_N}h_{\gamma_N}) = \sum_{j=1}^{k} g_{\Gamma,j}(g_{\gamma_1},...,g_{\gamma_N})k_{\Gamma,j}(h_{\gamma_1},...,h_{\gamma_N}) \tag{8.1}$$

The involution operation is given by

$$f^*_\Gamma(h_{\gamma_1},...,h_{\gamma_N}) = \overline{f_\Gamma(h^{-1}_{\gamma_1},...,h^{-1}_{\gamma_N})} \tag{8.2}$$

The set $AP(G^{|\Gamma|})$ equipped with the multiplication given by the convolution, the involution operation * presented in equation (8.2) and the supremum norm is a *-algebra, too. The completion of $AP(G^{|\Gamma|})$ w.r.t. the supremum norm is a Banach *-algebra.

On the other hand, the set $AP(G^{|\Gamma|})$ equipped with pointwise multiplication, complex conjugation as the involution operation and the supremum norm is also an algebra with involution. The involutive algebra $AP(G^{|\Gamma|})$ equipped with these structures is unital, since it contains the constant functions and the completion of $AP(G^{|\Gamma|})$ w.r.t. the supremum norm is a C^*-algebra and it is denoted by $\text{Cyl}(G^{|\Gamma|})$. This coincide with the C^*-algebra, which is used in Loop Quantum Gravity, where this algebra is usually called the algebra of cylindrical functions. Furthermore there exists an inductive limit algebra of the inductive family of algebras of cylindrical functions on $G^{|\Gamma|}$.

There is a Peter-Weyl theorem for the Banach *-algebra of almost periodic functions of a product of compact groups.

Theorem 8.1.1. ***Peter-Weyl theorem***
The set $AP(G^{|\Gamma|})$ as a Banach *-subalgebra of $\mathcal{C}(G^{|\Gamma|})$ is dense in $L^\infty(G^{|\Gamma|})$ with respect to the supremum norm.

For compact groups every finite dimensional representation is unitary. An unitary continuous representation π_σ of a group in a Hilbert space $\mathcal{H}_\sigma$ is a morphism of the group G into the unitary group of $\mathcal{H}_\sigma$. Hence let π_σ, where $\sigma := \{s,\Gamma\}$, in the following considerations be an continuous (w.r.t. the strong topology), unitary and, therefore, finite dimensional representation of $G^{|\Gamma|}$ on a Hilbert space $\mathcal{H}_\sigma$. A matrix element of a representation π_σ is defined by $T_\sigma(h_\Gamma) := \langle \pi_\sigma(h_\Gamma)\psi,\varphi\rangle_\sigma$ for all $\psi,\varphi\in\mathcal{H}_\sigma$. The map $h_\Gamma \mapsto T_\sigma(h_\Gamma)$ is an element of $L^\infty(G^{|\Gamma|})$ where $h_\Gamma := (h_{\gamma_1},...,h_{\gamma_N})$.

Consider for every $\sigma = (s,\gamma)$ and a chosen orthonormal basis $\{\xi_k\}_{k=1,..,\dim\pi_\sigma}$ in $\mathcal{H}_\sigma$. Then for all irreducible finite-dimensional representations of G matrix elements $\{T^i_{\sigma,j}\}$ are defined by $T^i_{\sigma,j}(h_\gamma) = \langle\pi_\sigma(h_\gamma)\xi_i,\xi_j\rangle_\sigma$ and are called coordinate functions.

Furthermore it is true that

$$T^i_{\sigma,j}(h_\gamma\tilde{h}_\gamma) = \sum_n T^i_{\sigma,n}(h_\gamma)T^n_{\sigma,j}(\tilde{h}_\gamma) \tag{8.3}$$

for all $h_\gamma,\tilde{h}_\gamma\in G$ and where $T^i_{\sigma,n}(h_\gamma)$ or $T^n_{\sigma,j}(\tilde{h}_\gamma)$ are coefficients of a irreducible representation π_σ of G on a finite-dimensional Hilbert space $\mathcal{H}_\sigma$. Notice, $T^i_{\sigma,j}(e_G) = \delta^i_{j,\gamma}$. Observe that for $\sigma = (\gamma,s)$

$$\begin{aligned}&(T^i_{\sigma,j} * T^k_{\sigma,l})(\tilde{h}_\gamma) = (\dim\pi_\sigma)^{-1}\delta^i_l T^k_{\sigma,j}(\tilde{h}_\gamma),\\ &(T^i_{\sigma,j} * T^i_{\sigma',j})(\tilde{h}_\gamma) = 0 \text{ if } \sigma\neq\sigma',\end{aligned} \tag{8.4}$$

and $\sigma_i = (\gamma_i,s)$ for $i=1,...,N$

$$\begin{aligned}&T^i_{\sigma,j}(h_\gamma) = T^i_{\sigma_1,j}(h_{\gamma_1})...T^i_{\sigma_N,j}(h_{\gamma_M}) \text{ if } \gamma=\gamma_1\circ...\circ\gamma_M \text{ and}\\ &T^i_{\sigma,j}(h^{-1}_\gamma) = \overline{T^i_{\sigma,j}(h_\gamma)}\end{aligned} \tag{8.5}$$

or, equivalently, the equations (8.4) read

$$\begin{aligned}&\int_{G^N} \mathrm{d}\,\mu_H(\mathfrak{h}_\Gamma)T^i_{\sigma,j}(h_\gamma)T^k_{\sigma,l}(h^{-1}_\gamma\tilde{h}_\gamma) = (\dim\pi_\sigma)^{-1}\delta^i_l T^k_{\sigma,j}(\tilde{h}_\gamma)\\ &\int_{G^N} \mathrm{d}\,\mu_H(\mathfrak{h}_\Gamma)T^i_{\sigma,j}(h_\gamma)T^k_{\sigma',l}(h^{-1}_\gamma\tilde{h}_\gamma) = 0 \text{ if } \sigma\neq\sigma'\\ &\int_G \mathrm{d}\,\mu(h_\gamma)T^i_{\sigma,j}(h_\gamma)T^k_{\sigma,l}(h^{-1}_\gamma) = (\dim\pi_\sigma)^{-1}\delta^j_k\delta^i_l\end{aligned} \tag{8.6}$$

where $\sigma=(s,\gamma)$, $\sigma_k = (s,\gamma_k)$ for all $k=1,...,N$ and $\gamma=\gamma_1\circ...\circ\gamma_N$.

Proposition 8.1.2. *The set $AP(G^{|\Gamma|})$ coincide with the linear span of the set of all matrix elements of irreducible and finite-dimensional representations of $G^{|\Gamma|}$.*

If $\mathcal{C}(G^{|\Gamma|})$ is equipped with the $L^2(G^{|\Gamma|})$-norm $\langle .,.\rangle$ then $\mathcal{C}(G^{|\Gamma|})$ is a Hausdorff pre-Hilbert space, which becomes a subspace of the Hilbert space $L^2(G^{|\Gamma|})$. Therefore $\mathcal{C}(G^{|\Gamma|})$ has the structure of a Hilbert algebra. The completion of $\mathcal{C}(G^{|\Gamma|})$ w.r.t. the $\|.\|_2$-norm induced by $\langle .,.\rangle$ is $L^2(G^{|\Gamma|})$, which is equivalent to the full Hilbert algebra[1] of a compact group $G^{|\Gamma|}$. In other words, $L^2(G^{|\Gamma|})$ is a Banach *-algebra with inner product

$$\langle f_\Gamma, k_\Gamma\rangle := \int_{G^N} \mathrm{d}\,\mu_H(h_{\gamma_1},...,h_{\gamma_N}) f_\Gamma(h_{\gamma_1},...,h_{\gamma_N}) k_\Gamma(h_{\gamma_1},...,h_{\gamma_N}) \tag{8.7}$$

such that several conditions presented in the book of Takesaki [92, Def 1.1] are satisfied. Finally it is true that $AP(G^{|\Gamma|})$ is dense in the Banach *-algebra $L^2(G^{|\Gamma|})$.

Dixmier show that the normalised coefficients

$$\dim(\pi_\sigma)^{1/2} T^i_{\sigma,j}(h_\gamma) = \dim(\pi_\sigma)^{1/2}\langle \pi_\sigma(h_\gamma)\xi_i, \xi_j\rangle_\sigma$$

for every σ and a chosen orthonormal basis $\{\xi_k\}_{k=1,..,\dim\pi_\sigma}$ in $\mathcal{H}_\sigma$ form an orthonormal basis of the full Hilbert algebra $L^2(G^{|\Gamma|})$ of $G^{|\Gamma|}$.

Proposition 8.1.3. *The *-algebra $AP(G^N)$ is isomorphic to the full matrix algebra*

$$\bigoplus_{\pi_{s,\gamma_i}\in\hat{G},\gamma_i\in\Gamma} M_{d_{s,\gamma_i}}(\mathbb{C})$$

Proof : Consider the following isomorphism

$$I\left(\sum_{i=1}^{\dim\pi_\sigma}\sum_{j=1}^{\dim\pi_\sigma} \alpha^i_j T^i_{\sigma,j}\right) = \sum_{i=1}^{\dim\pi_\sigma}\sum_{j=1}^{\dim\pi_\sigma} \alpha^i_j (\dim\pi_\sigma)^{-1} E^i_j \tag{8.8}$$

where E^i_j is a matrix with entry 1 in the i-th row and j-th column and entries 0 everywhere else.

■

Fortunately, irreducibile representations are in one-to-one correspondence with the characters

$$h_\gamma \mapsto \chi_\sigma(h_\gamma) := \mathrm{tr}(\pi_\sigma(h_\gamma)) = \sum_{i=1}^{\dim\pi_\sigma} T^i_{\sigma,i}(h_\gamma)$$

for $\sigma = (s,\gamma)$. A normalised character of $\pi_{s,\gamma}$ is the function $(\dim\pi_{s,\gamma})^{-1}\chi_{s,\gamma}$.

Definition 8.1.4. *A function f_Γ in $L^\infty(G^{|\Gamma|})$ is called **central** if f_Γ is invariant under inner automorphisms of the group, i.e.*

$$\begin{aligned}&(_{\breve{h}_\gamma} f_{\breve{h}_\gamma})(h_\gamma(\gamma_1),...,h_\gamma(\gamma_N))\\ &= f_\Gamma(\breve{h}_\gamma^{-1}(\gamma_1)h_\gamma(\gamma_1)\breve{h}_\gamma(\gamma_1),...,\breve{h}_\gamma^{-1}(\gamma_N)h_\gamma(\gamma_N)\breve{h}_\gamma(\gamma_N))\\ &= f_\Gamma(h_\gamma(\gamma_1),...,h_\gamma(\gamma_N))\end{aligned}$$

Denote the space of all central functions by $\mathcal{Z}(G^{|\Gamma|})$.

[1] A Hilbert algebra of a unimodular locally compact group $G^{|\Gamma|}$ is the involutive algebra $\mathcal{C}(G^{|\Gamma|})$ with scalar product $\langle .,.\rangle$. The Hilbert space $L^2(G^{|\Gamma|})$ is the completition w.r.t. $\langle .,.\rangle$. The full Hilbert algebra $\mathfrak{A}$ of $G^{|\Gamma|}$ is given by bounded elements in $L^2(G^{|\Gamma|})$. Consequently $\mathcal{C}(G^{|\Gamma|}) \subseteq \mathfrak{A} \subseteq L^2(G^{|\Gamma|})$.

Define the operator

$$(Pf_\Gamma)(h_\gamma) := \int_{G^{|\Gamma|}} f_\Gamma(\breve{h}_\gamma^{-1} h_\gamma \breve{h}_\gamma)\, \mathrm{d}\,\mu_H^N(\breve{h}_\gamma) \tag{8.9}$$

and call P the avarage over conjugacy classes operator on $L^\infty(G^{|\Gamma|})$.

The set of central functions is a closed linear subspace of $L^\infty(G^{|\Gamma|})$. Moroever a central function f_Γ satisfies

$$(Pf_\Gamma)(h_\gamma) = \int_{G^{|\Gamma|}} f_\Gamma(h_\gamma)\, \mathrm{d}\,\mu_H^N(\breve{h}_\gamma) \tag{8.10}$$

Observe that Pf_Γ is a central function for all $f_\Gamma \in L^\infty(G^{|\Gamma|})$. More precisely, f_Γ is central if and only if $Pf_\Gamma = f_\Gamma$.

Certainly, χ_σ is a positive central continuous function. Moroever $\langle \chi_\sigma, \chi_{\sigma'}\rangle = 0$ where $\langle .,.\rangle$ is the inner product of $L^2(G^{|\Gamma|})$, iff π_σ and $\pi_{\sigma'}$ are inequivalent representations. For inequivalent representations it is also true that $\chi_\sigma * \chi_{\sigma'} = 0$. In particular $\chi_\sigma * \chi_\sigma = (\pi_\sigma)^{-1}\chi_\sigma$ and $\|\chi_\sigma\|_2 = 1$. Denote the conjugate representation of σ by $\overline{\sigma}$. Then the element satisfies

$$\chi_\sigma^*(h_\gamma) = \overline{\chi}_\sigma(h_\gamma^{-1}) = \chi_\sigma(h_\gamma) \tag{8.11}$$

Moroever χ_σ is a central function on G.

Corollary 8.1.5. *[50, Ch.VII, §. 28] The closure w.r.t. the supremum norm of the linear span of* $\{\chi_\sigma : \sigma \in \hat{G}^{|\Gamma|}\}$ *is equal to* $\mathcal{Z}(G^{|\Gamma|})$.

Proof. Let $f_\Gamma \in L^\infty(G^{|\Gamma|})$ be a central function. For a $\epsilon > 0$ the Peter-Weyl theorem states that there is a function g_Γ in $AP(G^{|\Gamma|})$ such that

$$\|f_\Gamma - g_\Gamma\|_\infty < \epsilon \tag{8.12}$$

Furthermore,

$$\|f_\Gamma - Pg_\Gamma\|_\infty = \|Pf_\Gamma - Pg_\Gamma\|_\infty < \|f_\Gamma - g_\Gamma\|_\infty < \epsilon \tag{8.13}$$

Then use that every element of $AP(G^{|\Gamma|})$ is written as a linear span of matrix elements: $g_\Gamma(h_\gamma) = \sum_\sigma \sum_{i,j} c_\sigma T^i_{\sigma,j}(h_\gamma)$

$$\begin{aligned}
(PT^i_{\sigma,j})(h_\gamma) &= \int_{G^{|\Gamma|}} T^i_{\sigma,j}(\breve{h}_\gamma^{-1} h_\gamma \breve{h}_\gamma)\, \mathrm{d}\,\mu_H^N(\breve{h}_\gamma)\\
&= \sum_{k,l} \int_{G^{|\Gamma|}} T^i_{\sigma,k}(\breve{h}_\gamma^{-1}) T^k_{\sigma,l}(h_\gamma) T^l_{\sigma,j}(\breve{h}_\gamma)\, \mathrm{d}\,\mu_H^N(\breve{h}_\gamma)\\
&= \sum_{k,l} T^k_{\sigma,l}(h_\gamma) \int_{G^{|\Gamma|}} T^i_{\sigma,k}(\breve{h}_\gamma^{-1}) T^l_{\gamma,j}(\breve{h}_\gamma)\, \mathrm{d}\,\mu_H^N(\breve{h}_\gamma)\\
&= \sum_{k,l} T^k_{\sigma,l}(h_\gamma)\delta^l_k \delta^i_j\\
&= \dim(\pi_\sigma)^{-1}\delta^i_j \chi_\sigma(h_\gamma)
\end{aligned} \tag{8.14}$$

□

Finally a function f_Γ of the Hilbert algebra $L^2(G^{|\Gamma|})$ is of the form

$$f_\Gamma = \sum_{\pi_\sigma \in \hat{G}^{|\Gamma|}} (\dim \pi_\sigma)(f_\Gamma * \chi_\sigma) \tag{8.15}$$

If f_Γ is an element of the commutative Banach *-algebra $\mathcal{Z}(G^{|\Gamma|})$ of central functions, then

$$f_\Gamma = \sum_{\pi_\sigma \in \hat{G}^{|\Gamma|}} \langle f_\Gamma, \chi_\sigma\rangle \chi_\sigma \tag{8.16}$$

The operators $U(f_\Gamma)$ defined by $U(f_\Gamma)\psi_\Gamma = f_\Gamma * \psi_\Gamma$ for each $f_\Gamma \in L^\infty(G^N)$ are Hilbert-Schmidt operators on the Hilbert space $L^2(G^N)$. Equivalently, there is an operator $V(f_\Gamma)$ given by $V(f_\Gamma)\psi_\Gamma = \psi_\Gamma * f_\Gamma$ which is Hilbert-Schmidt, too.

Therefore Dixmier argued that χ_σ form, therefore, a basis of the center of the algebra of Hilbert-Schmidt operators on the Hilbert space $L^2(G^N)$ or, equivalently, of the center of the Hilbert algebra $L^2(G^N)$.

Proposition 8.1.6. *The center $\mathcal{Z}(G^{|\Gamma|})$ of the Banach *-algebra $L^\infty(G^{|\Gamma|})$ equipped with convolution, involution and supremum norm is a unital commutative Banach *-algebra. There is a homeomorphism of the discrete space $\hat{G}^{|\Gamma|}$ onto the Gel'fand space $\triangle(\mathcal{Z}(G^{|\Gamma|}))$ of $\mathcal{Z}(G^{|\Gamma|})$.*

Proof : The second part is derived from the following observations. The map

$$\tilde{\sigma}_\sigma(f_\Gamma) = \langle f_\Gamma, \chi_\sigma \rangle \tag{8.17}$$

such that

$$\tilde{\sigma}_\sigma(f_\Gamma * f'_\Gamma) = \tilde{\sigma}_\sigma(f_\Gamma)\tilde{\sigma}_\sigma(f'_\Gamma) \quad \forall f_\Gamma, f'_\Gamma \in \mathcal{Z}(G^{|\Gamma|}) \tag{8.18}$$

is a non-zero *-homomomorphism $\tilde{\sigma}$ from the center $\mathcal{Z}(G^{|\Gamma|})$ to $\mathbb{C}$.

Certainly, $\tilde{\sigma}_\sigma(\mathbb{1}_\Gamma) = 1$ for $\mathbb{1}_\Gamma := (\mathbb{1}_{\gamma_1}, ..., \mathbb{1}_{\gamma_N})$ being the identity function on $G^{|\Gamma|}$. Consider the set

$$\Delta(\mathcal{Z}(G^{|\Gamma|})) := \{\tilde{\sigma}_\sigma \in \text{Hom}(\mathcal{Z}(G^{|\Gamma|}), \mathbb{C}) : \quad \sigma \in \hat{G}^{|\Gamma|}\}$$

The Gel'fand transform $\widehat{.}$ is a homomorphism from $\mathcal{Z}(G^{|\Gamma|})$ to $C(X)$ given by $\widehat{f_\Gamma}(\tilde{\sigma}_\sigma) := \tilde{\sigma}_\sigma(f_\Gamma)$, where X is a compact Hausdorff space. With other words,

$$X := \{\widehat{f_\gamma}(\tilde{\sigma}_\sigma) : \tilde{\sigma}_\sigma \in \Delta(\mathcal{Z}(G^{|\Gamma|}))\}$$

Clearly the image of $\mathcal{Z}(G^{|\Gamma|})$ under the Gel'fand transform separates points in $\Delta(\mathcal{Z}(G^{|\Gamma|}))$, since for two inequivalent representations σ_1 and σ_2 the elements satisfies $\tilde{\sigma}_{\sigma_1} \neq \tilde{\sigma}_{\sigma_2}$ and, therefore, there is an element f_Γ such that $\widehat{f_\Gamma}(\tilde{\sigma}_{\sigma_1}) \neq \widehat{f_\Gamma}(\tilde{\sigma}_{\sigma_2})$. Therefore the set X is the spectrum of the unital commutative *-algebra $\mathcal{Z}(G^{|\Gamma|})$.

■

Summarising there is another *-algebra of almost periodic functions on the configuration space $G^{|\Gamma|}$ derivable. This algebra is ismorphic to a non-commutative matrix *-algebra, which is completed to a non-commutative matrix C^*-algebra $M_\Gamma(\mathbb{C})$ by the norm given by the matrix norm. The *-algebra $\mathcal{Z}(G^{|\Gamma|})$, which is generated by the characters, is a commutative Banach *-algebra and this *-algebra is completed w.r.t. the universal norm to a commutative C^*-algebra.

8.2 The holonomy-flux cross-product *-algebra

In [6] Ashtekar, Corichi and Zapata have introduced the concept of an algebra generated by Lie group-valued holonomies along paths and quantum fluxes associated to surfaces and paths, which take values in the Lie algebra of the Lie group. A further analysed *-algebra and representations of this *-algebra have been presented by Sahlmann [83, 84], or by Okołów and Lewandowski [71, 72]. Finally the holonomy-flux *-algebra has been presented by the project group Lewandowski, Okołów, Sahlmann and Thiemann [64].

In this dissertation the holonomy-flux *-algebra is reformulated in a slightly different way such that the resulting *-algebra is different from the *-algebra presented by Lewandowski, Okołów, Sahlmann and Thiemann. The aim of this revised version is to compare this *-algebra with the Heisenberg double, which has been introduced by Schmüdgen and Klimyk [53]. The Heisenberg double $\mathcal{H}(C^\infty(G), \mathcal{E})$ depends on a Lie algebra G and the enveloping algebra $\mathcal{E}$ of the Lie algebra $\mathfrak{g}$ associated to G. The universal enveloping algebra $\mathcal{E}$ of the complexified Lie algebra $\mathfrak{g}^\mathbb{C}$ is equipped with an antilinear and antimultiplicative involution $Y \mapsto Y^+$ such that $X^+ = -X$ for all $X \in \mathfrak{g}$. Therefore the universal enveloping flux algebra itself is a unital *-algebra, which is isomorphic to a particular O^*-algebra in the sense of Inoue [51] and Schmüdgen [89]. The holonomy-flux cross-product *-algebra associated to surfaces presented in section 8.2.1, is regarded as an abstract cross-product algebra, which is constructed from holonomies (G-valued) and quantum fluxes ($\mathfrak{g}$- or $\mathcal{E}$-valued). With no doubt there are a lot of different holonomy-flux cross-product *-algebras for surfaces, since the construction of the cross-product depends on the intersection behavoir of each path of a graph and a surface set. Consequently the fundamental *-algebra of holonomies and quantum fluxes is given by the multiplier algebra of the holonomy-flux cross-product *-algebra associated to a fixed surface set $\breve{S}$ and a graph. This algebra contains many holonomy-flux cross product *-algebras for different surface sets. Furthermore it is discovered that, the holonomy-flux cross-product *-algebra is not equivalent to the Heisenberg algebra $\mathcal{H}(C^\infty(\bar{G}_{\breve{S},\Gamma}), \mathcal{E}_{\breve{S}})$ associated to a surface set and a graph, which is presented in section 8.2.2. Finally the inductive limit *-algebra of an inductive family of holonomy-flux *-algebras for surfaces and graphs, and the multiplier algebra of this inductive limit *-algebra is constructed. There is an exceptional *-representation of the holonomy-flux cross-product *-algebra, which is presented in section 8.2.3, and which is connected to a $\bar{G}_{\breve{S},\Gamma_\infty}$-integrable *-representation of the universal enveloping flux algebra. Since there is a GNS-construction for *-representations, this representation is associated to a state on the holonomy-flux cross-product *-algebra. Finally it is shown that there is a unique representation of a certain holonomy-flux cross-product *-algebra, which has particular invariance properties. This result is comparable with the result of Lewandowski, Okołów, Sahlmann and Thiemann in [64]. For some technical reasons, the quantum flux operators are implemented by elements of the center of the universal enveloping flux algebra $\bar{\mathcal{E}}_{\breve{S}}$. Moroever for a restricted notion of graph-diffeomorphism invariance other *-representations and states are analyseds.

8.2.1 The construction of the holonomy-flux cross-product *-algebra

Recall that, there is a big bunch of actions on the naturally identified configuration space $\bar{\mathcal{A}}_\Gamma$, which is connected to the requirement of paths lying above or below the surface and are ingoing or outgoing w.r.t. the surface orientation of a surface S. Recall the Lie flux algebra $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$, which is given by the evluation of all maps for a fixed finite orientation preserved graph system associated to a graph Γ and a suitable surface set $\breve{S}$. Refer to section 3.4 for a precise definition. Let $\bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma}$ be the complexified Lie flux algebra, then $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ denotes the universal enveloping flux algebra.

For simplicity, the investigations start with a graph Γ, which contains only one path γ, and one surface S. Clearly the following definition generalises to a finite orientation preserved graph system associated to an arbitrary graph Γ and a suitable surface set $\breve{S}$.

Definition 8.2.1. *Let the graph Γ contain only a path γ and S be a surface such that the path lies below and outgoing w.r.t. the surface orientation of this surface. Set $E_S(\Gamma) =: X_S$. Then the **right-invariant flux vector field** $e^{\overrightarrow{L}}$ is defined by*

$$[E_S(\Gamma), f_\Gamma] := e^{\overrightarrow{L}}(f_\Gamma)$$

where

$$e^{\overrightarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\exp(tX_S)\mathfrak{h}_\Gamma(\gamma)) \text{ for } X_S \in \mathfrak{g}, \mathfrak{h}_\Gamma(\gamma) \in G, t \in \mathbb{R} \tag{8.19}$$

whenever $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$ and $E_S(\Gamma) \in \bar{\mathfrak{g}}_{S,\Gamma}$.

Respectively, for a path γ lying above and outgoing w.r.t. the surface orientation, it is true that

$$e^{\overleftarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\exp(-tX_S)\mathfrak{h}_\Gamma(\gamma)), \text{ for } X_S \in \mathfrak{g}, \mathfrak{h}_\Gamma(\gamma) \in G, t \in \mathbb{R} \tag{8.20}$$

if $-E_S(\Gamma) =: X_S$. Since $E_S \in \mathfrak{g}_{\breve{S},\Gamma}$ there exists a skew-adjoint operator $E_S(\Gamma)^+$ that satisfies

$$[E_S(\Gamma)^+, f_\Gamma] = e^{\overleftarrow{L}}(f_\Gamma)$$

where

$$[E_S(\Gamma)^+, f_\Gamma] = [E_{S^{-1}}(\Gamma), f_\Gamma]$$

A quantum flux operator of a surface $\tilde{S}$ and a path γ lying below and outgoing with respect to the surface orientation of S can be changed by a flip of the path orientation such that the path γ lie below and ingoing.

Recall the map $\breve{.}: C^\infty(\bar{\mathcal{A}}_\Gamma) \rightarrow C^\infty(\bar{\mathcal{A}}_\Gamma)$ s.t.

$$f_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_n)) \mapsto \breve{f}_\Gamma(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_n)) := f_\Gamma(\mathfrak{h}_\Gamma(\gamma)^{-1},...,\mathfrak{h}_\Gamma(\gamma_n)^{-1})$$

Definition 8.2.2. *Define the* ***surface and graph orientation flip operator*** *as a map* $\mathfrak{F}: C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathfrak{g}}_{\breve{S},\Gamma} \rightarrow C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathfrak{g}}_{\breve{S},\Gamma}$

$$\mathfrak{F}(f_\Gamma, E_S(\Gamma)) = (\breve{f}_\Gamma, E_{S^{-1}}(\Gamma)) = (\breve{f}_\Gamma, E_S^+(\Gamma)), \quad \mathfrak{F}(f_\Gamma, E_S(\Gamma)^+) = (\breve{f}_\Gamma, E_S(\Gamma))$$
$$\mathfrak{F}(f_\Gamma^*, E_S(\Gamma)) = (\breve{f}_\Gamma^*, E_{S^{-1}}(\Gamma))$$

$$(\mathfrak{F}\circ \mathrm{pr}_1)(f_\Gamma, E_S(\Gamma)) = \mathfrak{F}(f_\Gamma) = \breve{f}_\Gamma,$$
$$(\mathfrak{F}\circ \mathrm{pr}_1)(f_\Gamma, E_S(\Gamma)) = \mathfrak{F}(E_S(\Gamma)) = E_{S^{-1}}(\Gamma)$$

Notice that $f_\Gamma^* = \overline{f_\Gamma}$ whenever $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$.

Definition 8.2.3. *Let the graph Γ contain only a path γ and S be a surface such that the path lies below and outgoing w.r.t. the surface orientation of this surface. Set $s(\gamma) = v$ and $-E_S(\Gamma) =: Y_S$.*

The ***left-invariant flux vector field*** $e^{\overleftarrow{R}}$ *is realized as the following commutator*

$$[E_S(\Gamma), f_\Gamma] =: e^{\overleftarrow{R}}(f_\Gamma)$$

where

$$e^{\overleftarrow{R}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\exp(-tY_S))) \text{ for } Y_S \in \mathfrak{g}, t \in \mathbb{R} \tag{8.21}$$

whenever $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$ and $E_S \in \mathfrak{g}_{\breve{S},\Gamma}$. There exists a skew-adjoint operator $E_S(\Gamma)^+$ such that

$$[E_S(\Gamma)^+, f_\Gamma] := e^{\overrightarrow{R}}(f_\Gamma)$$

where

$$e^{\overrightarrow{R}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\mathfrak{h}_\Gamma(\gamma)\exp(tY_S))) \text{ for } Y_S \in \mathfrak{g}, t \in \mathbb{R} \tag{8.22}$$

holds.

Summarising the flux operators are implemented as differential operators $e^{\overrightarrow{L}}$ (or $e^{\overrightarrow{R}}$) on $\bar{\mathcal{A}}_\Gamma$ commuting with the right (or left) shifts.

Definition 8.2.4. *Let A be an (associative complex) algebra.*

A ***homomorphism of a Lie algebra*** $\mathfrak{g}$ *in A is a map $\tilde{\tau}: \mathfrak{g} \rightarrow A$ such that*

$$\tilde{\tau}(\alpha X + \beta Y) = \alpha\tilde{\tau}(X) + \beta\tilde{\tau}(Y),$$
$$\tilde{\tau}([X,Y]) = \tilde{\tau}(X)\tilde{\tau}(Y) - \tilde{\tau}(Y)\tilde{\tau}(X)$$

whenever $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{R}$.

There is a map $\tilde{\tau}_1 : \bar{\mathfrak{g}}_{\breve{S},\Gamma} \rightarrow C(\bar{\mathcal{A}}_\Gamma)$ defined by $\tilde{\tau}_1(E_S(\Gamma))f_\Gamma = [E_S(\Gamma), f_\Gamma]$ for $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$, which is indeed a homomorphism of a Lie flux algebra $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ associated to a suitable surface set $\breve{S}$ and a graph in $C^\infty(\bar{\mathcal{A}}_\Gamma)$.

Lemma 8.2.5. *Fix an element $E_S(\Gamma) \in \bar{\mathfrak{g}}_{\breve{S},\Gamma}$. Let $\check{\tau}_1 : C^\infty(\bar{\mathcal{A}}_\Gamma) \rightarrow C^\infty(\bar{\mathcal{A}}_\Gamma)$ be a map such that $\check{\tau}_1(E_S(\Gamma))(f_\Gamma) := [E_S(\Gamma), f_\Gamma]$ or, equivalently, $\check{\tau}_1(E_S(\Gamma))(f_\Gamma) := e^{\overrightarrow{L}}(f_\Gamma)$ for each function $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$.*

Then $\check{\tau}_1 \circ \mathfrak{F}$ defines an *-isomorphism $\mathcal{F}$ on $C^\infty(\bar{\mathcal{A}}_\Gamma)$ by

$$(\mathcal{F}\circ e^{\overrightarrow{L}})(f_\Gamma) = [E_{S^{-1}}(\Gamma), \breve{f}_\Gamma] = e^{\overleftarrow{R}}(\breve{f}_\Gamma)$$

which implements a flip of the path orientation.

Proof : This is true, since,

$$\begin{aligned} e^{\overrightarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) &= \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f_\Gamma(\exp(tX_S)\mathfrak{h}_\Gamma(\gamma)) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \breve{f}_\Gamma(\mathfrak{h}_\Gamma(\gamma)^{-1}\exp(-tX_S)) \\ &= e^{\overleftarrow{R}}(\breve{f}_\Gamma)(\mathfrak{h}_\Gamma(\gamma)^{-1}) \end{aligned}$$

and

$$\begin{aligned} (\check{\tau}_1 \circ \mathfrak{F})(E_{S_0}(\Gamma))([E_S(\Gamma), f_\Gamma]) &= e^{\overrightarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) = e^{\overleftarrow{R}}(\breve{f}_\Gamma)(\mathfrak{h}_\Gamma(\gamma)^{-1}) \\ &= [(\check{\tau}_1 \circ \mathfrak{F})(E_S(\Gamma))(\mathbb{1}_\Gamma), (\check{\tau}_1 \circ \mathfrak{F})(E_{S_0}(\Gamma))(f_\Gamma)] \end{aligned}$$

where S_0 is a surface, which does not intersect any path in Γ and $\mathbb{1}_\Gamma$ is the constant function for any Γ.

■

There is also an *-isomorphism $\tilde{\mathcal{F}}$ presented by

$$(\tilde{\mathcal{F}} \circ e^{\overrightarrow{L}})(f_\Gamma) = [E_{S^{-1}}(\Gamma), f_\Gamma] = e^{\overrightarrow{R}}(f_\Gamma)$$

connected to a flip of the path and surface orientation.

Now the focus lies on quantum fluxes, which takes values in the enveloping algebra of $\mathfrak{g}$.

Definition 8.2.6. *Let $\breve{S}$ be a surface set such that $\breve{S}$ has the same surface intersection property for a graph Γ. Then the* ***tensor algebra of flux operators*** *is defined by*

$$\mathcal{T}(\breve{S}) := \bigoplus_{k=0}^{\infty} \bar{\mathfrak{g}}_{\breve{S},\Gamma}^{\mathbb{C}\ \otimes_k}$$

There is a natural inclusion $j : \bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma} \to \mathcal{T}(\breve{S})$, $E_S(\Gamma) \mapsto (E_S(\Gamma))^{\otimes^1}$. Denote by $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ the ***universal enveloping*** ****-algebra for flux operators*** *generated by the quotient of $\mathcal{T}(\breve{S})$ and a two sided ideal I expressed by*

$$\begin{aligned} I = \Big\{ & j(E_{S_1}(\Gamma)) \otimes j(E_{S_2}(\Gamma)) - j(E_{S_2}(\Gamma)) \otimes j(E_{S_1}(\Gamma)) - j([E_{S_1}(\Gamma), E_{S_2}(\Gamma)]) : \\ & E_{S_1}, E_{S_2} \in \mathfrak{g}^{\mathbb{C}}_{\breve{S}}, S_K \in \breve{S}, K = 1,2 \Big\} \end{aligned}$$

The antilinear and antimultiplicative involution $^+$ is given by

$$\begin{aligned} &(E_{S_1}(\Gamma) \times ... \times E_{S_k}(\Gamma))^+ = E^+_{S_k}(\Gamma) \times ... \times E^+_{S_1}(\Gamma), \\ &E_{S_K}(\Gamma)^+ = -E_{S_K}(\Gamma) \text{ for } E_{S_K}(\Gamma) \in \bar{\mathfrak{g}}_{\breve{S},\Gamma} \text{ and } K = 1, ..., k \end{aligned}$$

Recall the structure of the enveloping algebra of $\mathfrak{g}$. Moroever there is a bilinear map

$$\tau_1 : C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma} \to C^\infty(\bar{\mathcal{A}}_\Gamma)$$

such that for $(f_\Gamma, E_S(\Gamma)) \in C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma}$ it is true that

$$\tau_1(f_\Gamma, E_S(\Gamma)) = [E_S(\Gamma), f_\Gamma] \tag{8.23}$$

which is further generalised to

$$\tau_2 : C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma} \otimes \bar{\mathfrak{g}}^{\mathbb{C}}_{\breve{S},\Gamma} \to C^\infty(\bar{\mathcal{A}}_\Gamma)$$

such that

$$\tau_2(f_\Gamma, E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma)) = -[E_{S_1}(\Gamma), [E_{S_2}(\Gamma), f_\Gamma]] \tag{8.24}$$

Hence in general there is a bilinear map $\tau : C^\infty(\bar{\mathcal{A}}_\Gamma) \times \bar{\mathcal{E}}_{\breve{S},\Gamma} \to C^\infty(\bar{\mathcal{A}})$

$$\tau(f_\Gamma, E_{S_1}(\Gamma) \cdot ... \cdot E_{S_n}(\Gamma)) = [E_{S_1}(\Gamma), [..., [E_{S_n}(\Gamma), f_\Gamma]]...] \tag{8.25}$$

and such that $\tilde{\tau} : \bar{\mathcal{E}}_{\breve{S},\Gamma} \to C^\infty(\bar{\mathcal{A}})$ where $\tilde{\tau}(E_S(\Gamma))f_\Gamma = [E_S(\Gamma), f_\Gamma]$ for $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$ is a unit-preserving homomorphism.

The following corollary implies that due to the universality structure of $\bar{\mathcal{E}}_{\breve{S}}$ this map $\tilde{\tau}$ is unique.

Corollary 8.2.7. *Let A be a unital algebra and $\tilde{\tau}$ be a homomomorphism of a Lie algebra $\mathfrak{g}$ into A. Then there exsits a unique unit-preserving homomorphism of the universal enveloping flux algebra $\mathcal{E}$ of $\mathfrak{g}$ into A which extends $\tilde{\tau}$.*

Lemma 8.2.8. *Let $\breve{S}$ be a set of surfaces which has the same intersection surface property for a finite orientation preserved graph system associated to Γ.*

Then $C^\infty(\bar{\mathcal{A}}_\Gamma)$ is a left $\bar{\mathcal{E}}_{\breve{S},\Gamma}$-module algebra. The action of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on $C^\infty(\bar{\mathcal{A}}_\Gamma)$ is given by $E_S(\Gamma) \triangleright f_\Gamma := e^{\overrightarrow{L}}(f_\Gamma)$.

Proof : This following from the fact that $C^\infty(\bar{\mathcal{A}}_\Gamma)$ is a left $\bar{\mathcal{E}}_{\breve{S},\Gamma}$-module, which is defined by the map

$$E_S(\Gamma) \triangleright f_\Gamma := e^{\overrightarrow{L}}(f_\Gamma) = \tau(f_\Gamma, E_S(\Gamma)) \text{ for } E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}, f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$$

which is obviously bilinear and $1 \triangleright f_\Gamma = f_\Gamma$ is satisfied. Moroever

$$E_{S_1}(\Gamma) \triangleright (E_{S_2}(\Gamma) \triangleright f_\Gamma) = (E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma)) \triangleright f_\Gamma$$

holds. Furthermore it turns out to be left $\bar{\mathcal{E}}_{\breve{S},\Gamma}$-module algebra, since additionally

$$\begin{aligned}
&E_S(\Gamma) \triangleright (f_\Gamma k_\Gamma) = (e^{\overrightarrow{L}}(f_\Gamma))k_\Gamma + f_\Gamma(e^{\overrightarrow{L}}(k_\Gamma)) \text{ and}\\
&E_S(\Gamma) \triangleright \mathbb{1}_\Gamma = 0 \text{ for all } E_S(\Gamma) \in \bar{\mathfrak{g}}_{\breve{S},\Gamma}
\end{aligned}$$

yields.

■

Lemma 8.2.9. *Let $\breve{S}$ be a set of surfaces which has the appropriate same intersection surface property for a finite orientation preserved graph system associated to Γ.*

Then $C^\infty(\bar{\mathcal{A}}_\Gamma)$ is a right $\bar{\mathcal{E}}_{\breve{S},\Gamma}$-module algebra. The action of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on $C^\infty(\bar{\mathcal{A}}_\Gamma)$ is given by $E_S(\Gamma) \triangleleft f_\Gamma := e^{\overrightarrow{R}}(f_\Gamma)$.

Finally the definition of the holonomy-flux *-algebra in LQG by the authors [64, Def.2.7] is rewritten for the case of a fixed graph Γ. The vector space $C^\infty(\bar{\mathcal{A}}_\Gamma) \otimes \bar{\mathcal{E}}_{\breve{S},\Gamma}$ is equipped with the multiplication

$$f^1_\Gamma \otimes E_{S_1}(\Gamma) \cdot f^2_\Gamma \otimes E_{S_2}(\Gamma) = -\tau(f^2_\Gamma, E_{S_1}(\Gamma)) \otimes E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma) \tag{8.26}$$

such that a Lie algebra bracket is derived

$$\begin{aligned}
&[f^1_\Gamma \otimes E_{S_1}(\Gamma), f^2_\Gamma \otimes E_{S_2}(\Gamma)] =\\
&\quad - \tau(f^2_\Gamma, E_{S_1}(\Gamma)) - \tau(f^1_\Gamma, E_{S_2}(\Gamma)) \otimes [E_{S_1}(\Gamma), E_{S_2}(\Gamma)]
\end{aligned} \tag{8.27}$$

Notice if S_1 and S_2 are disjoint the commutator on $C^\infty(\bar{\mathcal{A}}_\Gamma) \otimes \bar{\mathcal{E}}_{\breve{S},\Gamma}$ is zero. Calculate the commutator

$$[(f_\Gamma \otimes \mathbb{1}), (\mathbb{1} \otimes E_S(\Gamma))] = -\tau(f_\Gamma, E_S(\Gamma)) \otimes E_S(\Gamma) \tag{8.28}$$

Additionally the algebra is equipped with an involution such that this algebra is a unital associative *-algebra. In this dissertation the algebra is slightly modificated.

Definition 8.2.10. *Let $\breve{S}$ be a set of surfaces which has the appropriate same intersection surface property for a finite orientation preserved graph system associated to Γ.*

The ***holonomy-flux cross-product *-algebra for a graph Γ and a surface set $\breve{S}$*** *is given by the left or right cross-product *-algebra*

$$C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_L \bar{\mathcal{E}}_{\breve{S},\Gamma} \text{ or } C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_R \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

which are defined by the vector space $C^\infty(\bar{\mathcal{A}}_\Gamma) \otimes \bar{\mathcal{E}}_{\breve{S},\Gamma}$ with the multiplication given by

$$(f^1_\Gamma \otimes E_{S_1}(\Gamma)) \cdot_L (f^2_\Gamma \otimes E_{S_2}(\Gamma)) := f^1_\Gamma(E_{S_1}(\Gamma) \triangleright f^2_\Gamma) \otimes E_{S_2}(\Gamma) + f^1_\Gamma f^2_\Gamma \otimes E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma)$$

or respectively

$$(f^1_\Gamma \otimes E_{S_1}(\Gamma)) \cdot_R (f^2_\Gamma \otimes E_{S_2}(\Gamma)) := (E_{S_2}(\Gamma) \triangleleft f^1_\Gamma) f^2_\Gamma \otimes E_{S_1}(\Gamma) + f^1_\Gamma f^2_\Gamma \otimes E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma)$$

and the involution

$$(f_\Gamma \triangleright E_S(\Gamma))^* := \bar{f}_\Gamma \triangleright E_S(\Gamma)^+$$

or respectively

$$(f_\Gamma \triangleleft E_S(\Gamma))^* := \bar{f}_\Gamma \triangleleft E_S(\Gamma)^+$$

whenever $E_{S_1}(\Gamma), E_{S_2}(\Gamma), E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$ *and* $f^1_\Gamma, f^2_\Gamma, f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$.

The ***holonomy-flux cross-product *-algebra associated to a surface set*** $\breve{S}$ *is given by the left or right cross-product *-algebra*

$$C^\infty(\bar{\mathcal{A}}) \rtimes_L \bar{\mathcal{E}}_{\breve{S}} \text{ or } C^\infty(\bar{\mathcal{A}}) \rtimes_R \bar{\mathcal{E}}_{\breve{S}}$$

which are the inductive limit of the families $\{(C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_L \bar{\mathcal{E}}_{\breve{S},\Gamma}, \beta_{\Gamma,\Gamma'} \times \breve{\beta}_{\Gamma,\Gamma'})\}$ *or* $\{(C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_R \bar{\mathcal{E}}_{\breve{S},\Gamma}, \beta_{\Gamma,\Gamma'} \times \breve{\beta}_{\Gamma,\Gamma'})\}$ *where* $\breve{\beta}_{\Gamma,\Gamma'} : \bar{\mathcal{E}}_{\breve{S},\Gamma} \to \bar{\mathcal{E}}_{\breve{S},\Gamma'}$ *are suitable unit-preserving *-homomorphisms for a suitable set* $\breve{S}$ *of surfaces that preserve the left or right vector field structure.*

Summarising the unital holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_L \bar{\mathcal{E}}_{\breve{S}}$ may be thought of as the universal algebra generated by $C^\infty(\bar{\mathcal{A}}_\Gamma)$ and $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ with respect to the commutator relation

$$E_S(\Gamma) f_\Gamma = E_S(\Gamma) \triangleright f_\Gamma + f_\Gamma E_S(\Gamma) \tag{8.29}$$

Derive for suitable surface S and a graph Γ the following commutator relation between elements of the holonomy-flux cross-product *-algebra

$$\begin{aligned}
&[(f^1_\Gamma \otimes E_{S_1}(\Gamma)), (f^2_\Gamma \otimes E_{S_2}(\Gamma))] \\
&= f^1_\Gamma (E_{S_1}(\Gamma) \triangleright f^2_\Gamma) \otimes E_{S_2}(\Gamma) + f^1_\Gamma f^2_\Gamma \otimes E_{S_1}(\Gamma) \cdot E_{S_2}(\Gamma) \\
&\quad - f^2_\Gamma (E_{S_2}(\Gamma) \triangleright f^1_\Gamma) \otimes E_{S_1}(\Gamma) - f^2_\Gamma f^1_\Gamma \otimes E_{S_2}(\Gamma) \cdot E_{S_1}(\Gamma) \\
&= f^1_\Gamma (E_{S_1}(\Gamma) \triangleright f^2_\Gamma) \otimes E_{S_2}(\Gamma) - f^2_\Gamma (E_{S_2}(\Gamma) \triangleright f^1_\Gamma) \otimes E_{S_1}(\Gamma) + f^1_\Gamma f^2_\Gamma \otimes [E_{S_1}(\Gamma), E_{S_2}(\Gamma)]
\end{aligned} \tag{8.30}$$

which is comparable to the definition usually used in LQG, which is illustrated in (8.27). The definitions do not coincide, since in LQG the ACZ- holonomy-flux algebra $C^\infty(\bar{\mathcal{A}}_\Gamma) \otimes \bar{\mathcal{E}}_{\breve{S},\Gamma}$ is defined by the multiplication (8.26). This shows that, the holonomy-flux cross-product *-algebra is a slightly modificated holonomy-flux *-algebra if it is compared with the *-algebra presented in [64].

Notice

$$\begin{aligned}
&[(f^1_\Gamma \otimes \mathbb{1}), (f^2_\Gamma \otimes E_S(\Gamma))] \\
&= f^1_\Gamma f^2_\Gamma \otimes E_S(\Gamma) + f^1_\Gamma f^2_\Gamma \otimes E_S(\Gamma) - f^2_\Gamma (E_S(\Gamma) \triangleright f^1_\Gamma) \otimes \mathbb{1} - f^2_\Gamma f^1_\Gamma \otimes E_S(\Gamma) \\
&= f^1_\Gamma f^2_\Gamma \otimes E_S(\Gamma) - f^2_\Gamma (E_S(\Gamma) \triangleright f^1_\Gamma) \otimes \mathbb{1}
\end{aligned} \tag{8.31}$$

holds. Observe that the commutator

$$\begin{aligned}
&[(f_\Gamma \otimes \mathbb{1}), (\mathbb{1} \otimes E_S(\Gamma))] \\
&= f_\Gamma \otimes E_S(\Gamma) + f_\Gamma \otimes E_S(\Gamma) - (E_S(\Gamma) \triangleright f_\Gamma) \otimes \mathbb{1} - f_\Gamma \otimes E_S(\Gamma) \\
&= f_\Gamma \otimes E_S(\Gamma) - (E_S(\Gamma) \triangleright f_\Gamma) \otimes \mathbb{1}
\end{aligned} \tag{8.32}$$

is different from the commutator (8.28) of the holonomy-flux *-algebra.

Clearly for different surface sets there are a lot of different holonomy-flux cross-product *-algebras. For example let $\breve{S}$ be a set of N surfaces and let Γ be a graph with N independent edges such that every surface S_i in $\breve{S}$ intersects only one path γ_i of a graph Γ only once in the target vertex of the path γ_i, the path γ_i lies above and there are no other intersection points of each other path γ_j and the surface S_i in $\breve{S}$ $(i \neq j)$. Moroever let $\breve{T}$ be a set of N

surfaces and let Γ be a graph with N independent edges such that every surface T_i in $\breve{T}$ intersects only one path γ_i of a graph Γ only once in the source vertex of the path γ_i, the path γ_i lies below and there are no other intersection points of each other path γ_j and the surface T_i in $\breve{T}$ $(i \neq j)$.

Then the sets $\breve{S}$ and $\breve{T}$ have the simple surface intersection property for Γ. There exists two different holonomy-flux cross-product *-algebras $C^\infty(\bar{\mathcal{A}}) \rtimes_L \bar{\mathcal{E}}_{\breve{S}}$ and $C^\infty(\bar{\mathcal{A}}) \rtimes_R \bar{\mathcal{E}}_{\breve{T}}$.

The multiplier *-algebra of the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_L \bar{\mathcal{E}}_{\breve{S}}$ of a surface set $\breve{S}$ contain the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_R \bar{\mathcal{E}}_{\breve{T},\Gamma_\infty}$ for the surface set $\breve{T}$. This statement can be proven by similar arguments used in the proposition 7.2.12 in section 7.2.

Consider $C^\infty(\bar{\mathcal{A}}_\Gamma)$ as a *-subalgebra of the analytic holonomy C^*-algebra $\mathfrak{A}_\Gamma := C(\bar{\mathcal{A}}_\Gamma)$. Moroever refer to the appendix 12.2.4.9, Sakai [87] or Bratteli and Robinson [22] for the definition of *-derivations.

Lemma 8.2.11. *For any graph Γ and a surface S, which has the same intersection surface property for a finite orientation preserved graph system associated to Γ, the object*

$$i[E_S(\Gamma)^+ E_S(\Gamma), f_\Gamma] =: \delta^2_{S,\Gamma}(f_\Gamma) \tag{8.33}$$

*defines a unbounded symmetric *-derivation $\delta^2_{S,\Gamma}$ on $C(\bar{\mathcal{A}}_\Gamma)$ with domain $C^\infty(\bar{\mathcal{A}}_\Gamma)$, in other words $\delta^2_{S,\Gamma} \in \mathrm{Der}(C^\infty(\bar{\mathcal{A}}_\Gamma), C(\bar{\mathcal{A}}_\Gamma))$.*

Since multiplier algebra of a unital and commutative C^*-algebra is the algebra itself, the elements $E_S(\Gamma)$ are not contained in the multiplier algebra of $\mathfrak{A}_\Gamma = C(\bar{\mathcal{A}}_\Gamma)$. Hence the derivation defined in the lemma 8.2.11 is not inner.

Following the notion of infinitesimal representations $\mathrm{d}\,U$ in a C^*-algebra introduced by Woronowicz [115, p.8] the flux operators can be also understood in the following way. Recall the unbounded operators $e^{\overleftarrow{R}}, e^{\overleftarrow{L}}$ defined in (10.3), (8.22). These operators are infinitesimal representations or differentials of the Lie flux group $\bar{G}_{\breve{S},\Gamma}$ in $\mathcal{K}(L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma))$. They correspond to the set $\mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)))$ of unitary representations U of $\bar{G}_{\breve{S}}$, which are analysed in section 6.1. Therefore rewrite

$$e^{\overrightarrow{L}}(f_\Gamma) := \mathrm{d}\,U(E_S(\Gamma)) f_\Gamma \text{ for } f_\Gamma \in \mathcal{D}(\mathrm{d}\,U) \text{ and } E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

The domain of the infinitesimal representations $\mathrm{d}\,U$ is defined by

$$\begin{aligned}\mathcal{D}(\mathrm{d}\,U) := \{f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma) : &\text{ the mapping } \bar{G}_{\breve{S},\Gamma} \ni \rho_{S,\Gamma}(\Gamma) \mapsto U(\rho_{S,\Gamma}(\Gamma)) f_\Gamma \\ &\text{is a } C^\infty(\bar{G}_{\breve{S},\Gamma}) - \text{ function in norm-topology}\}\end{aligned}$$

which is a dense subset in $C(\bar{\mathcal{A}}_\Gamma)$.

The operators $\mathrm{d}\,U(E_S(\Gamma))$ are densely defined closed unbounded operators affiliated with $\mathcal{K}(L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma))$.

8.2.2 Heisenberg holonomy-flux cross-product *-algebras

The structures of Hopf algebras have been presented by Schmüdgen and Klimyk [53] (refer to appendix 12.2.2 for short overview).The authors have rewritten the algera of Quantum mechanics in terms of the Hopf *-algebra $(\mathrm{Pol}(\mathbb{R}^n), \triangle)$ of coordinate functions. Then $\mathrm{Pol}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ is the Heisenberg algebra of Quantum Mechanics, where the elements are differential operators with polynomial coefficients. In this section similar algebras for LQG are studied.

In mathematics further cross-product algebras, which are called the Heisenberg doubles, using the properties of bialgebras have been constructed. A particular bialgebra is a Hopf algebra. Let G be either a connected compact Lie group and $\mathbf{G}$ a simple matrix Lie group. Therefore consider either the Hopf *-algebra $(C^\infty(G), \triangle)$ or the Hopf *-algebra $(\mathrm{Pol}(\mathbf{G}), \triangle)$ of coordinate functions on the group $\mathbf{G}$ or the Hopf *-algebra $(\mathrm{Rep}(G), \triangle)$ of representative functions on the group G. Then restrict the *-algebra $\mathrm{Pol}(\mathbf{G})$ or $\mathrm{Rep}(G)$ to a *-subalgebra of $C^\infty(\mathbf{G})$ or $C^\infty(G)$, which is denoted by $\mathrm{Pol}^\infty(\mathbf{G})$ or $\mathrm{Rep}^\infty(G)$. Suppose that $\langle .,. \rangle : \mathcal{E} \times \mathrm{Pol}^\infty(\mathbf{G}) \to \mathbb{C}$ denotes the dual paring of $(\mathrm{Pol}^\infty(\mathbf{G}), \triangle)$ or respectively $(\mathrm{Rep}^\infty(G), \triangle)$ and Hopf algebra $(\mathcal{E}, \hat{\triangle})$, where $\mathcal{E}$ denote the universal enveloping flux algebra of G. The dual pairing is defined by

$$\langle E, f \rangle := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} f(e_G \exp(tE)) \text{ for } E \in \mathcal{E} \text{ and } f \in \mathrm{Rep}^\infty(G) \tag{8.34}$$

Then the Heisenberg double $\text{Rep}^\infty(G) \rtimes_H \mathcal{E}$ is defined by the bilinear map

$$E \triangleright_H f := \langle E, \mathbb{1}\rangle f + \langle E, f\rangle = \langle E, f\rangle \tag{8.35}$$

and the multiplication

$$(f_1, E_1) \cdot_H (f_2, E_2) := \langle E_1, \mathbb{1}\rangle f_1 f_2 \otimes E_2 + \langle \mathbb{1}, f_2\rangle f_1 \otimes E_1 E_2 \tag{8.36}$$

Definition 8.2.12. *Let G be either a connected compact Lie group or a simple matrix Lie group. Moroever let $\bar{\mathcal{A}}_\Gamma$ for a graph Γ be the set of generalised connections for G such that $\bar{\mathcal{A}}_\Gamma$ is identified with G^N naturally, where $N = |E_\Gamma|$. Suppose that $\breve{S}$ has the simple intersection surface property for a finite orientation preserved graph system associated to Γ. Then $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ is identified with $\mathfrak{g}^N$.*

*The **Heisenberg representation-holonomy-flux *-algebra of the graph Γ and the surface set $\breve{S}$** is given by*

$$\text{Rep}^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_H \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

*The **Heisenberg polynomial-holonomy-flux *-algebra of the graph Γ and the surface set $\breve{S}$** is given by*

$$\text{Pol}^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_H \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

*The **Heisenberg holonomy-flux *-algebra of the graph Γ and the surface set $\breve{S}$** is given by*

$$C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_H \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

Notice that, an element of $\text{Pol}^\infty(\mathbf{G}^N)$ is a matrix element $(\mathfrak{h}_\Gamma)_{ij}$ of a $M \times M$ matrix. This elements are called coordinate functions $v^i_j(\mathfrak{h}_\Gamma) = (\mathfrak{h}_\Gamma)_{ij}$ on a simple matrix Lie group $\mathbf{G}$. Then an element of $\text{Pol}^\infty(\mathbf{G}^N) \rtimes \bar{\mathcal{E}}^N$ is for example given by

$$(\mathfrak{h}_\Gamma)_{ij}(\mathfrak{h}_\Gamma)_{kl} \otimes E_S(\Gamma)$$

where by natural identification $h_\Gamma := \mathfrak{h}_\Gamma(\Gamma)$ is an element of $\mathbf{G}^N$ and $E_S(\Gamma)$ is an element of the universal enveloping flux algebra $\bar{\mathcal{E}}^N$ of the Lie group $\mathbf{G}^N$. Clearly these Heisenberg cross-product algebras defined above are not equivalent to a holonomy-flux cross-product *-algebra. They are in particular Heisenberg doubles in the sense of Schmüdgen and Klimyk.

Similarly to the different automorphic actions on the C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ there are a lot of different Heisenberg doubles depending on the number of intersections and the orientations of the surface and paths.

8.2.3 Representations and states of the holonomy-flux cross-product *-algebra

Surface-orientation-preserving graph-diffeomorphism-invariant states of the holonomy-flux cross-product *-algebra

A *-representations of the universal enveloping flux algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ is given by the infinitesimal representation $\mathrm{d}U$ of a unitary representation U of $\bar{G}_{\breve{S},\Gamma}$ in $C(\bar{\mathcal{A}}_\Gamma)$. In general *-representations is defined on arbitrary *-algebra, but there is no necessary condition that a unitary representation U of the Lie group $\bar{G}_{\breve{S},\Gamma}$ on a Hilbert space exists such that the commutator is equivalent to the infinitesimal representation. Mathematically *-representations of Lie algebras are required to recover the structure of the Lie algebra.

Definition 8.2.13. *Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. A ***-representation of a Lie algebra** $\mathfrak{g}$ on $\mathcal{D}$ is a mapping π of $\mathfrak{g}$ into $L(\mathcal{D})$ such that*

(i) $\pi(\alpha X + \beta Y) = \alpha\pi(X) + \beta\pi(Y)$

(ii) $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$

(iii) $\langle \pi(X)\phi, \varphi\rangle = \langle \phi, \pi(X^+)\varphi\rangle$

whenever $X, Y \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{R}$ and $\phi, \varphi \in \mathcal{D}$ where $L(\mathcal{D})$ vector space of linear mappings of X into X.

Notice that $\pi(X) \in L(\mathcal{D})$ and property (iii) it follows that $\pi(X) \in \mathfrak{L}^+(\mathcal{D})$ (refer to appendix 12.2.3).

Let $\breve{S}$ be a surface set with same surface intersection property for a finite orientation preserved graph system associated to a graph Γ.

In LQG the flux operators $E_S(\Gamma)$ are represented as differential operators $\mathrm{d}\, U(E_S(\Gamma))$ on the Hilbert space $\mathcal{H}_\Gamma := L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ of square integrable functions on $\bar{\mathcal{A}}_\Gamma$.

In the following the relation to the flux group $\bar{G}_{\breve{S},\Gamma}$ will be explained. The domain $\mathcal{D}(\mathrm{d}\, U)$ of the infinitesimal representations $\mathrm{d}\, U$ of the flux group $\bar{G}_{\breve{S},\Gamma}$ on the Hilbert space $\mathcal{H}_\Gamma$ is the set of all functions ψ_Γ in $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ such that for each $E_S(\Gamma) \in \bar{\mathfrak{g}}_{\breve{S},\Gamma}$ the limit

$$\mathrm{d}\, U(E_S(\Gamma))\psi_\Gamma := \lim_{t\to 0} \frac{(U(\exp(tE_S(\Gamma))) - \mathbb{1})\psi_\Gamma}{t}$$

exists weakly in $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. Notice that there is a domain $\mathcal{D}(\mathrm{d}\mathcal{U})$ for all infinitesimal representations that corresponds to unitary representations $U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)))$. The requirement of the existence of the limit is equivalent to the condition that the function $\rho_S(\Gamma) \mapsto \langle U(\rho_{S,\Gamma}(\Gamma)\psi_\Gamma, \phi_\Gamma\rangle$ is in $C^\infty(\bar{G}_{\breve{S},\Gamma})$ for each $\phi_\Gamma \in L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$. With no doubt $C^\infty(\bar{\mathcal{A}}_\Gamma) \subset \mathcal{D}(\mathrm{d}\, U)$ holds. This reformulation can be further used to understand the connection between the construction of the holonomy-flux *-algebra of Lewandowski, Okołów, Sahlmann and Thiemann [64] and the Weyl C^*-algebra of Fleischhack [39].

Now recognize a short remark. The domain of the unbounded operator $\mathrm{d}\, U$ is rewritten in the following way. Let $\{X_{S_1}, ..., X_{S_d}\}$ be a basis of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ where $S_1, ..., S_d \in \breve{S}$. Then by a corollary [89, Cor.10.1.10] the domain $D(\mathrm{d}\, U)$ is equivalent to the set of all elements $\psi_\Gamma \in L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ such that for all X_{S_k} where $k = 1, ..., d$ and $\phi_\Gamma \in L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ the function $\mathbb{R} \ni t \mapsto \langle U(\exp(tX_{S_k})\psi_\Gamma, \phi_\Gamma\rangle$ is in $C^\infty(\mathbb{R})$. Notice that, $t \mapsto U(\exp(tX_k))$ is a unitary representation of the Lie group $\mathbb{R}$ for each element X_{S_k} of the basis of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$, too. Consequently it is assumed that, $U \in \mathrm{Rep}(\mathbb{R}, \mathcal{K}(\mathcal{H}_\Gamma))$ for each X_{S_k}. The operators X_{S_k} corresponding infinitesimal representation $\mathrm{d}\, U(X_{S_K})$ are called the infinitesimal generators of U.

Summarising the operators $e^{\overleftarrow{L}}$ and $e^{\overleftarrow{R}}$ are defined on a dense linear subspace $\mathcal{D}(\mathrm{d}\mathcal{U})$ of the Hilbert space $\mathcal{H}_\Gamma := L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$, and their adjoint operators $e^{\overrightarrow{R}}, e^{\overrightarrow{L}}$ defined on $\mathcal{D}(\mathrm{d}\mathcal{U}^*)$. In particular, $e^{\overleftarrow{L}}$ and $e^{\overleftarrow{R}}$ are elements of the set

$$\begin{aligned}\mathfrak{L}^+_{\mathcal{U}}(\mathcal{D}(\mathrm{d}\mathcal{U})) := \{\mathrm{d}\, U \in \mathfrak{L}(\mathcal{D}(\mathrm{d}\mathcal{U})) :\ & U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma)), \\ & \mathcal{D}(\mathrm{d}\mathcal{U}) \subset \mathcal{D}(\mathrm{d}\mathcal{U}^*), \mathrm{d}\, U^*\mathcal{D}(\mathrm{d}\mathcal{U}) \subset \mathcal{D}(\mathrm{d}\mathcal{U})\}\end{aligned}$$

where $\mathfrak{L}^+_{\mathcal{U}}(\mathcal{D}(\mathrm{d}\mathcal{U})) \subset \mathfrak{L}(\mathcal{D}(\mathrm{d}\mathcal{U}))$ and $\mathfrak{L}(\mathcal{D}(\mathrm{d}\mathcal{U}))$ denotes the set of all linear operators from $\mathcal{D}(\mathrm{d}\mathcal{U})$ to $\mathcal{D}(\mathrm{d}\mathcal{U})$ and $\mathcal{D}(\mathrm{d}\mathcal{U}^*)$ the domain of the adjoint of the linear operator $\mathrm{d}\, U$.

Now it is obvious that $\mathrm{d}\, U$ is a *-representation of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ on $\mathcal{D}(\mathrm{d}\mathcal{U})$.

In analogy to the result of Schmüdgen in [89, Prop 10.1.6] the following proposition holds.

Proposition 8.2.14. *Let $\breve{S}$ be a surface set with same surface intersection property for a finite orientation preserved graph system associated to a graph Γ.*

*Let $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ be the universal enveloping Lie flux *-algebra for a surface set $\breve{S}$ and U a unitary representation of $\bar{G}_{\breve{S},\Gamma}$ on the Hilbert space $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.*

*Then $(\mathrm{d}\, U(E_S(\Gamma)))(\psi_\Gamma) := e^{\overrightarrow{L}}(\psi_\Gamma)$ for $\psi \in \mathcal{D}(\mathrm{d}\, U)$ and $E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$ define a *-repesentation $\mathrm{d}\, U$ of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on a dense subdomain $\mathcal{D}(\mathrm{d}\, U)$ of the Hilbert space $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.*

Proof : Let $\tilde{\tau}_1$ be a homomorphism of the Lie algebra $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ in $C^\infty(\bar{\mathcal{A}}_\Gamma)$. Furthermore $\mathrm{d}\, \tilde{U}$ is a *-homomorphism of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ into the O^*-algebra $\mathfrak{L}^+(\mathcal{D}(\mathrm{d}\, U))$ such that $\mathrm{d}\, \tilde{U}(\mathbb{1}) = 1$. Then $\mathrm{d}\, \tilde{U}$ defines a *-representation of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ on the domain $\mathcal{D}(\mathrm{d}\, U)$. There exists a unique extension of $\mathrm{d}\, \tilde{U}$ to an homomorphism $\mathrm{d}\, U$ of the *-algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ into the *-algebra $\mathfrak{L}^+(\mathcal{D}(\mathrm{d}\, U))$, which defines a *-representation of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ by corollary 8.2.7.

Consequently one shows that, the map $\bar{\mathfrak{g}}_{\breve{S},\Gamma} \ni X_S \mapsto \mathrm{d}\, U(X_S)$ is a *-representation of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ on $\mathcal{D}(\mathrm{d}\, U)$. For a suitable surface S and a graph Γ set $E_S(\Gamma) = X_S$. First derive that

$$\begin{aligned}\langle \mathrm{d}\, U(X_S)\varphi_\Gamma, \phi_\Gamma\rangle &= \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\langle U(\exp(tX_S))\psi_\Gamma, \varphi_\Gamma\rangle = \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\langle \psi_\Gamma, U^*(\exp(tX_S))\varphi_\Gamma\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\langle \psi_\Gamma, U(-\exp(tX_S))\varphi_\Gamma\rangle = -\frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\langle \psi_\Gamma, U(\exp(tX_S))\varphi_\Gamma\rangle = -\langle \varphi_\Gamma, \mathrm{d}\, U(X_S)\phi_\Gamma\rangle\end{aligned}$$

for $\psi_\Gamma, \varphi_\Gamma \in \mathcal{H}_\Gamma$ yields. Remember that $X_S^+ = -X_S$ for $E_S(\Gamma) \in \mathfrak{g}_{\breve{S},\Gamma}$ to conclude $\mathrm{d}\, U(X_S)^* = -\mathrm{d}\, U(X_S) = \mathrm{d}\, U(X_S^+)$.

Hence the crucial property is (ii). Let $X \mapsto U(\exp(tX))$ be weakly continuous, then derive

$$\begin{aligned}&\langle(\mathrm{d}\, U(X)\,\mathrm{d}\, U(Y) - \mathrm{d}\, U(Y)\,\mathrm{d}\, U(X))\varphi_\Gamma, \phi_\Gamma\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\Big(\frac{\mathrm{d}}{\mathrm{d}\, s}\Big|_{s=0}\langle U(\exp(-tX_S)\exp(-sY_S))\psi_\Gamma, \varphi_\Gamma\rangle\Big) \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}\, s}\Big|_{s=0}\Big(\frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\langle U(\exp(-sY_S)\exp(-tX_S))\psi_\Gamma, \varphi_\Gamma\rangle\Big) \\ &= \big\langle\, (Y_SX_S - X_SY_S)\,\psi_\Gamma, \varphi_\Gamma\big\rangle = \big\langle [Y_S, X_S]\psi_\Gamma, \varphi_\Gamma\big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\big\langle U(t[Y_S, X_S])\psi_\Gamma, \varphi_\Gamma\big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0}\big\langle U(-t[X_S, Y_S])\psi_\Gamma, \varphi_\Gamma\big\rangle\end{aligned}$$

■

Remark that, the unbounded operator $\mathrm{d}\, U$ of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ and the operator $\mathrm{d}\, U(E_S(\Gamma))$ for a fixed element $E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$ are not equivalent, since for example the domains are different. Observe that, for an infinitesimal generator $\mathrm{d}\, U(E_S(\Gamma))$ of the strongly continuous one-parameter unitary group $\mathbb{R} \ni t \mapsto U(\exp(tE_S(\Gamma)))$ on $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$ define the self-adjoint $i\,\mathrm{d}\, U(E_S(\Gamma))$ on the domain $D(\mathrm{d}\, U(E_S(\Gamma)))$. Clearly the subset $D(\mathrm{d}\, U)$ is contained in $D(\mathrm{d}\, U(E_S(\Gamma)))$. Therefore different special flux operators $E_S(\Gamma)$ or all flux operators $E_S(\Gamma)$ can be analysed. Significant examples for a flux operator are related to elliptic operators.

Let $\{X_{S_1}, ..., X_{S_d}\}$ be a basis of the Lie flux algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ where $S_1, .., S_d \in \breve{S}$. Then note that, any element $E_S(\Gamma)$ of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ is written as

$$E_S(\Gamma) = \sum_{k=0}^{m}\sum_{\substack{n\in\mathbb{N}_0^d\\ |n|=k}} \lambda_n X_i^n \text{ for all } m \in \mathbb{N}_0, \lambda_n \in \mathbb{C}^d \tag{8.37}$$

where $|n| := n_1 + ... + n_d$, $i = (1, ..., d)$ and $n := (n_1, ..., n_d) \in \mathbb{N}_0^d$

Elliptic elements are of the form (8.37) where $m \neq 0$ and $\sum_{|n|=m} \lambda_n t^n \neq 0$ for all non-zero vectors $t \in \mathbb{R}^d$. For example an elliptic element is given by the Nelson Laplacian

$$\triangle_{\mathcal{E}} = X_1^2 + ... + X_d^2 \quad \text{or the resolvent operator } (\mathbb{1} - \triangle_{\mathcal{E}})^k \text{ for every } k \in \mathbb{N} \tag{8.38}$$

Moroever the operators the $\mathrm{d}\, U$ and $\mathrm{d}\, U(E_S(\Gamma))$ have different self-adjointness properties. Indeed the *-representation $\mathrm{d}\, U$ on $D(\mathrm{d}\, U)$ is self-adjoint [89, Cor.10.2.3], whereas $\mathrm{d}\, U(E_S(\Gamma))$ for any hermitian elliptic element $E_S(\Gamma)$ of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on the domain $D(\mathrm{d}\, U)$ is essentially self-adjoint [89, Cor. 10.2.5].

Finally for general elliptic elements in $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ the adjoint operator $\mathrm{d}\, U(E_S(\Gamma))^*$ is equivalent to the closure w.r.t. the graph topology of $\mathrm{d}\, U(E_S^+(\Gamma))$, [89, Cor.10.2.7]. For an abelian or compact Lie group G it turns out that the adjoint $\mathrm{d}\, U(E_S(\Gamma))^*$ is equivalent to the closure w.r.t. the graph topology of $\mathrm{d}\, U(E_S^+(\Gamma))$ for all $E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$. Hence since in LQG literature often compact Lie groups are used one has to focus on the domains of the different operators.

According to the observations of the Lie flux group C^*-algebra, the universal enveloping flux *-algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ is considered. This algebra itself is shown to be equivalent to the algebra of differential operators on $C^\infty(\bar{G}_{\breve{S},\Gamma})$. Observe that, due to the different structure of $\bar{G}_{\breve{S},\Gamma}$ and $\bar{\mathcal{A}}_\Gamma$ the identification of both sets is valid only for suitable surface sets and graphs. Following Schmüdgen [89] one obtain the following.

Proposition 8.2.15. *Let G be a compact Lie group and the set $\breve{S}$ has the same intersection surface property for a finite orientation preserved graph system associated to a graph Γ. Set $N = |E_\Gamma|$ and identify $\bar{\mathcal{A}}_\Gamma$ with G^N naturally.*

Then the universal enveloping Lie flux *-algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ is *-isomorphic to the O^*-algebra $\mathcal{D}_{\breve{S}}(\bar{G}_{\breve{S},\Gamma})$ of differential operators on $C^\infty(G^N)$ in the Hilbert space $L^2(G^N,\mu_N)$, where $\mathcal{D}_{\breve{S}}(\bar{G}_{\breve{S},\Gamma})$ is the algebra of all right-invariant differential operators $\mathrm{d}\, U_{\overleftarrow{L}}(\bar{\mathcal{E}}_{\breve{S},\Gamma})\big|_{C^\infty(G^N)}$ on G^N.

Summarising there are different involutive algebras, like the analytic holonomy algebra for a graph, the universal enveloping Lie flux *-algebra for a graph and a surface set or the holonomy-flux cross-product *-algebra for a graph represented on the Hilbert space $\mathcal{H}_\Gamma$.

Theorem 8.2.16. *Let $\breve{S}$ be a surface set having the same intersection surface property for a finite orientation preserved graph system associated to Γ.*

There exists the following *-representations of the analytic holonomy C^*-algebra $C^\infty(\bar{\mathcal{A}}_\Gamma)$, the universal enveloping Lie flux *-algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ and the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes_L \bar{\mathcal{E}}_{\breve{S},\Gamma}$ for a graph Γ and a surface set $\breve{S}$ on the Hilbert space $\mathcal{H}_\Gamma = L^2(\bar{\mathcal{A}}_\Gamma,\mu_\Gamma)$ on $C^\infty(\bar{\mathcal{A}}_\Gamma)$.

$$\begin{aligned}
&\Phi_M(f_\Gamma)\psi_\Gamma = f_\Gamma\psi_\Gamma \text{ for } f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)\\
&\Phi_M(f_\Gamma^*)\psi_\Gamma = \overline{f_\Gamma}\psi_\Gamma \text{ for } f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)\\
&\mathrm{d}\,U(E_S(\Gamma))\psi_\Gamma = [E_S(\Gamma),\psi_\Gamma] \text{ for } E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}\\
&\mathrm{d}\,U(E_S(\Gamma)^+)\psi_\Gamma = [E_S(\Gamma)^+,\psi_\Gamma] \text{ for } E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}\\
&\pi(f_\Gamma\otimes E_S(\Gamma))\psi_\Gamma = \frac{1}{2}[E_S(\Gamma),f_\Gamma]\psi_\Gamma + \frac{1}{2}f_\Gamma[E_S(\Gamma),\psi_\Gamma] \text{ for } f_\Gamma\otimes E_S(\Gamma) \in C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}\\
&\pi((f_\Gamma\otimes E_S(\Gamma))^*)\psi_\Gamma = \frac{1}{2}[E_S(\Gamma)^+,f_\Gamma^*]\psi_\Gamma + \frac{1}{2}f_\Gamma[E_S(\Gamma)^+,\psi_\Gamma] \text{ for } f_\Gamma\otimes E_S(\Gamma) \in C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}
\end{aligned}$$

whenever $\psi_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$. For two surfaces $S_1\cap S_2 = \varnothing$ the representation satisfies

$$\begin{aligned}
&\pi([f_\Gamma^1\otimes E_{S_1}(\Gamma), f_\Gamma^2\otimes E_{S_2}(\Gamma)])\psi_\Gamma\\
&= \frac{1}{4}[E_{S_2}(\Gamma), f_\Gamma^1[E_{S_1}(\Gamma),f_\Gamma^2]]\psi_\Gamma + \frac{1}{4}f_\Gamma^1[E_{S_1}(\Gamma),f_\Gamma^2][E_{S_2}(\Gamma),\psi_\Gamma]\\
&\quad - \frac{1}{4}[E_{S_1}(\Gamma), f_\Gamma^2[E_{S_2}(\Gamma),f_\Gamma^1]]\psi_\Gamma - \frac{1}{4}f_\Gamma^2[E_{S_2}(\Gamma),f_\Gamma^1][E_{S_1}(\Gamma),\psi_\Gamma]
\end{aligned}$$

whenever $\psi_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$.

Proof : The following computations show that, π is a *-representation of $C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes_L\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on the domain $C^\infty(\bar{\mathcal{A}}_\Gamma)$:

$$\begin{aligned}
&\pi(\lambda_1 f_\Gamma\otimes\lambda_2 E_S(\Gamma))\psi_\Gamma = \frac{1}{2}\lambda_1\lambda_2\left([E_S(\Gamma),f_\Gamma]\psi_\Gamma - \frac{1}{2}f_\Gamma[E_S(\Gamma),\psi_\Gamma]\right)\\
&\pi(f_\Gamma^1\otimes E_{S_1}(\Gamma) + f_\Gamma^2\otimes E_{S_2}(\Gamma))\psi_\Gamma = \pi(f_\Gamma^1\otimes E_{S_1}(\Gamma)) + \pi(f_\Gamma^2\otimes E_{S_2}(\Gamma))\psi_\Gamma\\
&\pi((f_\Gamma\otimes E_S(\Gamma))^*)\psi_\Gamma = \frac{1}{2}[E_S(\Gamma)^+,f_\Gamma^*]\psi_\Gamma + \frac{1}{2}f_\Gamma[E_S(\Gamma)^+,\psi_\Gamma] = \pi(f_\Gamma\otimes E_S(\Gamma))^*\psi_\Gamma
\end{aligned}$$

for $f_\Gamma, f_\Gamma^1, f_\Gamma^2 \in C^\infty(\bar{\mathcal{A}}_\Gamma)$, $E_S(\Gamma), E_{S_1}(\Gamma), E_{S_2}(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$, $\lambda_1,\lambda_2\in\mathbb{C}$.

For two surfaces $S_1\cap S_2=\varnothing$ calculate

$$\begin{aligned}
&\pi([f_\Gamma^1\otimes E_{S_1}(\Gamma), f_\Gamma^2\otimes E_{S_2}(\Gamma)])\psi_\Gamma\\
&= \pi(f_\Gamma^1[E_{S_1}(\Gamma),f_\Gamma^2]\otimes E_{S_2}(\Gamma)) - \pi(f_\Gamma^2[E_{S_2}(\Gamma),f_\Gamma^1]\otimes E_{S_1}(\Gamma)) + \pi(f_\Gamma^1 f_\Gamma^2\otimes[E_{S_1}(\Gamma),E_{S_2}(\Gamma)])\\
&= \frac{1}{4}[E_{S_2}(\Gamma), f_\Gamma^1[E_{S_1}(\Gamma),f_\Gamma^2]]\psi_\Gamma + \frac{1}{4}f_\Gamma^1[E_{S_1}(\Gamma),f_\Gamma^2][E_{S_2}(\Gamma),\psi_\Gamma]\\
&\quad - \frac{1}{4}[E_{S_1}(\Gamma), f_\Gamma^2[E_{S_2}(\Gamma),f_\Gamma^1]]\psi_\Gamma - \frac{1}{4}f_\Gamma^2[E_{S_2}(\Gamma),f_\Gamma^1][E_{S_1}(\Gamma),\psi_\Gamma]
\end{aligned}$$

■

From another point of view the bracket $[E_S(\Gamma),.]$ defines a *-derivation of the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ for a graph Γ. Moroever in general such *-derivations are implemented by automorphisms on $C(\bar{\mathcal{A}}_\Gamma)$. This point of view is more general than the consideration of differential operators.

For a simplification restrict the following computations to a suitable surface S and a graph $\Gamma := \{\gamma\}$.

Lemma 8.2.17. *Let Φ_M be a representation of $C(\bar{\mathcal{A}}_\Gamma)$ on $\mathcal{H}_\Gamma$ and $\alpha \in \text{Act}(\bar{G}_{\breve{S},\Gamma}, C(\bar{\mathcal{A}}_\Gamma))$ defined in section 6.1, where $\rho_{S,\Gamma}(\Gamma) = \exp(tE_S(\Gamma)) \in \bar{G}_{\breve{S},\Gamma}$ for $t \in \mathbb{R}$. Let $\Gamma = \{\gamma\}$ and S be suitable and set $E_S(\Gamma) =: X_S$.*

Then it is true that

$$\begin{aligned}\omega_M^\Gamma(\alpha^t_{\exp(X_S)}(f_\Gamma)) &= \int_{\bar{\mathcal{A}}_\Gamma} f_\Gamma(\exp(tX_S)\mathfrak{h}_\Gamma(\gamma))\,\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\gamma))\\ &= \int_{\bar{\mathcal{A}}_\Gamma} f_\Gamma(\mathfrak{h}_\Gamma(\gamma))\,\mathrm{d}\,\mu_\Gamma(\mathfrak{h}_\Gamma(\gamma))\\ &= \omega_M^\Gamma(f_\Gamma)\end{aligned}$$

for all $t \in \mathbb{R}$ and $f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma)$.

There is a general property of states on an arbitrary (untial) C^*-algebra $\mathfrak{A}$ represented on a Hilbert space and a group of *-automorphisms α.

Recall from Lemma 8.2.11 the *-derivation on $C(\bar{\mathcal{A}}_\Gamma)$ given by

$$\delta^2_{S,\Gamma}(f_\Gamma) := i[E_S(\Gamma_i)^+E_S(\Gamma), f_\Gamma] \text{ for } E_S \in \mathcal{E}_{\breve{S},\Gamma}, f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$$

such that $\delta_{S,\Gamma} \in \text{Der}(C^\infty(\bar{\mathcal{A}}_\Gamma), C(\bar{\mathcal{A}}_\Gamma))$.

Corollary 8.2.18. *Let $\alpha \in \text{Act}(\bar{G}_{\breve{S},\Gamma}, C(\bar{\mathcal{A}}_\Gamma))$.*

Then for each element $E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$ the limit

$$\tilde{\delta}_{S,\Gamma}(f_\Gamma) := \lim_{t\to 0}\frac{\alpha^t_{i\exp(E_S(\Gamma_i)^+E_S(\Gamma))}(f_\Gamma) - f_\Gamma}{t} \textit{ for } f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma) \tag{8.39}$$

exists in norm topology and $\tilde{\delta}_{S,\Gamma} = \delta_{S,\Gamma}$.

Then the state ω_M^Γ on $C(\bar{\mathcal{A}}_\Gamma)$ presented in 6.1.41 associated to the GNS-representation $(\Phi_M, \mathcal{H}_\Gamma, \Omega_M^\Gamma)$ satisfies

$$\omega_M^\Gamma(\delta_{S,\Gamma}(f_\Gamma)) = 0 \tag{8.40}$$

for all $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$ and it is true that

$$iE_S(\Gamma_i)^+E_S(\Gamma)\Omega_M^\Gamma = 0 \textit{ for all } E_S \in \mathcal{E}_{\breve{S},\Gamma}$$

where $iE_S^+(\Gamma)E_S(\Gamma)$ is a self-adjoint and positive operator with domain $\mathcal{D}(E_S^+(\Gamma)E_S(\Gamma))$.

Notice that, the unbounded *-derivation given by $\delta_{S,\Gamma}$ is symmetric.

Proof : First observe that $\omega_M^\Gamma(\alpha^t_{i\exp(E_S(\Gamma_i)^+E_S(\Gamma))}(f_\Gamma)) = \omega_M^\Gamma(f_\Gamma)$ for all $t \in \mathbb{R}$ and $f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma)$. This follows from lemma 8.2.17. For $f_\Gamma \in C(\bar{\mathcal{A}}_\Gamma)$ the *-automorphisms α are implementable as a one-parameter group $\mathbb{R} \ni t \mapsto \alpha^t_{iE_S(\Gamma_i)^+E_S(\Gamma)}(f_\Gamma) \in C(\bar{\mathcal{A}}_\Gamma)$ for each $E_S(\Gamma) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$, which is weakly continuous.

Then the norm limit 8.39 exists for suitable f_Γ in the domain $C^\infty(\bar{\mathcal{A}}_\Gamma)$. The symmetric derivation is therefore given by

$$\delta_{S,\Gamma}(f_\Gamma) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha^t_{iE_S(\Gamma_i)^+E_S(\Gamma)}(f_\Gamma) \text{ for } f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$$

The operator $iE_S(\Gamma_i)^+E_S(\Gamma)$ is the generator of the unbounded symmetric *-derivation $\delta_{S,\Gamma}$ on $\mathcal{H}_\Gamma$ by definition.

Recall the state ω_M^Γ of $C(\bar{\mathcal{A}}_\Gamma)$ associated to the GNS-representation $(\Phi_M, \mathcal{H}_\Gamma, \Omega_M^\Gamma)$ presented in proposition 6.1.40, then the derivation $\delta_{S,\Gamma}$ satisfies

$$\begin{aligned}\omega_M^\Gamma(\delta_{S,\Gamma}(f_\Gamma)) &= \Big\langle \Omega_M^\Gamma, \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \Phi_M(\alpha^t_{\exp(iE_S(\Gamma_i)^+E_S(\Gamma))}(f_\Gamma))\Omega_M^\Gamma\Big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} \omega_M^\Gamma(\alpha^t_{\exp(iE_S(\Gamma_i)^+E_S(\Gamma))}(f_\Gamma)) = 0\end{aligned} \tag{8.41}$$

for $f_\Gamma \in C^\infty(\bar{\mathcal{A}}_\Gamma)$. There exists a covariant representation (Φ_M, U) of $(\bar{G}_{\breve{S},\Gamma}, \mathfrak{A}_\Gamma, \alpha)$ in $\mathcal{L}(\mathcal{H}_\Gamma)$ such that

$$\Phi_M(\alpha^t_{\exp(iE_S(\Gamma_i)^+E_S(\Gamma))})(f_\Gamma)) = U(it\exp(tE_S(\Gamma_i)^+E_S(\Gamma)))\Phi_M(f_\Gamma)U(\exp(-itE_S(\Gamma_i)^+E_S(\Gamma)))$$

and $U(\exp(itE_S(\Gamma_i)^+E_S(\Gamma)))\Omega_M^\Gamma = \Omega_M^\Gamma$ for all $t \in \mathbb{R}$. The one-parameter group $\mathbb{R} \ni t \mapsto U(\exp(itE_S(\Gamma_i)^+E_S(\Gamma)))$ is weakly continuous. Then there exists a self-adjoint operator $iE_S^+(\Gamma)E_S(\Gamma)$ such that $iE_S^+(\Gamma)E_S(\Gamma)\Omega_M^\Gamma = 0$.

■

Notice that

$$\begin{aligned}\Phi_M(\delta(f))\Omega_M^\Gamma &= \lim_{t\to 0} \frac{\Big(\Phi_M(\alpha^t_{iE_S(\Gamma)^+E_S(\Gamma)}(f)) - \Phi_M(f)\Big)\,\Omega_M^\Gamma}{t} \\ &= \lim_{t\to 0} \frac{(U(itE_S(\Gamma)^+E_S(\Gamma))\Phi_M(f) - \Phi_M(f)U(itE_S(\Gamma)^+E_S(\Gamma)))\,\Omega_M^\Gamma}{t} \\ &= i[E_S(\Gamma)^+E_S(\Gamma), \Phi_M(f)]\Omega_M^\Gamma\end{aligned} \tag{8.42}$$

The next derivation is defined only for a suitable family of graphs and a suitable surface set.

Define the *-derivation on the domain $C^\infty(\bar{\mathcal{A}})$ of the C^*-algebra $C(\bar{\mathcal{A}})$ by

$$\delta_S(f) := i[E_S(\Gamma_\infty)^+E_S(\Gamma_\infty), f] \text{ for } f \in C^\infty(\bar{\mathcal{A}}), E_S \in \mathcal{E}_{\breve{S}} \text{ and } \Gamma_\infty \in \mathcal{P}_{\Gamma_\infty}$$

Proposition 8.2.19. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Let $\breve{S}$ be a finite set of surfaces in Σ such that*

(i) such that the surface set $\breve{S}$ has the same surface intersection property for each graph of the family,

(ii) the inductive limit structure preserves the same surface intersection property for $\breve{S}$ and

(iii) each surface in $\breve{S}$ intersects the inductive limit graph Γ_∞ only in a finite number of vertices.

Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems. Let $\bar{\mathcal{A}}_{\Gamma_i}$ is identified in the natural way with G^{N_i}.

Then the limit

$$\alpha^t_{\exp(iE_S^+(\Gamma_\infty)E_S(\Gamma_\infty))}(f) := \lim_{j\to\infty} \alpha^t_{\exp(iE_S^+(\Gamma_j)E_S(\Gamma_j))}(f) \text{ for } f \in C^\infty(\bar{\mathcal{A}}), E_S \in \mathcal{E}_{\breve{S}} \text{ and } \Gamma_j \in \mathcal{P}_{\Gamma_\infty}$$

exists for each $t \in \mathbb{R}$ in norm topology. Consequently the limit

$$\tilde{\delta}_S(f) := \lim_{t\to 0} \frac{\alpha^t_{\exp(iE_S^+(\Gamma_\infty)E_S(\Gamma_\infty))}(f) - f}{t} \text{ for } f \in C^\infty(\bar{\mathcal{A}})$$

exists in norm topology and $\tilde{\delta}_S = \delta_S$ for $S \in \breve{S}$.

Finally for each *-derivation δ_S the state satisfies

$$\omega_M(\delta_S(f)) = 0$$

for all $f \in C^\infty(\bar{\mathcal{A}})$ and for $S \in \breve{S}$.

Proof : On the inductive limit of the family of C^*-algebras $\{(C(\bar{\mathcal{A}}_{\Gamma_i}),\beta_{\Gamma_i,\Gamma_j}) : \mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\leq\mathcal{P}^{\mathfrak{o}}_{\Gamma_j}\}$ the action of $\alpha(\rho_{S,\Gamma_i}(\Gamma_i))$ on $C(\bar{\mathcal{A}}_{\Gamma_i})$ for $\rho_{S,\Gamma_i} := \exp(iE_S(\Gamma_i)^+E_S(\Gamma_i))\in\bar{G}_{\breve{S},\Gamma_i}$ is non-trivial if each surface S of $\breve{S}$ intersect the graph Γ_i in vertices of the graph Γ_i. In particular, there is a graph Γ_j having the maximal number of intersection vertices with any surface S in $\breve{S}$. Consequently for a graph Γ_{j+1} that contain Γ_j the flux $E_S(\Gamma_{j+1}\setminus\Gamma_j)=0$. Furthermore derive

$$\begin{aligned}\alpha^t_{\exp(iE^+_S(\Gamma_\infty)E_S(\Gamma_\infty))}(f) &:= \lim_{k\rightarrow\infty}\alpha^t_{\exp(iE^+_S(\Gamma_k)E_S(\Gamma_k))}(f)\\ &= \lim_{k\rightarrow\infty}\alpha^t_{\exp(iE^+_S(\Gamma_k)E_S(\Gamma_k))}(\beta_{\Gamma'}f_{\Gamma'})\\ &= \alpha^t_{\exp(iE^+_S(\Gamma_j)E_S(\Gamma_j))}(\beta_{\Gamma'}f_{\Gamma'}) \text{ for } f\in C^\infty(\bar{\mathcal{A}}), f=\beta_{\Gamma'}f_{\Gamma'} \text{ and } E_S\in\mathcal{E}_{\breve{S}}\end{aligned}$$

whenever $\Gamma_k\leq\Gamma'$ for $1\leq k\leq j$. Furthermore conclude that,

$$\begin{aligned}\delta_S(f) &= i[E_S(\Gamma_j)^+E_S(\Gamma_j),f] + \lim_{i\rightarrow\infty} i[E^+_S(\Gamma_{j+i}\setminus\Gamma_j)E_S(\Gamma_{j+i}\setminus\Gamma_j),f]\\ &= \beta_{\Gamma_j}\circ(i[E_S(\Gamma_j)^+E_S(\Gamma_j),f_{\Gamma_j}]) \text{ for } f\in C^\infty(\bar{\mathcal{A}}), f=\beta_{\Gamma_j}f_{\Gamma_j} \text{ and } E_S\in\mathcal{E}_{\breve{S}}\end{aligned}$$

holds. Hence there is a *-homomorphism $\beta_{\Gamma,\Gamma'}$ from $C^\infty(\bar{\mathcal{A}}_\Gamma)$ to $C^\infty(\bar{\mathcal{A}}_{\Gamma'})$ such that $\beta_{\Gamma,\Gamma'}\circ\delta_{S,\Gamma}\circ\beta^{-1}_{\Gamma,\Gamma'}=\delta_{S,\Gamma'}$ is a *-derivation from $C^\infty(\bar{\mathcal{A}}_{\Gamma'})$ into $C(\bar{\mathcal{A}}_{\Gamma'})$ and

$$\omega^\Gamma_M(\delta_{S,\Gamma}(f_\Gamma)) = \omega^{\Gamma'}_M(\beta_{\Gamma,\Gamma'}\circ\delta_{S,\Gamma})(f_\Gamma)) = \omega^{\Gamma'}_M(\delta_{S,\Gamma'}(f_{\Gamma'}))=0 \quad (8.43)$$

Finally derive

$$\omega_M(\delta_S(f)) = \omega_M(\beta_{\Gamma_j}\circ\delta_{S,\Gamma_j})(f_{\Gamma_j})) = \beta^*_{\Gamma_j}\omega^{\Gamma_j}_M(\delta_{S,\Gamma_j}(f_{\Gamma_j}))=0 \quad (8.44)$$

whenever $f\in C^\infty(\bar{\mathcal{A}})$.

■

Recall that ω_M is graph-diffeomorphism invariant if the natural identification of $\bar{\mathcal{A}}_\Gamma$ with $G^{|\Gamma|}$ is used. Recall that a real-valued, linear and *-preserving functional ω on a *-algebra associated to a *-representation π has to be positive.

Theorem 8.2.20. *Let Γ_∞ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Let $\breve{S}$ be a finite set of surfaces in Σ such that*

(i) such that the surface set $\breve{S}$ has the same surface intersection property for each graph of the family,

(ii) the inductive limit structure preserves the same surface intersection property for $\breve{S}$ and

(iii) each surface in $\breve{S}$ intersects the inductive limit graph Γ_∞ only in a finite number of vertices.

Then $\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}$ is the inductive limit of an inductive family $\{\mathcal{P}^{\mathfrak{o}}_{\Gamma_i}\}$ of finite orientation preserved graph systems. Let $\bar{\mathcal{A}}_\Gamma$ is identified in the natural way with G^N. Denote the center of $\bar{\mathcal{E}}_{\breve{S}}$ by $\mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$.

The state $\bar{\omega}_M$ associated to the GNS-representation $(\mathcal{H}_\Gamma,\pi,\Omega)$ given in theorem 8.2.16 is a surface-orientation-preserving graph-diffeomorphism invariant state for a fixed set of surfaces $\breve{S}$ on the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}})\rtimes_L\mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ such that

$$\bar{\omega}_M(f\otimes E_S(\Gamma_\infty)) = \beta^*_{\Gamma_j}\bar{\omega}^{\Gamma_j}_M(f_{\Gamma_j}\otimes E_S(\Gamma_j))=0 \text{ for all } E_S\in\mathcal{Z}(\mathcal{E}_{\breve{S}})$$

Moreover the state $\bar{\omega}_M$ on $C^\infty(\bar{\mathcal{A}})\rtimes_L\mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ is the unique state, which is surface-orientation-preserving graph-diffeomorphism invariant.

Proof : First recall that

$$\zeta_\sigma\circ\alpha(\rho_{S,\Gamma}(\Gamma))\neq\alpha(\rho_{S,\Gamma}(\Gamma_\sigma))\circ\zeta_\sigma$$

for every $\sigma \in \mathfrak{B}(\mathcal{P}^{\mathfrak{o}}_\Gamma)$ and $\rho_{S,\Gamma} := \exp(iE_S(\Gamma)^+E_S(\Gamma)) \in G_{\breve{S},\Gamma}$. Hence the problem carry over to

$$\delta_{S,\Gamma} \circ \zeta_\sigma \neq \zeta_\sigma \circ \delta_{S,\Gamma}$$

Consequently in the following the center $\mathcal{Z}(\bar{\mathcal{E}}_{\breve{S},\Gamma})$ and the center $\mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ are considered.

Moroever a surface-orientation-preserving graph-diffeomorphism invariant state for a fixed set of surfaces $\breve{S}$ means that

$$\bar{\omega}_M(\zeta_\sigma(f \otimes E_S(\Gamma_\infty))) = \bar{\omega}_M(f \otimes E_S(\Gamma_\infty)) \text{ for all } \sigma \in \mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty}) \text{ and } E_S(\Gamma_\infty) \in \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$$

holds. To show that $\bar{\omega}_M$ satisfies this property derive the following

$$\begin{aligned}\bar{\omega}_M(f \otimes E_S(\Gamma_\infty)) &= \beta^*_{\Gamma_j}\omega^{\Gamma_j}_M(f_{\Gamma_j} \otimes E_S(\Gamma_j)) = \langle \Omega^{\Gamma_j}_M, \pi(f_{\Gamma_j} \otimes E_S(\Gamma_j))\Omega^{\Gamma_j}_M\rangle \\ &= \frac{1}{2}\langle \Omega^{\Gamma_j}_M, [E_S(\Gamma_j), f_{\Gamma_j}]\Omega^{\Gamma_j}_M\rangle + \frac{1}{2}\langle \Omega^{\Gamma_j}_M, f_{\Gamma_j}\,\mathrm{d}\,U(E_S(\Gamma_j))\Omega^{\Gamma_j}_M\rangle \\ &= \frac{1}{2}\langle \Omega^{\Gamma_j}_M, \delta_{S,\Gamma_j}(f_{\Gamma_j})\Omega^{\Gamma_j}_M\rangle + \frac{1}{2}\langle \Omega^{\Gamma_j}_M, f_{\Gamma_j}\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0} U(\exp(tE_S(\Gamma_j)))\Omega^{\Gamma_j}_M\rangle = 0\end{aligned}$$

for all $f \in C(\bar{\mathcal{A}})$ and $E_S \in \mathcal{E}_{\breve{S}}$. Recognize that

$$\bar{\omega}_M(f \otimes \mathbb{1}) = \omega_M(f)$$

where ω_M is a state on $C(\bar{\mathcal{A}})$ holds. With no doubt a *-derivation is also defined for all $E_S(\Gamma) \in \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S},\Gamma_j})$ for $j = 1, ..., \infty$ such that corollary 8.2.18 and proposition 8.2.19 hold. Clearly it is true that

$$\begin{aligned}&\bar{\omega}_M((f \otimes E_S(\Gamma_\infty))^*(f \otimes E_S(\Gamma_\infty))) = \langle \Omega_M, \pi_\Gamma((f^* \otimes E_S(\Gamma_\infty)^+(f \otimes E_S(\Gamma_\infty)))\Omega_M\rangle \\ &= \langle \Omega^\Gamma_M, \frac{1}{4}[E_S(\Gamma), f^*_\Gamma[E_S(\Gamma)^+, f_\Gamma]]\Omega^\Gamma_M\rangle + \langle \Omega^\Gamma_M, \frac{1}{4}f^*_\Gamma[E^+_S(\Gamma), f_\Gamma][E_S(\Gamma), \Omega^\Gamma_M]\rangle \\ &\quad - \langle \Omega^\Gamma_M, \frac{1}{4}[E^+_S(\Gamma), f_\Gamma[E_S(\Gamma), f^*_\Gamma]]\Omega^\Gamma_M\rangle - \langle \Omega^\Gamma_M, \frac{1}{4}f_\Gamma[E_S(\Gamma), f^*_\Gamma][E^+_S(\Gamma), \Omega^\Gamma_M]\rangle \\ &= 0\end{aligned}$$

yields and therefore $\bar{\omega}_M$ is a $\mathfrak{B}_{\breve{S},\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_\infty})$-invariant state on $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$.

Let $\bar{\omega}'_M$ be another state of the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_L \bar{\mathcal{Z}}(\mathcal{E}_{\breve{S}})$ such that $\bar{\omega}'_M(f \otimes \mathbb{1}) = \omega_M(f)$ for all $f \in C(\bar{\mathcal{A}})$. Recall corollary 6.4.3 that states that, ω_M is the unique state on $C(\bar{\mathcal{A}})$ being invariant under the translation of $\bar{G}_{\breve{S},\Gamma}$ and graph-diffeomorphisms $\text{Diff}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_i})$ for every $1 \leq i \leq \infty$. Then it is assumed that for some $t_0 \in \mathbb{R}$

$$\omega'_M(\alpha^{t_0}_{\exp(E_S(\Gamma_\infty)^+E_S(\Gamma_\infty))}(f)) \neq \omega'_M(f) \quad \forall f \in C(\bar{\mathcal{A}}) \tag{8.45}$$

Consequently $\omega'_M(\delta_S(f)) \neq 0$.

But from (8.45) it follows for a suitable graph-diffeomorphism $(\varphi, \Phi) \in \text{Diff}_{\breve{S},\text{or}}(\mathcal{P}^{\mathfrak{o}}_{\Gamma_i})$ such that $\varphi(S) = S'$ for $S, S' \in \breve{S}$ that

$$\begin{aligned}\omega'_M(\alpha_{(\varphi,\Phi)}(\alpha^{t_0}_{\exp(E_S(\Gamma_\infty))}f)) &= \omega'^{\,\Gamma}_M(\alpha_{(\varphi,\Phi)}(\alpha^{t_0}_{\exp(E_S(\Gamma))}(f_\Gamma))) \\ &= \omega'^{\,\Phi(\Gamma)}_M(\alpha^{t_0}_{\exp(E_{S\circ\varphi}(\Gamma\circ\Phi))}(f_{\Phi(\Gamma)})) \\ &= \omega'^{\,\Gamma'}_M(\alpha^{t_0}_{\exp(E_{S'}(\Gamma'))}(f_{\Gamma'})) = \omega'_M(\alpha^{t_0}_{\exp(E_{S'}(\Gamma_\infty))}f) \\ &\neq \omega'^{\,\Gamma}_M(\alpha^{t_0}_{\exp(E_S(\Gamma))}(f_\Gamma)) = \omega'_M(\alpha^{t_0}_{\exp(E_S(\Gamma_\infty))}f)\end{aligned} \tag{8.46}$$

yields, whenever $\Phi(\Gamma) = \Gamma'$ and $f \in C(\bar{\mathcal{A}})$. In other words the state ω' is only surface-preserving graph-diffeomorphism invariant. The state ω'_M is not invariant under surface-orientation-preserving graph-diffeomorphisms and hence general graph-diffeomorphisms. Consequently the state ω'_M is equal to ω_M where $\omega_M(\delta_S(f)) = 0$ for all $f \in C^\infty(\bar{\mathcal{A}})$ and for all $\delta_S \in \text{Der}(C^\infty(\bar{\mathcal{A}}), C(\bar{\mathcal{A}}))$.

■

The algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{D}_{\breve{S}}(\bar{G}_{\breve{S}})$ is an O^*-algebra and is called the **holonomy-flux cross-product O^*-algebra associated a surface set** $\breve{S}$ on $C^\infty(\bar{\mathcal{A}})$ in $\mathcal{H}_\infty$.

Conditions for a surface preserving graph-diffeomorphism-invariant state of the holonomy-flux cross-product *-algebra

In theorem 8.2.20 the uniqueness of the state is referred to the assumption that, ω_M is surface-orientation graph-diffeomorphism-invariant state of the holonomy-flux cross-product *-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$. Now this requirement is relaxed to surface preserving graph-diffeomorphisms. In the case of the holonomy-flux cross-product C^*-algebra $C^\infty(\bar{\mathcal{A}}) \rtimes_{\alpha_{\overline{L}}} \mathcal{Z}_{\breve{S}}$, a surface preserving graph-diffeomorphism-invariant state ω_E is presented in proposition 7.2.15. Hence the question arise if there exists another state on $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ satisfying weaker conditions.

It is assumed that, the flux operators are implemented as *-derivations δ_S on the domain $\mathcal{D}(\delta_S)$ of the unital C^*-algebra $C(\bar{\mathcal{A}})$, which are generators of a strongly continuous one-parameter group $t \mapsto \alpha(t)$ of *-automorphisms of $C(\bar{\mathcal{A}})$. In this case the derivation is of the form $\delta_S(f) = i[E_S(\Gamma_\infty)^+ E_S(\Gamma_\infty), f]$ for $E_S(\Gamma_\infty) \in \bar{\mathfrak{g}}_{\breve{S},\Gamma_\infty}$, and where $iE_S(\Gamma_\infty)^+ E_S(\Gamma_\infty)$ is some unbounded symmetric operator with domain $\mathcal{D}$ on the Hilbert space $\mathcal{H}_\infty$ such that $\mathcal{D}(\delta_S)\mathcal{D} \subset \mathcal{D}$. Then a new state $\tilde{\omega}$ on $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$, which is not of the form $\tilde{\omega}(\delta_S(f)) = 0$, is required to satisfy a set of three conditions:

First condition:
Require the state $\tilde{\omega}^\Gamma$ to be $\text{Diff}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma^\circ)$-invariant, i.o.w.

$$\begin{aligned}\tilde{\omega}(\alpha_{(\varphi_S,\Phi_{\Gamma_\infty})}(f)) &= \beta_\Gamma^* \tilde{\omega}^\Gamma(\alpha_{(\varphi_S,\Phi_\Gamma)}(f_\Gamma)) = \beta_\Gamma^*\tilde{\omega}^\Gamma(f_\Gamma) = \tilde{\omega}(f)\\ \tilde{\omega}(\alpha_{(\varphi_S,\Phi_{\Gamma_\infty})}(\delta_S(f))) &= \beta_\Gamma^*\tilde{\omega}^\Gamma(\alpha_{(\varphi_S,\Phi_\Gamma)}(\delta_{S,\Gamma}(f_\Gamma))) = \beta_\Gamma^*\tilde{\omega}^\Gamma(\delta_{S,\Gamma}(f_\Gamma)) = \tilde{\omega}(\delta_S(f))\end{aligned} \tag{8.47}$$

for all $f \in C(\bar{\mathcal{A}})$, $f = \beta_\Gamma \circ f_\Gamma$, $\delta_S \in \text{Der}(\mathcal{D}(\delta_S), C(\bar{\mathcal{A}}))$ and $(\varphi_S, \Phi_\Gamma) \in \text{Diff}_{\breve{S},\text{surf}}(\mathcal{P}_\Gamma^\circ)$ yields.

Second condition:
Furthermore the state need to have the property

$$\tilde{\omega}(\delta_S(f)) \neq 0 \quad \forall f \in C(\bar{\mathcal{A}}) \tag{8.48}$$

which is equivalent to the requirement that for some $t \in \mathbb{R}$

$$\tilde{\omega}(\alpha_{\exp(E_S(\Gamma)^+ E_S(\Gamma))}(t)(f)) \neq \omega(f) \quad \forall f \in C(\bar{\mathcal{A}}) \text{ and } E_S(\Gamma_\infty) \in \bar{\mathcal{E}}_{\breve{S},\Gamma}$$

yields.

Third condition:
The state is assumed to fulfill

$$|\tilde{\omega}(\delta_S(f))| \leq c\,(\tilde{\omega}(f^*f) + \tilde{\omega}(ff^*))^{1/2} \quad \forall f \in C(\bar{\mathcal{A}}) \text{ and some } c > 0 \tag{8.49}$$

Clearly the ansatz is to search for *-representations of $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ on $\mathcal{H}_\Gamma$ that are not $\bar{G}_{\breve{S},\Gamma}$-integrable, i.o.w. these representations are not equal to the infinitesimal representation $\mathrm{d}\,U$ of some unitary representation U of $\bar{G}_{\breve{S},\Gamma}$. The author does not know such a representation. Note that, in comparison to Dziendzikowski and Okołów [34] this representation is called the non-standard representation. But for representations of the universal enveloping algebra, which are not $\bar{G}_{\breve{S},\Gamma}$-integrable, the relation to the unitary Weyl elements, which define the Weyl algebra for surfaces, is not clear.

8.3 Tensor products of the holonomy-flux cross-product *-algebra

The structure of the holonomy-flux cross-product *-algebra is slightly modificated in the following way.

Definition 8.3.1. *The* ***modified holonomy-flux cross-product*** ****-algebra restricted to a graph*** Γ ***and a surface set*** $\breve{S}$ *is given by*

$$C(\bar{G}_{\breve{S},\Gamma}) \otimes \Big(C^\infty(\bar{\mathcal{A}}_\Gamma) \rtimes \bar{\mathcal{E}}_{\breve{S},\Gamma}\Big)$$

where the tensor product $\otimes$ *is the minimal tensor product of* C^**-algebras.*

The ***modified holonomy-flux cross-product** ***-algebra associated a surface set** $\breve{S}$ *is equivalent to the inductive limit of the family*

$$\left\{\left(C(\bar{G}_{\breve{S},\Gamma})\otimes\Big(C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}\Big),\mathring{\beta}_{\Gamma,\Gamma'}\times\beta_{\Gamma,\Gamma'}\right):\mathcal{P}_\Gamma\leq\mathcal{P}_{\Gamma'}\right\}$$

Then for the state $\breve{\omega}^\Gamma_M$ on $C(\bar{G}_{\breve{S},\Gamma})\otimes\Big(C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}\Big)$ it is true that

$$\begin{aligned}&\breve{\omega}^\Gamma_M\left(f(\rho_{S,\Gamma}(\Gamma))\delta_{S,\Gamma}(f_\Gamma)\right)\\&=\int_{\bar{G}_{\breve{S},\Gamma}}\mathrm{d}\,\mu_{\breve{S},\Gamma}(\rho_{S,\Gamma}(\Gamma))f(\rho_{S,\Gamma}(\Gamma))\delta(\rho_{S,\Gamma}(\Gamma),\exp(E_S(\Gamma)))\langle\Omega^\Gamma_M,\delta_{S,\Gamma}(f_\Gamma)\Omega^\Gamma_M\rangle\\&=f(\exp(E_S(\Gamma))\omega^\Gamma_M\left(\delta_{S,\Gamma}(f_\Gamma)\right)\end{aligned}\tag{8.50}$$

where $\delta(g_1,g_2)$ is the delta function on $\bar{G}_{\breve{S},\Gamma}$.

Definition 8.3.2. *The* ***modified intersection-holonomy-flux cross-product** ***-algebra restricted to a graph** Γ **and a surface set** $\breve{S}$ *is given by*

$$C(V^S_\Gamma)\otimes\Big(C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}\Big)$$

where $V^S_\Gamma=V_\Gamma\cap S$ *and the tensor product* $\otimes$ *is the minimal tensor product of* C^**-algebras.*

Then for the state $\hat{\omega}^\Gamma_M$ on $C(V^S_\Gamma)\otimes\Big(C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{\mathcal{E}}_{\breve{S},\Gamma}\Big)$ it is true that

$$\hat{\omega}^\Gamma_M\left(f(v_1,...,v_M)\delta_{S,\Gamma}(f_\Gamma)\right)=f(v_1,...,v_M)\omega^\Gamma_M\left(\delta_{S,\Gamma}(f_\Gamma)\right)\tag{8.51}$$

where $v_1,...,v_M\in V^S_\Gamma$.

Clearly these states are not surface-orientation-preserving graph-diffeomorphism invariant, but the states are surface preserving graph-diffeomorphism invariant.

8.4 The localised holonomy-flux cross-product *-algebra

In section 6.5 the issue of KMS-states in LQG framework has been studied. Since KMS-states are not available for the analytic holonomy C^*-algebra and the Weyl C^*-algebra for surfaces, other C^*-algebras are interesting. A suitable C^*-algebra is the non-commutative holonomy C^*-algebra presented in section 7.1. Furthermore the structure of fluxes and bisections indicates that, there is a semi-localised structure. The following construction is also available for non-localised objects and can be easily extracted from the development. The flux operators are always localised on surfaces. Remark that, if matter fields come into the play, they will be localised somewhere, too. The matter hamiltonian may be constructed from fluxes localised on surfaces and holonomies. The bisections are maps from a certain set of vertices in the manifold Σ to paths that start at a given vertex in the set of vertices. Consequently also bisections are somehow localised objects in a manifold. Hence a discretised surface set $\breve{S}_\mathrm{d}$, which contains only fixed sets of vertices, is used. The idea is to construct the localised holonomy-flux cross-product *-algebra by two components. One part, which does not interplay with fluxes localised on a given finite set of surfaces, is given by the analytic holonomy C^*-algebra on a localised configuration space and the second part is given by an infinite tensor product of a certain new holonomy-flux cross-product *-algebra. Then states on the localised holonomy-flux cross-product *-algebra, which are invariant under restricted diffeomorphisms, are found. In particular, there are KMS-states, which are $\mathfrak{B}_{\breve{S}_\mathrm{d},\mathrm{surf}}(\mathcal{P}_\Gamma)$-invariant, of a particular C^*-algebra. This C^*-algebra is a completetion of a *-subalgebra of the localised holonomy-flux cross-product *-algebra. Finally the modified quantum Hamilton constraint operator is implemented as a generator of an automorphism group on the localised holonomy-flux cross-product *-algebra. The aim is to find a *-algebra such that the modified quantum Hamilton constraint is contained in the algebra.

8.4.1 The localised holonomy *-algebra

The construction of the localised holonomy C^*-algebra

The construction of the new algebra of quantum configuration variables combine a lot of the structures presented before.

Assume that G is a compact connected Lie group, Γ be a graph, $\breve{S}$ a surface set and $\breve{S}_{\rm d}$ a discretised surface set (associated to $\breve{S}$). Recall the configuration spaces $\bar{\mathcal{A}}_\Gamma^{\rm d}$, $\bar{\mathcal{A}}_{\rm d}^\Gamma$ and $\bar{\mathcal{A}}_{\bar{\Gamma}}$, which have been defined in section 3.3.4.1. The convolution holonomy *-algebra associated to Γ is denoted by $\mathcal{C}(\bar{\mathcal{A}}_\Gamma^{\rm d})$ (resp. $\mathcal{C}(\bar{\mathcal{A}}_{\rm d}^\Gamma)$). This algebra is completed with respect to an appropriate norm to a C^*-algebra, which is called the *non-commutative holonomy C^*-algebra* $C^*(\bar{\mathcal{A}}_\Gamma^{\rm d})$ (resp. $C^*(\bar{\mathcal{A}}_{\rm d}^\Gamma)$) associated to a graph Γ. Moroever the C^*-algebra $C^*(\bar{\mathcal{A}}_\Gamma^{\rm d})$ (resp. $C^*(\bar{\mathcal{A}}_{\rm d}^\Gamma)$) is isomorphic to a infinite matrix C^*-algebra $M_\Gamma(\mathbb{C})$ (refer to proposition 7.1.8). The *analytic holonomy C^*-algebra* associated to the graph $\bar{\Gamma}$ is denoted by $C(\bar{\mathcal{A}}_{\bar{\Gamma}})$. Note that the graph $\bar{\Gamma}$ is defined such that there are no intersections with elements of $\breve{S}_{\rm d}$. Now new C^*-algebras are constructed from C^*-tensor product algebras.

Definition 8.4.1. *Let Γ be a graph, $\breve{S}$ a surface set and $\breve{S}_{\rm d}$ a discretised surface set (associated to $\breve{S}$). Then denote the subgraph of Γ such that, this graph contains all edges of the graph Γ that do not intersect with any vertex of the discretised surface set $\breve{S}_{\rm d}$, by $\bar{\Gamma}$.*

Define

$$C^*(\bar{\mathcal{A}}_{{\rm d},\Gamma}) := C^*(\bar{\mathcal{A}}_\Gamma^{\rm d}) \otimes C^*(\bar{\mathcal{A}}_{\rm d}^\Gamma) \text{ where}$$
$$C^*(\bar{\mathcal{A}}_\Gamma^{\rm d}) := \bigotimes_{i\in I} \bigotimes_{k=1,\ldots,N_k^i} C^*(\bar{\mathcal{A}}_{{\rm d},\gamma_{i,1}\circ\ldots\circ\gamma_{i,k}})$$

The ***localised holonomy C^*-algebra associated to a graph and a discretised surface set*** *is given by the tensor product $C^*(\bar{\mathcal{A}}_{{\rm d},\Gamma}) \otimes C(\bar{\mathcal{A}}_{\bar{\Gamma}})$ (with respect to the minimal C^*-norm).*

In the construction of localised *-algebras only certain graphs are studied. These graphs are assumed to decompose into two sets of graphs: one set contains disconnected graphs that contains only paths such that either the source or target vertex is an element of each surface $S_{\rm d}$ in $\breve{S}_{\rm d}$, and the other set of disconnected graphs contains graphs $\bar{\Gamma}_i$ that contains paths, which does not intersect any point of each discretised surface set $S_{\rm d}$ in $\breve{S}_{\rm d}$. Hence this property generalises to set of graphs. In particular such a decomposition exists for an inductive family of graphs.

Definition 8.4.2. *Let $\{\Gamma_i\}$ be an inductive family of graphs, which contain only paths such that either the source or target vertex is an element of each surface $S_{\rm d}$ in $\breve{S}_{\rm d}$. Moroever let $\{\bar{\Gamma}_i\}$ be inductive family $\{\bar{\Gamma}_i\}$ of graphs that contains no paths, which start or end in a vertex contained in any set of the discretised surface set $\breve{S}_{\rm d}$.*

There is a increasing family of matrix algebras $\{C^*(\bar{\mathcal{A}}_{\Gamma_i}^{\rm d}), \beta^{\rm d}_{\Gamma_i,\Gamma_{i+1}}\}_{i=1,..,\infty}$ with $\beta^{\rm d}_{\Gamma_i,\Gamma_{i+1}}$ unit-preserving *-homomorphisms such that the union of all matrix algebras is a normed *-algebra, which is completed by the minimal tensor product norm to a C^*-algebra

$$C^*(\bar{\mathcal{A}}^{\rm d}) := \bigcup_{m=1,\ldots,\infty} C^*(\bar{\mathcal{A}}_{\Gamma_m}^{\rm d})$$

There is a increasing family of matrix algebras $\{C^*(\bar{\mathcal{A}}_{{\rm d},\Gamma_i}), \beta_{{\rm d},\Gamma_i,\Gamma_{i+1}}\}_{i=1,..,\infty}$ with $\beta_{{\rm d},\Gamma_i,\Gamma_{i+1}}$ unit-preserving *-homomorphisms such that the union of all matrix algebras is a normed *-algebra, which is completed by the minimal tensor product norm to a C^*-algebra

$$C^*(\bar{\mathcal{A}}_{\rm d}^{\rm d}) := \bigcup_{m=1,\ldots,\infty} C^*(\bar{\mathcal{A}}_{{\rm d},\Gamma_m})$$

Furthermore there is an inductive limit C^*-algebra $C(\bar{\mathcal{A}}_{\rm loc})$, which is constructed from an inductive family $\{C(\bar{\mathcal{A}}_{\bar{\Gamma}_i}), \beta_{\bar{\Gamma}_i,\bar{\Gamma}_{i+1}})\}_{i=1,\ldots,\infty}$ of C^*-algebras.

In particular, an element of $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma})$ is for example given by

$$f^1_\Gamma(\mathfrak{h}_\Gamma(\gamma_{1,1}),...,\mathfrak{h}_\Gamma(\gamma_{N,1}))\otimes f^2_\Gamma(\mathfrak{h}_\Gamma(\gamma_{1,1}\circ\gamma_{1,2}),...,\mathfrak{h}_\Gamma(\gamma_{N,1}\circ\gamma_{N,2}))$$

Notice that, for $\Gamma_i\cap\Gamma_j=\varnothing$ for $i\neq j$ then $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_i\cup\Gamma_j})=C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_i})\otimes C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_j})$, $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_i})$ is isomorphic to the C^*-subalgebra $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_i})\otimes\mathbb{1}_{\Gamma_j}$ of $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_i})\otimes C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_j})$ where $\mathbb{1}_{\Gamma_j}$ is the identity operator in $C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma_j})$.

Definition 8.4.3. *The* ***localised holonomy*** C^****-algebra*** *is the C^*-tensor product algebra $C(\bar{\mathcal{A}}_{\mathrm{loc}})\otimes C^*(\bar{\mathcal{A}}^{\mathrm{d}}_{\mathrm{d}})$ (with respect to the minimal C^*-norm) associated to a discretised set of surfaces.*

In this definition the notion of localised is emphasized, since the elements of this new algebra really depend on a chosen discretised surface set associated to a surface set.

Actions of the group of bisections on the localised holonomy C^*-algebra associated to a graph and a discretised surface set

In this paragraph graph changing operations are studied. First observe that there are some certain bisections, which map target vertices of certain paths to suitable paths. The set of these bisections in a finite graph system $\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}$ has been introduced at the end of section 3.3.4.4 and is denoted by $\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$. Only these bisections restricted to a set $V^{\breve{S}_{\mathrm{d}}}$ are used to define an action of bisections on the localised analytic holonomy C^*-algebra associated to a graph and a discretised surface set. The action is for example given by

$$(\zeta_\sigma f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma'))=f_\Gamma(\mathfrak{h}_\Gamma({\Gamma'}_\sigma))\text{ for }\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$$

yields, whenever

- $f_\Gamma\in C^*(\bar{\mathcal{A}}_{\mathrm{d},\Gamma})$,
- $\Gamma':=\{{\gamma'}_i\},{\Gamma'}_\sigma:=\{{\gamma'}_i\circ\sigma(t({\gamma'}_i))\}$ are subgraphs of Γ.

Lemma 8.4.4. *There is an action α of the group $\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$ of bisections on the C^*-algebra $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$, which is defined by*

$$\zeta_\sigma(f_\Gamma):=f_\Gamma\circ R_\sigma$$

whenever $f_\Gamma\in C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$.

Proof : Let $\sigma\in\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$ then $\sigma\mapsto\zeta_\sigma$ is a group homomorphism and

$$(\zeta_{\sigma_1}\circ\zeta_{\sigma_2})(f_\Gamma)=\zeta_{\sigma_1*_2\sigma_2}(f_\Gamma)$$
$$\zeta_\sigma(f^*_\Gamma)=\zeta_\sigma(f_\Gamma)^*$$

for all $\sigma,\sigma_1,\sigma_2\in\mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$ and $f_\Gamma\in C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$.

■

Now focus paths, which do not have any intersection with a discretised surface in $\breve{S}_{\mathrm{d}}$. Then there is an action of $\mathrm{Diff}(\mathcal{P}_{\bar\Gamma})$ and hence $\mathfrak{B}(\mathcal{P}_{\bar\Gamma})$ on $C(\bar{\mathcal{A}}_{\bar\Gamma})$. This action is a point-norm continuous automorphic action of $\mathrm{Diff}(\mathcal{P}_{\bar\Gamma})$ on $C(\bar{\mathcal{A}}_{\bar\Gamma})$ for every graph $\bar\Gamma$ of the inductive family $\{\bar\Gamma_i\}$ of graphs.

Derivations defined by the discretised and localised flux operator for surfaces and graphs

In section 3.4 the discrete and localised flux operator $\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{i+1})^+\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{i+1})$ has been introduced in definition 3.4.14. The definition of this operator is chosen such that this operator acts non-trivial on elements of $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_{i+1}})$ and commute with all elements of $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$.

Definition 8.4.5. *Define the derivation $\tilde{\delta}_{S_{\rm d},\Gamma_j}$ on $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_j})$ with domain $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_{i+1}})$ by the following commutator*

$$\tilde{\delta}_{S_{\rm d},\Gamma_{i+1}}(f_{\Gamma_{i+1}}) \quad := [\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1}), f_{\Gamma_{i+1}}]$$

for a fixed $\tilde{E}_{S_{\rm d}}(\Gamma_{i+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{i+1}) \in \bar{\mathfrak{g}}^{\rm loc}_{\breve{S}_{\rm d},\Gamma_i}$ and $f_{\Gamma_{i+1}} \in \mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_{i+1}})$.

The domain $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_{i+1}})$ is a *-subalgebra of $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_j})$.

Lemma 8.4.6. *The linear operator $\tilde{\delta}_{S_{\rm d},\Gamma_i}$ is a symmetric unbounded *-derivation with the domain $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_i})$ of the unital C^*-algebra $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_i})$. The domain $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_i})$ is a dense *-subalgebra of $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_i})$.*

Proof : To show that, the domain $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_i})$ is a dense *-subalgebra of $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_i})$ recognize that, $\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_i}) := C^\infty(\bar{\mathcal{A}}^{\rm d}_{\Gamma_i})$ is indeed dense in $C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_i})$.

■

Corollary 8.4.7. *The limit*

$$\tilde{\delta}_{S_{\rm d}}(f) := i\lim_{j\to\infty}[\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}), f]$$

for every $f \in \mathcal{D}(\tilde{\delta}_{S_{\rm d}})$ and an element $\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}) \in \bar{\mathfrak{g}}^{\rm loc}_{\breve{S}_{\rm d},\Gamma_{j+1}}$ for every j, is well-defined in the norm topology. The domain is given by

$$\mathcal{D}(\tilde{\delta}_{S_{\rm d}}) = \bigcup_{j=1,\ldots,\infty}\mathcal{D}(\tilde{\delta}_{S_{\rm d},\Gamma_j})$$

Proof : Note that,

$$[\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}), f_{\Gamma_k}] = 0$$

yields whenever $\mathcal{P}_{\Gamma_k} \leq \mathcal{P}_{\Gamma_{j+1}}$ and $0 \leq k \leq j$ and $f_{\Gamma_k} \in C^*(\bar{\mathcal{A}}^{\rm d}_{\Gamma_k})$. Consequently derive

$$\begin{aligned}\tilde{\delta}_{S_{\rm d}}(f) &:= i\lim_{j\to\infty}[\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}), f]\\ &= i\lim_{j\to\infty}[\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}), f_{\Gamma_0}\otimes\ldots\otimes f_{\Gamma_j}\otimes f_{\Gamma_{j+1}}]\\ &= i\lim_{j\to\infty}[\tilde{E}_{S_{\rm d}}(\Gamma_{j+1})^+\tilde{E}_{S_{\rm d}}(\Gamma_{j+1}), f_{\Gamma_{j+1}\setminus\Gamma_j}]\\ &= 0\end{aligned}$$

■

Redefine the symmetric unbounded *-derivation for the discretised flux operator $E_{S_{\rm d}}(\Gamma_i)$ for a graph Γ_i, which is given by

$$\delta_{S_{\rm d},\Gamma_j}(f) = [E_{S_{\rm d}}(\Gamma_j)^+E_{S_{\rm d}}(\Gamma_j), f] \tag{8.52}$$

whenever $f \in \mathcal{D}(\delta_{S_{\rm d}})$ and for a fixed $E_{S_{\rm d}} \in \mathfrak{g}_{\breve{S}_{\rm d},\Gamma_j}$.

In contrast to the property of the *-derivation of the C^*-algebra $C(\bar{\mathcal{A}})$ presented in proposition 8.2.19, the *-derivation of $C^*(\bar{\mathcal{A}}^{\rm d})$ exists under weaker conditions for the surface set and the directed family of graphs. In proposition 8.2.19 the set $\breve{S}$ of surfaces has to be chosen such that, for each graph of the inductive family of graphs $\{\Gamma_i\}$ there is only a finite number of intersection vertices with each surface of the set $\breve{S}$.

Proposition 8.4.8. *Let $\breve{S}_{\rm d}$ be an arbitrary discretised surface set and $\{\Gamma_i\}_{i=1,\ldots,\infty}$ be an inductive family of graphs. Then the limit*

$$\delta_{S_{\rm d}}(f) := i\lim_{j\to\infty}\delta_{S_{\rm d},\Gamma_{j+1}}(f) \tag{8.53}$$

whenever $f \in \mathcal{D}(\delta_{S_{\rm d}})$ exists in norm. The domain of the limit is given by

$$\mathcal{D}(\delta_{S_{\rm d}}) = \bigcup_{j=1,\ldots,\infty}\mathcal{D}(\delta_{S_{\rm d},\Gamma_j})$$

and $\mathcal{D}(\delta_{S_{\rm d}})$ is a *-subalgebra of $C^*(\bar{\mathcal{A}}^{\rm d})$.

Proof : Derive

$$\begin{aligned}\delta_{S_{\mathrm{d}}}(f) &:= i\lim_{j\to\infty}\delta_{S_{\mathrm{d}},\Gamma_{j+1}}(f) = i\lim_{j\to\infty}\left(\tilde{\delta}_{S_{\mathrm{d}},\Gamma_{j+1}}(f)+\delta_{S,\Gamma_j}(f)\right)\\ &= i\lim_{j\to\infty}[\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{j+1})^+\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{j+1}),f] + i\lim_{j\to\infty}[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),f]\\ &= i\lim_{j\to\infty}[\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{j+1})^+\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{j+1}),f] + ... + i\lim_{j\to\infty}[\tilde{E}_{S_{\mathrm{d}}}(\Gamma_1)^+\tilde{E}_{S_{\mathrm{d}}}(\Gamma_1),f]\\ &\qquad + i\lim_{j\to\infty}[E_{S_{\mathrm{d}}}(\Gamma_0)^+E_{S_{\mathrm{d}}}(\Gamma_0),f]\end{aligned} \tag{8.54}$$

by using corollary 8.4.7.

■

In the following considerations the *-algebra $C^*(\bar{\mathcal{A}}^{\mathrm{d}})$ has to be restricted to functions in $C^\infty(\bar{\mathcal{A}}^{\mathrm{d}})$. The resulting *-subalgebra is denoted by $\mathbf{C}^*(\bar{\mathcal{A}}^{\mathrm{d}})$ and is called the localised analytic holonomy *-algebra again.

Recall the concept of abstract cross-product algebras, which has been presented by Schmüdgen and Klimyk [53]. This concept has been used in definition 8.2.10 for the definition of the holonomy-flux cross-product *-algebra associated to a surface set. In analogy a similar cross-product *-algebra is defined as follows.

Definition 8.4.9. *Let $\{\Gamma_i\}$ be an inductive family of graphs with inductive limit Γ_∞, $\breve{S}$ be a set of surfaces and $\breve{S}_{\mathrm{d}}$ a set of discretised surfaces associated to $\breve{S}$ such that the assumptions in definition 3.4.14 are satisfied.*

The ***general localised part of the localised holonomy-flux cross-product *-algebra*** $\mathbf{C}^*(\bar{\mathcal{A}}^{\mathrm{d}})\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}}}$ ***associated to a discretised surface set*** $\breve{S}_{\mathrm{d}}$ *is the cross-product *-algebra, which is defined by the localised analytic holonomy *-algebra $\mathbf{C}^*(\bar{\mathcal{A}}^{\mathrm{d}})$ and the localised enveloping flux algebra $\mathcal{E}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}}}$ associated a discretised surface set.*

8.4.2 A representation of the general localised part of the localised holonomy-flux cross-product *-algebra

In section 8.2.3 a certain *-representation of a Lie algebra has been studied. This *-representation is called the infinitesimal representation of a Lie algebra on a Hilbert space. Similarly the representation of the general localised part of the localised holonomy-flux cross-product *-algebra is presented as follows.

First the *-representation of the Lie flux algebra $\bar{\mathfrak{g}}_{\breve{S}_{\mathrm{d}},\Gamma}$ is implemented by the infinitesimal representation $\mathrm{d}\,u$ on the Hilbert space $\mathcal{H}^{\mathrm{d}}_\Gamma$, which is given by $L^2(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma,\mu^{\mathrm{d}}_\Gamma)$. Notice that, the configuration space $\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma$ is equivalent to G^m for a suitable $m\in\mathbb{N}$ and μ^{d}_Γ is the corresponding Haar measure on G^m. The infinitesimal representation corresponds to the unitary representation u of the Lie flux group $\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma}$ in the C^*-algebra $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$, i.o.w. $u\in\mathrm{Rep}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma},C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma))$. The *-representation of the general localised part of the localised holonomy-flux cross-product *-algebra is derived from this *-representation.

For generality observe the following. For a given unitary representation of a group there exists an infinitesimal representation of the associated Lie algebra. The unitary representation $\tilde{u}$ is a map from $\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma}$ into the multiplier algebra $M(\mathcal{K}(\mathcal{H}^{\mathrm{d}}_\Gamma))$, i.o.w. $\tilde{u}\in\mathrm{Rep}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma},\mathcal{K}(\mathcal{H}^{\mathrm{d}}_\Gamma))$. Notice that, this is similarly to the unitary representation U, which is contained in $\mathrm{Rep}(\bar{G}_{\breve{S},\Gamma},\mathcal{K}(\mathcal{H}_\Gamma))$ defined in section 6.1. The unitary representation $\tilde{u}$ is related to the infinitesimal representation $\mathrm{d}\,\tilde{u}$. For another unitary continuous representation u of $\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma}$ in the C^*-algebra $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$, Woronowicz and Napiórkowski have shown in [115] that, there exists a unique morphism Φ between the C^*-algebras $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$ and $\mathcal{K}(\mathcal{H}^{\mathrm{d}}_\Gamma)$, where $\mathcal{H}^{\mathrm{d}}_\Gamma := L^2(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma,\mu^{\mathrm{d}}_\Gamma)$, and such that $\tilde{u}(g)=\Phi(u(g))$ holds. Consequently $u\in\mathrm{Rep}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma},C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma))$.

The next definition will give *-representations of the following algebras:

· the *-algebra $\mathbf{C}^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$,

· the localised enveloping flux algebra $\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ and

· the general localised part of the localised holonomy-flux cross-product *-algebra $\mathbf{C}^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$

on the Hilbert space $\mathcal{H}_\Gamma^{\mathrm{d}}$.

Definition 8.4.10. *The *-representation of $\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ is defined by*

$$\begin{aligned}&\Phi_M(f_\Gamma)\psi_\Gamma = f_\Gamma\psi_\Gamma \text{ for } f_\Gamma\in\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}) \text{ and } \psi_\Gamma\in\mathcal{H}_\Gamma^{\mathrm{d}}\\&\Phi_M(f_\Gamma^*)\psi_\Gamma = f_\Gamma^*\psi_\Gamma \text{ for } f_\Gamma\in\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}) \text{ and } \psi_\Gamma\in\mathcal{H}_\Gamma^{\mathrm{d}}\end{aligned}$$

Then there exists a positive self-adjoint operator $\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma))$ or respectively the adjoint operator $\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)E_{S_{\mathrm{d}}}(\Gamma)^+)$ on the Hilbert space $\mathcal{H}_\Gamma^{\mathrm{d}}$ defined by

$$\begin{aligned}&\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma))\psi_\Gamma := i[E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma),\psi_\Gamma]\\&\qquad \text{for a fixed } E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)\in\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma} \text{ and } \psi_\Gamma\in\mathcal{D}(\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)))\\&\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)E_{S_{\mathrm{d}}}(\Gamma)^+)\psi_\Gamma := -i[E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma),\psi_\Gamma]\\&\qquad \text{for a fixed } E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)\in\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma} \text{ and } \psi_\Gamma\in\mathcal{D}(\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)))\end{aligned}$$

and $u\in\mathrm{Rep}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma},\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}))$.

The *-representation of the *-algebra $\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ on $\mathcal{H}_\Gamma^{\mathrm{d}}$ is defined by

$$\begin{aligned}&\hat{\pi}_\Gamma(f_\Gamma\otimes iE_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j))\psi_\Gamma := \frac{1}{2}i[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),f_\Gamma]\psi_\Gamma + \frac{1}{2}if_\Gamma[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),\psi_\Gamma]\\&\qquad \text{for } f_\Gamma\in\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}) \text{ and for a fixed } E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)\in\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}\\&\hat{\pi}_\Gamma(f_\Gamma\otimes iE_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j))\psi_\Gamma := \frac{1}{2}i[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),f_\Gamma]\psi_\Gamma + \frac{1}{2}if_\Gamma[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),\psi_\Gamma]\\&\qquad \text{for } f_\Gamma\in\mathbf{C}^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}) \text{ and for a fixed } E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)\in\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}\end{aligned}$$

whenever $\psi_\Gamma\in\mathcal{D}(\mathrm{d}\,u(E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j)))$.

8.4.3 C^*-dynamical systems, KMS-states and the localised holonomy-flux cross-product *-algebra

In this section different C^*-dynamical systems are constructed from different actions and algebras. The aim is to implement a strongly continuous one-parameter automorphism group such that a modified quantum Hamilton constraint is the generator of this automorphism group. This will be done in several steps. In this section the basic C^*-dynamical systems are introduced, which are used in the section 8.4.4 for the analysis of the modified quantum Hamilton constraint.

First notice the following result. In general, for every C^*-algebra $\mathfrak{A}$ and a point norm-continuous automorphic action β of $\mathbb{R}$ on $\mathfrak{A}$, there is a set $\mathfrak{A}^\beta$ defined by all element $A\in\mathfrak{A}$ such that $\beta_t(A)=A$ for every $t\in\mathbb{R}$. Then the set $\mathfrak{A}^\beta$ is a norm-dense *-subalgebra of $\mathfrak{A}$.

Set $\mathfrak{h}_\Gamma(\Gamma) =: \mathfrak{h}_\Gamma\in\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$. Let $\bar{a}_{\mathrm{d},\Gamma}$ be the enveloping Lie algebra of the Lie algebra associated to $\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$. Note that, $\mathfrak{a}_{\mathrm{d},\Gamma}^+ = -\mathfrak{a}_{\mathrm{d},\Gamma}$ holds for all elements of the Lie algebra of $\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$. Consider the C^*-subalgebra $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ of $C^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$, which is generated by all central functions, i.e. all functions $f_\Gamma\in C^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ such that $f_\Gamma(\mathfrak{h}_\Gamma) = f_\Gamma(\mathfrak{g}_\Gamma^{-1}\mathfrak{h}_\Gamma\mathfrak{g}_\Gamma)$ for all $\mathfrak{g}_\Gamma\in\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$.

Finally consider an action $\beta_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}$ of $\mathbb{R}$ on $C^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ defined by

$$(\beta_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(t)f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i}) := f_{\Gamma_i}(\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i}\exp(t\mathfrak{a}_{\mathrm{d},\Gamma_i}))$$

and notice that

$$(\beta_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(t)f_{\Gamma_i})^*(\mathfrak{h}_{\Gamma_i}) = \overline{(\beta_{\mathfrak{a}^+_{\mathrm{d},\Gamma_i}}(t)f^*_{\Gamma_i})(\mathfrak{h}^{-1}_{\Gamma_i})}$$

for a fixed Lie algebra element $\mathfrak{a}_{\mathrm{d},\Gamma_i}$ in $\bar{a}_{\mathrm{d},\Gamma}$ yields. Set $C^*(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ be equal to $\mathfrak{A}$. Then $\mathfrak{A}^\beta$ is isomorphic to $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$.

Proposition 8.4.11. *The triple $(C^*(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}}),\mathbb{R},\beta_{\mathfrak{a}_{\mathrm{d}},\Gamma_i})$ is a C^*-dynamical system.*

This is verifed easily, after the following considerations.

Furthermore there is an action $\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}$ of $\mathbb{R}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$ defined by

$$(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i}) := f_{\Gamma_i}(\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})$$

and

$$(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)f_{\Gamma_i})^*(\mathfrak{h}_{\Gamma_i}) := \overline{(\tilde{\alpha}_{\mathfrak{a}^+_{\mathrm{d},\Gamma_i}}(t)f_{\Gamma_i})(\mathfrak{h}^{-1}_{\Gamma_i})}$$

whenever $f_{\Gamma_i} \in \mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$ and $\mathfrak{a}_{\mathrm{d},\Gamma_i} \in \bar{a}_{\mathrm{d},\Gamma}$.

Proposition 8.4.12. *The triple* $(\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}), \mathbb{R}, \tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}})$ *is a* C^**-dynamical system.*

Proof : Derive

$$\begin{aligned}
&(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t_1+t_2)f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i})\\
&= f_{\Gamma_i}(\exp(-(t_1+t_2)\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})\\
&= f_{\Gamma_i}(\exp(-t_1\mathfrak{a}_{\mathrm{d},\Gamma_i})\exp(-t_2\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})\\
&= (\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t_1)\circ\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t_2)f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i})
\end{aligned}$$

and, since $f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}) = f_{\Gamma_i}(\mathfrak{g}^{-1}_{\Gamma_i}\mathfrak{h}_{\Gamma_i}\mathfrak{g}_{\Gamma_i})$ for all $\mathfrak{g}_{\Gamma_i} \in \bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}$ it follows that,

$$\begin{aligned}
(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)f_{\Gamma_i})^*(\mathfrak{h}_{\Gamma_i}) &= \overline{f_{\Gamma_i}((\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})^{-1})}\\
&= \overline{f_{\Gamma_i}(\mathfrak{h}^{-1}_{\Gamma_i}(\Gamma_i)\exp(t\mathfrak{a}_{\mathrm{d},\Gamma_i}))}\\
&= \overline{(\tilde{\alpha}_{\mathfrak{a}^+_{\mathrm{d},\Gamma_i}}(t)f_{\Gamma_i})(\mathfrak{h}^{-1}_{\Gamma_i}(\Gamma_i))}\\
&= (\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)f^*_{\Gamma_i})(\mathfrak{h}_{\Gamma_i})
\end{aligned}$$

yields whenever $t \in \mathbb{R}$ and $f_{\Gamma_i} \in \mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$.

$$\begin{aligned}
&(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)(k_{\Gamma_i} * f_{\Gamma_i}))(\mathfrak{h}_{\Gamma_i}) = (k_{\Gamma_i} * f_{\Gamma_i})(\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})\\
&= \int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i}(\Gamma_i))k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}(\Gamma_i))f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}(\Gamma_i)^{-1}\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})\\
&= \int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i}(\Gamma_i))k_{\Gamma_i}(\exp(-t\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{g}_{\Gamma_i}(\Gamma_i))f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i})\\
&= (\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)(k_{\Gamma_i}) * \tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)(f_{\Gamma_i}))(\mathfrak{h}_{\Gamma_i})
\end{aligned}$$

whenever $t \in \mathbb{R}$ and $k_{\Gamma_i}, f_{\Gamma_i} \in \mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$. Moroever $t \mapsto \tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}(t)(f_{\Gamma_i})$ is point-norm continuous.

■

Clearly the same calculations can be done to verify proposition 8.4.11.

Proposition 8.4.13. *There is a state* $\tilde{\omega}^\Gamma_{\mathcal{L}}$ *on* $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$ *associated to the GNS-triple* $(L^2(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma, \mu_{\breve{S}_{\mathrm{d}},\Gamma}), \Phi_M, \Omega^{\mathrm{d}}_\Gamma)$*, which consists of the* *-*representation* Φ_M *presented in definition 8.4.10, the Hilbert space* $L^2(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma, \mu_{\breve{S}_{\mathrm{d}},\Gamma})$ *and the cyclic vector* $\Omega^{\mathrm{d}}_\Gamma$*. The state is given by*

$$\tilde{\omega}^\Gamma_{\mathcal{L}}(f_\Gamma) := \int_{\bar{\mathcal{A}}_\Gamma} \mathrm{d}\,\mu_{\bar{\mathcal{A}}_\Gamma}(\mathfrak{h}_\Gamma)|f_\Gamma(\mathfrak{h}_\Gamma)|^2$$

whenever $f_\Gamma \in \mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$.

The set $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$ contains only entire analytic elements for $\beta_{\mathfrak{a}_{\mathrm{d},\Gamma_i}}$.

Proposition 8.4.14. *Let Γ_i be a graph, $(C^*(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}),\mathbb{R},\beta_{\mathfrak{a}_{\mathrm{d}},\Gamma_i})$ and $(\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}),\mathbb{R},\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma_i})$ be two C^*-dynamical systems defined above.*

Then the state $\tilde{\omega}^{\Gamma_i}_{\mathcal{L}}$ is a KMS-state at value $\beta\in\mathbb{R}$ on $C^*(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$ or respectively on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$.

Proof : Calculate for $k_{\Gamma_i}, f_{\Gamma_i}\in\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$ and $f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}) = f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}\mathfrak{h}_{\Gamma_i})$ for all $\mathfrak{g}_{\Gamma_i}\in\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}$ it is derived that,

$$\begin{aligned}
&(k_{\Gamma_i}\ast\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(i\beta)(f_{\Gamma_i}))(\mathfrak{h}_{\Gamma_i})\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i}) k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}^{-1}\exp(-i\beta\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{h}_{\Gamma_i})\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i}(\Gamma_i)) k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) f_{\Gamma_i}(\exp(i\beta\mathfrak{a}_{\mathrm{d},\Gamma_i})\mathfrak{g}_{\Gamma_i}^{-1}\mathfrak{h}_{\Gamma_i})\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i}) k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}^{-1}\mathfrak{h}_{\Gamma_i}\exp(-i\beta\mathfrak{a}_{\mathrm{d},\Gamma_i}))
\end{aligned}$$

is true. Then derive

$$\begin{aligned}
&\tilde{\omega}^{\Gamma_i}_{\mathcal{L}}(k_{\Gamma_i}\ast\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(i\beta)(f_{\Gamma_i}))\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{h}_{\Gamma_i}) \vert(k_{\Gamma_i}\ast\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(i\beta)(f_{\Gamma_i}))(\mathfrak{h}_{\Gamma_i})\vert^2\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{h}_{\Gamma_i})\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i})\vert k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}^{-1}\mathfrak{h}_{\Gamma_i}\exp(-i\beta\mathfrak{a}_{\mathrm{d},\Gamma_i}))\vert^2\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{h}_{\Gamma_i})\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i})\vert k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}^{-1}\mathfrak{h}_{\Gamma_i})\vert^2\\
&=\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{g}_{\Gamma_i})\int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i}} \mathrm{d}\,\mu_{\breve{S}_{\mathrm{d}},\Gamma_i}(\mathfrak{h}_{\Gamma_i})\vert f_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}) k_{\Gamma_i}(\mathfrak{g}_{\Gamma_i}^{-1}\mathfrak{h}_{\Gamma_i})\vert^2\\
&=\tilde{\omega}^{\Gamma_i}_{\mathcal{L}}(f_{\Gamma_i}\ast k_{\Gamma_i})
\end{aligned}$$

■

Clearly the state $\tilde{\omega}^{\Gamma}_{\mathcal{L}}$ is $\mathbb{R}$-invariant

$$\tilde{\omega}^{\Gamma}_{\mathcal{L}}(f_\Gamma) = \tilde{\omega}^{\Gamma}_{\mathcal{L}}(\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma_i}(t)(f_\Gamma)) \text{ for } f_\Gamma\in\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}) \text{ and all } t\in\mathbb{R}$$

Recall the *-derivation $\delta_{S_{\mathrm{d}},\Gamma_{i+1}}$ given in definition 8.4.8, then the state $\tilde{\omega}^{\Gamma}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma_i})$ satisfies

$$\begin{aligned}
\tilde{\omega}^{\Gamma_{i+1}}_{\mathcal{L}}(\delta_{S_{\mathrm{d}},\Gamma_{i+1}}(f_{\Gamma_{i+1}})) &= \tilde{\omega}^{\Gamma_{i+1}}_{\mathcal{L}}(\delta_{S_{\mathrm{d}},\Gamma_i}(f_{\Gamma_{i+1}}) + [\tilde{E}^{+}_{S_{\mathrm{d}}}(\Gamma_{i+1})\tilde{E}_{S_{\mathrm{d}}}(\Gamma_{i+1}), f_{\Gamma_{i+1}}])\\
&= \tilde{\omega}^{\Gamma_i}_{\mathcal{L}}(\delta_{S_{\mathrm{d}},\Gamma_i}(f_{\Gamma_i}))
\end{aligned}\tag{8.55}$$

Hence the limit state $\tilde{\omega}_{\mathcal{L}}$ of the states $\tilde{\omega}^{\Gamma_{i+1}}_{\mathcal{L}}$ on the *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}})$ is compatible with the family of *-derivations $\{\delta_{S_{\mathrm{d}},\Gamma_{i+1}}\}$. Recall the *-representations presented in definition 8.4.10.

Corollary 8.4.15. *The state $\tilde{\omega}^{\Gamma}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$ extends to a state $\hat{\omega}^{\Gamma}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$.*

Equivalently, the *-representation Φ_M on $\mathcal{H}^{\mathrm{d}}_{\Gamma}$ with cyclic vector $\Omega^{\mathrm{d}}_{\Gamma}$ constructed from $\tilde{\omega}^{\Gamma}_{\mathcal{L}}$ of $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$ extends to a *-representation $\hat{\pi}^{\mathrm{d}}_{\Gamma}$ on $\mathcal{H}^{\mathrm{d}}_{\Gamma}$ with cyclic vector $\hat{\Omega}^{\mathrm{d}}_{\Gamma}$ of $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$.

Proof : Notice that, it is true that $[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j),\Omega^{\mathrm{d}}_{\Gamma}]=0$ and, hence, derive

$$\begin{aligned}
\hat{\omega}^{\Gamma}_{\mathcal{L}}(f_\Gamma\otimes iE_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma)) &= \tilde{\omega}^{\Gamma}_{\mathcal{L}}\Big(\frac{1}{2}\delta_{S_{\mathrm{d}},\Gamma}(f_\Gamma)\Big) + \langle\Omega^{\mathrm{d}}_{\Gamma},\frac{1}{2}i[E_{S_{\mathrm{d}}}(\Gamma_j)^+E_{S_{\mathrm{d}}}(\Gamma_j)\Omega^{\mathrm{d}}_{\Gamma}]\rangle\\
&= \tilde{\omega}^{\Gamma}_{\mathcal{L}}\Big(\frac{1}{2}\delta_{S_{\mathrm{d}},\Gamma}(f_\Gamma)\Big)
\end{aligned}$$

whenever $E_{S_{\mathrm{d}}}\in\mathfrak{g}_{\breve{S}_{\mathrm{d}},\Gamma}$.

■

Notice that, the state $\tilde{\omega}^{\Gamma}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$ is not unique. Since for $K_\Gamma \in L^1(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}, \mu_{\mathrm{d},\Gamma})$ there is another state defined by

$$\tilde{\omega}^{\Gamma}_{\mathcal{L},K}(f_\Gamma) := \int_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}} K_\Gamma(\mathfrak{h}_\Gamma)\,\mathrm{d}\,\mu_{\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}}(\mathfrak{h}_\Gamma)|f_\Gamma(\mathfrak{h}_\Gamma)|^2$$

whenever $f_\Gamma \in \mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$.

There exists a limit state $\hat{\omega}_{\mathcal{L}}$ of the states $\{\hat{\omega}^{\Gamma_{i+1}}_{\mathcal{L}}\}$ on the *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}}}$.

Recall that, there is a group action of $\mathfrak{B}(\mathcal{P}^{\breve{S}_{\mathrm{d}}}_{\Gamma})$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}})$. This action is also action on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}}}$, since $E^+_{S_{\mathrm{d}}}(\Gamma_\sigma)E_{S_{\mathrm{d}}}(\Gamma_\sigma) = E^+_{S_{\mathrm{d}}}(\Gamma)E_{S_{\mathrm{d}}}(\Gamma)$ is true and, hence,

$$\zeta_\sigma(f_\Gamma \otimes iE^+_{S_{\mathrm{d}}}(\Gamma)E_{S_{\mathrm{d}}}(\Gamma)) = \zeta_\sigma(f_\Gamma) \otimes iE^+_{S_{\mathrm{d}}}(\Gamma)E_{S_{\mathrm{d}}}(\Gamma))$$

whenever $\sigma \in \mathfrak{B}(\mathcal{P}^{\breve{S}_{\mathrm{d}}}_{\Gamma})$ holds.

Definition 8.4.16. *Denote the center of the Lie flux group* $\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma}$ *by* $\hat{\mathcal{Z}}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma})$ *and the Lie flux algebra associated to* $\hat{\mathcal{Z}}(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma})$ *by* $\mathfrak{z}_{\breve{S}_{\mathrm{d}},\Gamma}$. *Finally the enveloping algebra of the Lie flux algebra* $\mathfrak{z}_{\breve{S}_{\mathrm{d}},\Gamma}$ *is denoted by* $\mathfrak{E}_{\breve{S}_{\mathrm{d}},\Gamma}$

Recall that, the space $\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma}$ is identified with $G^{|\Gamma|}$. The state $\hat{\omega}^{\Gamma}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$ is already $\mathrm{Diff}(\mathcal{P}^{\breve{S}_{\mathrm{d}}}_{\Gamma})$-invariant.

Definition 8.4.17. *Let* $\{\Gamma_i\}$ *be an inductive family of graphs with inductive limit* Γ_∞, $\breve{S}$ *be a set of surfaces and* $\breve{S}_{\mathrm{d}}$ *a set of discretised surfaces associated to* $\breve{S}$ *such that the assumptions in definition 3.4.14 are satisfied.*

Then the ***localised holonomy-flux cross-product *-algebra associated to a discretised surface set*** $\breve{S}_{\mathrm{d}}$ *is given by the following tensor product*

$$C(\bar{\mathcal{A}}_{\mathrm{loc}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$$

*The cross-product *-algebra* $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$ *is called the* ***localised part of the localised holonomy-flux cross-product *-algebra*** $C(\bar{\mathcal{A}}_{\mathrm{loc}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$.

Note that, the localised holonomy-flux cross-product *-algebra associated to a discretised surface set $\breve{S}_{\mathrm{d}}$ is abbreviated by the term localised holonomy-flux cross-product *-algebra for surfaces.

Theorem 8.4.18. *Let* $\{\Gamma_i\}$ *be an inductive family of graphs with inductive limit* Γ_∞, $\breve{S}$ *be a set of surfaces and* $\breve{S}_{\mathrm{d}}$ *a set of discretised surfaces associated to* $\breve{S}$ *such that the assumptions in definition 3.4.14 are satisfied.*

Then there exists a $\mathrm{Diff}(\mathcal{P}^{\breve{S}_{\mathrm{d}}}_{\Gamma})$*- and* $\mathrm{Diff}(\mathcal{P}_{\bar{\Gamma}})$*-invariant state on* $C(\bar{\mathcal{A}}_{\mathrm{loc}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$, *which is a product state of the state* ω_M *of* $C(\bar{\mathcal{A}}_{\mathrm{loc}})$ *and the state* $\hat{\omega}_{\mathcal{L}}$ *of* $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$. *The state* $\hat{\omega}^{\Gamma}_{\mathcal{L}}$ *is a KMS-state on* $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_{\Gamma})$ *at inverse temperature* $\beta \in \mathbb{R}$ *w.r.t. the automorphism* $\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d}},\Gamma}$.

Summarising one can conclude that, a modified holonomy-flux algebra is constructed, if the assumptions of diffeomorphism invariance of the state space of the modified algebra is relaxed to a surface-preserving graph-diffeomorphism invariance for a finite set $\breve{S}$ of surfaces and an arbitrary fixed graph Γ.

Finally if different surface sets are considered, the following is true. There is a family $\{C(\bar{\mathcal{A}}_{\bar{\Gamma}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}_{\Gamma}) \rtimes_L \mathfrak{E}_{\breve{S}_{\mathrm{d}},\Gamma}\}_\Gamma$ of localised holonomy-flux cross-product *-algebras associated to graphs and a surface set $\breve{S}$. Consider a subset $\breve{S}^1$ of $\breve{S}$ and $\breve{S}^1_{\mathrm{d}}$ of $\breve{S}_{\mathrm{d}}$ such that the assumptions in definition 3.4.14 are satisfied. Then for every surface S_1 in $\breve{S}^1$ there is a surface S in $\breve{S}$ with $S_1 \subset S$ and $S^{\mathrm{d}}_1 \subset S^{\mathrm{d}}$. Then it is true that, the algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}^1_{\mathrm{d}}}_{\Gamma}) \rtimes \mathfrak{E}_{\breve{S}^1_{\mathrm{d}},\Gamma}$ is a subalgebra of $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}^2_{\mathrm{d}}}_{\Gamma}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}},\Gamma}$. But this is not true for the full localised holonomy-flux cross-product *-algebras associated to a graph and a surface set $\breve{S}$.

For two disjoint surface sets $\breve{S}^1_{\mathrm{d}}$ and $\breve{S}^2_{\mathrm{d}}$ the elements of the localised holonomy-flux cross-product *-algebras satisfies some relations. But there is no easy locality relation such that two algebra elements commute, i.e. $A \in C(\bar{\mathcal{A}}_{\bar{\Gamma}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}^2_{\mathrm{d}}}_{\Gamma}) \rtimes \mathfrak{E}_{\breve{S}^2_{\mathrm{d}},\Gamma}$ and $B \in C(\bar{\mathcal{A}}_{\bar{\Gamma}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}^1_{\mathrm{d}}}_{\Gamma}) \rtimes \mathfrak{E}_{\breve{S}^1_{\mathrm{d}},\Gamma}$ it is not true that, $[A,B]=0$ yields. Notice that, the quantum flux operators $E^1_{S_{\mathrm{d}}}(\Gamma) \in \mathfrak{E}_{\breve{S}^1_{\mathrm{d}},\Gamma}$ and $E^2_{S_{\mathrm{d}}}(\Gamma) \in \mathfrak{E}_{\breve{S}^2_{\mathrm{d}},\Gamma}$ satisfy $[E^1_{S_{\mathrm{d}}}(\Gamma), E^2_{S_{\mathrm{d}}}(\Gamma)] = 0$.

8.4.4 The modified quantum Hamilton constraint operator

Recall the quantum Hamilton constraint operator defined in the introduction by equation (2.3), which is given for example in [98] by

$$\mathcal{Q}(C_T(N)) = \sum_{\Delta\in T} \operatorname{tr}\Big(\big(\mathfrak{h}_A(l_\Delta) - \mathfrak{h}_A(l_\Delta)^{-1}\big)\, \mathfrak{h}_A(\gamma_\Delta)[\mathfrak{h}_A(\gamma_\Delta)^{-1}, \mathcal{Q}(V)]\Big)$$

where $\mathfrak{h}_A(l_\Delta)$ denotes a holonomy along a loop l_Δ in a subset Δ of a triangulation T, γ_Δ denotes a path. Let $\breve{S} := \{S_1, S_2, S_3\}$ be a set of surfaces associated to the triangulation T. Then the quantum volume operator $\mathcal{Q}(V)$ is defined by

$$\mathcal{Q}(V) = \sum_{\gamma_1,\gamma_2,\gamma_3} E_{S_1}(\gamma_1)E_{S_2}(\gamma_2)E_{S_3}(\gamma_3)$$

the sum is over all triples of paths, which are build from three paths that intersects in a common vertex v. Consequently one can localise the quantum volume operator and the quantum Hamilton constraint operator on a set of surfaces $\breve{S}_{\mathrm{d}}$ and a graph system $\mathcal{P}_\Gamma$. The resulting operators are denoted by $\mathcal{Q}(C_T(N))_{\mathrm{d},\Gamma}$ or $\mathcal{Q}(V)_{\mathrm{d},\Gamma}$ and are called the **modified (or discretised) quantum Hamilton constraint** or the **discretised quantum volume operator** associated to graphs. But the operator $\mathcal{Q}(C_T(N))_{\mathrm{d},\Gamma}$ is neither an element of $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^{\mathrm{d}})$ nor $C(\bar{\mathcal{A}}_{\mathrm{loc}}) \otimes \mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}}) \rtimes \mathfrak{E}_{\breve{S}_{\mathrm{d}}}$.

Consequently in the following the quantum Hamilton constraint is restricted to the **quantum Hamilton constraint H_Γ restricted to a graph**, which is given by

$$\exp(H_\Gamma) := \big(\mathfrak{h}_\Gamma(l) - \mathfrak{h}_A(l)^{-1}\big)\, \mathfrak{h}_\Gamma(\gamma)[\mathfrak{h}_\Gamma(\gamma)^{-1}, \mathcal{Q}(V)_{\mathrm{d},\Gamma}]$$

whenever $l,\gamma\in\Gamma$. In particular, **the quantum Hamilton part $H_{\Gamma,P}$ restricted to a graph** is given by

$$\exp(H_{\Gamma,P}) := \big(\mathfrak{h}_\Gamma(l) - \mathfrak{h}_\Gamma(l)^{-1}\big)\, \mathfrak{h}_\Gamma(\gamma)$$

whenever $l,\gamma\in\Gamma$. The operator $[\mathfrak{h}_\Gamma(\gamma)^{-1}, \mathcal{Q}(V)_{\mathrm{d},\Gamma}]$ will be omitted first. Then the quantum Hamilton part is localised such that $H_{\Gamma,P}^+ H_{\Gamma,P}$ is an element of the enveloping Lie algebra of the Lie algebra $\bar{a}_{\mathrm{d},\Gamma}$ associated to $\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$.

The quantum Hamilton part $H_{\Gamma,P}$ defines an action of $\mathbb{R}$ on $\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}})$ by

$$(\alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}}(t) f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i}) := f_{\Gamma_i}(\exp(-tH_{\Gamma_i,P}^+ H_{\Gamma_i,P})\mathfrak{h}_{\Gamma_i})$$

and

$$(\alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}}(t) f_{\Gamma_i})^*(\mathfrak{h}_{\Gamma_i}) = \overline{(\alpha_{H_{\Gamma_i,P} H_{\Gamma_i,P}^+}(t) f_{\Gamma_i})(\mathfrak{h}_{\Gamma_i}^{-1})}$$

whenever $f_{\Gamma_i}\in\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}})$ and $H_{\Gamma,P}^+ H_{\Gamma,P}\in\bar{a}_{\breve{S}_{\mathrm{d}},\Gamma}$ yields.

Proposition 8.4.19. *The triple $(\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}}), \mathbb{R}, \alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}})$ is a C^*-dynamical system.*

The state $\tilde{\omega}_{\mathcal{L}}^{\Gamma_i}$ is a KMS-state at value $\beta\in\mathbb{R}$ on $\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}})$ such that

$$\tilde{\omega}_{\mathcal{L}}^{\Gamma_i}(A\alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}}(i\beta)(B)) = \tilde{\omega}_{\mathcal{L}}^{\Gamma_i}(BA)$$

holds for all $A,B\in\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}})$.

Proposition 8.4.20. *Let the triples $(\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}}), \mathrm{Diff}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}}), \zeta)$ and $(\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}}), \mathbb{R}, \alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}})$ be C^*-dynamical systems.*

Then the automorphisms ζ and $\alpha_{H_{\Gamma_i,P}^+ H_{\Gamma_i,P}}$ on $\mathcal{Z}(\bar{\mathcal{A}}_{\Gamma_i}^{\mathrm{d}})$ do not commute.

Definition 8.4.21. *Denote the center of the compact Lie group $\bar{\mathcal{A}}_\Gamma^{\mathrm{d}}$ by $\bar{z}_{\breve{S}_{\mathrm{d}},\Gamma}$.*

The problem in proposition 8.4.20 is solved, if it is assumed that, $\exp(tH^+_{\Gamma,P}H_{\Gamma,P}) \in \bar{z}_{\breve{S}_{\text{d}},\Gamma}$ holds for all $t\in\mathbb{R}$. The state $\tilde{\omega}^\Gamma_{\mathcal{L}}$ is $\text{Diff}(\mathcal{P}^{\breve{S}_{\text{d}}}_\Gamma)$-invariant. Furthermore this state is $\mathbb{R}$-invariant for all quantum Hamilton part $H_{\Gamma_i,P}$ such that $\exp(tH^+_{\Gamma_i,P}H_{\Gamma_i,P}) \in \bar{z}_{\breve{S}_{\text{d}},\Gamma}$ for all $t\in\mathbb{R}$ and Γ_i being a subgraph of Γ.

Notice that, $(\alpha_{H^+_{\Gamma,P}H_{\Gamma,P}})(A_{\bar{\Gamma}}) = A_{\bar{\Gamma}}$ yields for all $A_{\bar{\Gamma}}\in C(\bar{\mathcal{A}}_{\bar{\Gamma}})$ and

$$(\alpha_{H^+_{\Gamma,P}H_{\Gamma,P}})(f_\Gamma\otimes E^+_{\breve{S}_{\text{d}}}(\Gamma)E_{\breve{S}_{\text{d}}}(\Gamma)) = (\alpha_{H^+_{\Gamma,P}H_{\Gamma,P}})(f_\Gamma)\otimes E^+_{\breve{S}_{\text{d}}}(\Gamma)E_{\breve{S}_{\text{d}}}(\Gamma)$$

for all $f_\Gamma\in\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_\Gamma)$ and $E^+_{\breve{S}_{\text{d}}}(\Gamma)E_{\breve{S}_{\text{d}}}(\Gamma)\in\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma}$. Consequently $\alpha_{H^+_{\Gamma,P}H_{\Gamma,P}}\in\mathfrak{Aut}(\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_\Gamma)\rtimes\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma})$.

Theorem 8.4.22. *Let $\{\Gamma_i\}$ be an inductive family of graphs, $\breve{S}$ be a set of surfaces and $\breve{S}_{\text{d}}$ a set of discretised surfaces associated to $\breve{S}$ such that the assumptions in definition 3.4.14 are satisfied. For a fixed graph Γ let $(C(\bar{\mathcal{A}}_{\text{loc}}),\text{Diff}(\mathcal{P}_{\bar{\Gamma}}),\zeta)$ and $(\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}}),\text{Diff}(\mathcal{P}^{\breve{S}_{\text{d}}}_\Gamma),\zeta)$ be two C^*-dynamical systems.*

Moroever let $\{H_{\Gamma_i,P}\}$ a family of quantum Hamilton parts restricted to graphs such that each element $\exp(tH^+_{\Gamma',P}H_{\Gamma',P})\in\bar{z}_{\breve{S}_{\text{d}},\Gamma_i}$ for all $t\in\mathbb{R}$ and all graphs Γ' being a subgraph of Γ_i.

Let $\{(\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i}),\mathbb{R},\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i}})\}$ be a family of C^*-dynamical systems. Finally let $\hat{\omega}_{\mathcal{L}}$ be the limit state on the *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}})\rtimes_L\mathfrak{E}_{\breve{S}_{\text{d}}}$ of the states $\{\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\}$ of the families $\{\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i})\rtimes_L\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma_i}\}$ of *-algebras. The state $\hat{\omega}^{\Gamma_i}_{\mathcal{L}}$ is a KMS-state for $\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i})$ at value $\beta\in\mathbb{R}$ for $\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i,P}}$ and such that

$$\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i,P}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}$$

for a graph Γ_i and all $1\leq i<\infty$.

Then for a fixed graph Γ there exists a $\text{Diff}(\mathcal{P}^{\breve{S}_{\text{d}}}_\Gamma)$- and $\text{Diff}(\mathcal{P}_{\bar{\Gamma}})$-invariant state on $C(\bar{\mathcal{A}}_{\text{loc}})\otimes\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes_L\mathfrak{E}_{\breve{S}_{\text{d}}}$, which is a product state on a state ω_M of $C(\bar{\mathcal{A}}_{\text{loc}})$ and a state $\hat{\omega}_{\mathcal{L}}$ of $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes_L\mathfrak{E}_{\breve{S}_{\text{d}}}$.

There is an equivalent proposition to proposition 6.4.1.

Remark 8.4.23. *The state, which is defined by*

$$\hat{\omega}_{\mathcal{L},\mathfrak{B}_\Sigma}(A) := \lim_{\Gamma_i\to\Gamma_\infty}\frac{1}{k_{\Gamma_i}}\sum_{l=1}^{k_{\Gamma_i}}\omega^{\Gamma_i}_{\mathcal{L}}(\zeta_{\sigma_l}(A_{\Gamma_i}))\text{ for }\sigma_l\in\mathfrak{B}^{\Gamma_i}_{\breve{S},\text{surf}}(\mathcal{P}_{\Gamma_i})$$

whenever $A\in\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}}}$ and $A_{\Gamma_i}\in\mathcal{Z}(\bar{\mathcal{A}}_{\breve{S}_{\text{d}},\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma_i}$, which is $\mathfrak{B}_{\breve{S},\text{surf}}(\mathcal{P}_{\Gamma_\infty})$-invariant and where k_{Γ_i} is the maximal number of subgraphs in $\mathcal{P}_{\Gamma_i}$, which is generated by all edges and their compositions of the graph Γ_∞, does not converge in weak*-topology.

The state on the localised holonomy-flus cross product *-algebra depends on the family of KMS-states and the state ω_M of $C(\bar{\mathcal{A}}_{\text{loc}})$. Notice that, ω_M need not be $\bar{G}_{\breve{S}_{\text{d}},\bar{\Gamma}}$-invariant for any graph $\bar{\Gamma}$. This is indeed distinguishing from the results of the analytic holonomy C^*-algebra, where the state is required to be invariant. For example refer to corollary 6.1.41. But since there is no action of the fluxes on this part of the localised holonomy-flux cross-product *-algebra, this invariance is not required.

Contrary to corollary 8.4.7 consider the following remark.

Remark 8.4.24. *The limit*

$$\tilde{\delta}_{S_{\text{d}},P}(f) := i\lim_{j\to\infty}[H^+_{\Gamma'_j,P}H_{\Gamma'_j,P},f]$$

for every $f\in\mathcal{D}(\tilde{\delta}_{S_{\text{d}},P})$ and every element $\exp(H^+_{\Gamma'_j,P}H_{\Gamma'_j,P})\in\bar{z}_{\breve{S}_{\text{d}},\Gamma_k}$ and $\Gamma'_j<\Gamma_k$, $i,k\in\mathbb{N}$, is not well-defined in the norm topology.

Until now only quantum Hamilton parts restricted to certain subgraphs of a graph are considered. Hence the modified quantum Hamilton part for a family $\{\Gamma_i\}$ of graphs is given by

$$H^+_PH_P := \lim_{\Gamma_i\to\Gamma_\infty}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}H^+_{\Gamma',P}H_{\Gamma',P}$$

Proposition 8.4.25. *Let $\{\Gamma_i\}$ be an inductive family of graphs, $\breve{S}$ be a set of surfaces and $\breve{S}_{\text{d}}$ a set of discretised surfaces associated to $\breve{S}$ such that the assumptions in definition 3.4.14 are satisfied.*

Moroever, let $\{H_{\Gamma_i,P}\}$ a family of quantum Hamilton parts restricted to graphs such that each element $\exp(tH^+_{\Gamma',P}H_{\Gamma',P}) \in \bar{z}_{\breve{S}_{\text{d}},\Gamma_i}$ for all $t\in\mathbb{R}$ and each subgraph Γ' of Γ_i.

Let $\{(\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i}),\mathbb{R},\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i,P}})\}$ be a family of C^*-dynamical systems. Finally let $\{\hat{\omega}^{\Gamma_i}_{\mathcal{L},\mathfrak{B}}\}$ be a family of states such that $\hat{\omega}^{\Gamma_i}_{\mathcal{L},\mathfrak{B}}$ is a state of the *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma_i}$.

Then the limit state $\hat{\omega}_{\mathcal{L},P}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}}}$, which is given by

$$\hat{\omega}_{\mathcal{L},P}(A) := \lim_{\Gamma_i\to\Gamma_\infty}\frac{1}{|\mathcal{P}_{\Gamma_i}|}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\Big(\alpha_{H^+_{\Gamma',P}H_{\Gamma',P}}(t)(A)\Big)$$

and where $|\mathcal{P}_{\Gamma_i}|$ denotes the number of subgraphs of a graph Γ_i, and which is $\mathbb{R}$-invariant w.r.t. the automorphism group $t\mapsto\alpha_{H^+_PH_P}(t)$, for $A\in\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}}}$, does not converge in weak *-topology.

This proposition implies that, the one-parameter group $t\mapsto\alpha_{H^*_PH_P}(t)$ of *-automorphisms is not strongly continuous. Consequently the derivation δ_P on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})$, which is given by

$$\delta_P(f) := \lim_{t\to 0}\frac{1}{t}\Big(\alpha_{H^*_PH_P}(t)(f)-f\Big) = \lim_{t\to 0}\lim_{\Gamma_i\to\Gamma_\infty}\frac{1}{t|\mathcal{P}_{\Gamma_i}|}\Big(\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\alpha_{H^*_{\Gamma',P}H_{\Gamma',P}}(t)(f)-f\Big)$$

for $f\in\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})$, is not converging in norm.

Now, recall the operator $[\mathfrak{h}_A(\gamma)^{-1},\mathcal{Q}(V)_{\text{d},\Gamma}]$, whenever $\gamma\in\Gamma$ and where $\mathcal{Q}(V)_{\text{d},\Gamma}$ is sum over finite products of discretised flux operators for a surface S_{d} and a graph Γ. Then the quantum Hamilton constraint restricted to a graph contains also elements of $\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}},\Gamma}$.

Consequently define the discretised flux operator associated to a surface S_{d} and for a family $\{\Gamma_i\}$ of graphs by

$$E^+_{S_{\text{d}}}E_{S_{\text{d}}} := \lim_{\Gamma_i\to\Gamma_\infty}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}E^+_{S_{\text{d}},\Gamma'}E_{S_{\text{d}},\Gamma'}$$

where $E_{S_{\text{d}},\Gamma'} := E_{S_{\text{d}}}(\Gamma')\in\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}},\Gamma_i}$ for every subgraph Γ' of Γ_i.

Proposition 8.4.26. *Let $\{\Gamma_i\}$ be an inductive family of graphs, $\breve{S}$ be a set of surfaces and $\breve{S}_{\text{d}}$ a set of discretised surfaces associated to $\breve{S}$ such that the assumptions in definition 3.4.14 are satisfied.*

Let $\Big\{(\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i}),\mathbb{R},\alpha_{E^+_{S_{\text{d}},\Gamma_i}E_{S_{\text{d}},\Gamma_i}})\Big\}$ be a family of C^*-dynamical systems. Morever, let $\{\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\}$ be a family of states such that $\hat{\omega}^{\Gamma_i}_{\mathcal{L}}$ is a state of the *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\text{d}}_{\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}},\Gamma_i}$.

Then the limit $\hat{\omega}_{\mathcal{L},E}$ on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}}}$, which is given by

$$\hat{\omega}_{\mathcal{L},E}(A) := \lim_{\Gamma_i\to\Gamma_\infty}\frac{1}{|\mathcal{P}_{\Gamma_i}|}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\Big(\alpha_{E^+_{S_{\text{d}},\Gamma'}E_{S_{\text{d}},\Gamma'}}(t)(A)\Big)$$

whenever $A\in\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\text{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\text{d}}}$ and where $|\mathcal{P}_{\Gamma_i}|$ denotes the number of subgraphs of a graph Γ_i. The state $\hat{\omega}_{\mathcal{L},E}$ is $\mathbb{R}$-invariant w.r.t. the automorphism group $t\mapsto\alpha_{E^+_{S_{\text{d}}}E_{S_{\text{d}}}}(t)$ and converges in weak *-topology.

Proof : Derive

$$\lim_{\Gamma_i\longrightarrow\Gamma_\infty}\Big|\hat{\omega}^{\Gamma_i}_{\mathcal{L}}(A)-\frac{1}{|\mathcal{P}_{\Gamma_i}|}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\Big(\alpha_{E^+_{S_{\text{d}},\Gamma'}E_{S_{\text{d}},\Gamma'}}(t)(A)\Big)\Big| = \Big|\hat{\omega}^{\Gamma_0}_{\mathcal{L}}(A)-\hat{\omega}^{\Gamma_0}_{\mathcal{L}}\Big(\alpha_{E^+_{S_{\text{d}},\Gamma_0}E_{S_{\text{d}},\Gamma_0}}(t)(A)\Big)\Big|$$

$$=0$$

■

Recall proposition 8.4.8. Furthermore the last proposition imply that the derivation δ_E on $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_\mathrm{d}})$, which is given by

$$\delta_E(f) := \lim_{t\to 0}\frac{1}{t}\Big(\alpha_{E^+_{S_\mathrm{d},\Gamma'}E_{S_\mathrm{d},\Gamma'}}(t)(f) - f\Big) = \lim_{t\to 0}\ \lim_{\Gamma_i\to\Gamma_\infty}\frac{1}{t|\mathcal{P}_{\Gamma_i}|}\Big(\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\alpha_{E^+_{S_\mathrm{d},\Gamma'}E_{S_\mathrm{d},\Gamma'}}(t)(f) - f\Big)$$

for $f\in\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_\mathrm{d}})$, converges in norm.

Problem 8.4.1: Let $\{\Gamma_i\}$ be an inductive family of graphs, $\breve{S}$ be a set of surfaces and $\breve{S}_\mathrm{d}$ a set of discretised surfaces associated to $\breve{S}$ such that the assumptions in definition 3.4.14 are satisfied. For a fixed graph Γ let $(C(\bar{\mathcal{A}}),\mathfrak{B}(\mathcal{P}_{\bar{\Gamma}}),\zeta)$ and $(\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_\mathrm{d}}),\mathfrak{B}_{\breve{S}_\mathrm{d},\mathrm{surf}}(\mathcal{P}_\Gamma^{\breve{S}_\mathrm{d}}),\zeta)$ be two C^*-dynamical systems.

The discetrised quantum volume operator is explicity defined by

$$\mathcal{Q}(V^*V)_{\mathrm{d},\Gamma} := \sum_{\substack{(\gamma_1,\gamma_2,\gamma_3)\\ \in\mathcal{P}^v_\Gamma\times\mathcal{P}^v_\Gamma\times\mathcal{P}^v_\Gamma}} E_{S_3^\mathrm{d}}(\gamma_3)^+E_{S_2^\mathrm{d}}(\gamma_2)^+E_{S_1^\mathrm{d}}(\gamma_1)^+E_{S_1^\mathrm{d}}(\gamma_1)E_{S_2^\mathrm{d}}(\gamma_2)E_{S_3^\mathrm{d}}(\gamma_3)$$

such that $\mathcal{Q}_{\mathrm{d},\Gamma}(V^*V)\in\mathfrak{E}_{\breve{S}_\mathrm{d},\Gamma}$. Recall the quantum Hamilton constraint H_Γ restricted to a graph is presented by

$$\exp(H_\Gamma) := \exp(H_{\Gamma,P})[\mathfrak{h}_\Gamma(\gamma),\mathcal{Q}(V)_{\mathrm{d},\Gamma}]$$

Moreover let $\{H_{\Gamma_i}\}$ be a family of quantum Hamilton constraints restricted to graphs such that each element $\exp(tH^+_{\Gamma'}H_{\Gamma'})\in C^*(\bar{\mathcal{A}}^\mathrm{d}_{\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_\mathrm{d},\Gamma_i}$ for all $t\in\mathbb{R}$ and all graphs $\{\Gamma'\}$ being subgraphs of Γ_i.

Recall the family $\{\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\}$ of states of the family $\{\mathcal{Z}(\bar{\mathcal{A}}^\mathrm{d}_{\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_\mathrm{d},\Gamma_i}\}$ of *-algebras, which are KMS-states for $\mathcal{Z}(\bar{\mathcal{A}}^\mathrm{d}_{\Gamma_i})$ at value $\beta\in\mathbb{R}$ and such that the states satisfy

$$\begin{aligned}
&\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{H^+_{\Gamma',P}H_{\Gamma',P}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}\\
&\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i,P}}(t)\circ\zeta_\sigma = \hat{\omega}^{\Gamma_i}_{\mathcal{L}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\zeta_\sigma\circ\alpha_{H^+_{\Gamma_i,P}H_{\Gamma_i,P}}(t)\\
&\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{E^+_{S_\mathrm{d},\Gamma_i}E_{S_\mathrm{d},\Gamma_i}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}\\
&\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\zeta_\sigma\circ\alpha_{E^+_{S_\mathrm{d},\Gamma_i}E_{S_\mathrm{d},\Gamma_i}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{E^+_{S_\mathrm{d},\Gamma_i}E_{S_\mathrm{d},\Gamma_i}}\circ\zeta_\sigma\\
&\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{H^+_{\Gamma',P}H_{\Gamma',P}}(t)\circ\alpha_{E^+_{S_\mathrm{d},\Gamma_i}E_{S_\mathrm{d},\Gamma_i}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}} = \hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{E^+_{S_\mathrm{d},\Gamma_i}E_{S_\mathrm{d},\Gamma_i}}\circ\alpha_{H^+_{\Gamma',P}H_{\Gamma',P}}(t)
\end{aligned}$$

for all $\sigma\in\mathrm{Diff}(\mathcal{P}^{\breve{S}_\mathrm{d}}_{\Gamma_i})$, $t\in\mathbb{R}$, a subgraph Γ' of Γ_i and all $1\leq i<\infty$.

There is a problem of convergence of the limit state on the localised holonomy-flux cross-product *-algebra presented in proposition 8.4.25. Consequently the limit state $\hat{\omega}_{\mathcal{L}}$ on $\mathcal{Z}(\bar{\mathcal{A}}^\mathrm{d}_{\Gamma_i})\rtimes\mathfrak{E}_{\breve{S}_\mathrm{d},\Gamma_i}$ has to be analysed further. The hope is that for a suitable modified quantum Hamilton constraint derived from

$$\hat{H}^+\hat{H} := \lim_{N\to\infty}\sum_{i=1}^N H^+_{\Gamma_i}H_{\Gamma_i} = \lim_{\Gamma_i\to\Gamma_\infty}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}} H^+_{\Gamma'}H_{\Gamma'}$$

the state $\omega_{\mathcal{L}}$ satisfies

$$\hat{\omega}_{\mathcal{L}}\circ\alpha_{\hat{H}^+\hat{H}} = \lim_{\Gamma_i\to\Gamma_\infty}\sum_{\Gamma'\in\mathcal{P}_{\Gamma_i}}\hat{\omega}^{\Gamma_i}_{\mathcal{L}}\circ\alpha_{H^+_{\Gamma'}H_{\Gamma'}} = \hat{\omega}_{\mathcal{L}} \tag{8.56}$$

Summarising in this situation the state $\hat{\omega}_{\mathcal{L}}$ would be invariant under the automorphisms inherited by the modified quantum Hamilton H, but the state is only invariant under a finite set of exceptional graph-diffeomorphisms. Despite this fact a total localised finite quantum diffeomorphism is defined as follows. First recall the construction presented in section 7.3. There some certain operators are developed in the situation of C^*-algebras. Apart from C^*-properties the following objects can be analysed. Similarly define an operator, which depend on a bisection in $\mathfrak{B}(\mathcal{P}_{\bar{\Gamma}})$ and which is $C(\bar{\mathcal{A}}_{\bar{\Gamma}})$-valued, and denote this operator by $D^\sigma_{\bar{\Gamma}}$. The set of all these operators is denoted by

$\mathfrak{D}_{\breve{S}_{\mathrm{d}},\Gamma}$. Furthermore there is an operator, which depends on a bisection in $\mathfrak{B}_{\breve{S}_{\mathrm{d}},\mathrm{surf}}(\mathcal{P}_\Gamma^{\breve{S}_{\mathrm{d}}})$ and which is $\mathcal{Z}(\bar{\mathcal{A}}^{\breve{S}_{\mathrm{d}}})\rtimes\mathfrak{E}_{\breve{S}_{\mathrm{d}}}$-valued, and this operator is denoted by $D^\sigma_{\breve{S}_{\mathrm{d}},\Gamma}$. The set of all these operators is denoted by $\mathfrak{D}_{\bar{\Gamma}}$. For each graph Γ_i of a family of graphs there exists a generating system $\mathfrak{B}^{\Gamma_i}_{\breve{S},\mathrm{surf}}(\mathcal{P}_{\Gamma_i}^{\breve{S}_{\mathrm{d}}})$ of bisections for this graph. Then set

$$D^+_{\breve{S},\Gamma_i}D_{\breve{S}_{\mathrm{d}},\Gamma_i} := \sum_{\sigma_l\in\mathfrak{B}^{\Gamma_i}_{\breve{S}_{\mathrm{d}},\mathrm{surf}}(\mathcal{P}_{\Gamma_i}^{\breve{S}_{\mathrm{d}}})} D^{\sigma_l,*}_{\breve{S}_{\mathrm{d}},\Gamma'}D^{\sigma_l}_{\breve{S}_{\mathrm{d}},\Gamma'}$$

for every subgraph Γ' of Γ_i. The sum over all graphs of a family of graphs defines the **localised quantum diffeomorphism constraint**. The linear hull over all graphs of a family of graphs of all elements of the set $\mathfrak{D}_{\breve{S}_{\mathrm{d}},\Gamma}$, the set $\mathfrak{D}_{\bar{\Gamma}}$ and the set of all quantum Hamilton constraints restricted to a graph Γ forms the *-algebra $\mathfrak{C}$ of quantum constraints. Note that, this algebra is not a subalgebra of the localised holonomy-flux cross-product *-algebra associated to a discretised surface set. Finally, the **modified quantum Master constraint M** is defined by the sum of the modified quantum Hamilton constraint and the localised quantum diffeomorphism constraint.

The localised holonomy-flux cross-product *-algebra can be enlarged such that this algebra will be a subalgebra. This algebra will be based on the cross-product construction once more and consequently will be called the **localised holonomy-flux-graph-diffeomorphism cross-product *-algebra** associated to a discretised surface set. It will contain all finite graph-diffeomorphisms. Note that, the modified quantum Hamilton constraint is not contained in this algebra, but it will be in a suitable sense be affilliated with. Now, Dirac states and Dirac observables have to be analysed.

Assume that, $\mathcal{S}_D$ denotes a set of Dirac states on the localised holonomy-flux-graph-diffeomorphism cross-product *-algebra $\mathfrak{A}$. It is not obvious that Dirac observables can be easily defined, since the set generated by all quantum constraints in $\mathfrak{C}$ defines a closed left and right ideal in $\mathfrak{A}$. Assume that $\mathcal{O}_D$ is the algebra of Dirac observables, which is a subalgebra of the localised holonomy-flux-graph-diffeomorphism cross-product *-algebra. Then

$$\mathcal{O}^\alpha_D := \{A\in\mathcal{O}_D : \alpha_{\mathbf{M}}(t)(A) = A, \forall t\in\mathbb{R}\}$$

defines a **localised *-algebra of complete quantum observables for surfaces**.

■

Finally, a short remark with respect to C^*-algebras is stated. The **localised holonomy-flux cross-product C^*-algebra for surfaces** is constructable as the inductive limit C^*-algebra of the inductive family of C^*-algebras $\{C(\bar{\mathcal{A}}_{\bar{\Gamma}})\otimes C(\bar{G}_{\breve{S}_{\mathrm{d}},\Gamma})\rtimes\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma\}$ for a suitable set $\breve{S}_{\mathrm{d}}$ of discretised surfaces associated to a surface set $\breve{S}$ with appropriate properties with respect to the inductive limit of the family of graphs. The ideas are derived from to the holonomy-flux cross-product C^*-algebra presented in chapter 7. In section 7.3 an enlargement of the holonomy-flux cross-product C^*-algebra, which contains finite diffeomorphisms, has been given. The generators defined by the quantum diffeomorphisms are not contained in this algebra but affiliated with. This idea will be used in a future work for a similar enlargment of the localised holonomy-flux cross-product C^*-algebra for surfaces.

8.5 The holonomy-flux Nelson transform C^*-algebra

In [115, section 3] Woronowicz has shown that, a particular set of unbounded elements generate an C^*-algebra. This formalism is used to define a new C^*-algebra, which is generated by holonomies and special flux operators. The hope is to relate the set of quantum constraints to a set of unbounded and bounded operators which define a C^*-algebra. Consequently the following ansatz for a modification of a holonomy-flux *-algebra is used. For simplicity assume that, the surface $\breve{S}$ has the simple surface intersection property for a graph Γ. Recall the Lie algebra $\mathfrak{g}_{\breve{S},\Gamma}$, which is isomorphic to $\mathfrak{g}^N$, where N counts the number of paths in Γ.

Definition 8.5.1. *For a basis* $\{X_1^{v_k},...,X_j^{v_k},...,X_N^{v_k}\}_{v_k=S_k\cap\gamma,S_k\in\breve{S},\gamma\in\Gamma}$ *of the Lie algebra* $\mathfrak{g}_{\breve{S},\Gamma}$ *the **flux Nelson operator** is defined by*

$$\triangle^v_{S_1,S_2} = \sum_{1\leq j\leq N} X_j^{v+}X_j^v,\ \textit{where } v = S_1\cap S_2\cap\gamma, S_1,S_2\in\breve{S},\gamma\in\Gamma$$

or respectively

$$\triangle^v_{S_1,S_2,\Gamma} = \sum_{1\leq j\leq N,\gamma\in\Gamma} X^{v+}_j X^v_j, \text{ where } v = S_1\cap S_2\cap\gamma, S_1,S_2\in\breve{S}$$

Notice that, the elements $\{X^{v_k}_1,...,X^{v_k}_j,...,X^{v_k}_N\}_{v_k=S_k\cap\gamma,S_k\in\breve{S},\gamma\in\Gamma}$ generate the C^*-algebra $C^*(\bar{G}_{\breve{S},\Gamma})$.

Definition 8.5.2. *Define the **flux Nelson transform** of the flux Nelson operator to be*

$$z_{S_1,S_2,S_3} := \sum_{1\leq k\leq N} X^v_k(I+\triangle^v_{S_1,S_2})^{-1} \text{ where } v = S_1\cap S_2\cap S_3\cap\gamma$$

or respectively

$$z^{S_1,S_2,S_3}_\Gamma := \sum_{1\leq k\leq N,\gamma\in\Gamma} X^v_k(I+\triangle^v_{S_1,S_2})^{-1} \text{ where } v = S_1\cap S_2\cap S_3\cap\gamma$$

Then the representation of the flux Nelson transform on the Hilbert space $\mathcal{H}_{s,\gamma}$ is given by

$$\pi_{s,\gamma}(z_{S_1,S_2,S_3})\psi_{s,\gamma} := \sum_{1\leq k\leq N}[X^v_k(I+\triangle^v_{S_1,S_2})^{-1},\psi_{s,\gamma}]$$

where $\psi_{s,\gamma}\in\mathcal{H}_{s,\gamma}$ and $v = S_1\cap S_2\cap S_3\cap\gamma$. Moreover

$$(\pi_{s,\gamma}(z_{S_1,S_2,S_3})\psi_{s,\gamma})\,(\mathfrak{h}_\gamma(\gamma)) = \sum_{1\leq k\leq N}\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\psi_{s,\gamma}(\exp(tX^v_k(I+\triangle^v_{S_1,S_2})^{-1})\mathfrak{h}_\gamma(\gamma))$$

holds. Observe that, the flux Nelson operator is a unbounded, self-adjoint, i.e. $\triangle^{v*}_{S_1,S_2} = \triangle^v_{S_1,S_2}$, and positive operator on $\mathcal{H}_{s,\gamma}$. Now for a continuous function f_Γ the canonical commutator relations read

$$[X^v_k(I+\triangle^v_{S_1,S_2})^{-1},f_\gamma] = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\alpha(\exp(tX^v_k(I+\triangle^v_{S_1,S_2})^{-1}))(f_\gamma) \tag{8.57}$$

where

$$(\alpha(\exp(tX^v_k(I+\triangle^v_{S_1,S_2})^{-1}))f_\gamma)(\mathfrak{h}_\gamma(\gamma)) = f_\gamma(\exp(tX^v_k(I+\triangle^v_{S_1,S_2})^{-1})\mathfrak{h}_\gamma(\gamma))$$

and $X^v_S\in\mathfrak{g}$ and $v := S_1\cap S_2\cap S_3\cap\gamma$

Clearly this generalises such that the continuous functions in $C(\bar{\mathcal{A}}_\Gamma)$ and the flux Nelson transform z^{S_1,S_2,S_3}_Γ, which satisfy canonical commutator relations equivalent to (8.57), generate a *-algebra for all $\Gamma\in\mathcal{P}_{\Gamma_\infty}$ and a suitable surface set $\breve{S}$. This algebra is called the **holonomy-flux Nelson transform *-algebra $\mathfrak{R}_{\breve{S},\Gamma}$ for a surface set $\breve{S}$ and a graph Γ**.

Assume that, the sets

$$\Big\{\|\pi_{s,\Gamma}(R_{S,\Gamma})\|_{\mathcal{H}_{s,\Gamma}} : \pi_{s,\Gamma} \text{ is a unital }^*\text{-representation of } \mathfrak{R}_{\breve{S},\Gamma} \text{ on a Hilbert space } \mathcal{H}_{s,\Gamma}\Big\}$$

is bounded. Then the C^*-seminorm on $\mathfrak{W}_{S,\Gamma}$ is defined by

$$\|R_{S,\Gamma}\| = \sup\Big\{\|\pi_{s,\Gamma}(R_{S,\Gamma})\|_{\mathcal{H}_{s,\Gamma}} : \pi_{s,\Gamma} \text{ is a unital }^*\text{-representation of } \mathfrak{R}_{\breve{S},\Gamma} \text{ on a Hilbert space } \mathcal{H}_{s,\Gamma}\Big\}$$

for all $R_{S,\Gamma}\in\mathfrak{R}_{\breve{S},\Gamma}$. Define the two-sided ideal $\mathcal{J} = \{R_{S,\Gamma}\in\mathfrak{R}_{\breve{S},\Gamma} : \|R_{S,\Gamma}\| = 0\}$. Then the unital **universal holonomy-flux Nelson transform C^*-algebra** $\mathcal{R}_{\breve{S},\breve{\Gamma}}$ is the completition of $\mathfrak{R}_{\breve{S},\Gamma}$ w.r.t. this norm. In fact, the flux Nelson transform belong to the multiplier algebra of the holonomy-flux cross-product C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{G}_{\breve{S},\Gamma}$. Consequently the universal unital flux Nelson transform C^*-algebra $\mathcal{R}_{\breve{S},\breve{\Gamma}}$ is always a C^*-subalgebra of $M(C(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{G}_{\breve{S},\Gamma})$.

With no doubt one can also define the holonomy-flux resolvent *-algebra constructed from the holonomies $\mathfrak{h}_\Gamma(\gamma)$ along paths in a graph Γ and the set of resolvents, which are defined by

$$R^{S_1,S_2}(\lambda) := (i\lambda-\triangle^v_{S_1,S_2})^{-1} \text{ for } \lambda\in\mathbb{R}\setminus\{0\}$$

The resolvents are bounded operators on $\mathcal{H}_{s,\gamma}$. Furthermore the resolvents are contained in the multiplier algebra of the holonomy-flux cross-product C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)\rtimes\bar{G}_{\breve{S},\Gamma}$.

Chapter 9

Holonomy groupoid and holonomy-flux groupoid C^*-algebras for gauge theories

In this chapter some ideas for a new construction of algebras in the holonomy groupoid formulation of LQG is presented. This part of the dissertation is work in progress and a detailed analysis will be studied in a further project. The aim is to find a suitable algebra such that the curvature is contained in this new algebra. In the section 3.2 it has been argued that, curvature and the infinitesimal connection have the same base, since their values are contained in the Lie algebroid of the holonomy groupoid. This is the new starting point of the construction of algebras in section 9.1. The fundamental idea of Barrett has been to declare the set of all holonomy maps to be the configuration space of the theory. Consequently a set of algebras depending on holonomy groupoids associated to path connections generalises this choice. The implementation of the flux operators is indicated in section 9.2 by a cross-product construction, which is similar to the definition of the holonomy-flux cross-product C^*-algebra presented in section 7. Furthermore there are morphisms between holonomy Lie groupoids associated to different path connections, which correspond to relations between Lie algebroids. These relations are used to define morphisms between algebras. Since the construction depends on the choice of the base manifold Σ a new covariant formulation is suggested. The author proposes in section 9.3 to use for the covariant holonomy groupoid formulation of LQG the ideas, which have been presented by Brunetti, Fredenhagen and Verch [24] in the context of algebraic quantum field theory.

9.1 The construction of the holonomy groupoid C^*-algebra for gauge theories

In this section a couple of different approaches for a construction are summarised. Some of the ideas are not available for LQG. The first idea is to use the a theory of locally compact Hausdorff groupoids, which is influenced by the theory of locally compact groups. Indeed Renault [79] has presented C^*-algebras constructed from locally compact Hausdorff groupoids. The idea is the following. In comparison with the group algebra of a locally compact group, a similar groupoid algebra is constructed. The space of continuous functions with compact support on a groupoid, which is equipped with a convolution multiplication and an involution, form a *-algebra. There is a C^*-norm such that the representations of that algebra are continuous. In analogy to Haar measures on locally compact groups a system of Haar measures is derived. Now recall the holonomy groupoid $\mathrm{Hol}_\Lambda(\Sigma)$, which has been presented in section 3.2.2, and the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ for a gauge theory, which has been given in section 3.3.3. Then in general it is not clear, if the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ is locally compact and Hausdorff. Consequently this approach by Renault cannot directly being used in this context.

The second idea is to consider the *-algebra $\mathcal{C}^*(\mathcal{G})$ of continuous functions on the Lie groupoid $\mathcal{G}$, which is for example given by the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. This algebra is the analog of the convolution *-algebra $\mathcal{C}^*(G)$ for a locally compact group G. Then the groupoid C^*-algebra $C^*(\mathcal{G})$ is isomorphic to $\mathcal{K}(L^2(P)) \otimes C^*(G)$. This result has been stated by Landsmann [57]. But the C^*-algebra $C^*(\mathcal{G})$ is not the right choice, since the particular holonomy groupoid structure is absent and the full knowledge of the manifold P, or the base manifold Σ and a section $s : \Sigma \to P$, is needed. From this point of view, this algebra is maybe not the favoured algebra. Consequently one might use this idea for gravitational theories instead of a pure gauge theory.

The third idea is based on transitive Lie groupoids. Assume that the holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ associated to a principal bundle $P(\Sigma, G, \pi)$ is a transitive Lie groupoid. Then another construction of a C^*-algebra is available. Notice that, $\mathrm{Hol}^P_\Lambda(\Sigma)$ is a Lie subgroupoid of the gauge groupoid $\frac{P\times P}{G} \rightrightarrows \Sigma$. Then the holonomy groupoid C^*-algebra for a gauge theory is defined as follows. Landsman [56, Definition 3.3.2] has defined a family of measures for a Lie groupoid.

Definition 9.1.1. *A* ***left Haar system on a Lie groupoid*** $\mathcal{G} \rightrightarrows \mathcal{G}^0$ *is a family* $\{\mu_v^{t_\mathcal{G}}\}_{v\in\mathcal{G}^0}$ *of positive measures, where* $\mu_v^{t_\mathcal{G}}$ *is a measure on the manifold* $t_\mathcal{G}^{-1}(v)$ *such that*

(i) the family is invariant under the left-translation in a Lie groupoid $\mathcal{G}$

(ii) each $\mu_v^{t_\mathcal{G}}$ *is locally Lebesque (i.e. a Lebesque measure in every co-ordinate chart)*

(iii) for each $f \in \mathcal{C}(\mathcal{G})$ *the map* $v \mapsto \int_{t_\mathcal{G}^{-1}(v)} \mathrm{d}\,\mu_v^{t_\mathcal{G}}(\gamma)f(\gamma)$ *from* $\mathcal{G}^0$ *to* $\mathbb{C}$ *is smooth.*

Equivalently a right Haar system on a Lie groupoid is defined. Consequently there is a left Haar measure on $\mathrm{Hol}^P_\Lambda(\Sigma)$. Set $\mathcal{G} := \mathrm{Hol}^P_\Lambda(\Sigma)$. Note that, for a Lie groupoid $\mathcal{G}$ over $\mathcal{G}^0$ the property (i) induces

$$\int_{t_\mathcal{G}^{-1}(v)} \mathrm{d}\,\mu_v^{t_\mathcal{G}}(\gamma)f(\gamma) = \int_{t_\mathcal{G}^{-1}(v)} \mathrm{d}\,\mu_v^{t_\mathcal{G}}(L_\theta(\gamma))f(L_\theta(\gamma))$$

for every $\theta \in \mathcal{G}_v^{\varphi_0(v)}$ and $(\varphi, \varphi_0) \in \mathrm{Diff}(\mathcal{G})$. Furthermore it is possible to consider the reduced groupoid C^*-algebra $C_r^*(\mathcal{G})$, which is defined in analogy to the reduced group C^*-algebra of a Lie group. Clearly the algebra depends on the choice of the left Haar system on $\mathcal{G} \rightrightarrows \mathcal{G}^0$. The convolution product of two functions $f, k \in \mathcal{C}(\mathcal{G})$ is given by

$$(f * k)(\gamma) := \int_{t_\mathcal{G}^{-1}(s_\mathcal{G}(\gamma))} \mathrm{d}\,\mu_{s_\mathcal{G}(\gamma)}^{t_\mathcal{G}}(\tilde{\gamma})f(\gamma\circ\tilde{\gamma})k(\tilde{\gamma}^{-1})$$

and involution is presented by $f^*(\gamma) := \overline{f(\gamma^{-1})} = \overline{\breve{f}(\gamma)}$.

Definition 9.1.2. *The groupoid* C^**-algebra for the holonomy Lie groupoid* $\mathrm{Hol}^P_\Lambda(\Sigma)$ *is called the* ***holonomy groupoid*** C^****-algebra for a gauge theory associated to a path connection*** Λ *and is denoted by* $C^*(\mathrm{Hol}^P_\Lambda(\Sigma))$.

Recall from section 3.1.4 that for every Lie groupoid $\mathcal{G}$ there exists an associated Lie algebroid $A\mathcal{G}$. This is a vector bundle over $\mathcal{G}^0$, which is equipped with a vector bundle map $\mathcal{G} \to T\mathcal{G}^0$, a Lie bracket $[.,.]_\mathcal{G}$ on the space $\Gamma(\mathcal{G})$ of smooth sections of $\mathcal{G}$, satisfying certain compatibility conditions. Similarly to the exponentiated map from a Lie algebra $\mathfrak{g}$ associated to a Lie group G to G a generalised exponentiated map has been mentioned in section 3.2.

Now remember that, it has been assumed that, $\mathrm{Hol}^P_\Lambda(\Sigma)$ defines a holonomy Lie groupoid for each path connection Λ in $\breve{\Lambda}$. There exists an associated Lie algebroid $\mathcal{A}\,\mathrm{Hol}^P_\Lambda(\Sigma)$ for each path connection Λ in $\breve{\Lambda}$. This Lie algebroid contains the values of the infinitesimal Lie algebroid connections and the curvature. Moreover a Lie algebroid morphism $\mathfrak{a}_l$ between two Lie algebroids and the groupoid morphism $\mathfrak{m}_l$ between two holonomy Lie groupoids associated to different path connections have been presented in corollary 3.3.11. Let Λ and Λ_l be two path connections in $\breve{\Lambda}$. Then there is a morphism, which depends on the Lie algebroid morphism $\mathfrak{a}_l$, between the C^*-algebra $C^*(\mathrm{Hol}^P_\Lambda(\Sigma))$ and another C^*-algebra $C^*(\mathrm{Hol}^P_{\Lambda_l}(\Sigma))$. This morphism is defined by

$$(\alpha_{\mathfrak{a}_l} f)(\mathfrak{h}_\Lambda(\gamma)) := f^l((\mathfrak{m}_l \circ \mathfrak{h}_\Lambda)(\gamma)) = f^l(\mathfrak{h}_{\Lambda'}(\gamma)) \tag{9.1}$$

whenever $f \in C^*(\mathrm{Hol}^P_\Lambda(\Sigma))$ and $f^l \in C^*(\mathrm{Hol}^P_{\Lambda_l}(\Sigma))$.

The left generalised exponentiated map $\mathrm{Exp}_L : \Gamma A(\mathrm{Hol}^P_\Lambda(\Sigma)) \to \Gamma\,\mathrm{Hol}^P_\Lambda(\Sigma)$, which is defined for all right-invariant vector fields in a vector subspace of $\frac{TP}{G}$, leads to an action on $C^*(\mathrm{Hol}^P_\Lambda(\Sigma))$. This action is defined by

$$(\alpha_{\mathrm{Exp}_L(t(\gamma_A(X))(v))}f)(\mathfrak{h}_\Lambda(\gamma)) := f(L_{\mathrm{Exp}_L(t(\gamma_A(X))(v))}(\mathfrak{h}_\Lambda(\gamma))) = f(\mathrm{Exp}_L(t(\gamma_A(X))(v))\mathfrak{h}_\Lambda(\gamma))$$

where $X \in T_v\Sigma$, $v = t(\gamma)$ such that $\mathrm{Exp}_L(t(\gamma_A(X))(v)) \in \mathrm{Hol}^P_\Lambda(\Sigma)^v$ and $f \in C^*(\mathrm{Hol}^P_\Lambda(\Sigma))$.

Similarly an action of a bisection σ_A of $\frac{P\times P}{G} \rightrightarrows \Sigma$ is given. Since it is true that $\text{Exp}(\gamma_A(X)(v)) = \Lambda(\varphi, v)(1)$ holds and $\tilde{\varphi}_t(\phi_t, \text{id}_G)$ defines a gauge and diffeomorphism transformation on $\frac{P\times P}{G} \rightrightarrows \Sigma$ such that

$$(\alpha_{\sigma_A} f)(\mathfrak{h}_\Lambda) := f(L_{\tilde{\varphi}(\phi,\text{id}_G)}\mathfrak{h}_\Lambda(\gamma)) := f((\text{id}_\Lambda \circ \mathfrak{h}_\Lambda)(\varphi\circ\gamma)) = f(\mathfrak{h}_\Lambda(\varphi)\mathfrak{h}_\Lambda(\gamma))$$

where $\phi(v) = s(\varphi) =: w$, $t(\gamma) = k$, $\gamma \in \mathcal{P}\Sigma_k^v$, $\varphi \in \mathcal{P}\Sigma_v^w$, $\text{id}_\Lambda : \text{Hol}_\Lambda^P(\Sigma) \longrightarrow \text{Hol}_\Lambda^P(\Sigma)$ is the identity morphism and $f \in C^*(\text{Hol}_\Lambda^P(\Sigma))$. This action is generalised to an action of a bisection $\sigma_{A'}$ of $\frac{P\times P}{G} \rightrightarrows \Sigma$ by the definition

$$(\alpha_{\sigma_{A'}} f)(\mathfrak{h}_\Lambda) := f(L_{\tilde{\varphi}'(\phi,\text{id}_G)}\mathfrak{h}_\Lambda(\gamma)) := f((\mathfrak{m}_l \circ \mathfrak{h}_\Lambda)(\varphi\circ\gamma)) = f(\mathfrak{h}_{\Lambda'}(\varphi\circ\gamma))$$

whenever $\Lambda' = \mathfrak{m}_l \circ \Lambda$, $\mathfrak{a}_l := \mathfrak{a}'$, $f \in C^*(\text{Hol}_\Lambda^P(\Sigma))$ and $\gamma_{A'} = \mathfrak{m}'_* \circ \gamma_A$.

9.2 Cross-product C^*-algebras for gauge theories

In the last section the holonomy groupoid C^*-algebra for a gauge theory associated to a path connection has been studied. This algebra is the algebra of quantum configuration variables, which contains the curvature in an appropriate sense. The full algebra is derived analogously to the procedure presented in chapter 7. There the flux operators are implemented by cross-product algebras. Indeed Masuda [69] has invented cross-product C^*-algebras in the context of groupoids. The aim is to transfer his idea to the holonomy groupoid formulation.

Definition 9.2.1. *The triplet $(\mathfrak{A}, \mathcal{G}, \rho)$ is called a C^***-groupoid dynamical system** if $\mathfrak{A}$ is a C^*-algebra, $\mathcal{G}$ is a locally compact groupoid with a faithful transverse function $\mu = \{\mu^v\}_{v\in\mathcal{G}^0}$ and $\rho : \mathcal{G} \longrightarrow \mathfrak{Aut}(\mathfrak{A})$ is a continuous morphism.*

First recall that in section 3.4.3 the flux operators are implemented as elements of a Lie group G. Concern the holonomy Lie groupoid is $\mathcal{G} := \text{Hol}_\Lambda^P(\Sigma)$. Then the following C^*-groupoid dynamical system $(C_0(G), \mathcal{G}, \rho)$, where $\rho : \mathcal{G} \longrightarrow G$ is a continuous groupoid morphism, is studied. The associated cross-product $C_0(G) \rtimes \mathcal{G}$ is defined as the completition of the set $\mathcal{C}(\mathcal{G}, \mathfrak{A})$ of all $\mathfrak{A}$-valued continuous functions over $\mathcal{G}$ with compact support with respect to a appropriate C^*-norm. Clearly there are left actions ρ_L and right actions ρ_R of $\mathcal{G}$ on the algebra $C_0(G)$ associated to left or right Haar systems on the holonomy Lie groupoid $\mathcal{G}$.

Consequently the following algebra contains holonomies and flux operators for a gauge theory.

Definition 9.2.2. *The **holonomy-flux groupoid C^*-algebra for a gauge theory associated to a path connection** Λ is defined by the cross-products $C_0(G) \rtimes_{\rho_L} \text{Hol}_\Lambda^P(\Sigma)$ or $C_0(G) \rtimes_{\rho_R} \text{Hol}_\Lambda^P(\Sigma)$.*

Summarising these algebras are the algebras of quantum configuration and momentum variables for a gauge theory. But recognize that there is a big bunch of these algebras, since each algebra is associated to a path connection. Moreover each algebra really depends on the chosen gauge theory and hence on the principal fibre bundle $P(\Sigma, G, \pi)$. This leads directly to sets of algebras.

9.3 Covariant holonomy groupoid formulation of LQG

In this section two categories, which arise naturally in the holonomy groupoid formulation of LQG are presented.

First recall the set $\text{Hol}_{\breve{\Lambda}}^P(\Sigma)$ of holonomy Lie groupoids, which has been analysed in section 3.3.3.1. Then one category is given by the following objects and morphisms. The set of objects is formed by all holonomy groupoids $\text{Hol}_\Lambda(\Sigma)$ associated to each manifold Σ in a set $\breve{\Sigma}$ of 3-dimensional spatial manifolds and to each path connection Λ in a set $\breve{\Lambda}$ of path connections w.r.t. a principal fibre-bundle $P(\Sigma, G_\Sigma, \pi)$. Notice that the structure group G_Σ vary for principal fibre bundles associated to different base manifolds. The set of morphisms of the category are Lie groupoid morphism between two holonomy Lie groupoids. Denote this category by $\mathfrak{Hol}$ and call it the **holonomy category**.

The second category is given by the objects, which are given by all unital C^*-algebras $C_0(G_\Sigma) \rtimes_{\rho_X} \text{Hol}_\Lambda(\Sigma)$ for every principal fibre bundle $P(\Sigma, G_\Sigma, \pi)$ containing a manifold Σ in $\breve{\Sigma}$ and a Lie group G_Σ associated to Σ, and defined by the left and right actions ($X = R, L$). The set of morphisms are faithful unit-preserving *-morphisms between these algebras. The category is denoted by $\mathfrak{Alg}$ and called the **holonomy-flux category**.

Hence a covariant holonomy groupoid formulation of LQG is an assignment of C^*-algebras to holonomy groupoids in such a way that the algebras are identifyable if the holonomy Lie groupoids are connected by a Lie groupoid morphism.

The last sections have given an overview about a new formulation of LQG. This is a starting point for a new detailed study, which will be done in a future work.

Chapter 10

Conclusion and Outlook

In this dissertation it has been achieved that, there is an operator algebraic formulation of the theory of LQG. In comparison to the Weyl algebra of Quantum Geometry [39] and the holonomy-flux algebra [64], the Weyl algebra for surfaces and the holonomy-flux cross-product *-algebra have been developed. Form the study of quantum constraints, KMS-Theory and dynamics a set of conditions for an algebra to be a physical algebra is derived. In the following it is argued why the Weyl algebra for surfaces and the holonomy-flux cross-product *-algebra are not physical in this context. For this reason other algebras, which are constructed from the basic quantum variables of the theory, are constructed and analysed with respect to this question. It turns out that, the localised holonomy-flux cross-product *-algebra satisfies more conditions for a physical algebra than other algebras derived from holonomies and fluxes. Briefly the *set of conditions for a physical algebra* is given by

(i) the quantum constraint operators are affiliated or contained in the physical algebra and

(ii) the physical algebra contains complete observables.

Finally, for the quantisation of the classical Hamilton constraint some classical transformations, which simplify the constraint, has been used. Consequently, a quantum analogue of the non-modified classical Hamilton constraint need not of be the form (2.4). The main difficulty is to find an quantum analogue of the curvature. In this dissertation the ideas for the quantisation of the classical connections, holonomy along paths, fluxes and curvature have been reviewed. Some new modifications of the development of the quantum variables have been presented. The new quantum variables, which have been developed, have been used for a first attempt of a construction of a new algebra of quantum gravity. The first step in this direction is given by the holonomy groupoid algebra for a gauge theory, which has to be generalised for a gravitational theory. This algebra is not comparable with the Weyl algebra for surfaces or the other holonomy-flux algebras, since the algebra is constructed from different quantum variables. The full detailed study of the formulation of quantum gravity in terms of this new ansatz is a new project, which will extend this dissertation.

In the following a more detailed review about the achievements of this work is presented.

The quantum configuration variables: holonomies along paths

The fundamental geometric objects for a theory of Loop Quantum Gravity have been (semi-) analytic paths and loops that form graphs. In chapter 3 the following main objects have been introduced and are shortly reviewed in the next paragraph.

A *graph* contains a finite set of independent edges. A set of edges is called *independent* if the edges only intersect each other in the source or target vertices. A *finite groupoid* is a finite set of paths equipped with a groupoid structure. The *finite graph system* associated to a graph Γ is given by all subgraphs of Γ. A *finite path groupoid* associated to the graph Γ is generated by all compositions of elements or their inverse elements of the set of edges that defines the graph Γ. Note that an element of a finite path groupoid is not necessarily an independent path. Clearly, for all these objects there exists an ordering such that

(i) an *inductive family of graphs*

(ii) an *inductive family of finite path groupoids* and

(iii) an *inductive family of finite graph systems* can be studied.

Furthermore, a *holonomy map* is a groupoid morphism from the path groupoid to the compact structure group G. If a graph is considered, then the holonomy map maps each edge of the graph to an element of the structure group G. For generality it is assumed that G is a locally compact unimodular group. In section 3.3.4 two ways of an identification of the holonomy map evaluated for a subgraph of Γ with elements in $G^{|\Gamma|}$ have been presented. One distinguishes between the *natural* or *the non-standard identification of the configuration space* $\bar{\mathcal{A}}_\Gamma$ with $G^{|\Gamma|}$. Recall that a subgraph of Γ is a set of independent paths, which are generated by the edges of the graph Γ. In the natural identification these paths are decomposed into the edges, which define the graph Γ. In the non-standard identification only graphs that contain only non-composable paths are considered. In both cases the holonomy maps evaluated on a subgraph Γ' of Γ are elements of G^M, where M is the number of paths in Γ'. One obtains a product group G^M for $M \leq |\Gamma|$, and which is embedded into $G^{|\Gamma|}$ by $G^M \times \{e_G\} \times ... \times \{e_G\}$. Hence, in both cases the holonomy evaluated on a subgraph of a graph Γ is an element of $G^{|\Gamma|}$. In LQG [8, 10, 104] a holonomy map evaluated at the graph Γ is an element of $G^{|\Gamma|}$, too.

The *analytic holonomy C^*-algebra restricted to a finite graph system associated to a graph* has been given by the commutative unital C^*-algebra $C(\bar{\mathcal{A}}_\Gamma)$ of continuous functions on the configuration space $\bar{\mathcal{A}}_\Gamma$ vanishing at infinity and supremum norm.

The inductive limit C^*-algebra has been constructed from an inductive family of C^*-algebras, which depend on finite graph systems. The reason is the following: Consider *graph-diffeomorphisms* of the finite graph system associated to a graph Γ. These objects are pairs of maps and have been presented section 3.4. For short such a pair consists of a bijective map from vertices to vertices, which are situated in the manifold Σ, and a map that maps subgraphs to subgraphs of Γ. Then there are actions of these graph-diffeomorphisms on the analytic holonomy C^*-algebra restricted to a finite graph system associated to the graph Γ. There is no well-defined action of these graph-diffeomorphisms on the analytic holonomy C^*-algebra restricted to a fixed graph in general. This can be verified as follows. Assume that $\Gamma := \{\gamma_1, \gamma_2, \gamma_3\}$ is a graph and $\Gamma' := \{\gamma_1\}$, $\Gamma'' := \{\gamma_1 \circ \gamma_3\}$ are subgraphs of Γ . Then consider a graph-diffeomorphism (φ, Φ) such that $\Phi(\Gamma') = \Gamma''$. Now the action $\zeta_{(\varphi,\Phi)}$ on the analytic holonomy C^*-algebra restricted to the graph Γ, which is defined by

$$(\zeta_{(\varphi,\Phi)} f_\Gamma)(\mathfrak{h}_\Gamma(\Gamma)) = f_{\Phi(\Gamma)}(\mathfrak{h}_{\Phi(\Gamma)}(\Phi(\Gamma))) = f_{\Gamma'''}(\mathfrak{h}_{\Gamma'''}(\Gamma'''))$$

whenever $\Phi(\Gamma) = \Gamma''' = \{\gamma_1 \circ \gamma_3, \gamma_2, \gamma_3\}$ is not well-defined. The reason is: Γ''' is not a graph. If $\Phi(\Gamma)$ is a subgraph of Γ, then in particluar $f_{\Phi(\Gamma)}$ is an element of the analytic holonomy C^*-algebra restricted to the subgraph $\Phi(\Gamma)$. The analytic holonomy C^*-algebra restricted to every subgraph of Γ is a C^*-subalgebra of the analytic holonomy C^*-algebra restricted to the graph Γ. Hence, the last C^*-algebra is in particular a C^*-algebra, which is characterised by the finite graph system associated to Γ. An action of graph-diffeomorphisms is an automorphism of the analytic holonomy C^*-algebra restricted to finite graph system associated to Γ. Summarising, the concepts of the limit of C^*-algebras restricted to finite graph systems, and actions of graph-diffeomorphisms on the holonomy C^*-algebra restricted to finite graph systems engage with each other.

Finally note that, the inductive limit C^*-algebra of the inductive family of C^*-algebras $\{C(\bar{\mathcal{A}}_\Gamma), \beta_{\Gamma,\Gamma'}\}$ defines the *projective limit configuration space* $\bar{\mathcal{A}}$. The inductive limit C^*-algebra $C(\bar{\mathcal{A}})$ has been called the *analytic holonomy C^*-algebra* in this dissertation.

The idea of using families of graph systems has been influenced by the work of Giesel and Thiemann [42] in the LQG framework. They have used particular cubic graphs instead of sets of paths in a groupoid and their inductive limit has been constructed from families of cubic graph systems. In this dissertation the *inductive limit Hilbert space* $\mathcal{H}_\infty$ has been derived from the natural or non-standard identified configuration spaces, the Haar measure on the structure group G and an inductive limit of finite graph systems. It has been assumed that, the inductive limit graph system only contains a countable set of subgraphs of an inductive limit graph Γ_∞. This is contrary to the Hilbert space used in LQG literature [104], which is given by the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$. The Hilbert space $\mathcal{H}_{\text{AL}}$ is manifestly non-separable, since the limit is taken over all sets of paths in Σ and, hence over an infinite and uncountable set of all graphs. Clearly, the Hilbert space $\mathcal{H}_\infty$ is constructed by using certain identification of the configuration space and the countable set of subgraphs. In this simplified formulation some important aspects of the theory have been studied in this dissertation. It is possible to generalise partly the results for the Ashtekar-Lewandowski Hilbert space.

The *classical configuration space* in the context of LQG and Ashtekar variables is the space of smooth connections $\breve{\mathcal{A}}_s$ on an arbitrary principal fibre bundle $P(\Sigma, G)$. In this dissertation the quantum operator $\mathcal{Q}(A)$ of the infinitesimal connection A has been given by the holonomy $\mathfrak{h}$ along a path γ. The operator $\mathcal{Q}(A)$ has been represented as a multiplication operator on the inductive limit Hilbert space $\mathcal{H}_\infty$.

For a construction of a completely new algebra of quantum variables, which is derived from holonomies, fluxes and curvature, the setup of the configuration variables has to be changed. This is described later.

The quantum momentum variables: group-valued or Lie algebra-valued flux operators

In this dissertation the quantum operator $\mathcal{Q}(E^i)$ of the classical flux E^i has been either a group- or Lie algebra-valued operator, which depend on a surface S and a path γ or a graph Γ. The idea of this definition is the following: Consider a surface S and a path γ that intersets each other in the source vertex of γ and the path lies below the orientated surface S. Let $\mathfrak{g}$ be the Lie algebra of a compact connected (linear) Lie group G. The *Lie algebra-valued quantum flux operator* $E_S(\gamma)$ is given by the value of a map $E_S : P\Sigma \rightarrow \mathfrak{g}$ evaluated for a path γ in the set $P\Sigma$ of paths in Σ. This definition does not coincide with the usual definition presented in LQG literature completely. In this dissertation the flux-like variables introduced by Lewandowski, Okołów, Sahlmann and Thiemann [64] have been replaced and generalised to Lie algebra-valued quantum flux operators. The *group-valued quantum flux operator* $\rho_S(\gamma)$ are defined similarly by suitable maps $\rho_S : P\Sigma \rightarrow G$.

In general the idea is to obtain algebras, which are generated by

(i) the group-valued quantum flux operators and the holonomies along paths in a graph, or

(ii) the group-valued quantum flux operators and functions depending on holonomies along paths in a graph, or

(iii) the Lie algebra-valued quantum flux operators and the holonomies along paths in a graph, or

(iv) the Lie algebra-valued quantum flux operators and functions depending on holonomies along paths in a graph.

In the following algebras, which are generated among other operators by the Lie algebra-valued quantum flux operators, are presented. Therefore consider either (iii) or (iv) and some certain canonical commutator relations.

The $\mathfrak{g}$-valued quantum flux operator $E_S(\gamma)$ and the holonomy $\mathfrak{h}$ along a path γ satisfy the canonical commutator relation, which is given by

$$[E_S(\gamma), \mathfrak{h}(\gamma)] = \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0} \exp(tE_S(\gamma))\mathfrak{h}(\gamma) - \mathfrak{h}(\gamma)E_S(\gamma) \tag{10.1}$$

whenever $t \in \mathbb{R}$. Set

$$E_S(\gamma)\mathfrak{h}(\gamma) := \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0} \exp(tE_S(\gamma))\mathfrak{h}(\gamma)$$

Furthermore the *right-invariant flux vector field* $e^{\overrightarrow{L}}$ is defined by

$$[E_S(\gamma), f_\Gamma] = e^{\overrightarrow{L}}(f_\Gamma) \tag{10.2}$$

where

$$e^{\overrightarrow{L}}(f_\Gamma)(\mathfrak{h}_\Gamma(\gamma)) := \frac{\mathrm{d}}{\mathrm{d}\, t}\Big|_{t=0} f_\Gamma(\exp(tX_S)\mathfrak{h}_\Gamma(\gamma)) \text{ for } X_S \in \mathfrak{g}, \mathfrak{h}_\Gamma(\gamma) \in G, t \in \mathbb{R} \tag{10.3}$$

whenever $f_\Gamma \in C_0^\infty(\bar{\mathcal{A}}_\Gamma)$.

The quantum flux operator $E_S(\Gamma)$ is represented as the differential operator $\frac{\mathrm{d}}{\mathrm{d}\, t} \exp(tE_S(\gamma))$ on the Hilbert space $\mathcal{H}_\Gamma$. The holonomies along paths or the functions depending on holonomies along paths are represented as multiplication operators on the Hilbert space $\mathcal{H}_\Gamma$.

Until now, a suitable set of surfaces in Σ and a path γ in the finite path groupoid $\mathcal{P}_\Gamma\Sigma$ are fixed. For a general situation the following maps have been studied in section 3.4:

(i) a certain map $E_S : \mathcal{P}_\Gamma\Sigma \rightarrow \mathfrak{g}$

(ii) a certain map $E_S : \mathcal{P}_\Gamma\Sigma \rightarrow \mathcal{E}$

(iii) a certain map $\rho_S : \mathcal{P}_\Gamma\Sigma \rightarrow G$

(iv) a cetrain map $\rho_S : \mathcal{P}_\Gamma\Sigma \rightarrow \mathcal{Z}$, where $\mathcal{Z}$ denotes the center of the group G, and

(v) a certain map $\varrho : \mathcal{P}_\Gamma\Sigma \rightarrow G$ and this map ϱ has been called *admissible* in analogy to Fleischhack [39].

Then the maps of the form E_S given by (i) (or (ii)) define a Lie algebra (or an enveloping algebra), which depends on a fixed path γ in $\mathcal{P}_\Gamma\Sigma$ and on surfaces in a suitable fixed surface set $\breve{S}$. Note that, the surface set always contains at least one surface in Σ. This Lie algebra has been called the *Lie flux algebra associated to a surface set and a path.* The maps ρ_S given by (iii) (or (iv)) define a group, which depends on the fixed path γ and a suitable fixed surface set $\breve{S}$. This group has been called *flux group* $\bar{G}_{\breve{S},\gamma}$ *associated to a surface set* $\breve{S}$ *and a path* γ. Clearly, for each suitable surface set there exist a flux group associated to this surface set. The maps of the form ϱ given by (v) have been used to define a more complicated structure. Furthermore, this concept generalises to holonomies of a graph Γ, which are maps from graphs to products of the structure group G. Then for example the *flux group* $\bar{G}_{\breve{S},\Gamma}$ *associated to a surface set and a graph* has been defined in definition 3.4.14.

Now, for the group-valued or the Lie algebra-valued quantum flux operators different actions on the configuration space have been explicitly considered in section 6.1. In particular the left, right and inner actions have been studied independently from each other and have been denoted by L,R or I. Furthermore, only the maps (iv) and (v) define groupoid morphisms by composition of the action L (or R, or I) and the holonomy map. For an overview about which maps define groupoid morphisms consider table 11.2 in chapter 11. Note that, using admissible maps (maps of the form (v)) particular morphisms are defined. These morphisms have been called *equivalent groupoid morphisms* in analogy to Mackenzie [66] and have been related to gauge transformations on the configuration space. The flux groups constructed from the maps (iii) and (iv), the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ and the actions L, R or I define C^*-dynamical systems. If admissible maps are taken into account, the C^*-dynamical systems are very complicated.

The starting point of Fleischhack's construction [39] of an algebra has been the analysis of homeomorphisms on the projective limit configuration space $\bar{\mathcal{A}}$. He has assumed that, G is a compact connected Lie group. The analytic holonomy algebra has been given by the unital commutative C^*-algebra $C(\bar{\mathcal{A}})$ and has been represented on the Hilbert space $\mathcal{H}_{\text{AL}}$ as multiplication operators. The Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{\text{AL}}$ is equivalent to $L^2(\bar{\mathcal{A}}, \mu_{\text{AL}})$, where μ_{AL} is a measure on $\bar{\mathcal{A}}$. Measure preserving transformations have been implemented by certain homeomorphisms on the configuration space $\bar{\mathcal{A}}$. They correspond to unitary operators on the Hilbert space $\mathcal{H}_{\text{AL}}$. The Weyl algebra of Quantum Geometry [39] has been generated by functions in $C(\bar{\mathcal{A}})$ and these unitaries. Hence, elements of the Weyl algebra are for example of the form f, fU or U, if f is an element of $C(\bar{\mathcal{A}})$ and U is a unitary operator on the Hilbert space $\mathcal{H}_{\text{AL}}$. On the other hand, homeomorphisms on the projective limit Hilbert space define automorphisms on the C^*-algebra $C(\bar{\mathcal{A}})$. In this dissertation these automorphisms have played a fundamental role.

But for example the parameter group of automorphism, which is defined from arbitrary group-valued quantum flux operators $\rho_S(\gamma)$ for every surface S and a fixed path γ to the group of automorphisms, i.e. $\rho_S(\gamma) \mapsto \alpha(\rho_S(\gamma)) \in \mathfrak{Aut}(C(\bar{\mathcal{A}}_\gamma))$, does not define a group homomorphism to the group of automorphisms in $C(\bar{\mathcal{A}}_\gamma)$. This is only true for certain group-valued quantum flux operators, which form a flux group associated to a certain surface set. Furthermore, the analytic holonomy C^*-algebra can be restricted to certain subgraphs of a graph Γ. Therefore, the following object is important. A *finite orientation preserved graph system* is a set of certain subgraphs of a graph Γ such that all paths in a subgraph are generated by compositions of the edges that generate the graph Γ. Note that in this definition the composition of edges and inverses of this edges are excluded. Then clearly there is an action of the flux group associated to the graph Γ and a surface set on the analytic holonomy C^*-algebra restricted to the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$. Furthermore, there is an action of the flux group associated to every subgraph of the finite orientation preserved graph system $\mathcal{P}^{\mathfrak{o}}_\Gamma$ and a surface set on the analytic holonomy C^*-algebra restricted to a finite orientation preserved graph system. There is a set of exceptional C^*-dynamical systems, which is defined by these automorphisms of the flux groups associated to suitable surface sets and graphs on the analytic holonomy algebras restricted to finite orientation preserved graph systems. The restriction to orientation preserved subgraphs is necessary to obtain either a purely left or right action of the flux group associated to a fixed surface set and subgraphs of a particular graph system on the holonomy C^*-algebra restricted to suitable graph systems.

In general there are C^*-dynamical systems, which are constructed from left and right actions of the flux group associated to a surface set and a graph on the analytic holonomy C^*-algebra restricted to the finite graph system.

The Gelfand-Naĭmark theorem implies that there is an isomorphism between commutative C^*-algebras and continuous function algebras on configuration spaces. If other in particular non-abelian C^*-algebras are studied, then automorphisms of the algebras do not correspond to certain homeomorphisms on the configuration spaces. More generally, covariant representations of the C^*-dynamical systems replace the construction of Fleischhack. A covariant representation is a pair of maps, which is given by a representation of the C^*-algebra on the Hilbert space and a unitary representation of the flux group, and these maps satisfy a certain canonical commutator relation. In this dissertation the *Weyl C^*-algebra for surfaces* has been constructed from all C^*-dynamical systems, which contains all actions of the flux groups associated to all different surface sets on the analytic holonomy C^*-algebra. In particular an element of the *Weyl algebra of a surface set $\breve{S}$ restricted to a finite graph system* $\mathcal{P}_\Gamma$ is for example of the form

$$\sum_{l=1}^{L} \mathbb{1}_\Gamma U_{S_1}(\rho^l_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K}\sum_{i=1}^{M} f^k_\Gamma U_{S_2}(\rho^i_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K}\sum_{i=1}^{M} U_{S_3}(\rho^i_{S,\Gamma}(\Gamma)) f^l_\Gamma U_{S_3}(\rho^i_{S,\Gamma}(\Gamma))^* + \sum_{p=1}^{P} f^p_\Gamma$$

whenever $f^k_\Gamma, f^l_\Gamma, f^p_\Gamma \in C_0(\bar{\mathcal{A}}_\Gamma)$, $U_{S_i} \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$. The notion $U_{S_i} \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ means that the unitary operators are represented on the C^*-algebra $\mathcal{K}(\mathcal{H}_\Gamma)$ of compact operators on the Hilbert space $\mathcal{H}_\Gamma$. Furthermore, the unitaries and products of these unitaries, which satisfy the canonical commutator relation, have been called *Weyl elements* in this dissertation. Some further comments on the Weyl algebras and the relation to the holonomy-flux cross-product *-algebra are given in the next section.

The use of C^*-dynamical systems have several advantages in comparison to the ansatz of Fleischhack, which are given by:

(i) The operator algebraic formulation in terms of C^*-dynamical systems is independent of a particular Hilbert space.

(ii) From C^*-dynamical systems new algebras have been constructed. One example has been constructed in chapter 7 and has been called the holonomy-flux cross-product C^*-algebra. Furthermore, the framework allows to replace for example the C^*-algebra of quantum configuration variables.

(iii) The Weyl C^*-algebra for surfaces and the holonomy-flux cross-product *-algebras have been constructed in the same framework such that the uniqueness of the state, which is invariant under certain diffeomorphisms, has been obtained easily in both cases. For a comparison of the constructions refer to table 11.1 in chapter 11.

(iv) The operator algebraic framework can be used to study KMS-theory in LQG (which has not been considered in the LQG framework until now).

The quantum spatial diffeomorphisms

In this dissertation the classical spatial diffeomorphisms have been replaced by new quantum diffeomorphism constraints. The classical diffeomorphism constraints are certain diffeomorphisms in the spatial hypersurface Σ. In Mackenzie [66] a concept of translations in a general Lie groupoid has been presented. The ideas have been used for a redefinition of the diffeomorphism constraints. The new operators have been called bisections. The idea of the definition of a bisection is presented in the next paragraph.

In the theory of groupoids the following object is often used: the groupoid isomorphism in a path groupoid, which consists of the classical diffeomorphism in Σ and an additional bijective map from paths to paths in the path groupoid over Σ. This pair of maps is called the *path-diffeomorphisms of a path groupoid.* The path-diffeomorphisms extend the notion of graphomorphisms, which have been introduced by Fleischhack [39]. There is only a slight difference betweeen these objects: A graphomorphism is a map from Σ to Σ that preserves additionally the path groupoid structure, whereas a path-diffeomorphism is a pair of maps. In particular *finite path-diffeomorphisms* are given by a pair of maps, which contains a map that maps paths to paths in a finite path groupoid $\mathcal{P}_\Gamma\Sigma$ and a bijective map that maps vertices to vertices of the vertex set of the graph Γ. Moreover, since graph systems are used in this project, a pair of maps that contains a map, which maps subgraphs to subgraphs, plays a fundamental role and is called *finite graph-diffeomorphism.* Graphomorphisms define in particular groupoid isomorphisms and, hence, they transform non-trivial paths to non-trivial paths. To define maps that transform a trivial path at a vertex in Σ to a non-trivial path other objects have to be considered. Translations in a finite path groupoid are naturally given by adding or deleting edges, which generate the graph Γ. One distinguishes between three translations in a path groupoid. One is given by adding a path γ to a path θ at the source vertex $s(\theta)$ of the path θ. The other case is given by composition of a path γ to a path θ at the target vertex $t(\theta)$ of the path θ. Finally, two paths can be composed with a path at the source and target vertices simultaneously. Hence, there is a natural map from the set of vertices to the set of paths, which is called a *bisection of a finite path-groupoid.* For such a bisection σ the map $t \circ \sigma$ is assumed to be bijective, where t denotes the target map of the finite path groupoid. In the definition of a *bisection of a path groupoid* the map $t \circ \sigma$ is required to be a diffeomorphism from Σ to Σ and the map σ maps vertices to paths in a path groupoid. Furthermore a *right-translation R_σ of a bisection σ* is a map that composes a path γ with the path $\sigma(t(\gamma))$, i.e. $R_\sigma(\gamma) = \gamma \circ \sigma(t(\gamma))$. Furthermore a *left-translation L_σ* and an *inner-translation I_σ of a bisection σ* can be defined similarly. The pair consisiting of the composition $t \circ \sigma$ of the bisection and the target map and the right translation R_σ define in general no groupoid isomorphism. Nevertheless there are particular translations of suitable bisections that define path-diffeomorphisms. There is no doubt that the notion of a bisection can be generalised to a *bisection of a path groupoid* or a *bisection of a finite graph system.* Moreover, the bisections of a path groupoid form a group and there is a group homomorphism between this group and the group of diffeomorphisms in Σ. Moreover, the bisections of a finite path groupoid or a finite graph system equipped with a sophisticated group multiplication form groups, too. Finally, a quantum diffeomorphism is assumed to be an element of the group of bisections of a path groupoid, a finite path groupoid or a finite graph system.

Actions of the group of bisections on the analytic holonomy C^*-algebra restricted to a finite graph system have been used in section 6.2 to construct C^*-dynamical systems. If the group $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections of a finite graph system $\mathcal{P}_\Gamma$ is considered, then the right-, left- or inner-translation of the bisections define three different C^*-dynamical systems. For example, there is a C^*-dynamical system $(C_0(\bar{\mathcal{A}}_\Gamma), \mathfrak{B}(\mathcal{P}_\Gamma), \zeta)$, where the action ζ is defined by the right-translation of the bisections. For each C^*-dynamical system there exists a covariant representation on the Hilbert space $\mathcal{H}_\Gamma$. Hence, the right- , left- or inner-translation of the bisections define unitary operators on the Hilbert space $\mathcal{H}_\Gamma$ associated to a graph. The main advantage of translations of bisections is that, they define graph changing operators. In particular these maps transfrom subgraphs into subgraphs of a graph Γ such that the number of edges of the subgraphs can change.

Both actions, which are the action of the group of bisections of a finite graph system and the action of the flux group on the configuration space, lead to automorphisms on the analytic holonomy C^*-algebra. A comparison of the actions can be found in table 11.2 in chapter 11. Similarly to actions of the flux group, the actions of the group of bisections composed with holonomy maps do not define groupoid morphisms in general. This causes no problems, since the configuration space restricted to a finite graph system $\mathcal{P}_\Gamma$ is identified (naturally or in non-standard way) with $G^{|\Gamma|}$ and the right-, left- or inner-translation in the finite path groupoid transfer to *right-translation R_σ, left-translation L_σ* or *inner-translation I_σ in the groupoid G over $\{e_G\}$.* Finally, notice that only actions of certain bisections preserve the flux operators associated to a surface S. For example consider the bisection σ of a path groupoid and recall the diffeomorphism $t \circ \sigma$. Then for example the diffeomorphism $t \circ \sigma$ is required to preserve the surface S. This particular bisection is called the *surface-preserving bisection of a path groupoid.* There exists a similar description for a *surface-preserving bisection for a finite path groupoid or a finite graph system.* Then the concept can be extended to bisections of a finite graph system that map surfaces to surfaces in a certain surface set and preserve the orientation of the surfaces with respect to the transformed subgraph. In this situation the bisections are called *surface-orientation-preserving bisections for a finite graph system* and they form a subgroup of the group of bisection of a finite graph system. Finally both actions on the analytic holonomy C^*-algebra restricted to a finite graph system:

(i) the action of the group of surface-orientation-preserving bisections for a finite graph system and

(ii) the action of the center of the flux group associated to a surface set

commute. In analogy to the surface-orientation-preserving bisections of a finite graph system the *surface-orientation-preserving graph-diffeomorphisms* can be constructed.

Finally there is an action of bisections of the path groupoid $\mathcal{P}$ over Σ or the inductive limit graph system $\mathcal{P}_{\Gamma_\infty}$ on the analytic holonomy C^*-algebra $C(\bar{\mathcal{A}})$. This automorphism is not point-norm continuous. Consequently, the infinitesimal diffeomorphism constraint is not implemented as a Hilbert space operator.

Algebras in Loop Quantum Gravity

The Weyl C^*-algebras and the holonomy-flux *-algebras

The main objects, which have been introduced in this dissertation, are given by

- the flux groups or the Lie flux algebras of Lie flux groups associated to surface sets,
- the analytic holonomy C^*-algebra, which is given by the inductive limit C^*-algebra of an inductive family of analytic holonomy C^*-algebras restricted to finite graph systems.

They have been used for the definition of the Weyl C^*-algebra for surfaces, the holonomy-flux cross-product C^*-algebra and the holonomy-flux cross-product *-algebra. These three algebras have been constructed by using different representations of the flux group, or of functions depending on elements of the flux group or of the Lie flux algebra and the representation of the analytic holonomy C^*-algebra in the C^*-algebra $\mathcal{L}(\mathcal{H}_\infty)$ of bounded operators on the inductive limit Hilbert space $\mathcal{H}_\infty$. The ideas for the development of the Weyl C^*-algebra for surfaces and the holonomy-flux cross-product *-algebra are presented in the next paragraphs. A detailed overview about the correspondence between the two algebras is presented in table 11.4 in chapter 11. The following degrees of freedom have been used for a construction of these algebras in LQG: (i), (v) and (vi) given in section 1.5.

In the last section the construction of the Weyl algebra has been introduced. The Weyl algebra of Quantum Geometry [39] has been constructed from the analytic holonomy C^*-algebra and unitary operators, which are defined by weakly continuous one-parameter unitary groups of $\mathbb{R}$ on the Hilbert space $\mathcal{H}_{\text{AL}}$. The unitaries have been called Weyl operators by Fleischhack. The Weyl C^*-algebra for the surface set and restricted to a finite graph system has been generated by the analytic holonomy C^*-algebra restricted to a finite graph system and Weyl elements. Assume that G is a compact connected Lie group. Then consider a strongly continuous one-parameter unitary group of $\mathbb{R}$, which is given by $\mathbb{R} \ni t \mapsto U(\exp(tE_S(\gamma))$, on the Hilbert space $\mathcal{H}_\infty$. Then each unitary $U(\exp(tE_S(\gamma))$ defines a *Weyl element*, too.

To obtain a uniqueness result of a representation of a C^*-algebra the following general facts have been used. Since irreducible representations of a C^*-algebra on a Hilbert space correspond one-to-one to pure states on the C^*-algebra, the uniqueness of a particular representation of the C^*-algebra on a Hilbert space corresponds to a unique state. The inductive limit of an inductive family of C^*-algebras corresponds one-to-one to a projective limit on the projective family of state spaces of the C^*-algebras. The GNS-representation associated to a state of a C^*-algebra consists of a cyclic vector Ω on a Hilbert space and a representation of the C^*-algebra on the Hilbert space.

The uniqueness of a finite surface-orientation-preserving graph-diffeomorphism invariant pure state of the commutative Weyl C^*-algebra for surfaces has been obtained in theorem 6.4.6 by several steps. The *commutative Weyl C^*-algebra for surfaces* has been constructed similarly to the Weyl C^*-algebra for surfaces with the difference that the group G is replaced by the center of the group G. Then graph-diffeomorphism invariant states of the commutative Weyl algebra for surfaces restricted to a graph system $\mathcal{P}_\Gamma$ have been analysed. It turns out that a difference occur, if either the natural or if the non-standard identification of the configuration space $\bar{\mathcal{A}}_\Gamma$ is taken into account. In particular, for the natural identification one state is a sum over states, which are indexed by bisections. For the commutative Weyl algebra for surfaces the difference has disappeared. There exists a pure and unique state, which is invariant under finite graph-diffeomorphisms. This result is similar to the uniqueness of the representation of the Weyl algebra of Quantum Geometry and has been obtained in a complete new operator algebraic formulation.

Furthermore, a comparable uniqueness result of the holonomy-flux cross-product *-algebra has been achieved in the operator algebraic framework, too. The uniqueness is directly related to the uniqueness result of the Weyl algebra for surfaces. The holonomy-flux cross-product *-algebra presented in section 8.2 is related to the holonomy-flux *-algebra [64]. This new *-algebra is, in particular, an abstract cross-product algebra. This mathematical object has been presented by Schmüdgen and Klimyk [53] in the context of Hopf algebras. Similarly to the *-algebras in Quantum Mechanics, which have been presented in section 1.3, the new *holonomy-flux cross-product *-algebra* is generated by the identity $\mathbb{1}$, the holonomies along paths and the Lie algebra-valued quantum flux operators satisfying the canonical commutator relations (10.2). If the surfaces are restricted to a certain set of surfaces, then this algebra has been called the *holonomy-flux cross-product *-algebra associated to a surface set*. In contrast to the holonomy-flux *-algebra the construction of the holonomy-flux cross-product *-algebra is independent of the Hilbert space and the representation of the operators on the Hilbert space. For the definition of the *holonomy-flux* *cross-product* **-algebra for a graph* Γ *and a surface set* $\breve{S}$ the enveloping flux algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ associated to a surface set $\breve{S}$ and a graph Γ is necessary. This abstract cross-product *-algebra is given by the tensor vector space of the analytic holonomy C^*-algebra restricted to a graph and the enveloping flux algebra associated to a surface set equipped with a multiplication operation, which is derived from a certain action of enveloping flux algebra associated to a surface set on the analytic holonomy C^*-algebra restricted to a graph. In particular, it has been used that the analytic holonomy C^*-algebra restricted to a graph is a right (or left) $\bar{\mathcal{E}}_{\breve{S},\Gamma}$-module algebra.

The *-representation of the enveloping flux algebra associated to a surface set and a graph has been given by *infinitesimal representation of the flux group associated to the surface set* $\breve{S}$ *and the graph* Γ on the Hilbert space $\mathcal{H}_\Gamma$. The *-representation π of the holonomy-flux cross-product *-algebra associated to the graph Γ and the surface set $\breve{S}$ is the representation $\mathrm{d}\,U_{\overrightarrow{L}}$ of the enveloping flux algebra associated to the surface set and the graph and the representation Φ_M of the analytic holonomy C^*-algebra restricted to the graph. Consequently, an element $f_\Gamma \otimes E_S(\gamma)$ is represented on the Hilbert space $\mathcal{H}_\Gamma$ by

$$\pi(f_\Gamma \otimes E_S(\gamma)) := \frac{1}{2}\Phi_M(e^{\overrightarrow{L}}(f_\Gamma)) + \frac{1}{2}\Phi_M(f_\Gamma)\,\mathrm{d}\,U_{\overrightarrow{L}}(E_S(\gamma))$$

where $e^{\overrightarrow{L}}$ denotes the right-invariant vector field and $E_S(\gamma)$ is an element of the enveloping flux algebra associated to a surface set $\breve{S}$ and a graph Γ. The representation extends to a representation of the holonomy-flux cross-product *-algebra associated to a surface set $\breve{S}$. In theorem 8.2.20 it has been shown that, the corresponding state is the unique surface-orientation-preserving graph-diffeomorphism invariant state of the holonomy-flux cross-product *-algebra associated to the surface set $\breve{S}$.

In particular, the analysis have shown the reason for the difficulty of a construction of other representations of this *-algebra. One searches for other *-representation of the enveloping flux algebra associated to a surface set and a graph, which satisfy a certain graph-diffeomorphism invariance condition and which are distinguished from an infinitesimal representation. In particular, since the right-invariant vector fields associated to a surface set and a graph define a *-derivation δ on the analytic holonomy C^*-algebra restricted to a graph, the corresponding state ω of the representation of the analytic holonomy C^*-algebra is assumed to satisfy $\omega(\delta(f_\Gamma)) \neq 0$ for every $f_\Gamma \in C_0^\infty(\bar{\mathcal{A}}_\Gamma)$. Consequently the state associated to the *-representation of the holonomy-flux cross-product *-algebra restricted to the analytic holonomy *-algebra is required to satisfy a similar condition. In this dissertation the conditions for such states associated to new *-representations have been presented, but the states, or respectively the representations, are not explicitly constructed.

The same problem of finding other representations of the algebras have been occured for the Weyl C^*-algebra for surfaces or the holonomy-flux cross-product C^*-algebra, too. For example for the Weyl C^*-algebra for surfaces the important fact is that the flux group associated to a surface set has been represented on the Hilbert space $\mathcal{H}_\Gamma$ by a unitary representation in $\mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$. The only naturally or satisfactory representations of the group- (or enveloping algebra-)valued quantum flux operators have been given by:

(i) Weyl elements, which are given by unitary representation of the flux group on the Hilbert space $\mathcal{H}_\Gamma$,

(ii) the differential operators, which are given by the infinitesimal representation of the enveloping flux algebra on the Hilbert space $\mathcal{H}_\Gamma$.

These representations define the natural representations of the following algebras:

(i) the Weyl C^*-algebra for surfaces,

(ii) the holonomy-flux cross-product *-algebra.

Remark that, the problem of finding other representations can be solved if the generating set of operators for the algebras does not contain unitary or differential operators derived from the flux group associated to a surface set. A new idea for a solution has been introduced by Buchholz and Grundling in [26]. They have proposed a resolvent C^*-algebra in the context of QFT. This C^*-algebra is generated by the resolvents of the Segal operators instead of unitaries, which generate the original Weyl C^*-algebra. In this dissertation a similar construction of C^*-algebras generated by a set of operators and relations among them have been presented in section 8.5 and is reviewed briefly in the next section.

Finally the same objects, which define the Weyl algebra for surfaces, have generated the *holonomy-flux von Neumann algebra* in section 6.5. Note that, this feature is related to the degree of freedom (iii).

The holonomy-flux cross-product C^*-algebras, other cross-product C^*-algebras and other *- or C^*-algebras

The cross-product C^*-algebras for holonomies, fluxes and graph-diffeomorphisms

There is no obvious reason why the Weyl algebra of Quantum Geometry or the Weyl C^*-algebra for surfaces are the exceptional C^*-algebras of quantum configuration and momentum operators in LQG. New C^*-algebras of quantum variables in LQG have been given by the holonomy-flux cross-product C^*-algebra for a surface set and the multiplier algebra of the holonomy-flux cross-product C^*-algebra for a surface set. These algebras are introduced briefly in the next paragraphs and have been constructed in section 7.2. For a comparison of the Weyl C^*-algebra for surfaces and the new holonomy-flux cross-product C^*-algebra refer to table 11.4 in chapter 11. In particular the new algebras have been defined by using the degrees of freedom (iii), (iv) and (v) given in section 1.5.

Recall for a moment the construction of the Weyl C^*-algebra for surfaces of the last section. There the requirement of the group-valued flux operators to be unitary Hilbert space operators has been the important starting point. If this choice is not made, then the group-valued quantum flux operators can be represented on the Hilbert space $\mathcal{H}_\Gamma$ by the *generalised group-valued flux operators*, which are given by the integrated representations of the flux group associated to a surface set and the graph Γ on the Hilbert space $\mathcal{H}_\Gamma$. Furthermore, consider instead of the group-valued quantum flux operators contained in the flux group $\bar{G}_{\breve{S},\Gamma}$ for a surface set and a graph, certain functions which depend on the flux group and which map to the algebra $C_0(\bar{\mathcal{A}}_\Gamma)$. These functions form the *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$. This *-algebra equipped with the L^1-norm is, in particluar, a Banach *-algebra. Then a representation of the Banach *-algebra $L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$ on the Hilbert space $\mathcal{H}_\Gamma$ is derived from the unitary representations $\text{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$ and has been called the *Weyl-integrated holonomy-flux representation* on $\mathcal{H}_\Gamma$. This new representation has been used for the definition of the *holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and the graph* Γ. The construction of this algebra uses a particluar action of a fixed flux group $\bar{G}_{\breve{S},\Gamma}$ on the analytic holonomy C^*-algebra $C_0(\bar{\mathcal{A}}_\Gamma)$ restricted to a finite graph system $\mathcal{P}_\Gamma$, and hence a particular C^*-dynamical system and depends highly on the fixed choice of the surface set $\breve{S}$. If another surface set $\breve{T}$ is considered, then a new holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{T}$ and the graph Γ can be constructed. An element of this algebra is not contained in general in the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and the graph Γ.

Indeed there have been many distinguished C^*-dynamical systems for different surface sets (refer to section 6.1) and consequently there exists a set of holonomy-flux cross-product C^*-algebras associated to different surface sets. A particular surface set $\breve{S}$ is chosen such that $\breve{S}$ has the *simple surface intersection property for a graph* Γ. This means that each path in Γ intersects only one surface in $\breve{S}$ only once in the target vertex of the path. There are no other intersection points between paths and surfaces. Then it have been proved in 7.2.12 with main arguments for the proof given in remark 7.2.10 that, the *multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and the graph* Γ contains all operators of the holonomy-flux cross-product C^*-algebras associated to other suitable surface sets and the graph Γ. The idea of the proof is the following.

An element of this multiplier algebra is a linear map from the holonomy-flux cross-product C^*-algebra associated to a surface set $\breve{S}$ and the graph to the holonomy-flux cross-product C^*-algebra associated to a surface set $\breve{S}$ and the graph that satisfies a certain condition, which is connected to the existence of an adjoint operator. Hence, one has to show that particular maps are multipliers. Such linear maps can be for example given by the (left) multiplication of an element of the holonomy-flux cross-product C^*-algebra associated to a suitable surface set $\breve{T}$ and the graph. Note that $\breve{T}$ can be chosen to be equal to $\breve{S}$. It is clear that if the multiplier algebra of another holonomy-flux cross-product C^*-algebra associated to a surface set $\breve{R}$ and the graph is considered, then an element of the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and the graph can be an element of this multiplier algebra, too. But an element of the multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and the graph is in general not an element of the multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{R}$ and the graph.

In section 7.2 states on a holonomy-flux cross-product C^*-algebra that depend on the choice of the surface set have been presented. Hence these states are not generally path- or graph-diffeomorphism invariant.

Furthermore assume G to be compact. Then there exists an inductive family of holonomy-flux cross-product C^*-algebras associated to the fixed surface set $\breve{S}$ and graphs, which defines the inductive limit C^*-algebra. This

C^*-algebra has been called the *holonomy-flux cross-product C^*-algebra for the surface set $\breve{S}$*. This C^*-algebra is in particular constructed from the analytic holonomy C^*-algebra, the flux group associated to the fixed surface set and the particluar action of this group on the analytic holonomy C^*-algebra. Similarly to the multiplier algebra of a cross-product C*-algebra associated to the surface set and a fixed graph, the *multiplier algebra of the holonomy-flux cross-product C^*-algebra for the surface set $\breve{S}$* contains all elements of the holonomy-flux cross-product C^*-algebras for other suitable surface sets. Moreover, the multiplier algebra of the holonomy-flux cross-product C^*-algebra for the suitable fixed surface set contains the holonomy-flux cross-product C^*-algebra for the surface set and the holonomy-flux cross-product C^*-algebra for the surface set and a graph.

In the last paragraphs new C^*-algebras of a special kind have been constructed. All these algebras are based on new operators, which are more general than group-valued quantum flux operators and which take, in particluar, values in the analytic holonomy C^*-algebra. Until now the quantum diffeomorphisms are implemented only as automorphisms on these algebras. In the following paragraphs one of the previous algebras is extended such that functions on the group of bisections of a finite graph system to the holonomy-flux cross-product C^*-algebra, form this new C^*-algebra.

The cross-product C^*-algebra construction is particularly based on C^*-dynamical systems. It has been argued that, the action of the group of bisections of a finite graph system on the analytic holonomy C^*-algebra restricted to a finite graph system define a C^*-dynamical system, too. Furthermore, there is also an action of the group of certain bisections of a finite graph system on the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ and a graph. These objects define another C^*-dynamical system and a new cross-product C^*-algebra, which has been called the *holonomy-flux-graph-diffeomorphism cross-product C^*-algebra* in section 7.3.

There exists a covariant representation of this C^*-dynamical system on a Hilbert space. This pair is given by a unitary representation of the group of surface-orientation-preserving bisections of a finite graph system on the Hilbert space $\mathcal{H}_\Gamma$ and the multiplication representation Φ_M of the analytic holonomy C^*-algebra restricted to the finite graph system $\mathcal{P}_\Gamma$ on $\mathcal{H}_\Gamma$. The unitaries have been called the *unitary bisections of a finite graph system and surfaces* in this dissertation. Then each unitary bisections of a finite graph system and surfaces is contained in the *multiplier algebra of the holonomy-flux-graph-diffeomorphism cross-product C^*-algebra associated to a graph and the surface set*. The remarkable point is that the multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to a graph and the surface set does not contain these unitaries.

In general, the multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to a fixed surface set contains all operators of holonomy-flux cross-product C^*-algebra for other suitable surface sets, elements of the analytic holonomy C^*-algebra and all Weyl elements associated to other suitable surface sets. The Weyl C^*-algebra for surfaces contains elements of the analytic holonomy C^*-algebra and all Weyl elements. The multiplier algebra of the holonomy-flux cross-product C^*-algebra associated to the surface set $\breve{S}$ contains the Weyl algebra for suitable surface sets. The Lie algebra-valued quantum flux operators and the right-invariant vector fields are affiliated with the holonomy-flux cross-product C^*-algebra, but they are not affiliated with the Weyl C^*-algebra for surfaces. For a detailed overview about the multiplier algebras and affiliated elements with the C^*-algebras of quantum variables refer to table 11.6 in chapter 11.

The cross-product C^*-algebras for holonomies or fluxes

The cross-product C^*-algebra construction depends on the choice of the quantum configuration and momentum variables. The quantum configuration and momentum variables and the algebras are studied separately from each other in the next paragraphs and have been studied explictly in section 7.1.

First consider only the group-valued quantum flux operators that define a flux group associated to a graph and a surface set. Then there exists a certain cross-product C^*-algebra, which is only derived from quantum flux operators and which has been therefore called the *flux transformation group C^*-algebra associated to a graph and a surface set*.

A cross-product C^*-algebra derived only from holonomies along paths of a graph has been called the *heat-kernel-holonomy C^*-algebra*. The name of this algebra has been influenced by the work of Ashtekar and Lewandowski [9, section 6.2], where the authors have studied heat kernels. This algebra is distinguished from the analytic holonomy C^*-algebra. The heat-kernel-holonomy C^*-algebra associated to a graph contains certain functions on the configuration space $\bar{\mathcal{A}}_\Gamma$ to the analytic holonomy C^*-algebra.

All algebras defined in the previous paragraph have been constructed from the basic quantum variables, which are given by the holonomies along paths and the quantum fluxes. Some of them have been even indirectly proposed in LQG literature before. Hence, they are possible algebras of a quantum theory of gravity.

Simplified cross-product C^*-algebras for holonomies and fluxes

If both quantum variables: the quantum configuration and momentum variables restricted to a fixed graph Γ and a fixed suitable surface set $\breve{S}$ are considered simultaneously, then the following simplifications can be studied.

The flux group of a fixed graph Γ and a fixed suitable surface set $\breve{S}$ and the configuration space $\bar{\mathcal{A}}_\Gamma$ are identified with $G^{|\Gamma|}$. Moreover, the corresponding cross-product C^*-algebra $C_0(G^{|\Gamma|}) \rtimes_\alpha G^{|\Gamma|}$ is Morita equivalent to the C^*-algebra of compact operators on the Hilbert space $L^2(G^{|\Gamma|}, \mu_\Gamma)$, where μ_Γ denotes the product of $|\Gamma|$ Haar measures. Therefore, the representation theory of both C^*-algebras is the same and, hence, there is only one irreducible representation of the cross-product C^*-algebra up to unitary equivalence.

But this identification is only true for certain surface sets. The cross-product C^*-algebra is derived from the quantum momentum variables, which depend on the surface sets. In particular the flux groups associated to a suitable surface set can be identified with a product group G^M where $M \leq |\Gamma|$. Then there exists a left (or right) action of G^M on the C^*-algebra $C_0(G^{|\Gamma|})$. For $M < |\Gamma|$ a Morita equivalent C^*-algebra has been not found in this project. In theorem 7.1.11 a Morita equivalent algebra for the C^*-algebra $C_0(G^N) \rtimes_\alpha G^M$ whenever $N < M$, has been given.

In this dissertation the general case of arbitrary surfaces has been studied. Hence, the quantum configuration and the momentum variables of the theory are manifestly distinguished from each other. The quantum configuration variables only depends on graphs and holonomy mappings, whereas the quantum momentum variables depend on graphs, maps from graphs to products of the structure group, and the intersection behavior of the paths of the graphs and surfaces. But, nevertheless, the elements of the holonomy-flux cross-product C^*-algebra are understood as compact operators on the flux group associated to a surface set with values in the analytic holonomy C^*-algebra restricted to a graph, which are acting on the Hilbert space $L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$.

Other *- or C^*-algebras for holonomies, fluxes and other flux operators

There are two different *-algebras, which are completed to C^*-algebras, which contains certain continuous functions on a locally compact group. The difference between the *-algebras are related to the choice of the pointwise multiplication or the convolution multiplication operation. These two different *-algebras are completed to two different C^*-algebras. In this dissertation the analytic holonomy C^*-algebra has been obtained from the pointwise multiplication and the *non-commutative holonomy C^*-algebra* has been obtained from the convolution operation. Note that, this is related to the degrees of freedom (ii) and (iii) given in section 1.5. For a compact group this issue has been studied in 8.1 explicitly. In Quantum Mechanics the locally compact group is replaced by the abelian locally compact group $\mathbb{R}^n$. Then these two C^*-algebras are isomorphic. The two C^*-algebras obtained by the two different multiplication operations are not isomorphic for arbitrary non-abelian locally compact groups. Hence, in general in this dissertation the analytic holonomy *-algebra and the non-commutative holonomy *-algebra have not been isomorphic.

In the LQG literature, the compact group $SU(2)$ has been often used, but it is not the only structure group, which has been studied. For example, the non-compact group $\mathrm{SL}(2, \mathbb{C})$ in [7] or the compact quantum group $SU_q(2)$ in [73] have been used, too. In particular, Lewandowski and Okołów [73] have used the non-commutative holonomy C^*-algebra, which they construct from the quantum group. Hence in the LQG framework, the difficulties, which arise by replacing the compact connected Lie group by other groups or quatum groups, have to be analysed. This is the reason for the choice of a locally compact structure group G in this dissertation for the construction of the Weyl algebras for surfaces and the holonomy-flux cross-product C^*-algebras. Clearly, for more general groups and in particular for $\mathrm{SL}(2, \mathbb{C})$ the development has to be studied in detail once more. The starting point of the construction of the holonomy-flux cross-product *-algebra and other *-algebras has been a compact Lie group.

Furthermore, new ideas for a construction of C^*-algebras are available by the concept of affiliated operators. For example Woronowicz [110] has developed a construction of C^*-algebras by a finite set of bounded or unbounded operators. Moreover, Buchholz and Grundling [26] have introduced a new C^*-algebra in the context of QFT. These ideas have been used for the definition of new algebras, which are presented briefly in the next paragraphs.

On the one hand, the Lie algebra-valued quantum flux operators for a surface in a surface set and a fixed graph have been replaced once more by new operators. The new operators have been given by the *flux Nelson transforms for a surface set and a fixed graph*, which are similarly to the resolvents of Buchholz and Grundling or the Z-transforms of Schmüdgen or Woronowicz. The *holonomy-flux Nelson transform C^*-algebra associated to a surface set and a fixed graph* in section 8.5 has been generated by the operators: holonomies along paths of a fixed graph and the flux Nelson transform for a surface set and a fixed graph and canonical commutator relations similarly to (10.2).

On the other hand, instead of the Lie algebra-valued quantum flux operators for a surface in a surface set and a fixed graph, the functions depending on holonomies along paths are replaced by polynomials of, or respectively by representative functions depending on, holonomies along paths of a fixed graph. Then for example a new *-algebra have been constructed by the operators: polynomials of holonomies along paths of a fixed graph and Lie algebra-valued quantum flux operators for a surface in a surface set and the fixed graph and canonical commutator relations similarly to (10.1). This new *-algebra has been called the *Heisenberg polynomial-holonomy-flux *-algebra associated to a graph and a surface set* in section 8.2.2. The name of the algebra has been influenced by Schmüdgen and Inoue, who have defined the Heisenberg O^*-algebra in Quantum Mechanics for example in [51]. The important degree of freedom for this construction is (ii) given in section 1.5. If all continuous functions in $C^\infty(\bar{\mathcal{A}}_\Gamma)$ are considered, then it is possible to construct the *Heisenberg holonomy-flux *-algebra associated to a graph and a surface set*, which contains these functions and the quantum fluxes associated to a surface set $\breve{S}$ and a graph Γ with values in the enveloping algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ and which satisfy some canonical commutator relation. This relation is distinguished from the canonical commutator relation of the holonomy-flux *-algebra associated to a graph and a surface set.

Notice that these algebras have been derived from the basic quantum variables of LQG and have been completely new in this framework.

The important question

Finally, the important question is the following: Which algebra is the algebra of a quantum theory of gravity? Hence, physical reasons have to be taken into account to answer the question why some algebras are more suitable than others.

The quantum constraints of Loop Quantum Gravity

In the LQG framework the quantum constraint algebra is generated by the quantum gauge constraints, the quantum diffeomorphism constraints and the quantum Hamilton constraint. The quantum gauge constraints have been replaced by elements of the *local flux group*. The elements of the *fixed point algebra associated to the action of the local flux group* given in section 6.2 are invariant under the action of the local flux group. In the next subsection the algebra, which is generated by the quantum diffeomorphism constraints, is analysed.

The quantum spatial diffeomorphism constraints

In one of the last sections the quantum spatial diffeomorphism constraints have been introduced as bisections of a path groupoid, a finite path groupoid or a finite graph system. It has been argued that, the group $\mathrm{Diff}(\Sigma)$ of classical diffeomorphisms in the spatial manifold Σ is replaced by the group $\mathfrak{B}(\mathcal{P})$ of bisections in a path groupoid $\mathcal{P}$ over Σ. In the next paragraph this issue is treated once more.

It has been discussed that, the unitary bisections of a finite graph system and surfaces are neither contained in the Weyl C^*-algebra for surfaces nor in the holonomy-flux cross-product C^*-algebra. Similarly, the finite surface-orientation-preserving graph-diffeomorphisms, or respectively the surface-orientation-preserving bisections of a finite graph system, are not contained in these algebras. But they are affiliated with a larger C^*-algebra, which has been given by the holonomy-flux-graph-diffeomorphism cross-product C^*-algebra given in section 7.3.

The quantum Hamilton constraint

In section 2.1.3 the quantum Hamilton constraint of LQG has been shortly introduced. The question that arises is the following: If the quantum Hamilton constraint operator is well-defined, then which algebra presented before, this operator is contained in or affiliated with? The quantum Hamilton constraint operator is not contained in or affiliated with any of these algebras. Even the simpliest version of the quantum Hamilton constraint operator, which is given by

$$\lim_{T\to\Sigma}\sum_{\Delta\in T}\left(\mathfrak{h}_A(l_\Delta)-\mathfrak{h}_A(l_\Delta)^{-1}\right)\mathfrak{h}_A(e_\Delta)[\mathfrak{h}_A(e_\Delta)^{-1},\mathcal{Q}(V)] \tag{10.4}$$

is not contained in or affiliated with any of the algebras presented before. In the next section new algebras are developed. Then a certain modificated quantum Hamilton constraint operator is related to a new *-algebra, which is called the localised holonomy-flux cross-product *-algebra, and which is introduced in the next section.

KMS-Theory and a physical algebra for Loop Quantum Gravity

The physical algebra of quantum variables for LQG

In particluar in Loop Quantum Gravity the theory of KMS-states is inseparable from the problem of time evolution, the implementation of the quantum constraints and the issue of the physical algebra. Briefly the *set of conditions for a physical algebra of quantum variables* is assumed to be given by

(i) the quantum constraint operators are affiliated with or contained in the physical algebra and

(ii) the physical algebra contains complete quantum observables.

In particular, complete quantum observables are derived from Dirac states and Dirac observables, which are defined by the constraint operators. Furthermore, KMS-states, states that define time averages, or states that define expectations of the time of occurence of an event of the physical algebra of quantum variables have to be studied.

In the next paragraphs the issue of KMS-states of the algebras presented before is analysed. In the mathematical theory of KMS-states modular objects play a fundamental role. These objects are given by the modular automorphism group and the modular conjugation. Hence, in particular modular automorphisms of the Weyl C^*-algebra for surfaces are studied.

KMS-Theory for algebras of quantum variables for LQG

The holonomy-flux von Neumann algebra given in section 6.5 has not allowed a fruitful implementation of Tomita-Takesaki theory, since for this von Neumann algebra a cyclic and separating vector has not been available. For the Weyl C^*-algebra for surfaces, a KMS-theory has been studied for different automorphsims. The simpliest automorphism is generated by the exponentiated Lie algebra-valued quantum flux operator $\exp(E_S(\Gamma)^+E_S(\Gamma))$ associated to a surface S and a graph, or to the limit graph Γ_∞ of an inductive family $\{\Gamma\}$ of graphs. But, it has been shown in section 6.5.8 that, there are no KMS-states of the Weyl C^*-algebra for surfaces associated to this automorphism.

Since there are no other natural automorphisms on the Weyl C^*-algebra for surfaces, the theory of KMS-states in LQG has been very hard to investigate. The non-existence of KMS-states is related to the fact that for example on the Weyl C^*-algebra for surfaces the automorphism group defined by the flux operator $E_S(\Gamma)^+E_S(\Gamma)$ is inner. Since, modular automorphisms characterise the C^*-algebra by outer norm-continuous automorphisms, a good ansatz is to change the automorphisms. This issue is treated in the next paragraphs.

The automorphisms related to finite path- or graph-diffeomorphisms have been introduced. The automorphisms on the analytic holonomy C^*-algebra, which has been constructed from inductive families of finite graph systems, are not very sensitive on the choice of the particular graphs in the following sense. This is due to the identification of the quantum configuration space $\bar{\mathcal{A}}_\Gamma$ restricted to a finite graph system $\mathcal{P}_\Gamma$ with some products of the compact group G. Consequently, the automorphisms on the analytic holonomy C^*-algebra restricted to a finite graph system $\mathcal{P}_\Gamma$ are maps, that map functions depending on $G^M \times \{e_G\} \times ... \times \{e_G\}$-valued operators to functions depending on $G^K \times \{e_G\} \times ... \times \{e_G\}$-valued operators, where $M, K \leq |\Gamma|$. There is also an automorphism on the analytic holonomy C^*-algebra restricted to a finite graph system $\mathcal{P}_\Gamma$, which maps functions depending on $H^M \times \{e_G\} \times ... \times \{e_G\}$-valued operators to functions depending on $H^K \times \{e_G\} \times ... \times \{e_G\}$-valued operators, where $M, K \leq |\Gamma|$ and H is a closed subgroup of G. But there have not been any KMS-states of the analytic holonomy C^*-algebra restricted to a finite graph system $\mathcal{P}_\Gamma$ associated to any of these automorphisms (refer to theorem 6.5.10).

Furthermore, Tomita-Takesaki theory have been analysed for the Weyl C^*-algebra for surfaces. This C^*-algebras has been constructed from inductive families of finite graph systems, too. In the previous paragraph it has been argued that, the modular automorphism is independent of transformations of the graph systems. Consequently, the choice of the graph system has to be such that there are suitable automorphisms, which are generated by graph-diffeomorphisms and which leave all graphs globally invariant. Then the modular automorphism should implement the dynamics of the theory, and consequently this automorphism is related to the one-parameter velocity transformation along the foliation parameter. Until now all quantum variables are implemented on a fixed Cauchy surface Σ. If the algebra of quantum variables is enlarged such that the algebra elements depend on different Cauchy surfaces, then the one-parameter group of automorphisms maps algebra elements, which depend on a certain Cauchy surface, to algebra elements, which depend on the transformed Cauchy surface. These automorphisms are suggested to map quantum configuration variables, which are defined on a fixed Cauchy surface, to quantum operators that are derived from quantum configuration and momentum variables, which are defined on the fixed Cauchy surface, too. Note that, such automorphisms are not defined by either the holonomies along paths or the group-valued (or the Lie algebra-valued) quantum flux operators. These automorphisms are derived from both quantum operators. Indeed such automorphisms should be related to the quantum Hamilton constraint or a modified quantum Hamilton constraint and are required to commute with the automorphisms related to graph-diffeomorphisms.

But the Weyl C^*-algebra for surfaces has not admitted a KMS-state even for the simple automorphism, which has been derived from the exponentiated Lie algebra-valued quantum flux operator. Hence, the author suggests that, the Weyl C^*-algebra for surfaces does not admit a KMS-state for any automorphism derived from the untraced version of the quantum Hamilton constraint. Consequently, the last possibility is to change the C^*-algebra of quantum variables or to consider O^*-algebras.

Summarising, the author proposes the following ansätze for a KMS-theory in LQG. The analysis of the quantum constraints and their relation to the algebras of quantum variables implies that, the modular objects in Loop Quantum Gravity have to be implemented

(i) on a *-subalgebra of the holonomy-flux cross-product *-algebra, or

(ii) on a new *-algebra, which is called the localised holonomy-flux cross-product *-algebra, or a new C^*-algebra derived from this new *-algebra.

Furthermore,

(iii) the group of quantum spatial diffeomorphisms has to be restricted to a subgroup of this group, and

(iv) the flux group associated to a surface set has to be restricted to a closed subgroup of this flux group.

The localised holonomy-flux cross-product *-algebra

The idea of the construction of the new algebra in section 8.4 has been influenced by the work of Thiemann and Giesel [42, 43, 44, 45]. They have considered the quantum Hamiltonian operator in the framework of cubic lattices and infinite C^*-tensor algebras. A comparison of the localised holonomy-flux cross-product *-algebra and the holonomy-flux cross-product *-algebra can be found in table 11.5 in chapter 11. In particular in comparison with the algebras presented before the degree of freedom (vi) has been used for the construction of the localised holonomy-flux cross-product *-algebra.

For the definition of the localised holonomy-flux cross-product *-algebra it has been used that the flux operators are manifestly localised by the surfaces in the manifold Σ. The new configuration space has been divided into two parts. One of them has been constructed from holonomies along paths that start or end at some given surface and has been called the *localised part of the configuration space*. The other part has been built from holonomies along paths that do not intersect any surface in this surface set. Hence, the first configuration space is localised on surfaces, while the second is not. Furthermore, there are two different *-algebras of quantum holonomy variables. One *-algebra is constructed on the localised part of the configuration space and a convolution product between functions depending on this space. In particular, the *-subalgebra of central functions on the localised part of the configuration space has been used. The other *-algebra is given by the original analytic holonomy *-algebra, but is restricted to non-localised paths. These *-algebras are completed to different C^*-algebras and the C^*-tensor product of these two C^*-algebras defines the new *localised analytic holonomy C^*-algebra*. The C^*-algebra of central functions on the localised part of the configuration space has been called the localised part of the localised analytic holonomy C^*-algebra. This certain C^*-algebra admits KMS-states.

There are some new flux operators, which have been defined as difference operators between Lie algebra-valued quantum flux operators on different graphs, and which have been called the *localised and discretised flux operators associated to surfaces*. The main difference between the original Lie algebra-valued quantum flux operator and the localised and discretised Lie algebra-valued flux operator both restricted to a fixed graph is that, the second operator is only non-trivial on paths, which are not contained in a certain subgraph. The *localised enveloping flux algebra associated to a surface set* has been derived from the localised and discretised flux operators. Furthermore, there exists an action of this new localised and discretised flux operator on the C^*-algebra of central functions on the localised part of the configuration space.

In this dissertation the theory of an abstract cross-product *-algebra have been used to define a new holonomy-flux cross-product *-algebra. This construction have been also used for the definition of two new localised algebras. One algebra is based on the *-algebra of central functions on the localised part of the configuration space and the other is derived from the localised analytic holonomy *-algebra. The abstract cross-product *-algebra, which is obtained from the localised enveloping flux algebra associated to a surface set and the *-algebra of central functions on the localised part of the configuration space, has been called the *localised part of the localised holonomy-flux cross-product *-algebra*. The *localised holonomy-flux cross-product *-algebra* has been given by the abstract cross-product *-algebra, which has been obtained by the the localised analytic holonomy *-algebra and the localised enveloping flux algebra associated to a surface set. There have been several localised holonomy-flux cross-product *-algebras for different surface sets. It has been also possible to construct a *localised holonomy-flux cross-product C^*-algebra* associated to a surface set similarly to the holonomy-flux cross-product C^*-algebra.

In this dissertation the *discretised quantum volume operator* $\mathcal{Q}(V)_\mathrm{d}$ has been constructed as a sum over Lie algebra-valued quantum flux operators indexed by a triple of paths in a graph that start at a common vertex, which is an intersection of three surfaces.

Consider the *Lie holonomy algebra*, which is constructed from the localised configuration space restricted to a graph Γ and the non-standard identification of this space with the product group $G^{|\Gamma|}$ of a compact connected Lie group G. This Lie algebra acts on the C^*-algebra of central functions on the localised configuration space restricted to a graph, too. Then the C^*-algebra of central functions has admitted KMS-states with respect to this action. The *modified quantum Hamilton constraint restricted to a graph* has been given by

$$\exp(H^+_{\Gamma_i} H_{\Gamma_i}) := \big(\mathfrak{h}_A(\alpha) - \mathfrak{h}_A(\alpha)^{-1}\big)\,\mathfrak{h}_A(\gamma)[\mathfrak{h}_A(\gamma)^{-1}, \mathcal{Q}(V)_\mathrm{d}]$$

The *modified quantum Hamilton constraint* has been defined in this dissertation as the limit

$$H := \lim_{i\to\infty} \sum_{\Gamma_i} \exp(H^+_{\Gamma_i} H_{\Gamma_i})$$

of a sum over subgraphs of a graph of the modified quantum Hamilton constraint restricted to a graph. Note that, the limit graph has been assumed to contain an infinite countable set of subgraphs.

The next step is to show that this modified quantum Hamilton constraint is well-defined and is given as the generator of a strongly continuous one-parameter group of automorphisms on the localised part of the localised analytic holonomy C^*-algebra. The analysis of parts of the modified quantum Hamilton constraint have shown that, the convergence of the limit in the norm-topology is not obvious and is related to the structure of the discretised quantum volume operator $\mathcal{Q}(V)_\mathrm{d}$. Summarising, the norm-convergence of the limit of H is not easy to derive. The author conjectures that, this limit converges and does not depend on a particular Hilbert space representation of the modified quantum Hamilton constraint.

After the consideration of the quantum Hamilton constraint, which has been given in this dissertation by the modified quantum Hamilton constraint, a quantum Master constraint is studied in the following paragraphs.

In the previous sections translations defined by bisections of finite path groupoids or finite graph systems have played a fundamental role. The most general operators, which depend on bisections of finite graph systems that preserve a surface set $\breve{S}_{\rm d}$, have been denoted by $D^{\sigma}_{\breve{S}_{\rm d},\Gamma}$ and have been called the *localised finite quantum diffeomorphism constraints*. For example such operators can be defined similarly to elements of the holonomy-flux-graph-diffeomorphism cross-product C^*-algebra. The idea for these quantum constraints is to implement the complicated relations between the classical spatial diffeomorphism constraints and the classical Hamilton constraints on the quantum level.

Then the *modified quantum Master constraint* is defined to be sum of the *localised quantum diffeomorphism constraint*, which is given by

$$D_{\breve{S}_{\rm d}} := \lim_{N\rightarrow\infty} \sum_{i=1}^{N} \sum_{\sigma_l\in\mathfrak{B}(\mathcal{P}_{\Gamma_i})} D^{\sigma_l,*}_{\breve{S}_{\rm d},\Gamma_i} D^{\sigma_l}_{\breve{S}_{\rm d},\Gamma_i}$$

and the modified quantum Hamilton constraint

$$H := \lim_{N\rightarrow\infty} \sum_{i=1}^{N} H^{+}_{\Gamma_i} H_{\Gamma_i}$$

This modified Master constraint generalises the Master constraint, which has been studied by Thiemann [104].

Then the following issues has been partly studied in section 8.4, and will be further completed in a new extension of this dissertation:

- the Dirac state space of the localised holonomy-flux cross-product *- or C^*-algebra with respect to the localised quantum Master constraint,
- the KMS-states of the localised analytic holonomy C^*-algebra and the localised holonomy-flux cross-product *- or C^*-algebra associated to the automorphism group generated by the modified quantum Hamilton constraint,
- the time avarage (2.9) defined by a KMS-state,
- the *localised holonomy-flux-graph-diffeomorphism cross-product *-algebra for surfaces* that contains the localised finite quantum diffeomorphism constraints and all elements of the localised holonomy-flux cross-product *-algebra for surfaces, and the modified quantum Hamilton constraint is affiliated (in an appropriate sense) with this algebra;
- the *localised *-algebra of complete quantum observables for surfaces*, which is derived from the localised holonomy-flux-graph-diffeomorphism cross-product *-algebra for surfaces.

Indeed there is a KMS-theory for O^*-algebras, which has been studied for example by Inoue [51]. One can show that, the localised holonomy-flux cross-product *-algebra is an O^*-algebra. Until now only KMS-states for the localised part of the localised analytic holonomy C^*-algebra have been presented in section 8.4. The author suggests that, there are also KMS-states on the localised holonomy-flux cross-product *-algebra, which are similar to the KMS-states, which have been found. The first suggestion for a physical algebra is given by the localised holonomy-flux-graph-diffeomorphism cross-product *-algebra for surfaces.

The thermal Hamiltonian for LQG

The difficulty of the implementation of the quantum constraints is related to the fact that there is an algebra of constraints, which do not mutually commute and which are not described by a simple algebra. In particular, the quantum constraints are not contained in the algebras of quantum variables, which are usually used in LQG. Consequently, modifications or enlargements of the algebra have to be developed for the implementation of constraints on the operator algebraic level. Moreover, for the construction the expectation of the time of occurence of an event (2.11), the concept of clocks that generate a new enlarged algebra, is necessary.

On the other hand, for a KMS-theory of the algebras of quantum variables, new automorphisms have to be considered. But in the previous sections it has been argued that, there aren't many natural candidates for automorphisms on the Weyl C^*-algebra for surfaces or the holonomy-flux cross-product *-algebra. It is not obvious that, the quantum Hamilton constraint, or respectively a modification of this operator, is the generator of the modular group associated to a KMS-state. In contrast to QFT, the thermal Hamiltonian of this theory is not a constraint of the physical system. Consequently, the Hamiltonian of a clock can be a generator of an automorphism group, too. This is related to the problem of time in Loop Quantum Gravity, since the Hamiltonian is not a true Hamiltonian, it is a constraint. The dynamics of the theory is not implemented by the Hamilton constraint, it is given by an evolution with respect to a clock. Due to the Connes cocycle theorem for von Neumann algebras, there is only one preferred time evolution and, hence the author suggests that the thermal time of the system is related to the clock Hamiltonian instead of the quantum Hamilton constraint. Note that, there is a generalised Connes cocycle theorem even for O^*-algebras, which has been derived by Inoue [51]. The thermal equilibrium state with respect to the clock is not a thermal state with respect to the automorphisms implemented by the Hamilton constraint. The thermal states with respect to clocks have to be Dirac states. But in LQG there have not been any obvious quantised observables contained in the Weyl C^*-algebra or the holonomy-flux von Neumann- or *-algebra, which can be used as clocks. Therefore one may ask, which automorphisms on these algebras lead to self-adjoint operators and which of these operators are thermal Hamiltonians that can be physically interpreted as a clock. Note that, the automorphisms is uniquely determined up to inner automorphisms. Consequently it is possible that, there is some relation between these certain automorphisms associated to the thermal Hamiltonian of the clock and automorphisms associated to the quantum Hamilton constraint.

But, until now, there are no natural physical clocks contained in the algebra of quantum variables. Usually matter fields are used as clocks. Since matter fields are localised objects, an idea is to study the localised algebra of complete quantum observables on surfaces. The theory, which is described by such an algebra, is not completely diffeomorphism invariant, but this invariance can be relaxed. Then only certain diffeomorphisms, which preserve the localised surfaces in which the matter fields are situated, are taken into account. Note that the surfaces, which has been studied in the context of localised algebras, are always discretised in a suitable sense. Finally the full physical algebra can be for example given by a tensor product of a matter field algebra and the localised algebra of complete quantum observables for (discretised) surfaces.

A summary for a KMS-Theory and a suggestion for a physical algebra for LQG

The following important issues have been presented in the previous sections:

- a modification of the quantum configuration space,
- a modification of the quantum spatial diffeomorphism constraints, the quantum Hamilton and the quantum Master constraint and
- a suggestion for a physical *-algebra.

The new algebra is proposed to be the physical algebra of quantum gravity, if the quantum constraint Hamiltonian (2.4) is taken into account. But this quantum operator is constructed from the classical Hamiltonian by several classical transformations. The original classical Hamiltonian contains the classical variables holonomy along paths, fluxes and the curvature. But until now a quantum analogue of a curvature has not been suggested. In the next section the ideas for a construction of an algebra, which is generated by curvature, connections and fluxes for a given principal fibre bundle, ares presented.

Holonomy groupoid C^*-algebra for a gauge and gravitational theory

However, none of the *- or C^*-algebras of the previous sections contain a quantum analogue of the classical variable curvature. This is related to the degree of freedom (vii) given in section 1.5. In particular the classical Hamilton constraint, which has beens given by (2.2), contains curvature. This Hamilton constraint cannot be quantised without changing this operator by some classical modifications until now. The aim of this dissertation about *Algebras of Quantum Variables in LQG* is to find a suitable algebra of quantum variables of the theory. This algebra is specified by the fact that, the quantum Hamilton constraint operator is an element of (or affiliated with) the algebra, which is generated by certain holonomies along paths, quantum fluxes and the quantum analogue of curvature. The certain holonomies are given by generalised holonomy maps, which are a further development of the holonomy maps, which have been presented by Barrett [16].

Barrett has presented a roadmap for the construction of the configuration space of Yang-Mills or gravitational theories. In this project these ideas will play a fundamental role. In general, the quantisation of a gravitational theory in the context of LQG uses substantially the duality between infinitesimal objects like connections and curvature and integrated objects like holonomies or parallel transports. The ideas have been further developed by Mackenzie [66] in a more general context of Lie groupoids. In the context of LQG, this duality has been reviewed briefly section 3.2 by using the theory of Mackenzie. The next paragraphs give a short outline about these objects and how they can be used to construct a new algebra.

In this dissertation a smooth connection has been encoded in terms of a holonomy map. This object has been derived from a new object, which is called a *path connection in a Lie groupoid*. The path connections have been studied originally by Mackenzie [66]. The concept of Mackenzie fits into the framework of holonomy mappings. The new holonomy map has been called the *general holonomy map in a Lie groupoid* and this map is, in particular, a continuous groupoid morphism from the path groupoid to a general Lie groupoid, which satisfies some new conditions.

Let G be a Lie group, then G over $\{e_G\}$ is a simple Lie groupoid. Furthermore, consider a certain path groupoid, which has been called a *path groupoid along tangent germs*. In section 3.3.4 it has been argued that, the general holonomy map for a path groupoid along tangent germs in the groupoid G over $\{e_G\}$ corresponds one-to-one to a path connection. Notice that the original holonomy map has been defined in section 3.3.4 by a groupoid morphism from the path groupoid to the groupoid G over $\{e_G\}$, which satisfies no additional conditions.

In particular a gauge theory has been studied in section 3.1.4. The *generalised holonomy maps for a gauge theory* are certain continuous maps from a path groupoid to the gauge groupoid. The *gauge groupoid* w.r.t. a principal fibre bundle $P(\Sigma, G, \pi)$ is indeed a particular Lie groupoid. These generalised holonomy maps correspond uniquely to a path connection, which is given as the integrated infinitesimal smooth connection over a lifted path in the gauge groupoid. Therefore, in this context the generalised holonomy map for a gauge theory defines a parallel transport in a fixed principal fibre bundle. A fixed holonomy map for a gauge theory defines the *holonomy groupoid for a gauge theory*. Notice that, the original holonomy map is defined by a groupoid morphism from the path groupoid to the groupoid G over $\{e_G\}$. Consequently, this holonomy map does not define a parallel transport in $P(\Sigma, G)$.

The author of this dissertation suggests to generalise the idea of Barrett. Then the set of all general holonomy maps along loops or paths in a Lie groupoid is chosen as the configuration space of the theory. The new configuration space for example in the context of a pure gauge theory has been given by the *set of holonomy maps for a gauge theory* in section 3.3.3. In a more general context the *set of holonomy maps along tangent germs for a arbitrary Lie groupoid* has been defined in section 3.3.4.

A new C^*-algebra has been given by the *holonomy groupoid C^*-algebra for a gauge theory associated to a path connection* in section 9.1. In this framework the configuration space is given by the holonomy groupoid of a gauge theory associated to a path connection. The algebra is defined in analogy to the group algebra of a locally compact group. The measure on this groupoid is inherited from the measure defined on the gauge groupoid. This is similar to the original approach in LQG, where the measure on the quantum configuration space is inherited from the Lie group G.

The next step is to find a replacement of the curvature. The problem of implementing infinitesimal structures like infinitesimal diffeomorphisms and curvature arises from the special choice of the configuration variables. For the construction of the analytic holonomy C^*-algebra in chapter 6, or the non-commutative analytic holonomy C^*-algebra in section 7.1 the original holonomy maps along paths have been used. These maps are, in particular, not necessarily continuous groupoid morphisms from the path groupoid to the structure group G. In this usual approach infinitesimal objects like curvature cannot be treated as operators, which are contained in the algebra of quantum variables. In the new approach by using the theory of Mackenzie, the quantum curvature can be implemented as such an operator.

There exists a generalised Ambrose-Singer theorem given by Mackenzie [66] which states that the *Lie algebroid of the holonomy groupoid* of a path connection is the smallest Lie algebroid, which is generated from the connections and the curvature. In section 9.1 it has been used that there exists a left (or right) action of the exponentiated Lie algebroid elements on the holonomy groupoid C^*-algebra for a gauge theory. Hence, there is an action related to infinitesimal connections and curvature, since both objects are encoded as elements of this Lie algebroid. Hence, the quantum algebra of a gauge theory is generated by the G-valued quantum flux operators, the holonomy groupoid C^*-algebra for a gauge theory and the Lie algebroid of the holonomy groupoid for a path connection. The construction of this algebra has been influenced by the cross-product construction given in section 7.

There exists a cross-product algebra constructed from the left (or right) action of the Lie holonomy groupoid on the algebra $C(G)$, where G denotes the structure group. The development in section 9.2 has been based on Masuda [68, 69]. There ta new cross-product algebra has been shortly proposed. This new algebra contains the holonomy groupoid and G-valued quantum flux operators. The quantum curvature and the connections are not contained in this algebra. But the author of this dissertation suggests that these operators are affiliated in a proper sense with the new algebra, which has been called the *holonomy-flux groupoid C^*-algebra for a gauge theory.*

Summarising new basic quantum variables have been introduced. One of the new quantum variables are the generalised holonomy maps. Consequently a new configuration space of the quantum theory of gravity is given by these maps. Finally, this modification allows a new development of algebras of quantum gravity, which are not comparable to the algebras presented in the previous sections.

In an extension of this dissertation these constructions have to be generalised to a concrete principal fibre bundle of a gravitational theory. Furthermore, the new construction implies new problems, which can be studied in the future. Some of them are the following:

- The new quantum analogue of the classical Hamilton constraint, which contains curvature, can be derived. In particular the theory of constraints and KMS-theory can be studied with respect to this new quantum Hamilton constraint.
- The algebras depend manifestly on the chosen principal fibre bundle. Consequently there is a problem of background independence of the theory (the gauge groupoid depends on the principal fibre bundle). The author of this project suggests to use the ideas of Brunetti, Fredenhagen and Verch [24] to study this issue.

Part III

Comparison tables, Appendix, Symbols, Index and References

Chapter 11

Comparison tables

In chapter 1.3 the author has argued that, for example for Quantum Mechanics different algebras are obtained by using different generating sets of abstract operators. The aim of the construction of the Weyl C^*-algebra, which has been invented in chapter 6, and the holonomy-flux cross-product *-algebras defined in chapter 8, is to use a common setup. Both algebras are generated by functions depending on holonomies along paths, and group- or Lie algebra-valued quantum flux operators. These abstract operators satisfy some canonical commutator relations, which are called Heisenberg relations if the unbounded configuration and momentum operators are studied, or Weyl relations if the bounded configuration and momentum operators are used. Clearly by choosing different sets of operators other *-algebras or respectively C^*-algebas can be constructed. For example if the functions depending on the quantum flux group associated to surfaces, and the functions depending on holonomies along paths, are considered, then these operators generate the holonomy-flux cross-product C^*-algebra, which has been presented in chapter 7. The abstract operators are represented on a common Hilbert spaces as self-adjoint Hilbert space operators. The exponentiated Lie algebra-valued quantum flux operator are implemented by an unitary weakly continuous representation of the group $\mathbb{R}$ on the Hilbert space. The Lie algebra-valued quantum flux operator is related to the infinitesimal representation of this unitary weakly continuous representation. This flux operator is unbounded and self-adjoint. In a Hilbert space independent framework automorphisms of and derivations for the algebra of quantum variables play a fundamental role. In particular strongly continuous one-parameter group of *-automorphisms defines a derivation, which is given by the commutator of two self-adjoint operators. The comparison between the algebras of Quantum Mechanics and the algebras of LQG, which are given by the Weyl C^*-algebra for surfaces and the holonomy-flux cross-product *-algebra, is presented in table 11.1.

Table 11.1: Comparison QM and LQG

	Quantum Mechanics	Weyl alg. for surfaces and holonomy-flux cross-prod. *-alg.
Configuration space	$\mathbb{R}^n$	$\bar{\mathcal{A}} := \varprojlim_{\mathcal{P}_\Gamma \in \mathcal{P}_{\Gamma_\infty}} \bar{\mathcal{A}}_\Gamma$
Momentum space	$\mathbb{R}^n$	$\bar{\mathcal{E}}_{\breve{S}}$ or $\bar{G}_{\breve{S}}$ (where G compact connected Lie group)
Configuration variable I	x_i	$\mathfrak{h}(\gamma_i)$ for $\mathfrak{h} \in \text{Hom}(\mathcal{P}, G)$, $\gamma_i \in \mathcal{P}$
Configuration variable II	$f(x_i)$ for $f \in C_0(\mathbb{R}^n)$	$f(\mathfrak{h}(\gamma_i))$ for $f \in C(\bar{\mathcal{A}})$
Momentum variable I	p_i	$E_{S_j}(\gamma_i)$ for $E_{S_j} \in \mathcal{E}_{\breve{S}}$, $\gamma_i \in \mathcal{P}$, $S_j \in \breve{S}$
Momentum variable II	$\exp(tp_i)$	$\rho_{S_j}(\gamma_i)$ for $\rho_{S_j} \in G_{\breve{S}}$, $\gamma_i \in \mathcal{P}$, $S_j \in \breve{S}$
Dynamical Hamiltonian	$H = \sum_i \frac{p_i^2}{2m} + V(x_1, .., x_n)$	$H = \sum_i Tr\big((\mathfrak{h}(\alpha_i) - \mathfrak{h}(\alpha_i)^{-1}\big)\,\mathfrak{h}(\gamma_i)[\mathfrak{h}(\gamma_i)^{-1}, V])$ $V = \sum_{jk,l} E_{S_1}(\gamma_j)E_{S_2}(\gamma_k)E_{S_3}(\gamma_l)$ for $\alpha_i, \gamma_x \in \mathcal{P}$, $S_m \in \breve{S}$
Hilbert space	$\mathcal{H} := L^2(\mathbb{R}^n, \times_{1\leq k\leq n} \mathrm{d}\, x_k)$	$\mathcal{H}_\infty := L^2(\bar{\mathcal{A}}, \mathrm{d}\, \mu_\infty)$
self-adjoint Hilbert space operator	$\pi(x_i) = x_i$	$\pi(\mathfrak{h}(\gamma_i)) = \mathfrak{h}(\gamma_i)$
unitary Hilbert space operator	$\pi(\exp(tp_i)) = U_{p_i}(t)$ $\big(U_{p_i}(t)\psi\big)(x_i) := \psi(x_i - tp_i)$ for $\psi \in \mathcal{H}$	$\pi(\exp(tE_{S_j}(\gamma_i))) = U_t(E_{S_j}(\gamma_i))$ $\big(U_t(E_{S_j}(\gamma_i))\psi\big)(\mathfrak{h}(\gamma_i)) := \psi(\exp(tE_{S_j}(\gamma_i))\mathfrak{h}(\gamma_i))$ for $\psi \in \mathcal{H}$
unitary Hilbert space operator		$\pi(R_\sigma) = V_\sigma$ for $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$, $t \circ \sigma \in \text{Diff}(V_\Gamma)$
*-automorphism		$\pi(\zeta_\sigma(f)) = V_\sigma \pi(f) V_\sigma^*$, $\zeta_\sigma \in \mathfrak{Aut}(C(\bar{\mathcal{A}}))$
unitary transformation	Fourier transform $\mathcal{F}$	
self-adjoint Hilbert space operator	$\pi(p_i) = -i\frac{\partial}{\partial x_i}$ such that $\pi(p_i)\psi = \mathcal{F}p_i\mathcal{F}^{-1}\psi = -i\frac{\partial}{\partial x_i}\psi$ for $\psi \in D(p_i)$	
self-adjoint Hilbert space operators	$\pi(H)$, $\pi(\exp(tH) =: U_H(t)$	
strongly continuous 1-parameter unitary group	$\mathbb{R} \ni t \mapsto U_H(t)$ such that	$\mathbb{R} \ni t \mapsto U_t(E_{S_j}(\gamma_i))$
self-adjoint Hilbert space operator		$\pi(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i)) = -i\frac{\mathrm{d}}{\mathrm{d}\, t} U_t(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i))$ such that
Stone's theorem	$\frac{\mathrm{d}}{\mathrm{d}\, t}\Big\|_{t=0} U_H(t)\psi = iU_H(t)\Big\|_{t=0}\pi(H)\psi$ for $\psi \in D(H)$ $\frac{\mathrm{d}}{\mathrm{d}\, t}\psi_t = i\pi(H)\psi_t$ for $\psi_t := U_H(t)\psi$ Schrödinger equation	$\frac{\mathrm{d}}{\mathrm{d}\, t}\Big\|_{t=0} U_t(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i))\psi = i\pi(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i))\psi$ for $\psi \in D(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i))$

unitary Hilbert space operator	$\pi(\exp(sx_i)) = V_{x_i}(s)$	
self-adjoint Hilbert space operator		$\pi(f) = f$ for $f \in C(\bar{\mathcal{A}})$
strongly continuous 1-parameter group of *-automorphisms		$\mathbb{R} \ni t \mapsto \alpha_t(E_{S_j}(\gamma_i)) \in \mathfrak{Aut}(C(\bar{\mathcal{A}}))$
Canonical Commutatur Relations	$p_i x_j - x_j p_i = -i\delta_{ij}$ (Heisenberg relations)	$[E_{S_j}(\gamma_i), \mathfrak{h}(\gamma_i)] = i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0} \exp(tE_{S_j}(\gamma_i))\mathfrak{h}(\gamma_i) - \mathfrak{h}(\gamma_i)E_{S_j}(\gamma_i)$
	$V_{x_i}(s)U_{p_j}(t) = \exp(st\delta_{ij})U_{p_j}(t)V_{x_i}(s)$ (Weyl relations)	$[E_{S_j}(\gamma_i), f] = i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0} \alpha_t(E_{S_j}(\gamma_i))(f)$
*-automorphism		$\pi(\alpha(\rho_{S_j}(\gamma_i))(f)) = U(\rho_{S_j}(\gamma_i))\pi(f)U^*(\rho_{S_j}(\gamma_i))$ $\alpha(\rho_{S_j}(\gamma_i)) \in \mathfrak{Aut}(C(\bar{\mathcal{A}}))$
strongly contin. 1-parameter group of *-automorphism		$\mathfrak{B}(\mathcal{P}_\Gamma) \ni \sigma \mapsto \zeta_\sigma \in \mathfrak{Aut}(C(\bar{\mathcal{A}}))$ $\alpha(\rho_{S_j}(\gamma_i)) \circ \zeta_\sigma = \zeta_\sigma \circ \alpha(\rho_{S_j}(\gamma_i))$ for all $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and $\rho_{S_j} \in \mathcal{Z}_{\breve{S},\gamma_i}$
Uniqueness of the GNS-representation	$(\mathcal{H}, \pi, \Omega)$ irreducible, cyclic and regular representation of $C_0(\mathbb{R}^n)$ such that $\omega(f) = \langle \Omega, \pi(f)\Omega\rangle$ and $U_H(t)\Omega = \Omega$	$(\mathcal{H}_\infty, \pi, \Omega)$ irreducible and regular GNS-representation of $C(\bar{\mathcal{A}})$ such that $\omega_M(f) = \langle \Omega, \pi(f)\Omega\rangle$, $V_\sigma\Omega = \Omega$ for all $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and $U(\rho_{S_j}(\gamma_i))\Omega = \Omega$ for all $\rho_{S_j} \in \mathcal{Z}_{\breve{S}}$ and $\gamma_i \in \mathcal{P}$ $(\mathcal{H}_\infty, \Phi, \Omega_M)$ irreducible and regular GNS-repr. of $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ such that $\bar{\omega}_M(W) = \langle \Omega_M, \Phi(W)\Omega_M\rangle$ $V_\sigma\Omega_M = \Omega_M$ for all $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma)$ and $U(\rho_{S_j}(\gamma_i))\Omega_M = \Omega_M$ for all $\rho_{S_j} \in \mathcal{Z}_{\breve{S}}$ and $\gamma_i \in \mathcal{P}$ (w.r.t. natural or non-standard identification of conf. space)
symmetric *-derivation with domain $D(\delta)$		$\delta_{S_j}(f) := i[E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i), f]$ $\omega_M(\delta_{S_j}(f)) = 0$
Uniqueness of the state		$\tilde{\omega}_M$ is the unique state on $C^\infty(\bar{\mathcal{A}}) \rtimes_L \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ such that $\tilde{\omega}_M \circ \alpha_\sigma = \tilde{\omega}_M$ for all $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^{\mathrm{o}})$ $\tilde{\omega}_M \circ \alpha_t(E_{S_j}(\gamma_i)^+ E_{S_j}(\gamma_i)) = \tilde{\omega}_M$ for all $E_{S_j}(\gamma_i) \in \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$

Table 11.2: Actions on the configuration space

Actions of the flux group $\bar{G}_{S,\gamma}$	Actions of the group $\mathfrak{B}(\mathcal{P})$ of bisections
$L(\rho_{S_1}(\gamma))(\mathfrak{h}(\gamma)) = \rho_{S_1}(\gamma)\mathfrak{h}(\gamma) =: (L(\rho_{S_1}) \circ \mathfrak{h})(\gamma)$ $R(\rho_{S_2}(\gamma))(\mathfrak{h}(\gamma)) = \mathfrak{h}(\gamma)\rho_{S_2}(\gamma)^{-1} =: (R(\rho_{S_2}) \circ \mathfrak{h})(\gamma)$ $I(\rho_{S_3}(\gamma))(\mathfrak{h}(\gamma)) = \rho_{S_3}(\gamma)\mathfrak{h}(\gamma)\rho_{S_3}(\gamma)^{-1} =: (I(\rho_{S_3}) \circ \mathfrak{h})(\gamma)$ $\rho_{S_i} \in G_{S_i,\gamma}$ for a suitable surface S_i and path γ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $L(\rho_{S_1}) \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$ $R(\rho_{S_2}) \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$ $I(\rho_{S_3}) \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$	$(\mathfrak{h} \circ L_\sigma)(\gamma) = \mathfrak{h}(\sigma^{-1}(s(\gamma)))\mathfrak{h}(\gamma)$ $(\mathfrak{h} \circ R_\sigma)(\gamma) = \mathfrak{h}(\gamma)\mathfrak{h}(\sigma(t(\gamma)))$ $(\mathfrak{h} \circ I_\sigma)(\gamma) = \mathfrak{h}(\sigma^{-1}(s(\gamma)))\mathfrak{h}(\gamma)\mathfrak{h}(\sigma(t(\gamma)))$ $\sigma \in \mathfrak{B}(\mathcal{P})$ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $\mathfrak{h} \circ L_\sigma \notin \mathrm{Hom}(\mathcal{P}, G)$ $\mathfrak{h} \circ R_\sigma \notin \mathrm{Hom}(\mathcal{P}, G)$ $\mathfrak{h} \circ I_\sigma \in \mathrm{Hom}(\mathcal{P}, G)$
$\varrho \in \mathrm{Map}^{\mathrm{A}}(\mathcal{P}, G)$ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $L(\varrho) \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$ $R(\varrho) \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$ $I(\varrho) \circ \mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$	$(\Phi, \varphi) \in \mathrm{Diff}(\mathcal{P})$ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $(\mathfrak{h} \circ \alpha_{(\Phi,\varphi)})(\gamma) = \Phi(\gamma)$ $\mathfrak{h} \circ \alpha_{(\Phi,\varphi)} \in \mathrm{Hom}(\mathcal{P}, G)$
$\rho_{S_i} \in \mathcal{Z}_{S_i,\gamma}$ for a suitable surface S_i and path γ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $L(\rho_{S_1}) \circ \mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $R(\rho_{S_2}) \circ \mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $I(\rho_{S_3}) \circ \mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$	$v \in V_\Gamma$ fixed s.t. $v = t(\gamma)$ $w \in V_\Gamma$ fixed s.t. $w = s(\gamma)$ $\mathfrak{h} \in \mathrm{Hom}(\mathcal{P}, G)$ $(\mathfrak{h} \circ L_{\sigma(w)})(\gamma) = \mathfrak{h}(\sigma^{-1}(w))\mathfrak{h}(\gamma)$ $(\mathfrak{h} \circ L_{\sigma(v)})(\gamma) = \mathfrak{h}(\gamma)$ $(\mathfrak{h} \circ R_{\sigma(v)})(\gamma) = \mathfrak{h}(\gamma)\mathfrak{h}(\sigma(v))$ $(\mathfrak{h} \circ R_{\sigma(w)})(\gamma) = \mathfrak{h}(\gamma)$ such that $(L_{\sigma(v)}, t \circ \sigma) \in \mathrm{Diff}(\mathcal{P})$ $(R_{\sigma(v)}, t \circ \sigma) \in \mathrm{Diff}(\mathcal{P})$ $\mathfrak{h} \circ L_{\sigma(v)} \in \mathrm{Hom}(\mathcal{P}, G)$ $\mathfrak{h} \circ R_{\sigma(v)} \in \mathrm{Hom}(\mathcal{P}, G)$

Table 11.3: Comparison of C^*-algebras

	Weyl algebra of Quantum Geometry	Weyl algebra for surfaces	holonomy groupoid algebra for a gauge theory
ingredients	principal fibre bundle $P(\Sigma, G, \pi)$	principal fibre bundle $P(\Sigma, G, \pi)$	gauge groupoid w.r.t $P(\Sigma, G, \pi)$
	surfaces with codim. ≥ 1	set $\mathbb{S}$ of finite set $\breve{S}$ of surfaces with codim. 1	
assumption	G compact connected Lie group	G locally compact group	G locally compact group
path space	semi-analytic paths	semi-analytic paths	smooth paths
ingredients	fin. path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ	fin. path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ	
	path groupoid $\mathcal{P}$ over Σ	path groupoid $\mathcal{P}$ over Σ	path groupoid $\mathcal{P}$ over Σ
ingredients	hyph ν (for example a graph)	fin. graph system $\mathcal{P}_\Gamma$ associated to Γ	
inductive limit of	inductive family of hyphs	inductive family of fin. graph systems	
holonomy map	groupoid morph. $\mathfrak{h}$ from $\mathcal{P}$ to G	holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma\Sigma$ to G, or	contin. groupoid morphism $\mathfrak{h}_\Lambda$ from $\mathcal{P}$ to $\mathcal{G}$
		holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma$ to $G^{\vert\Gamma\vert}$	holonomy groupoid $\mathrm{Hol}^P_\Lambda(\Sigma)$ over Σ
config. space	$\bar{\mathcal{A}}_\nu$	$\bar{\mathcal{A}}_\Gamma$	space of holonomy maps $\mathcal{K}$
assumption	identification of	natural or non-standard identification of	
	a set of paths in $\mathcal{P}_\Gamma\Sigma$	a set of independent paths in $\mathcal{P}_\Gamma\Sigma$	
projective limit	$\bar{\mathcal{A}} := \varprojlim_{\nu} \bar{\mathcal{A}}_\nu$	$\bar{\mathcal{A}} := \varprojlim_{\mathcal{P}_\Gamma\in\mathcal{P}_{\Gamma_\infty}} \bar{\mathcal{A}}_\Gamma$	
Hilbert space	$L^2(\bar{\mathcal{A}}, \mu_{AL})$ non-separable	$L^2(\bar{\mathcal{A}}, \mu_\infty)$ separable	L^2-space constructed from families of measures constr. from gauge groupoid
C^*-algebra	$C(\bar{\mathcal{A}}_\nu)$ & sup-norm	$C_0(\bar{\mathcal{A}}_\Gamma)$ & sup-norm	$\mathcal{C}(\mathcal{K})$ & universal-norm
induct. limit alg.	$C(\bar{\mathcal{A}}) := \varinjlim_{\nu} C(\bar{\mathcal{A}}_\nu)$	$C_0(\bar{\mathcal{A}}) := \varinjlim_{\mathcal{P}_\Gamma\in\mathcal{P}_{\Gamma_\infty}} C_0(\bar{\mathcal{A}}_\Gamma)$	
momentum space	suitable admissible maps	the flux groups $\bar{G}_{\breve{S},\Gamma}$ or $\bar{G}_{\breve{S},\Gamma_\infty}$	the algebroid corresp. to the holonomy groupoid
transformations	suitable homeomorphism on $\bar{\mathcal{A}}$	left-, right- or inner- group actions L, R or I	actions of exp. Lie algebroid on holonomy groupoid
	define quasi-flux action w on $\bar{\mathcal{A}}$	on $\bar{\mathcal{A}}_\Gamma$ or $\bar{\mathcal{A}}$	
C^*-dynamical systems		$(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma}, \alpha_X)$, where $X = L, R$ or I	
Hilbert space operators	unitary Weyl-type operators	covariant representation of $(C_0(\bar{\mathcal{A}}_\Gamma), \bar{G}_{\breve{S},\Gamma}, \alpha_X)$	
groupoid morphism	$w \circ \mathfrak{h} \notin \mathrm{Hom}(\mathcal{P}, G)$	$X(\rho_{S,\gamma}) \circ \mathfrak{h}_\Gamma \notin \mathrm{Hom}(\mathcal{P}_\Gamma\Sigma, G)$ for $\rho_{S,\gamma} \in \bar{G}_{\breve{S},\Gamma}$	

	Weyl algebra of Quantum Geometry	Weyl algebra for surfaces	holonomy groupoid algebra for a gauge theory
diffeo. group	diffeomorphism φ in Σ	diffeomorphism φ in Σ	diffeomorphism φ in Σ
diffeomorphisms	graphomorphism φ in Σ	fin. path-diffeomorphism $(\Phi_\Gamma, \varphi_\Gamma)$	path-diffeomorphism (Φ, φ)
		fin. graph-diffeomorphism	
group of quant. diffeo.		group $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ of bisections in $\mathcal{P}_\Gamma\Sigma$	
group of quant. diffeo.		group $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections in $\mathcal{P}_\Gamma$	
actions		left-, right- or inner-actions L,R or I of $\mathfrak{B}(\mathcal{P}_\Gamma)$ on $\bar{\mathcal{A}}_\Gamma$	
C^*-dynamical system		$(C_0(\bar{\mathcal{A}}_\Gamma), \mathfrak{B}(\mathcal{P}_\Gamma), \zeta)$	
groupoid morphism	$\Phi \circ \mathfrak{h} \in \text{Hom}(\mathcal{P}, G)$	$\mathfrak{h}_\Gamma \circ X_\sigma \notin \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$, where $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$, $X = L$ or R	
		$\mathfrak{h}_\Gamma \circ I_\sigma \in \text{Hom}(\mathcal{P}_\Gamma\Sigma, G)$, where $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$	

The Weyl C^*-algebra for surfaces and the holonomy-flux cross-product C^*-algebra are constructed from the functions depending on holonomies along paths of a graph, and the strongly continuous unitary representation of the quantum flux group for surfaces. In contrast to the Weyl algebra, where the G-valued quantum flux operators are implemented as unitary operators, the elements of the holonomy-flux cross-product C^*-algebra are operator-valued functions depending on G-valued quantum flux variables for surfaces. In both cases these operators are represented on Hilbert spaces. For a comparison consider the next table.

Table 11.4: Comparison of C^*-algebras

	Weyl C^*-algebra for surfaces	holonomy-flux cross-product C^*-algebra
ingredients	set of fin. set of surfaces $\breve{S}$	set of fin. set of surfaces $\breve{S}$
	G locally compact group	G locally compact group
inductive family of	fin. graph systems	fin. orientation-preserved graph systems
config. space	$\bar{\mathcal{A}}_\Gamma$	$\bar{\mathcal{A}}_\Gamma$
assumption	natural or non-standard identification of	natural identification of
	a set of independent paths in $\mathcal{P}_\Gamma\Sigma$	a set of independent paths in $\mathcal{P}_\Gamma\Sigma$
mom. space	the flux groups $\bar{G}_{\breve{S},\Gamma}$ or $\bar{G}_{\breve{S},\Gamma_\infty}$	the flux groups $\bar{G}_{\breve{S},\Gamma}$ or $\bar{G}_{\breve{S},\Gamma_\infty}$
Hilbert space	$\mathcal{H}_\Gamma := L^2(\bar{\mathcal{A}}_\Gamma, \mathrm{d}\,\mu_\Gamma)$ or $\mathcal{H}_\infty := L^2(\bar{\mathcal{A}}, \mathrm{d}\,\mu_\infty)$	$\mathcal{H}_\Gamma := L^2(\bar{\mathcal{A}}_\Gamma, \mathrm{d}\,\mu_\Gamma)$ or $\mathcal{H}_\infty := L^2(\bar{\mathcal{A}}, \mathrm{d}\,\mu_\infty)$
		$\mathcal{H}^\Gamma_{E(\breve{S})} := L^2(\bar{G}_{\breve{S},\Gamma}, \mu_{\breve{S},\Gamma}) \otimes \mathcal{H}_\Gamma$
representations	$\Phi_M \in \mathrm{Rep}(C_0(\bar{\mathcal{A}}_\Gamma), \mathcal{L}(\mathcal{H}_\Gamma))$	$\Phi_M \in \mathrm{Rep}(C_0(\bar{\mathcal{A}}_\Gamma), \mathcal{L}(\mathcal{H}_\Gamma))$
	left regular representation	Weyl-integrated holonomy-flux repres.
	of the flux group	of the holonomy-flux cross-product *-alg.
	$U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))$	$\pi^{I,\Gamma}_{E(\breve{S})} \in \mathrm{Rep}(L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma)), \mathcal{K}(\mathcal{H}_\Gamma))$
*-algebra	Weyl algebra generated by	
	$C_0(\bar{\mathcal{A}}_\Gamma)$ and $\{U \in \mathrm{Rep}(\bar{G}_{\breve{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma))\}$	$L^1(\bar{G}_{\breve{S},\Gamma}, C_0(\bar{\mathcal{A}}_\Gamma))$
completion w.r.t.	$L^2(\bar{\mathcal{A}}_\Gamma, \mu_\Gamma)$-norm	universal-norm
C^*-algebra	$\mathfrak{Weyl}(\breve{S}, \Gamma)$	$C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{G}_{\breve{S},\Gamma}$
		multiplier algebra of $C_0(\bar{\mathcal{A}}_\Gamma) \rtimes_\alpha \bar{G}_{\breve{S},\Gamma}$
inductive limit C^*-alg.	$\mathfrak{Weyl}(\breve{S})$	$C(\bar{\mathcal{A}}) \rtimes_\alpha \bar{G}_{\breve{S}}$ (G compact)
state	unique and pure state $\bar{\omega}_M$ on $\mathfrak{Weyl}_{\mathcal{Z}}(\breve{S})$ s.t.	state $\omega_{E(\breve{S})}$ on $C(\bar{\mathcal{A}}) \rtimes_\alpha \bar{\mathcal{Z}}_{\breve{S}}$ s.t.
	$\bar{\omega}_M \circ \zeta_{(\Phi,\varphi)} = \bar{\omega}_M$	$\omega_{E(\breve{S})} \circ \zeta_{(\Phi,\varphi)} = \omega_{E(\breve{S})}$
	$\forall(\Phi,\varphi)$ certain diffeomorphism	$\forall(\Phi,\varphi)$ certain diffeomorphism, which
		preserve the surfaces in $\breve{S}$

In section 8.2 the holonomy-flux cross-product *-algebra has been presented. This algebra is comparable with the holonomy-flux *-algebra, which has been developed in [64]. The localised holonomy-flux cross-product *-algebra presented in section 8.4 is compared with the holonomy-flux cross-product *-algebra in the next table 11.5. Summarising the construction is based on the algebra of continuous functions depending on holonomies along paths, which is a left (or right-) module for the enveloping flux algebra for surfaces. Consequently certain algebras can be derived as abstract cross-product algebras. The differences appear by the choice of the set of paths, and hence the construction of the quantum configuration space. Therefore different holonomy algebras are considered. In particluar the algebras distinguish with respect to the multiplication operation of the elements of these algebras, and their localisation or non-localisation with respect to a set of discretised surfaces associated to surface sets.

Table 11.5: Comparison of *-algebras

	holonomy-flux algebra	holonomy-flux cross-product algebra	localised holonomy-flux cross-product algebra
ingredients	principal fibre bundle $P(\Sigma, G, \pi)$	principal fibre bundle $P(\Sigma, G, \pi)$	principal fibre bundle $P(\Sigma, G, \pi)$
	surfaces with codim. 1	set of finite set $\breve{S}$ of surfaces with codim. 1	set of finite set $\breve{S}_{\text{d}}$ of discretised surfaces
assumption	G compact connected Lie group	G compact connected Lie group	G compact connected Lie group
ingredients		fin. path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ	fin. path groupoid $\mathcal{P}_\Gamma\Sigma$ over V_Γ
	path groupoid $\mathcal{P}$ over Σ	path groupoid $\mathcal{P}$ over Σ	path groupoid $\mathcal{P}$ over Σ
ingredients	graph Γ	fin. orient.-preserv. graph sys. $\mathcal{P}_\Gamma^{\mathfrak{o}}$ assoc. to Γ	sets of paths starting or ending at disc. surfaces.
			graphs not located at disc. surfaces
inductive limit	inductive family of fin. path groupoids	ind. family of fin. orient.-preserv. graph sys.	inductive family of graphs
holonomy map	groupoid morph. $\bar{A}$ from $\mathcal{P}$ to G	holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma\Sigma$ to G	holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma\Sigma$ to G
		holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma$ to $G^{\vert\Gamma\vert}$	holonomy map $\mathfrak{h}_\Gamma$ from $\mathcal{P}_\Gamma$ to $G^{\vert\Gamma\vert}$
config. space	$\bar{\mathcal{A}}_\Gamma$ and proj. limit space $\bar{\mathcal{A}}$	$\bar{\mathcal{A}}_\Gamma$ and proj. limit space $\bar{\mathcal{A}}$	$\bar{\mathcal{A}}_{\text{d},\Gamma} \times \bar{\mathcal{A}}_{\bar{\Gamma}}$
assumption	identification of sets of paths in $\mathcal{P}_\Gamma\Sigma$	natural identif. of sets of indep. paths in $\mathcal{P}_\Gamma\Sigma$	non-stand. identif. of sets of paths
			located at disc. surfaces,
			natural identif. of sets of indep. paths
			not located at disc. surfaces
			enveloping alg. $\bar{a}_{\text{d},\Gamma}$ of Lie alg. assoc. to $\bar{\mathcal{A}}_\Gamma^{\text{d}}$
Hilbert space	$\mathcal{H}_{\text{AL}}$	$\mathcal{H}_\Gamma$ and ind. limit Hilbert space $\mathcal{H}_\infty$	$\mathcal{H}_\Gamma^{\text{d}} = L^2(\bar{\mathcal{A}}_\Gamma^{\text{d}}, \mu_\Gamma^{\text{d}})$ and $\mathcal{H}_{\bar{\Gamma}} = L^2(\bar{\mathcal{A}}_{\bar{\Gamma}}, \mu_{\bar{\Gamma}})$
			and limit Hilbert spaces $\mathcal{H}_{\text{d}}$ and $\mathcal{H}_{\text{loc}}$
diffeomorphisms	φ diffeomorphism on Σ, (Φ, φ)	fin. path- or graph- diffeom. $(\Phi_\Gamma, \varphi_\Gamma)$	fin. path- or graph- diffeom. $(\Phi_\Gamma, \varphi_\Gamma)$
		group $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ or group $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections	group $\mathfrak{B}(\mathcal{P}_\Gamma\Sigma)$ or group $\mathfrak{B}(\mathcal{P}_\Gamma)$ of bisections
mom. space	expon. smearing vector field $E_{S,F}$	the Lie flux algebra $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ or $\bar{\mathfrak{g}}_{\breve{S},\Gamma_\infty} =: \bar{\mathfrak{g}}_{\breve{S}}$	the localised Lie flux algebra $\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}},\Gamma}$ or $\bar{\mathfrak{g}}^{\text{loc}}_{\breve{S}_{\text{d}}}$
	on a fibre of P	the flux enveloping algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ or $\bar{\mathcal{E}}_{\breve{S},\Gamma_\infty} =: \bar{\mathcal{E}}_{\breve{S}}$	the localised flux enveloping algebra $\bar{\mathcal{E}}^{\text{loc}}_{\breve{S}_{\text{d}},\Gamma}$ or $\bar{\mathcal{E}}^{\text{loc}}_{\breve{S}_{\text{d}}}$
C^*-algebra	$C(\bar{\mathcal{A}}_\Gamma)$ & sup-norm	$C(\bar{\mathcal{A}}_\Gamma)$ & sup-norm	$C(\bar{\mathcal{A}}_{\bar{\Gamma}})$ & L^2-norm
	inductive limit C^*-algebra $C(\bar{\mathcal{A}})$	inductive limit C^*-algebra $C(\bar{\mathcal{A}})$	inductive limit C^*-algebra $C(\bar{\mathcal{A}}_{\text{loc}})$

C^*-algebra			$\bigotimes_i C^*(\bar{\mathcal{A}}_{\mathrm{d},\gamma_i})$ & L^2-norm infinite C^*-tensor algebra $C^*(\bar{\mathcal{A}}^{\mathrm{d}})$
Hilbert space operators	$\pi(f)\psi = f\cdot\psi$ for $f\in C(\bar{\mathcal{A}})$ for $\psi\in\mathcal{H}_{\mathrm{AL}}$ $\pi(E_{S,F})\psi = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}\psi\circ\theta_t(F) =: X_S\psi$ for $\psi\in D(E_{S,F})$	$\Phi_M(f)\psi = f\cdot\psi$ for $f\in C(\bar{\mathcal{A}})$ and for $\psi\in\mathcal{H}_\infty$ $\pi(\exp(tE_S(\gamma)))\psi = U_t(E_S(\gamma))\psi$ for $\psi\in\mathcal{H}_\infty$ $\pi(E_S(\gamma)^+E_S(\gamma))\psi = -i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E_S(\gamma)^+E_S(\gamma))\psi$ $\pi(E_S(\gamma)^+E_S(\gamma)) =: \mathrm{d}\,U(E_S(\gamma))$ for $\psi\in D(E_S(\gamma)^+E_S(\gamma))$	$\Phi_M(f)\psi = f\cdot\psi$ for $f\in C^*(\bar{\mathcal{A}}^{\mathrm{d}})$ and for $\psi\in\mathcal{H}_{\mathrm{d}}$ $\Phi_M(f)\psi = f\cdot\psi$ for $f\in C(\bar{\mathcal{A}}_{\mathrm{loc}})$ and for $\psi\in\mathcal{H}_{\mathrm{loc}}$ $\pi(\exp(tE^\gamma_{\breve{S}_{\mathrm{d}}}))\psi = U_t(E^\gamma_{\breve{S}_{\mathrm{d}}})\psi$ for $\psi\in\mathcal{H}_{\mathrm{d}}$ $\pi(\exp(tE^\gamma_{\breve{S}_{\mathrm{d}}}))\psi = \psi$ for $\psi\in\mathcal{H}_{\mathrm{loc}}$ where $E^\gamma_{\breve{S}_{\mathrm{d}}} := E_{S_{\mathrm{d}}}(\gamma)^+E_{S_{\mathrm{d}}}(\gamma)$ $\pi(E^\gamma_{\breve{S}_{\mathrm{d}}})\psi = -i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E^\gamma_{\breve{S}_{\mathrm{d}}})\psi =: \mathrm{d}\,U(E^\gamma_{\breve{S}_{\mathrm{d}}})\psi$ for $\psi\in D(\mathrm{d}\,U(E^\gamma_{\breve{S}_{\mathrm{d}}}))$ self-adj. quantum Hamilton part $H^+_{\Gamma,P}H_{\Gamma,P}$ on $\mathcal{H}^{\mathrm{d}}_\Gamma$
*-algebras	*-algebra $\mathcal{D}$ of differential op. on $\mathcal{H}_{\mathrm{AL}}$ enveloping algebra $\mathfrak{A}_{\mathrm{HF}}$ of $C(\bar{\mathcal{A}})\times\mathcal{D}$ $C^\infty(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S},\Gamma}$ with multiplication $\cdot$	enveloping algebra $\bar{\mathcal{E}}_{\breve{S},\Gamma}$ of $\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ with involution $^+$ s.t. $E_S(\Gamma)^+ = -E_S(\Gamma)$ for all $E_S(\Gamma)\in\bar{\mathfrak{g}}_{\breve{S},\Gamma}$ *-algebra $\mathcal{D}(\bar{G}_{\breve{S},\Gamma})$ of differential op. on $\mathcal{H}_\Gamma$ $C^\infty(\bar{\mathcal{A}}_\Gamma)\rtimes_X\bar{\mathcal{E}}_{\breve{S},\Gamma}$ for $X=L,R$ with multiplication $\cdot_X$ $C^\infty(\bar{\mathcal{A}})\rtimes_X\bar{\mathcal{E}}_{\breve{S},\Gamma_\infty}$ for $X=L,R$	enveloping algebra $\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ of $\bar{\mathfrak{g}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ with involution $^+$ s.t. $E_{S_{\mathrm{d}}}(\Gamma)^+ = -E_{S_{\mathrm{d}}}(\Gamma)$ for all $E_{S_{\mathrm{d}}}(\Gamma)\in\bar{\mathfrak{g}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ *-algebra $\mathcal{Z}(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$ of central functions on $\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma$ *-algebra $\mathcal{D}(\bar{G}^{\mathrm{loc}}_{\breve{S},\Gamma})$ of differential op. on $\mathcal{H}^{\mathrm{loc}}_\Gamma$ $C(\bar{\mathcal{A}}_{\bar{\Gamma}})\otimes_{\min}C^\infty(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}},\Gamma}$ with multiplication $\cdot_L$ $C(\bar{\mathcal{A}}_{\mathrm{loc}})\otimes_{\min}C^\infty(\bar{\mathcal{A}}^{\mathrm{d}})\rtimes\bar{\mathcal{E}}^{\mathrm{loc}}_{\breve{S}_{\mathrm{d}}}$
*-representation	$\pi(f,X_S)\psi = f\cdot X_S\psi$ for $\psi\in D(E_{S,F})$	$\pi(f_\Gamma, E_S(\Gamma)^+E_S(\Gamma))\psi$ $= -i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E_S(\Gamma)^+E_S(\Gamma))f_\Gamma\cdot\psi$ $-if_\Gamma\cdot\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E_S(\Gamma)^+E_S(\Gamma))\psi$ for $\psi\in C^\infty(\bar{\mathcal{A}}_\Gamma)$	$\pi(f_\Gamma, E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma))\psi$ $= -i\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma))f_\Gamma\cdot\psi$ $-if_\Gamma\cdot\frac{\mathrm{d}}{\mathrm{d}\,t}\Big\vert_{t=0}U_t(E_{S_{\mathrm{d}}}(\Gamma)^+E_{S_{\mathrm{d}}}(\Gamma))\psi$ for $\psi\in C^\infty(\bar{\mathcal{A}}^{\mathrm{d}}_\Gamma)$

automorphisms	$\zeta_{(\varphi,\Phi)} \in \mathfrak{Aut}(\mathfrak{A}_{\mathrm{HF}})$	$\alpha_{E_S(\Gamma)^+E_S(\Gamma)} \in \mathfrak{Aut}(C^\infty(\bar{\mathcal{A}}))$ for $E_S(\Gamma)^+E_S(\Gamma) \in \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ $\zeta_\sigma \in \mathfrak{Aut}(C^\infty(\bar{\mathcal{A}}) \rtimes_X \bar{\mathcal{E}}_{\breve{S}})$ for certain $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^{\mathfrak{o}})$	$\alpha_{E_{S_\mathrm{d}}(\Gamma)^+E_{S_\mathrm{d}}(\Gamma)} \in \mathfrak{Aut}(C^*(\bar{\mathcal{A}}))$ for $E_{S_\mathrm{d}}(\Gamma)^+E_{S_\mathrm{d}}(\Gamma) \in \mathfrak{z}_{\breve{S}_\mathrm{d}}$ $\beta_{\mathfrak{a}_{\mathrm{d},\Gamma}} \in \mathfrak{Aut}(C^*(\bar{\mathcal{A}}_\Gamma^\mathrm{d}))$ for $\exp(\mathfrak{a}_{\mathrm{d},\Gamma})$ in $\bar{\mathcal{A}}_\Gamma^\mathrm{d}$ $\tilde{\alpha}_{\mathfrak{a}_{\mathrm{d},\Gamma}} \in \mathfrak{Aut}(\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^\mathrm{d}))$ for $\exp(\mathfrak{a}_{\mathrm{d},\Gamma})$ in $\bar{\mathcal{A}}_\Gamma^\mathrm{d}$ $\alpha_{H_{\Gamma,P}^+H_{\Gamma,P}} \in \mathfrak{Aut}(\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^\mathrm{d}) \rtimes \mathfrak{z}_{\breve{S}_\mathrm{d},\Gamma})$ for $\exp(H_{\Gamma,P}^+H_{\Gamma,P})$ in center of $\bar{\mathcal{A}}_\Gamma^\mathrm{d}$ $\zeta_\sigma \in \mathfrak{Aut}(C^\infty(\bar{\mathcal{A}}^\mathrm{d}) \rtimes \mathfrak{z}_{\breve{S}_\mathrm{d}})$ for certain $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_\mathrm{d}})$
states	unique state ω on $\mathfrak{A}_{\mathrm{HF}}$ s.t. $\omega \circ \zeta_{(\varphi,\Phi)} = \omega$ and $\omega(f, X_S) = 0$	unique state $\bar{\omega}_M$ on $C^\infty(\bar{\mathcal{A}}) \rtimes_X \mathcal{Z}(\bar{\mathcal{E}}_{\breve{S}})$ s.t. $\bar{\omega}_M \circ \zeta_\sigma = \bar{\omega}_M$ for certain $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^{\mathfrak{o}})$ and $\bar{\omega}_M(f, E_S(\Gamma)) = 0$	state ω_M on $C(\bar{\mathcal{A}}_{\mathrm{loc}})$ s.t. $\omega_M \circ \zeta_\sigma = \omega_M$ for $\sigma \in \mathfrak{B}(\mathcal{P}_{\bar{\Gamma}})$ KMS-state $\tilde{\omega}_\mathcal{L}^\Gamma$ on $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^\mathrm{d})$ w.r.t. automorph. $\tilde{\alpha}_{\mathfrak{a}_\mathrm{d},\Gamma}$ KMS-state $\tilde{\omega}_\mathcal{L}^\Gamma$ on $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^\mathrm{d})$ w.r.t. automorph. $\alpha_{H_{\Gamma,P}^+H_{\Gamma,P}}$ and s.t. $\tilde{\omega}_\mathcal{L}^\Gamma \circ \alpha_{H_{\Gamma,P}^+H_{\Gamma,P}} = \tilde{\omega}_\mathcal{L}^\Gamma$ state $\hat{\omega}_\mathcal{L}$ on $\mathcal{Z}(\bar{\mathcal{A}}_\Gamma^\mathrm{d}) \rtimes \mathfrak{z}_{\breve{S}_\mathrm{d},\Gamma}$ s.t. $\hat{\omega}_\mathcal{L} \circ \zeta_\sigma = \hat{\omega}_\mathcal{L}$ for certain $\sigma \in \mathfrak{B}(\mathcal{P}_\Gamma^{\breve{S}_\mathrm{d}})$ and $\hat{\omega}_\mathcal{L}(f, E_{S_\mathrm{d}}(\Gamma)) = 0$

In the following comparison table assume that G is a locally compact group. Let $\mathcal{H}$ be a suitable separable Hilbert space. All algebras are norm-closed *-subalgebras of $\mathcal{L}(\mathcal{H})$ and they are non-degenerate. The multiplier algebra of a C^*-algebra contains all affiliated operators that are bounded operators on the Hilbert space $\mathcal{H}$. The set of all affiliated elements is a subset of the set $\mathfrak{C}(\mathcal{H})$ of all densely defined closed operators on a Hilbert space $\mathcal{H}$. Such elements can be unbounded Hilbert space operators. An element T is affiliated with a C^*-algebra $\mathfrak{A}$ if the Z_T-transform is an element of the multiplier algebra and $(\mathrm{id} - Z_T^* Z_T)\mathfrak{A}$ is dense in $\mathfrak{A}$. One writes $T\eta\mathfrak{A}$.

Table 11.6: Comparison of C^*-algebras

C^*-algebra	properties	Multiplier algebra	algebra of affiliated elem.	quant. flux op.
analytic holonomy alg. $C_0(\bar{\mathcal{A}})$	abelian	$C_b(\bar{\mathcal{A}})$	$C(\bar{\mathcal{A}})$	
non-comm. holonomy alg. $C^*(\bar{\mathcal{A}}_\gamma)$	non-abelian	$M(C^*(G))$	$(C^*(G))^\eta$	$E_S(\gamma)\ \eta\ C^*(G)$
Weyl alg. $\mathfrak{Weyl}(\breve{S})$ for surfaces	unital, non-abelian	$\mathfrak{Weyl}(\breve{S})$	$\mathfrak{Weyl}(\breve{S})$	
hol-flux cross-product C^*-alg. $C_0(\bar{\mathcal{A}}) \rtimes \bar{G}_{\breve{S}}$	non-abelian	$M(C_0(\bar{\mathcal{A}}) \rtimes \bar{G}_{\breve{S}})$	$(C_0(\bar{\mathcal{A}}) \rtimes \bar{G}_{\breve{S}})^\eta$	$e^{\overrightarrow{L}}\ \eta\ C_0(\bar{\mathcal{A}}) \rtimes \bar{G}_{\breve{S}}$
$C_0(G) \rtimes \bar{G}_{S,\gamma} \simeq \mathcal{K}(\mathcal{H}_\gamma)$ (let γ and S intersect in $s(\gamma)$, γ is outgoing w.r.t. S)	non-abelian	$\mathcal{L}(\mathcal{H}_\gamma)$	$\mathfrak{C}(\mathcal{H}_\gamma)$	$e^{\overrightarrow{L}}\ \eta\ \mathcal{K}(\mathcal{H}_\gamma)$

Chapter 12

Appendix

12.1 Some mathematical objects in Differential Geometry

12.1.1 Infinitesimal connections on principal bundles

In this section a very short overview over important structures in differential geometry is given. The standard references are [66] and [70].

12.1.1.1 Analysis on differentiable Manifolds

Let Σ be a differentiable manifold and $v \in \Sigma$. Let γ be a curve in Σ is a differentiable map $\gamma : I \to \Sigma$ and $\gamma(t_0) = v$, $t_0 \in I$. Define the directional derivative as a linear map $D_\gamma : C^\infty(\Sigma) \to \mathbb{R}$ via

$$D_\gamma f := \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} f \circ \gamma(t) \tag{12.1}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \gamma_1(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=t_0} \gamma_2(t) \Leftrightarrow D_{\gamma_1} = D_{\gamma_2} \tag{12.2}$$

Then there is a equivalence relation for two paths γ_1, γ_2 given by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow D_{\gamma_1} = D_{\gamma_2} \tag{12.3}$$

The equivalence class is called the **tangent space** $T_v\Sigma$ at v. Moreover $T_v\Sigma$ has the structure of a vector space.

12.1.1.2 Principal bundles

In this section a principal bundle and infinitesimal connection theory is shortly presented.

Definition 12.1.1. *A* ***principal fibre bundle*** *P over a connected Hausdorff manifold Σ with structure group G consisting of*

(i) the following differentiable manifolds: the total space P, the base space Σ,

(ii) a fibre F_v at $v \in \Sigma$ given by a principal homogeneous space, which is a homogeneous space H for a group G such that the stabilizer subgroup[1] of any point is trivial,

(iii) a surjection $\pi : P \to \Sigma$ such that the inverse $\pi^{-1}(v) = F_v \simeq F$,

[1]For every $h \in H$, the stabilizer subgroup of h is $G_h := \{g \in G : gh = g\}$.

(iv) *a structure group G acting continuously on F on the right,*

(v) *a set of open coverings $\{U_i\}$ of Σ with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(v, f) = v$ which is called local trivialization and*

(vi) *a transition function $t_{ij} : U_i \cap U_j \to G$ for $U_i \cap U_j \neq \varnothing$ and $\phi_j(v, f) = \phi_i(v, t_{ij}(v)f)$ where for the diffeomorphisms $\phi_{i,v} : F \to F_v, \quad \phi_i(v, f) = \phi_{i,v}(f)$ the function $t_{ij}(p) = \phi^{-1}_{i,v} \circ \phi_{j,v}$ is an element of G.*

Short hand notation for principal fibre bundles is $P \xrightarrow{\pi} \Sigma$.

Principal homogeneous spaces can be topological spaces with a right continuous action of a topological group or a smooth manifold and a smooth right action of a Lie group (then π is assumed to be a smooth map between smooth manifolds). Note that the action is free[2] and transitiv[3] along the fibres. Hence, for example for a smooth manifold P the orbits of the action $R : P \times G \to P$ are the fibres of $P \xrightarrow{\pi} \Sigma$ and the orbit space $\frac{P}{G}$ is homeomorphic to Σ.

In this section it is assumed that, G is a compact, connected Lie group. Moreover let Σ be a 3-dimensional, real, smooth, connected, orientable and differentiable spatial manifold.

Definition 12.1.2. *A* ***section of a principal fibre bundle*** *$P \xrightarrow{\pi} \Sigma$ is a (smooth) map $s : \Sigma \to P$ which satisfies $\pi \circ s = \mathrm{id}_\Sigma$. The set of sections is denoted by $\Gamma(\Sigma, P)$.*

The fundamental automorphisms of the principal G-bundle are vertical automorphisms formed by gauge group G transformations and horizontal automorphisms implemented by diffeomorphisms $\mathrm{Diff}(\Sigma)$ on the spatial manifold.

Definition 12.1.3. *A* ***vector bundle*** *$E \xrightarrow{q} \Sigma$ are locally trivial fibre bundles with*

- *fibre type of a k-dimensional $\mathbb{K}$-vector space V,*
- *each fibre F_v is a k-dimensional $\mathbb{K}$-vector space,*
- *for every atlas $\{\phi_I : U_I \times G \to F_{U_I}\}$ the map $\phi_{I,v} : F_v \to V$ is a vector space isomorphism.*

For example the **tangent bundle** $T\Sigma$ over a smooth n-dimensional manifold Σ is a vector bundle with fibre $\mathbb{R}^n$.

12.1.1.3 Associated fibre bundles of principal bundles

Definition 12.1.4. *Let K be a manifold and consider a left action of a Lie group G on K. Then the* ***associated fibre bundle*** *$(E, \tilde{\pi}, \Sigma)$* ***of a principal bundle*** *$P(\Sigma, G)$* ***w.r.t. an action*** *G* ***on*** *K is given by the quotient manifold given by $E := P \times K/R$ where R is the equivalence relation*

$$g(u, k) = (ug, g^{-1}k), \tag{12.4}$$

the projection $\tilde{\pi}(u, k) = \pi(u)$ and the base manifold Σ. The total space of the fibre bundle $(E, \tilde{\pi}, \Sigma)$ is also denoted by $\frac{P\times K}{G}$.

Consider $(\frac{P\times G}{G}, \tilde{\pi}, \Sigma)$ w.r.t. inner automorphism action α of G on itself, i.e. $g \mapsto h^{-1}gh$, then this is a smooth fibre bundle in which each fibre $K_v = G$ has a Lie group structure, and for an atlas $\{\phi_I : U_I \times G \to K_{U_I}\}$ the map $\phi_{I,v} : G \to K_v$, $v \in U_I$ is an isomorphism of Lie groups. The elements are of the form $< u, g >$ and

$$< uh, g >= (uh, g) = (uh, h^{-1}hgh^{-1}h) = \alpha_h(u, hgh^{-1}) =< u, hgh^{-1} > \tag{12.5}$$

The fibre bundle $(\frac{P\times G}{G}, \tilde{\pi}, \Sigma)$ is an example of a Lie group bundle.

A vector bundle (E, q, Σ) is a locally trivial fibre bundle with a fibre being a vector space over $\mathbb{R}^n$. A **frame bundle** $F(E)$ **of a vector bundle** E is principal fibre bundle such that the fibre at a point of Σ consists of all ordered bases of the vector space attached to the point of Σ with structure group $\mathrm{GL}(V)$.

[2] An action of a group G on a space H is called free, if $g \cdot x = x$ for some $x \in H$ then $g = e_G$.

[3] An action of a group G on a space H is called transitiv, if for any two $x, y \in H$ there exists a g in G such that $g \cdot x = y$.

The **tangent frame bundle** $F(\Sigma)$ of a smooth manifold Σ is the frame bundle associated to the tangent bundle $T\Sigma$ of Σ. Each fibre is a manifold of all ordered bases of the tangent space $T_v\Sigma$, i.e. all isomorphisms $e : \mathbb{R}^n \to T_v\Sigma$. The right action of $\mathrm{GL}(n, \mathbb{R})$ makes the frame bundle $F\Sigma$ into a principal $\mathrm{GL}(n, \mathbb{R})$- bundle on Σ.

For a Riemannian manifold Σ, the **orthogonal frame bundle** $O(\Sigma)$ is a smooth fibre bundle on Σ for which the fibre $\pi^{-1}(v)$ for each $v \in \Sigma$ is the manifold of ordered bases of the tangent space $T_v\Sigma$, i.e. all orthogonal isomorphisms $e : \mathbb{R}^n \to T_v\Sigma$. The orthogonal group $O(n)$ leaves $O(\Sigma)$-invariant. Hence $O(\Sigma)$ is a principal $O(n)$-bundle.

12.1.1.4 Invariant horizontal distributions

Definition 12.1.5. *An **invariant horizontal distribution** Γ on a principal bundle P is a subbundle HP of the tangent bundle TP such that for each $u \in P$ the tangent space T_uP is a direct sum of horizontal and vertical tangent subspaces,*

$$T_u(P) = H_u(P) \oplus V_u(P) \tag{12.6}$$

and the right action R of the structure group G on T_uP commutes with all horizotal subspaces $H_u(P)$,

$$H_{r(u)}(P) = R_g H_u(P) \text{ for all } g \in G \text{ and } u \in P \tag{12.7}$$

where $r : P \times G \to P$ denote the right action $(u, g) \mapsto ug$.

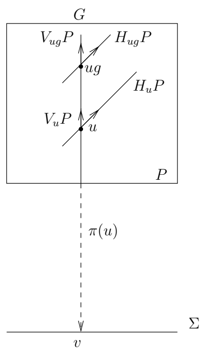


Figure 12.1: horizontal and vertical distributions of P

The right action of G on the tangent bundle $TP \to P$ is given by the induced action of P,

$$X_g = T_u(R_g)(X) \text{ for } X \in T_u(P) \tag{12.8}$$

Moreover the action restricts to an action of G on the vertical subbundle $VP \to P$. Finally the quotient space $\frac{TP}{G}$ is considered such that

$$\frac{T_u(P)}{G} = \frac{V_u(P)}{G} \oplus \frac{H_u(P)}{G}$$

yields.

The smooth sections in $\Gamma(T\Sigma)$ of the tangent bundle $T\Sigma$ are vector fields. The set of all vector fields $\mathfrak{X}(\Sigma)$ on Σ is a $C^\infty(\Sigma)$-module and form a Lie algebra such that the Lie bracket on $\mathfrak{X}(\Sigma)$ satisfies the Leibniz identity

$$[X, fY] = f[X, Y] + X(f)Y$$

for $f \in C^\infty(\Sigma)$ and $X, Y \in \mathfrak{X}(\Sigma)$.

There exists a Lie algebra homomorphism $X : \mathfrak{g} \to \mathfrak{X}(\Sigma)$ where the infinitesimal generator map $\xi \mapsto X_\xi$ and $X_\xi : \Sigma \to T\Sigma$ such that $v \mapsto X_\xi(v) = T_{e_G}(R_v)\xi$. The images of X are called **fundamental vector fields on** Σ.

12.2 Some mathematical objects in Operator Algebra Theory

12.2.1 Topological spaces and groups

Some basic definitions

Let X be a topological space. Then the space of continuous complex valued functions on X is denoted by $C(X)$. There exists the following subspaces of $C(X)$:

- $C_b(X)$... the space of bounded continuous functions on X
- $C_0(X)$... the space of all complex-valued continuous functions on X such that for all $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact
- $C_c(X)$... the space of all continuous functions on X with compact supports

Notice

$$C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$$

and for X being compact all spaces are identical.

Pontryagin duality

Let G, H be a commutative locally compact group. The dual group of G is given by the set of characters, i.o.w.

$$\hat{G} := \{\chi : G \to \mathbb{T} : \chi \text{ linear and continuous } \}$$

The group $\hat{G}$ is abelian locally compact on its own. There is a homomorphism $\epsilon_G : G \to \widehat{\widehat{G}}$ given by $g \mapsto \langle g, . \rangle$ such that for a morphism $f : G \to H$ and the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{\epsilon_G} & \widehat{\widehat{G}} \\ f\downarrow & & \downarrow f^{**} \\ H & \xrightarrow[\epsilon_H]{} & \widehat{\widehat{H}} \end{array}$$

Theorem 12.2.1. ***Pontryagin duality***
For every abelian locally compact group G the morphism ϵ_G is an isomorphism.

12.2.2 Hopf *-algebras

The theory of Hopf algebras has been studied for example by Kustermann and Tuset [54], Klimyk and Schmüdgen [53]. The important objects studied by these authors are presented in this section.

12.2.2.1 Finite quantum algebras

Definition 12.2.2. *The* ***function algebra*** *$K(G)$ of a finite group G is the set of all complex-valued functions on G is a unital *-algebra under the following operations:*

- $(\lambda f)(g) = \lambda f(g)$

- $(f+h)(g) = f(g) + h(g)$
- $(fh)(g) = f(g)h(g)$
- $f^*(g) = \overline{f(g)}$
- $\mathbb{1}(g) = \mathbb{1}$

where $f, h \in K(G)$,$\lambda \in \mathbb{C}$ *and* $g \in G$.

Definition 12.2.3. *For a unital* **-algebra* $\mathfrak{A}$ *with multiplication*

$$m : \mathfrak{A} \odot \mathfrak{A} \to \mathfrak{A}, a \otimes b \mapsto ab \tag{12.9}$$

and a unital **-homomorphism* $\triangle : \mathfrak{A} \to \mathfrak{A} \odot \mathfrak{A}$ *satisfying*

$$(\triangle \odot \iota)\triangle = (\iota \odot \triangle)\,\triangle \text{ (co-associativity)} \tag{12.10}$$

where ι *is the identity map on* $\mathfrak{A}$.

Let $\epsilon : \mathfrak{A} \to \mathbb{C}$ *and* $S : \mathfrak{A} \to \mathfrak{A}$ *be linear maps such that*

$$\begin{aligned} &(\epsilon \odot \iota)\triangle = (\iota \odot \epsilon)\triangle = \iota \\ &m(S \odot \iota)\triangle = m(\iota \odot S)\triangle = 1\epsilon(.) \end{aligned} \tag{12.11}$$

Then the pair $(\mathfrak{A}, \triangle)$ *is called a* ***Hopf*** ****-algebra****. The map* ϵ *is called co-unit,* S *is the antipode and* $\triangle$ *is the co-multiplication.*

Example 12.2.1: For $\mathfrak{A} = K(G)$ the maps

$$\begin{aligned} \iota(f) &= f \\ \triangle(f)(g_1, g_2) &= f(g_1 g_2) \\ \epsilon(f) &= f(e) \\ S(f)(g) &= f(g^{-1}) \end{aligned} \tag{12.12}$$

defines a commutative Hopf *-algebra. The relations (12.10) and (12.11) are

$$e^{-1} = e, \quad (gk)^{-1} = k^{-1}g^{-1}, \quad (g^{-1})^{-1} = g, \quad g, k \in G \tag{12.13}$$

Notice that, the algebras $K(G) \odot K(G)$ and $K(G \times G)$ can be identified via $(f \otimes h)(g_1, g_2) = f(g_1)h(g_2)$ for all $f, h \in K(G)$ and $g_1, g_2 \in G$.

The set of characters $\{\chi_i\}$ (which are unital *-homomorphisms) on $K(G)$ is a group isomorphic to the finite group G under the multiplication

$$\chi_1\chi_2 = (\chi_1 \otimes \chi_2)\triangle \tag{12.14}$$

For a finite group G, $K(G)$ is a finite-dimensional commutative C^*-algebra $C(G)$ with respect to the supremum norm.

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Definition 12.2.4. *Two Hopf* **-algebras* $(\mathfrak{A}_1, \triangle_1)$ *and* $(\mathfrak{A}_2, \triangle_2)$ *are isomorphic iff there exists a* **-isomorphism* $\pi : \mathfrak{A}_1 \to \mathfrak{A}_2$ *such that*

$$(\pi \odot \pi)\triangle_1 = \triangle_2\pi \tag{12.15}$$

If G_1 and G_2 are finite groups, then they are isomorphic iff the associated Hopf *-algebras $(K(G_1), \triangle_1)$ and $(K(G_2), \triangle_2)$ are isomorphic.

Definition 12.2.5. *Let $\mathbb{C}(G)$ be the unital *-algebra of complex-valued functions on a discrete group G with finite support and under the operations:*

- $(\lambda f)(g) = \lambda f(g)$
- $(f + h)(g) = f(g) + h(g)$
- $(fh)(k) = \sum_{g \in G} f(g)h(g^{-1}k)$
- $f^*(g) = \overline{f(g^{-1})}$

where $f, h \in \mathbb{C}(G)$,$\lambda \in \mathbb{C}$ and $g, k \in G$.

Formally an element of $\mathbb{C}(G)$ is given by $\sum_{g \in G} c_g \cdot g$ where $c_g \in \mathbb{C}$. It is useful to interpret c_g as a function on G. Then there is a multiplication operation given by

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(h) f_2(h^{-1}g) \tag{12.16}$$

and the involution is

$$f^*(g) = \overline{f(g^{-1})} \tag{12.17}$$

For every complex representation T of G , there is a representation π of the algebra $\mathbb{C}(G)$ given by

$$\pi(f) = \sum_{g \in G} f(g)T(g) \tag{12.18}$$

which extends to a *-representation such that $\pi(f)^* = \pi(f^*)$. The space V of the representation T is a $\mathbb{C}(G)$-module. The category of unitary representations is identifiable with the category of non-degenerate *-representations of $\mathbb{C}(G)$.

For an irreducible representation T_λ of class $\lambda \in \hat{G}$ the Fourier transform is given by

$$\mathcal{F}(f) = \sum_{g \in G} f(g) T_\lambda^*(g) \tag{12.19}$$

For $d_\lambda = \dim T_\lambda$ and a fixed basis in the space V_λ of the representation T_λ the element $\mathcal{F}(f)$ is contained in the matrix algebra $M_{d_\lambda}(\mathbb{C})$.

Definition 12.2.6. *A* ***finite quantum group*** $(\mathfrak{A}, \triangle)$ *is a Hopf *-algebra such that*

$$A^*A = 0 \Leftrightarrow A = 0 \quad \forall A \in \mathfrak{A}$$

Definition 12.2.7. *A linear functional ω on $\mathfrak{A}$ is called a* ***Haar functional*** *on a Hopf *-algebra $(\mathfrak{A}, \triangle)$ if $\omega(1) \neq 0$ and $(\omega \odot \iota)\triangle = 1\omega(.)$.*

A finite quantum group admits a unique (up to a positive scalar) Haar functional ω, which is faithful.

Example 12.2.2: Let G be discrete. The function δ_g, which is 1 at the point g and 0 elsewhere, is an element of $\mathbb{C}(G)$. The set $\{\delta_g\}_{g \in G}$ form a basis of the vector space $\mathbb{C}(G)$. The element δ_e is the unit element of $\mathbb{C}(G)$.

For $\mathfrak{A} = \mathbb{C}(G)$ the maps

$$\begin{aligned} \hat{\triangle}(\delta_g) &= \delta_g \otimes \delta_g \\ \hat{\epsilon}(\delta_g) &= 1 \\ \hat{S}(\delta_g) &= \delta_{g^{-1}} \end{aligned} \tag{12.20}$$

where $g \in G$. The pair $(\mathbb{C}(G), \hat{\triangle})$ is a co-commutative Hopf *-algebra, which means that $F\hat{\triangle} = \hat{\triangle}$ where F is the flip automorphism. The group multiplication $\hat{m}$ is encoded in the convolution product. The Hopf *-algebra

$(\mathbb{C}(G), \hat{\triangle})$ is dual to $(K(G), \triangle)$ for some discrete and finite group G. There is a linear positive and faithful Haar functional $\hat{\omega}$ on $\mathbb{C}(G)$ is represented by

$$\hat{\omega}(f^*f) := (f^*f)(e) = \sum_{g\in G} |f(g)|^2 \text{ for } f \in \mathbb{C}(G) \tag{12.21}$$

There is a bilinear form $\langle .,\rangle : K(G) \times \mathbb{C}(G) \to \mathbb{C}$ presented by

$$\langle f, h\rangle = \sum_{g\in G} f(g)h(g) = \omega(fh) \tag{12.22}$$

where ω is the Haar functional on $(K(G), \triangle)$ given by

$$\omega(f) = \sum_{g\in G} f(g) \text{ for all } f \in K(G) \tag{12.23}$$

▪

Definition 12.2.8. *An element A of $\mathfrak{A}$, which defines a Hopf*-algebra $(\mathfrak{A}, \triangle)$, is called **group-like** if $\triangle(A) = A\otimes A$ for all $A \neq 0$ and $A \in \mathfrak{A}$.*

Consequently, all group-like f in $\mathbb{C}(G)$ are of the form

$$f = \sum_{g\in G} f(g)\delta_g \tag{12.24}$$

Example 12.2.3: For G being an abelian finite group the dual group $\hat{G}$ is a subset of $K(G)$. The Fourier transform is

$$\mathcal{F}(h)(\chi) = \langle \chi, h\rangle = \omega(\chi h) \tag{12.25}$$

for $h \in \mathbb{C}(G)$ and $\chi \in \hat{G}$. The Fourier transformation is an isomorphism of Hopf *-algebras, which means that $K(G)$ and $\mathbb{C}(G)$ are equal as vector spaces (not as algebras).

For an abelian finite group there is a map $\tilde{\mathcal{F}} : K(G) \to K(\hat{G})$ such that $\tilde{\mathcal{F}}(f) = \mathcal{F}(f)$, where $f \in K(G)$. Then the Plancherel formula states that $\tilde{\mathcal{F}}$ is an isometry w.r.t. the L^2-norm on G and $\hat{G}$

$$c \sum_{g\in G} |f(g)|^2 = \sum_{\chi\in\hat{G}} |\tilde{\mathcal{F}}(f)(\chi)|^2 \quad \forall f \in K(G) \tag{12.26}$$

▪

12.2.2.2 Compact quantum groups

The theory of a compact quantum group is developed by Woronowicz.

Definition 12.2.9. *A **compact quantum group** is a pair $(\mathfrak{A}, \triangle)$, where $\mathfrak{A}$ is a unital C^*-algebra and $\triangle : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$[4] is a unital *-homomorphism such that*

- $(\triangle \otimes \iota)\triangle = (\iota \otimes \triangle)\triangle$
- $\{\triangle(A)(B \otimes \mathbb{1}) : A, B \in \mathfrak{A}\}$ *is a dense subspace of* $\mathfrak{A} \otimes \mathfrak{A}$
- $\{\triangle(A)(\mathbb{1} \otimes B) : A, B \in \mathfrak{A}\}$ *is a dense subspace of* $\mathfrak{A} \otimes \mathfrak{A}$

[4]the tensor product of C^*-algebras is the completion of the algebraic tensor product $\mathfrak{A} \odot \mathfrak{B}$ w.r.t. the minimal tensor product norm.

Example 12.2.4: Let G be a compact topological group and let $C(G)$ be the set of continuous functions on G. Then $C(G)$ is a unital C^*-algebra if it is equipped with pointwise multiplication, complex conjugation and uniform norm. The linear map $\pi : C(G) \odot C(G) \to C(G \times G)$ where $\pi(f \otimes h)(g_1, g_2) = f(g_1)h(g_2)$ for $f, h \in C(G)$ and $g_1, g_2 \in G$ can be uniquely and continuously extended to a *-isomorphism from $C(G) \otimes C(G)$ to $C(G \times G)$. The unital *-homopmorphism $\triangle$ is defined by $\triangle(f)(g_1, g_2) = f(g_1 g_2)$ for all $f \in C(G)$ and $g_1, g_2 \in G$. The pair $(C(G), \triangle)$ is a compact quantum group.

There are two *-homomorphisms $S : C(G) \to C(G)$ and $\epsilon : C(G) \to \mathbb{C}$ defined by the formulas $S(f)(g) = f(g^{-1})$ and $\epsilon(f) = f(e)$ for $f \in C(G)$ and $g \in G$.

▪

Example 12.2.5: For a discrete group G the unital *-algebra $\mathbb{C}(G)$ is equipped with an inner product defined by

$$(f, h) := \hat{\omega}(h^* f) \text{ for } f, h \in \mathbb{C}(G) \tag{12.27}$$

The Hilbert space completition of $\mathbb{C}(G)$ w.r.t. this norm is denoted by $L^2(\hat{\omega})$. Then for $f \in \mathbb{C}(G)$

$$\|f\|_2^2 = \sum_{g \in G} |f(g)|^2 \tag{12.28}$$

Left multiplication operators L_f in $L^2(\hat{\omega})$ are defined by

$$L_f(\psi) = f\psi \text{ for } f \in \mathbb{C}(G) \text{ and } \psi \in L^2(\hat{\omega}) \tag{12.29}$$

and are bounded operators on the dense subspace $\mathbb{C}(G)$ of $L^2(\hat{\omega})$. Hence, there is a unital injective *-homomorphism $f \mapsto \pi(f)$ where π is the GNS-representation on $L^2(\hat{\omega})$ of the state $\hat{\omega}$. Observe that $\|f\|_r = \|\pi(f)\|_2$ for $f \in \mathbb{C}(G)$ is a C^*-norm. The completition of $\mathbb{C}(G)$ in this norm is $C_r^*(G)$.

One can show that the co-multiplication $\hat{\triangle}$ on $\mathbb{C}(G)$ is bounded w.r.t. $\|.\|_r$, and therefore, there is an extension to a co-multiplication $\hat{\triangle}_r$ on $C_r^*(G)$. Then it follows that $(C_r^*(G), \hat{\triangle}_r)$ is a compact quantum group.

▪

Theorem 12.2.10. *Let $(\mathfrak{A}, \triangle)$ be a compact quantum group. Then there exists a unique state ω on $\mathfrak{A}$ such that*

$$(\omega \otimes \iota) \triangle (A) = (\iota \otimes \omega) \triangle (A) = \omega(A)\mathbb{1} \quad \forall A \in \mathfrak{A} \tag{12.30}$$

*This state is called **Haar state** on $(\mathfrak{A}, \triangle)$.*

The Haar state does not have to be faithful. The Haar state $\hat{\omega}_r$ of $(C_r^*(G), \hat{\triangle}_r)$ for a discrete group G is faithful.

Example 12.2.6: Let G be a discrete group. If $\mathbb{C}(G)$ is endowed with the universal norm $\|.\|_u$ the completion is $C^*(G)$ and the pair $(C^*(G), \hat{\triangle}_u)$ is a compact quantum group.

There exists a unital surjective *-homomorphism $\pi_u : C^*(G) \to C_r^*(G)$ such that $f \mapsto \pi_u(f) := f$ for all $f \in \mathbb{C}(G)$, which is an isomorphism iff G is amenable (G is amenable if G is abelian, finite or a compact Lie group).

▪

The standard examples of a compact quantum group is a one-parameter q family of compact quantum group, which is neither commutative nor co-commutative. The deformed or twisted $SU(2)$ quantum group is due to Woronowicz [112].

Example 12.2.7: Let G be a compact Lie group like $SU(2)$ and

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} : a, c \in \mathbb{C} \quad |a|^2 + |c|^2 = 1 \right\}$$

Then $\alpha, \gamma \in C(SU(2))$ defined by

$$\alpha \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = a, \quad \gamma \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = c \tag{12.31}$$

are called the coordinate functions and generate a *-algebra $\mathrm{Pol}(SU(2))$. There is a dense unital *-subalgebra $\mathrm{Pol}(SU(2))$ of the C^*-algebra $C(SU(2))$. Moroever,

$$\triangle(\alpha) = \alpha \otimes \alpha - \gamma^* \otimes \gamma, \quad \triangle(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma \tag{12.32}$$

The compact group $SU(2)$ can be recovered from the commutative **Hopf *-algebra** $(\mathrm{Pol}(SU(2)), \triangle)$ **of coordinate functions**, which are regular functions on the group $SU(2)$. Clearly, the multiplication operation in $\mathrm{Pol}(SU(2))$ is commutative.

▪

Theorem 12.2.11. *Let $(\mathfrak{A}, \triangle)$ be a compact group algebra.*

There existsa unique Hopf *-algebra $(\mathfrak{B}, \Phi)$ such that $\mathfrak{B}$ is a dense unital *-subalgebra of $\mathfrak{A}$ such that $\triangle(\mathfrak{B}) \subseteq \mathfrak{B} \odot \mathfrak{B}$ and such that Φ is the restriction of $\triangle$ to $\mathfrak{B}$.

If $\mathfrak{A}$ is commutative, then $\mathfrak{B}$ consists of the linear space of coefficient functions of the finite-dimensional unitary representations of the compact group.

Recall the algebra $\mathrm{Pol}(SU(2))$ of example 12.2.2.7, which is also called the algebra of matrix coefficients.

Example 12.2.8: Let G be the compact Lie group $SU(2)$ with Lie algebra $\mathfrak{g}$ and universal enveloping algebra $\mathcal{E}$. Then define the co-multiplication $\hat{\triangle} : \mathcal{E} \to \mathcal{E} \odot \mathcal{E}$

$$\hat{\triangle}(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X \quad \forall X \in \mathfrak{g} \tag{12.33}$$

Moreover,

$$\hat{\epsilon}(X) = 0, \quad S(X) = -X \text{ for } X \in \mathfrak{g} \tag{12.34}$$

Notice that $\mathcal{E}$ defines the vector space of left-invariant vector fields on a Lie group. Moreover, $\mathcal{E}$ defines left-invariant differential operators on G. Then $(\mathcal{E}, \hat{\triangle})$ is a co-commutative Hopf algebra.

The linear form $\langle ., . \rangle : \mathcal{E} \times C^\infty(SU(2)) \to \mathbb{C}$ expressed by

$$\langle X, f \rangle = X(f)(e_G) \quad \forall X \in \mathcal{E}, f \in C^\infty(G) \tag{12.35}$$

Hence, X is a linear functional on the algebra $C^\infty(SU(2))$ of smooth functions on $G = SU(2)$.

Then the Hopf algebras $(\mathcal{E}, \hat{\triangle})$ and $(\mathrm{Pol}^\infty(SU(2)), \triangle)$ form a dual pair, where $\mathrm{Pol}^\infty(SU(2))$ is the restriction of $\mathrm{Pol}(SU(2))$ to a *-subalgebra of $C^\infty(SU(2))$.

▪

Example 12.2.9: Let G be a compact Lie group and let π be a continuous representation of G on a finite-dimensional complex vector space V. Denote by $\mathcal{R}(\pi)$ the linear subspace of $C(G)$ spanned by the matrix elements ϕ^i_j, $i, j = 1, ..., \dim \pi$ of π relative to some basis of V.

Then there is a linear mapping

$$\triangle(\phi^i_j) = \sum_k \phi^i_k \otimes \phi^k_j, \quad \epsilon(\phi^i_j) = \delta^i_j \tag{12.36}$$

The linear span of spaces $\mathcal{R}(\pi)$ for all continuous finite-dimensional representations π of G is denoted by $\mathrm{Rep}(G)$. The space $\mathrm{Rep}(G)$ is a subalgebra of $C(G)$. The pair $(\mathrm{Rep}(G), \triangle)$ is a Hopf algebra, called the **Hopf algebra of representative functions** on G. Notice that the multiplication operation is commutative.

Then the Hopf algebras $(\mathcal{E}, \hat{\triangle})$ and $(\mathrm{Rep}^\infty(G), \triangle)$ form a dual pair, where $(\mathrm{Rep}^\infty(G), \triangle)$ is the restriction of $\mathrm{Rep}(G)$ to a *-subalgebra of $C^\infty(G)$.

▪

The antipode S in both examples 12.2.7 and 12.2.9 is an algebra homomorphism such that $S^2 = \mathrm{id}$. This is not true for general quantum groups.

12.2.2.3 Cross product construction for Hopf algebras

For simplicity assume that $\mathfrak{A}$ is a Hopf algebra instead of a more general object called bialgebra.

Klimyk and Schmüdgen [53] have defined the Sweedler notation for the comultiplication $\triangle$ given by

$$\triangle(A) = \sum_{i=1}^{n} A_{1i} \otimes A_{2i}$$

for $A_{1i}, A_{2i} \in \mathfrak{A}$. Notice that the indices 1, 2 suppress the corresponding tensor factors.

Definition 12.2.12. *Let $\mathcal{X}$ be an algebra.*

A ***left $\mathfrak{A}$-module*** *$\mathcal{X}$ is given by a bilinear map $\mathfrak{A} \times \mathcal{X} \ni (A, X) \mapsto A \triangleright X \in \mathcal{X}$ such that $A \triangleright (B \triangleright X) = (AB) \triangleright X$ and $\mathbb{1} \triangleright X = X$ for $A, B \in \mathfrak{A}$, $X \in \mathcal{X}$.*

A ***left $\mathfrak{A}$-module algebra*** *$\mathcal{X}$ is given by a left $\mathfrak{A}$-module $\mathcal{X}$ such that*

$$A(XY) = \sum_{i=1}^{n} (A_{1i} \triangleright X)(A_{2i} \triangleright Y), \quad A \triangleright \mathbb{1} = \epsilon(A)\mathbb{1}$$

for $A_{1i}, A_{2i} \in \mathfrak{A}$.

Proposition 12.2.13. *Let $\mathcal{X}$ be a left $\mathfrak{A}$-module algebra. Then the vector space $\mathcal{X} \otimes \mathfrak{A}$ is a unital associative algebra, called the* ***left cross-product algebra*** *$\mathcal{X} \rtimes \mathfrak{A}$, with multiplication defined by*

$$(X \otimes A)(Y \otimes B) = \sum_{i=1}^{n} X(A_{1i} \triangleright Y) \otimes (A_{2i}B)$$

whenever $X, Y \in \mathcal{X}$ and $A, B \in \mathfrak{A}$.

Example 12.2.10: Consider an algebra $\mathcal{X}$ and the trivial action $AX := \epsilon(A)X$. Then the product of the algebra $\mathcal{X} \rtimes \mathfrak{A}$ is

$$(X \otimes A)(Y \otimes B) = XY \otimes AB$$

▪

Proposition 12.2.14. *Let $\varphi_{\mathfrak{A}}$ and $\varphi_{\mathcal{X}}$ be representations of $\mathfrak{A}$ and $\mathcal{X}$ on the same vector space V satisfying the condition*

$$\varphi_{\mathfrak{A}}(A)\varphi_{\mathcal{X}}(X) = \sum_{i=1}^{n} \varphi_{\mathcal{X}}(A_{1i} \triangleright X)\varphi_{\mathfrak{A}}(A_{2i}) \tag{12.37}$$

whenever $A \in \mathfrak{A}$ and $X \in \mathcal{X}$, then

$$\varphi(AX) := \varphi_{\mathfrak{A}}(A)\varphi_{\mathcal{X}}(X) \tag{12.38}$$

defines a representation of $\mathcal{X} \rtimes \mathfrak{A}$ on V.

Example 12.2.11: The algebra $\mathcal{X} := C^\infty(G)$ of differentiable and continuous functions on a Lie group G is a left $\mathcal{E}(\mathfrak{g})$-module algebra, where $\mathcal{E}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Recall the co-multiplication $\triangle(X) = X \otimes 1 + 1 \otimes X$. The left $\mathcal{E}(\mathfrak{g})$-module algebra is given by

$$X \triangleright f = \sum_{i=1}^{N} \langle X_i, f \rangle Y_i$$

for any two orthonormal basis $\{X_i\}_{1 \leq i \leq N}$ and $\{Y_i\}_{1 \leq i \leq N}$ of $\mathcal{E}(\mathfrak{g})$ and $f \in C^\infty(G)$ where $\langle .,. \rangle$ is the bilinear form (12.35) presented in section 12.2.2.2.

Consider the following representations on $V := C^\infty(G)$

$$\varphi_{\mathcal{X}}(f)Y = fY, \quad \varphi_{\mathfrak{A}}(X)f = X \triangleright f$$
$$\varphi(fX)k = f(X \triangleright k)$$

▪

12.2.3 Operator Theory in the O^*-algebra framework

The theory of O^*-algebras has been developed by Schmüdgen [89] and Inoue [51]. In this section the basic definitions are collected.

Definition of O^*-algebras

Let $\mathcal{D}$ be a dense subspace in a Hilbert space with inner product $\langle .,. \rangle$. By $\mathfrak{L}(\mathcal{D})$ (respect. $\mathfrak{L}_c(\mathcal{D})$) denote the set of all (closable) linear operators from $\mathcal{D}$ to $\mathcal{D}$ and set

$$\mathfrak{L}^+(\mathcal{D}) := \{A \in \mathfrak{L}(\mathcal{D}) : \mathcal{D} \subset \mathcal{D}(A^*), A^*\mathcal{D} \subset \mathcal{D}\}$$

Then with the operations AB, $A+B$ and λA the set $\mathfrak{L}(\mathcal{D})$ form an algebra, $\mathfrak{L}^+(\mathcal{D})$ form a *-algebra with involution $A \mapsto A^+ = A^*|_{\mathcal{D}}$.

Proposition 12.2.15. *[89, Prop 2.1.10] Let $A \in \mathfrak{L}^+(\mathcal{D})$ and let A be closed. Then $\mathfrak{L}^+(\mathcal{D})$ is equal to the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$.*

Observe

$$\mathfrak{L}^+(\mathcal{D}) \subset \mathfrak{L}_c(\mathcal{D}) \subset \mathfrak{L}(\mathcal{D})$$

Definition 12.2.16. *A subalgebra of $\mathfrak{L}(\mathcal{D})$ contained in $\mathfrak{L}_c(\mathcal{D})$ is said to be an **O-algebra on $\mathcal{D}$ in $\mathcal{H}$**, and a *-subalgebra of $\mathfrak{L}^+(\mathcal{D})$ is said to be an **O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$**.*

Representations of *-algebras

Definition 12.2.17. *Let $\mathfrak{A}$ be a *-algebra with unit $\mathbb{1}$ and let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$.*

*The map $\pi : \mathfrak{A} \to \mathfrak{L}(\mathcal{D})$ is a ***-representation of a *-algebra** $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ if*

(i) there exists a dense subset $\mathcal{D}$ of $\mathcal{H}$ such that

$$\mathcal{D} \subset \bigcap_{A \in \mathfrak{A}} (D(\pi(A)) \cap D(\pi(A)^*))$$

(ii) for every $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$

$$\pi(A+B) = \pi(A) + \pi(B), \quad \pi(\lambda A) = \lambda \pi(A)$$
$$\pi(AB) = \pi(A)\pi(B), \quad \pi(A^*) = \pi(A)^*$$
$$\pi(\mathbb{1}) = I$$

Definition 12.2.18. *Let $\mathfrak{A}$ be a *-algebra and $\mathcal{H}$ a Hilbert space.*

*A *-representation π of $\mathfrak{A}$ on $\mathcal{H}$ is called **non-degenerate** if*

$$\{\pi(A)\psi : A \in \mathfrak{A} \text{ and } \psi \in \mathcal{H}\} \tag{12.39}$$

*spans a dense subset of $\mathcal{H}$. Furthermore, a *-representation π is called **faithful** if π is an injective map.*

*If the inequality $\|\pi(A)\|_2 \leq \|A\|_1$ for any $A \in \mathfrak{A}$ is true, then π is called **L^1-norm decreasing**.*

12.2.4 Operator Theory in the C^*-algebra framework

The standard reference for general theory of C^*-algebras is given by Bratteli and Robinson [22].

12.2.4.1 Representations and states of C^*-algebras

Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of bounded operators on the Hilbert space $\mathcal{H}$. If $\mathfrak{A}$ is a C^*-algebra, then π is a ***-representation** of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ if $\pi : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism.

Lemma 12.2.19. *A *-representation $(\pi, \mathcal{H})$ of a C^*-algebra $\mathfrak{A}$ is **non-degenerate** iff the set $\{\psi \in \mathcal{H} : \pi(A)\psi = 0$ for all $A \in \mathfrak{A}\}$ only contain the vector 0.*

Definition 12.2.20. *A linear positive functional ω over a C^*-algebra with $\|\omega\| = 1$ is called a **state**. A state is **pure** if the positive linear functional majorized by ω are of the form $\lambda\omega$ with $0 \leq \lambda \leq 1$.*

Proposition 12.2.21. *Let ω be a state on the C^*-algebra $\mathfrak{A}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the associated GNS-representation.*

Then the following conditions are equivalent

(i) $(\mathcal{H}_\omega, \pi_\omega)$ is irreducible;

(ii) ω is pure;

(iii) ω is an extremal point of the set of states over $\mathfrak{A}$.

Proposition 12.2.22. *Let $\mathcal{M}$ be a self-adjoint set of bounded operators on $\mathcal{H}$.*

Then the following conditions are equivalent

(i) $\mathcal{M}$ is irreducible on $\mathcal{H}$;

(ii) the commutant $\mathcal{M}' := \{A \in \mathcal{L}(\mathcal{H}) : [A, B] = 0 \quad \forall B \in \mathcal{M}\}$ is equivalent to $\{\lambda \cdot \mathbb{1} : \lambda \in \mathbb{R}\}$;

(iii) every nonzero vector $\psi \in \mathcal{H}$ is cyclic for $\mathcal{M}$ in $\mathcal{H}$, or $\mathcal{M} = 0$ and $\mathcal{H} = \mathbb{C}$.

Since for an abelian C^*-algebra $\pi_\omega(\mathfrak{A}) \subset \pi_\omega(\mathfrak{A})'$ the representation $(\pi_\omega, \mathcal{H}_\omega)$ is irreducible if $\mathcal{H}_\omega$ is one-dimensional, the state ω factorise.

Proposition 12.2.23. *Let ω be a state over an abelian C^*-algebra. Then ω is pure iff*

$$\omega(AB) = \omega(A)\omega(B) \quad \forall A, B \in \mathfrak{A}.$$

Definition 12.2.24. *Let G be a locally compact group and α_g for all $g \in G$ form a group of *-automorphisms on a C^*-algebra $\mathfrak{A}$. A state ω is **G-invariant**, if*

$$\omega = \omega \circ \alpha_g \quad \forall g \in G$$

*Then the set of all G-invariant states is denoted by $\mathcal{S}^G(\mathfrak{A})$. The subset of $\mathcal{S}^G(\mathfrak{A})$ containing only extremal G-invariant states is denoted by $\mathcal{S}^G_P(\mathfrak{A})$ and are called **G-ergodic states**.*

Proposition 12.2.25. *Let $\mathfrak{A}$ be a C^*-algebra and ω be a state on $\mathfrak{A}$. Let E_ω be the orthogonal projection on the subspace of $\mathcal{H}_\omega$ formed by vectors being invariant under $U_\omega(G)$ for a locally compact group G.*

Then the following properties are eqivalent:

(i) $E_\omega \mathcal{H}_\omega$ is one-dimensional;

(ii) $\omega \in \mathcal{S}^G_P(\mathfrak{A})$;

(iii) $\{\pi(\mathfrak{A}) \cup U_\omega(G)\}$is irreducible and

(iv) $\inf_{g\in G} \left|\omega(\alpha_g(A)B) - \omega(A)\omega(B)\right| = 0$ (mixing property).

12.2.4.2 Hilbert C^*-modules

Lance [55] has presented the concept of Hilbert C^*-modules. In this section the basic notions are summarised.

Definition 12.2.26. *Let $\mathfrak{A}$ be a C^*-algebra.*

*An **inner-product $\mathfrak{A}$-module** is a linear space $\mathcal{E}$ which is a right $\mathfrak{A}$-module compatible with the scalar multiplication, i.o.w. $\lambda(xA) = (\lambda x)A = x(\lambda A)$ for $x \in \mathcal{E}$, $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, together with a map $(x, y) \mapsto \langle x, y\rangle_{\mathfrak{A}} : \mathcal{E} \times \mathcal{E} \to \mathfrak{A}$ such that*

(i) $\langle x, \alpha y + \beta z\rangle_{\mathfrak{A}} = \alpha\langle x, y\rangle_{\mathfrak{A}} + \beta\langle x, z\rangle_{\mathfrak{A}}$

(ii) $\langle x, yA\rangle_{\mathfrak{A}} = \langle x, y\rangle_{\mathfrak{A}} A$

(iii) $\langle x, y\rangle_{\mathfrak{A}} = \langle y, x\rangle^*_{\mathfrak{A}}$

(iv) $\langle x, x\rangle_{\mathfrak{A}} \geq 0$*; if* $\langle x, x\rangle_{\mathfrak{A}} = 0$ *then* $x = 0$

for $x, y, z \in \mathcal{E}$, $A \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$ and where $xA := \pi_R(A)x$ and π_R is a right action of $\mathfrak{B}$ on $\mathcal{E}$.

Lemma 12.2.27. *Let $\mathcal{E}$ be an inner-product $\mathfrak{A}$-module, then for $x \in \mathcal{E}$ set $\|x\|_{\mathcal{E}} := \|\langle x, x\rangle_{\mathfrak{A}}\|^{1/2}_{\mathfrak{A}}$, which defines a (scalar-valued) norm on $\mathcal{E}$.*

One can show the Cauchy-Schwarz inequality that,

$$\|\langle x, y\rangle_{\mathfrak{A}}\|_{\mathfrak{A}} \leq \|x\|_{\mathcal{E}}\|y\|_{\mathcal{E}} \text{ for } x, y \in \mathcal{E} \tag{12.40}$$

and

$$\|xA\|_{\mathcal{E}} \leq \|x\|_{\mathcal{E}}\|A\|_{\mathfrak{A}} \text{ for } x \in \mathcal{E}, A \in \mathfrak{A} \tag{12.41}$$

holds. With no doubt $\mathcal{E}$ is a normed $\mathfrak{A}$-module.

Definition 12.2.28. *An inner-product $\mathfrak{A}$-module which is complete w.r.t. its norm is called a (right) **Hilbert $\mathfrak{A}$-module or Hilbert C^*-module over the C^*-algebra** $\mathfrak{A}$.*

Remark that if $\mathfrak{A}_0$ is a pre-C^*-algebra and $\mathfrak{A}$ is its completion, then there is an inner-product $\mathfrak{A}_0$-module. Since (12.41) holds, the module action of $\mathfrak{A}_0$ can be extended by continuity to a module action of $\mathfrak{A}$ on $\mathcal{E}$.

Definition 12.2.29. *A Hilbert $\mathfrak{A}$-module $\mathcal{E}$ is said to be **full** if the two-sided ideal*

$$\langle \mathcal{E}, \mathcal{E}\rangle := \mathrm{Lin}\{\langle x, y\rangle_{\mathfrak{A}} : x, y \in \mathcal{E}\}$$

is dense in $\mathfrak{A}$.

Example 12.2.12: Any C^*-algebra $\mathfrak{A}$ is a $\mathfrak{A}$-module $\mathfrak{A}$ with $\langle A, B\rangle_{\mathfrak{A}} := A^*B$ and the C^*-norm of $\mathfrak{A}$.

▪

Example 12.2.13: Any Hilbert space $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module $\mathcal{H}$ with the inner product of $\mathcal{H}$.

▪

12.2.4.3 Morita equivalence of C^*-algebras

For a short overview refer to Schmüdgen [88]. For a detailed reference notice Lance [55] or Raeburn and Williams [76].

Definition 12.2.30. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two C^*-algebras.*

*A $\mathfrak{A}$-$\mathfrak{B}$-**imprimitivity bimodule** is a vector space $\mathcal{E}$ which is a full right Hilbert $\mathfrak{A}$-module and a full left Hilbert $\mathfrak{B}$-module such that*

(i) $\langle B\psi, \phi\rangle_{\mathfrak{A}} = \langle \psi, B^*\phi\rangle_{\mathfrak{A}}$ and $\langle \psi A, \phi\rangle_{\mathfrak{B}} = \langle \psi, \phi A^*\rangle_{\mathfrak{B}}$ for all $\psi, \phi \in \mathcal{E}$, $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$.

(ii) $x\langle y, z\rangle_{\mathfrak{A}} = \langle x, y\rangle_{\mathfrak{B}} z$ for $x, y, z \in \mathcal{E}$.

Definition 12.2.31. *Two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are called **(strongly) Morita equivalent** if there exists a $\mathfrak{A}$-$\mathfrak{B}$-imprimitivity bimodule.*

There are other possibilities to define this equivalence by using the theory of adjointable operators on Hilbert C^*-modules.

Definition 12.2.32. *Let $\mathcal{E}$ be a (right) Hilbert C^*-module over $\mathfrak{B}$.*

Consider a map $A : \mathcal{E} \to \mathcal{E}$ for which there exists a map $A^* : \mathcal{E} \to \mathcal{E}$ such that

$$\langle A\psi, \phi\rangle_{\mathfrak{B}} = \langle \psi, A^*\phi\rangle_{\mathfrak{B}} \tag{12.42}$$

*for all $\psi, \phi \in \mathcal{E}$. Then A is called **adjointable**. The space of all adjointable maps is denoted by $\mathcal{L}(\mathcal{E}, \mathfrak{B})$.*

Let $\Theta_{x,y} : \mathcal{E} \to \mathcal{E}$ be a map

$$\Theta_{x,y}(z) := x\langle y, z\rangle_{\mathfrak{B}} \text{ for } z \in \mathcal{E} \tag{12.43}$$

and denote the space, which is spanned by $\Theta_{x,y}$ where $x, y \in \mathcal{E}$, by $\mathcal{K}(\mathcal{E}, \mathfrak{B})$.

Lemma 12.2.33. *An adjointable map A is $\mathbb{C}$-linear, $\mathfrak{B}$-linear, bounded and unique.*

Lemma 12.2.34. *The space $\mathcal{K}(\mathcal{E}, \mathfrak{B})$ is a closed linear subspace of $\mathcal{L}(\mathcal{E}, \mathfrak{B})$.*

Theorem 12.2.35. *Let $\mathcal{E}$ be a Hilbert C^*-module over $\mathfrak{B}$.*

The algebra $\mathcal{L}(\mathcal{E}, \mathfrak{B})$ (or $\mathcal{K}(\mathcal{E}, \mathfrak{B})$) equipped with the map $A \mapsto A^*$ as the involution operation and with the norm defined by

$$\|A\| := \sup\{\|Ax\| : x \in \mathcal{E}, \|x\| = 1\} \tag{12.44}$$

where $\|x\|$ for $x \in \mathcal{E}$ is the norm on $\mathcal{E}$ is a C^*-algebra.

Moreover, for each $A \in \mathcal{L}(\mathcal{E}, \mathfrak{B})$ (or $A \in \mathcal{K}(\mathcal{E}, \mathfrak{B})$) it is true that

$$\langle A\psi, A\psi\rangle_{\mathfrak{B}} \leq \|A\|^2 \langle \psi, \psi\rangle_{\mathfrak{B}} \text{ for all } \psi \in \mathcal{E} \tag{12.45}$$

There is a non-degenerate action of $\mathcal{L}(\mathcal{E}, \mathfrak{B})$ (or $\mathcal{K}(\mathcal{E}, \mathfrak{B})$) on $\mathcal{E}$ given by $\pi_R(A)\psi = \psi A$.

Definition 12.2.36. *Let $\mathfrak{A}$ be a C^*-subalgebra of $\mathfrak{B}$.*

*Then $\mathfrak{A}$ is an **essential ideal** in $\mathfrak{B}$ if $\mathfrak{A}$ is an ideal and from $B \in \mathfrak{B}$ and $B\mathfrak{A} = \{0\}$ it follows that $B = 0$.*

Lemma 12.2.37. *$\mathcal{K}(\mathcal{E}, \mathfrak{B})$ is an essential ideal in $\mathcal{L}(\mathcal{E}, \mathfrak{B})$*

Proposition 12.2.38. *Two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are (strongly) Morita equivalent if there exists a full Hilbert C^*-module $\mathcal{E}$ over the C^*-algebra $\mathfrak{B}$ under which $\mathfrak{A}$ is isomorphic to $\mathcal{K}(\mathcal{E}, \mathfrak{B})$.*

Summarising, Morita equivalent C^*-algebras have isomorphic categories of representations on a Hilbert space.

Example 12.2.14: Remember the example 12.2.4.12, then $\mathcal{K}(\mathfrak{A}, \mathfrak{A})$ is equivalent to $\mathfrak{A}$. The C^*-algebra $\mathcal{L}(\mathfrak{A}, \mathfrak{B})$, which is constructed from $\mathfrak{A}$ via example 12, is indeed the multiplier algebra of $\mathfrak{A}$.

■

Example 12.2.15: The example given in 12.2.4.13 the algebra $\mathcal{K}(\mathcal{H}, \mathbb{C})$is equivalent to $\mathcal{K}(\mathcal{H})$.

■

12.2.4.4 Multiplier C^*-algebras

There are several definitions of multiplier algebras in the literature. One has been given briefly by Schmüdgen [88] in terms of double centralizer algebras. Another description has been presented in detail by Lance [55] for Hilbert C^*-modules. The third definition, which is related to linear operators and their adjoints on a C^*-algebra, has been given by Woronowicz [113], Woronowicz and Napiórkowski [115].

Briefly speaking the multiplier algebra $M(\mathfrak{A})$ is the largest unital C^*-algebra containing $\mathfrak{A}$ as an essential ideal, i.e. $M \in M(\mathfrak{A}), M\mathfrak{A} = \{0\}$ it follows that $M = 0$.

Double centralizer for a C^*-algebra

Definition 12.2.39. *Let $\mathfrak{A}$ be a semiprime*[5] *complex algebra. Then the set $DC(\mathfrak{A})$ is defined by all pairs (L, R) of maps $R, L : \mathfrak{A} \to \mathfrak{A}$ such that*

$$AL(B) = R(A)B \text{ for all } A, B \in \mathfrak{A}$$

and the module relations

$$\begin{aligned} R(AB) &= AR(B), \\ L(AB) &= L(A)B \quad \forall A, B \in \mathfrak{A} \end{aligned}$$

holds.

The set $DC(\mathfrak{A})$ can be equipped with some suitable operations such that it is a unital algebra.

Proposition 12.2.40. *Let $\mathfrak{A}$ be a C^*-algebra. Then the set $\mathcal{DC}(\mathfrak{A})$ of pairs (L, R) of maps $R, L : \mathfrak{A} \to \mathfrak{A}$ such that*

$$AL(B) = R(A)B \text{ for all } A, B \in \mathfrak{A}$$

holds. For $(L, R) \in \mathcal{DC}(\mathfrak{A})$ define the adjoint

$$(L(A), R(A))^* = (R^*(A), L^*(A)) = (R(A^*)^*, L(A^*)^*) \tag{12.46}$$

*whenever $A \in \mathfrak{A}$. The norm is given by $\|(L(A), R(A))\| := \|L(A)\| = \|R(A)\|$ where $\|.\|$ is the C^*-norm. Then the set $\mathcal{DC}(\mathfrak{A})$ is a unital C^*-algebra and it is called the **double centralizer C^*-algebra**.*

*The **multiplier M in** $\mathfrak{A}$ is defined by the relation*

$$(AM)B = A(MB)$$

The C^*-algebra of adjointable operators

In the language of Hilbert modules an anologue of the double centralizder algebra has been derived.

Let $\mathfrak{A}$ be a C^*-algebra. Consider $\mathfrak{A}$ as a left and right $\mathfrak{A}$-module. Then consider the C^*-algebra of adjointable operators $\mathcal{L}(\mathfrak{A}, \mathfrak{A})$ and the map $w_B : \mathfrak{A} \to \mathfrak{A}$ defined by $w_B(A) := AB$ such that $w_B \in \mathcal{L}(\mathfrak{A}, \mathfrak{A})$.

For any $B \in \mathfrak{A}$ let $R_B : \mathfrak{A} \to w_B(\mathfrak{A})$ be an isomorphism such that $R_B(A) = w_B(A)$ for all $A \in \mathfrak{A}$. There is also an isometric map w from $\mathfrak{A}$ into $\mathcal{L}(\mathfrak{A}, \mathfrak{A})$ defined by $B \mapsto w_B$. Observe

$$Aw_B(C) = w_B(AC) \quad \forall A, C \in \mathfrak{A} \tag{12.47}$$

and therefore conclude that, $w_{\mathfrak{A}}$ is an ideal in $\mathcal{L}(\mathfrak{A}, \mathfrak{A})$. Then the representation $R : \mathfrak{A} \to \mathcal{L}(\mathfrak{A}, \mathfrak{A})$ of $\mathfrak{A}$ given by $B \mapsto w_B$ is an injective *-homomorphism onto the closed essential ideal $\mathcal{K}(\mathfrak{A}, \mathfrak{A})$.

Otherwise for $B \in \mathfrak{A}$ the operator $\hat{w}_B : \mathfrak{A} \to \mathfrak{A}$ defined by $\hat{w}_B(A) = BA$ is adjointable. The map $L_B : \mathfrak{A} \to \hat{w}_B(\mathfrak{A})$ is an isomorphism. It is true that

$$\hat{w}_B(A)C = \hat{w}_B(AC) \quad \forall A, C \in \mathfrak{A} \tag{12.48}$$

[5] A complex algebra $\mathfrak{A}$ is called semiprime, if $\mathfrak{A}A = 0$ implies $A = 0$.

and

$$A\hat{w}_B(C) = w_B(A)C \tag{12.49}$$

holds. The pair $(\hat{w}_B, w_B)$ is an element of $\mathcal{DC}(\mathfrak{A})$ and $B \mapsto (\hat{w}_B, w_B)$ is an embedding of $\mathfrak{A}$ in $\mathcal{DC}(\mathfrak{A})$.

Consequently for a $D \in \mathfrak{A}$ define the adjoint map $w_D^* : \mathfrak{A} \to \mathfrak{A}$ by $w_D^*(B) := D^*B$ for all $B \in \mathfrak{A}$ such that a $C \in \mathfrak{A}$ exists with

$$A^* w_D^*(B) = A^*(D^*B) = C^*B$$

Then C is unique and $C^* = A^*D^*$. Set $\hat{w}_D^*(A^*) = A^*D^*$ and conclude

$$A^* w_D^*(B) = A^*(D^*B) = (A^*D^*)B = \hat{w}_D^*(A^*)B$$

Consequently, D^* is a multiplier of $\mathfrak{A}$. Notice that

$$\hat{w}_D^*(B) = w_D(B^*)^* \text{ and } w_D^*(B) = \hat{w}_D(B^*)^*$$

and consequently $(w_B, \hat{w}_B)^* = (\hat{w}_B^*, w_B^*)$. Thus the multipliers of $\mathfrak{A}$ can be understood as adjointable operators in $\mathcal{L}(\mathfrak{A}, \mathfrak{A})$.

Multipliers of a C^*-algebra Finally a natural definition of a multiplier algebra is given by the following.

Definition 12.2.41. *A linear operator $M : \mathfrak{A} \to \mathfrak{A}$ is called a* ***multiplier of*** $\mathfrak{A}$ ***(in the sense of Woronowicz)*** *if for every $A \in \mathfrak{A}$ there exists a $C \in \mathfrak{A}$ such that for all $B \in \mathfrak{A}$ it is true that*

$$A^*MB = C^*B$$

The set of multipliers of $\mathfrak{A}$ is denoted by $M(\mathfrak{A})$.

Proposition 12.2.42. *For each $M \in M(\mathfrak{A})$ and for any $A \in \mathfrak{A}$ the element C such that $A^*MB = C^*B$ holds for all $B \in \mathfrak{A}$ is unique. Set $M^* : \mathfrak{A} \to \mathfrak{A}$ be an operator such that $M^*(A) = C$. Then $M^* \in M(\mathfrak{A})$ and $M(\mathfrak{A})$ endowed with *-operation is a *-algebra.*

Proposition 12.2.43. *The algebra $M(\mathfrak{A})$ is unital. The *-algebra $M(\mathfrak{A})$ with operator norm is a unital C^*-algebra.*

Proposition 12.2.44. *If $\mathfrak{A}$ is unital, then $M(\mathfrak{A}) = \mathfrak{A}$.*

Recall that a *-representation π of a C^*-algebra $\mathfrak{A}$ on $\mathcal{H}$ is called non-degenerate, if the set

$$\{\pi(A)\phi : A \in \mathfrak{A}, \phi \in \mathcal{H}\}$$

is dense in $\mathcal{H}$.

Notice that, for a degenerate *-representation π there is an invariant subset of $\mathcal{H}$ given by

$$\mathcal{H}_0 = \{\psi \in \mathcal{H} : \pi(A)\psi = 0 \ \forall A \in \mathfrak{A}\}$$

Definition 12.2.45. *Let $\mathfrak{A}$ be a *-subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.*

Then $\mathfrak{A}$ act ***non-degenerately*** *on $\mathcal{H}$ if the set $\{\phi \in \mathcal{H} : A\phi = 0 \quad \forall A \in \mathfrak{A}\}$ contains only the zero vector.*

Definition 12.2.46. *Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{A}$ be a (separable) C^*-subalgebra of the C^*-algebra $\mathcal{L}(\mathcal{H})$ acting non-degenerately on $\mathcal{H}$.*

Then the ***multiplier algebra*** *$M(\mathfrak{A})$ of the C^*-algebra $\mathfrak{A}$ is the set*

$$M(\mathfrak{A}) := \{B \in \mathcal{L}(\mathcal{H}) : BA \in \mathfrak{A}, AB \in \mathfrak{A} \quad \forall A \in \mathfrak{A}\} \tag{12.50}$$

The natural topology on $M(\mathfrak{A})$ is almost uniform convergence, i.e. a net (A_α) of elements in $M(\mathfrak{A})$ converges almost uniformly to 0 if $\|A_\alpha X\| \to 0$ and $\|A_\alpha^* X\| \to 0$ for any $X \in \mathfrak{A}$. $\mathfrak{A}$ is dense in $M(\mathfrak{A})$.

Clearly $\mathfrak{A}$ is an ideal in $M(\mathfrak{A})$.

For example

(i) $\mathfrak{A} = \mathcal{K}(\mathcal{H})$ then $M(\mathfrak{A}) = \mathcal{L}(\mathcal{H})$,

(ii) $\mathfrak{A} = C_0(X)$ for X being a locally compact Hausdorff space, then $M(\mathfrak{A}) = C_b(X)$ (X compact then $M(\mathfrak{A}) = C(X)$),

(iii) $\mathfrak{A}$ commutative then $M(\mathfrak{A}) = C_b(\Delta(\mathfrak{A}))$ where $\Delta(\mathfrak{A})$ is the spectrum of $\mathfrak{A}$.

Proposition 12.2.47. *The unital C^*-algebra $\mathcal{DC}(\mathfrak{A})$ is isomorphic to $M(\mathfrak{A})$.*

Definition 12.2.48. *[110, p.402]Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. A **morphism between C^*-algebras** from $\mathfrak{A}$ to $\mathfrak{B}$ is a *-homomorphism $\Phi : \mathfrak{A} \to M(\mathfrak{B})$ such that $\Phi(\mathfrak{A})\mathfrak{B} := \{\Phi(A)B : A \in \mathfrak{A}, B \in \mathfrak{B}\}$ is dense in $\mathfrak{B}$. Denote the set of morphisms by $\mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$.*

Lemma 12.2.49. *For each $\Phi \in \mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$ there is a unique extension of Φ to a unital *-homomorphism from $M(\mathfrak{A})$ to $M(\mathfrak{B})$.*

Definition 12.2.50. *A **isomorphism between C^*-algebras** $\mathfrak{A}$ and $\mathfrak{B}$ is a morphism $\Phi \in \mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$ if there is another morphism $\Psi \in \mathrm{Mor}(\mathfrak{B}, \mathfrak{A})$ such that for any $A \in \mathfrak{A}$ the morphisms satisfy $\Psi(\Phi(A)) = A$ and for any $B \in \mathfrak{B}$ it is true that $\Phi(\Psi(B)) = B$.*

12.2.4.5 The Gelfand-Naĭmark theorems

Theorem 12.2.51. ***First Gelfand-Naĭmark theorem***
Let $\mathfrak{A}$ be a commutative C^*-algebra.

Then there is a unique locally compact space X such that $\mathfrak{A}$ is isometrically isomorphic to the C^*-algebra $C_0(X)$.

Equivalently the theorem is reformulated as follows.

Proposition 12.2.52. *Let $\mathfrak{A}$ be a commutative C^*-algebra.*

Then there is a unique locally compact space X and a linear isomorphism $\Phi : C_0(X) \to \mathfrak{A}$ such that

$$\begin{aligned} \Phi(f_1 f_2) &= \Psi(f_1)\Psi(f_2) \\ \Psi(f^*) &= \Psi(f)^* \end{aligned}$$

for all $f_1, f_2, f \in \mathfrak{A}$. The map Ψ is an isometry of Banach space $C_0(X)$ onto $\mathfrak{A}$.

Theorem 12.2.53. ***Second Gelfand-Naĭmark theorem***
For any C^*-algebra there is a Hilbert space $\mathcal{H}$ and a linear isomorphism Φ from $\mathfrak{A}$ onto a norm-closed *-subalgebra of $\mathcal{L}(\mathcal{H})$. This isomorphism is multiplicative, *-preserving and isometric.

Proposition 12.2.54. *For every C^*-algebra $\mathfrak{A}$ there is a Hilbert space $\mathcal{H}$ and a faithful non-degenerate *-representation of $\mathfrak{A}$ on $\mathcal{H}$.*

12.2.4.6 Elements affiliated with C^*-algebras

In this section the athor refers to private notes collected from talks given by A.Hertsch at the University of Leipzig and Woronowicz [110, p.402].

Note that, for a linear operator T on a Hilbert space $\mathcal{H}$ with dense domain $D(T)$ the domain of the adjoint operator is equivalent to the set

$$D(T^*) = \{\phi \in \mathcal{H} : \exists \varphi \in \mathcal{H} \text{ s.t. } \langle \phi, T\psi \rangle = \langle \varphi, \psi \rangle \forall \psi \in \mathcal{H}\}$$

Since $D(T)$ is dense, $T^*\phi = \varphi$ is uniquely defined.

Definition 12.2.55. *Let $\mathfrak{A}$ be a C^*-algebra and $T : \mathfrak{A} \supseteq D(T) \to \mathfrak{A}$ a linear operator with dense domain $D(T)$.*

The define the linear operator $T^* : \mathfrak{A} \supseteq D(T^*) \to \mathfrak{A}$ such that $D(T^*)$ consists of all $A \in \mathfrak{A}$ for which there exists an element $B \in \mathfrak{A}$ satisfying $T^*(C)A = C^*B$ for all $C \in D(T)$.

*For such an $A \in D(T^*)$, this element B is unique and $T^*(A) = B$ holds. The linear operator T^* is called the **adjoint of** T.*

Note that $T(AB) = T(A)B$.

Definition 12.2.56. *Let $\mathfrak{A}$ be a C^*-algebra and $T : \mathfrak{A} \supseteq D(T) \to \mathfrak{A}$ a closed linear operator with dense domain $D(T)$ such that $D(T^*)$ is dense in $\mathfrak{A}$ and the linear operator $1 + T^*T$ has dense range in $\mathfrak{A}$.*

*Then the element T is said to be **affiliated with the C^*-algebra** $\mathfrak{A}$, shortly $T\eta\mathfrak{A}$.*

Let $T\eta\mathfrak{A}$ then T is called self-adjoint iff $T = T^*$.

The following definition is due to Woronowicz.

Definition 12.2.57. *Let $\mathfrak{A}$ be a C^*-algebra and $T_1, ..., T_N$ affiliated elements with $\mathfrak{A}$.*

*Then $\mathfrak{A}$ **is generated by a finite set of unbounded elements** $T_1, ..., T_N$ if for any Hilbert space $\mathcal{H}$, any non-degenerate *-representation π of $\mathfrak{A}$ on $\mathcal{H}$ and every C^*-subalgebra $\mathfrak{B}$ of the algebra of bounded operators $\mathcal{L}(\mathcal{H})$ acting non-degenerately on $\mathcal{H}$ the following condition holds.*

If $\pi(T_i)\eta B$ for all $i \in \{1, ..., n\}$, then $\pi \in \mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$

where $\mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$ is the set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$.

12.2.4.7 Representations and actions of groups in C^*-algebras

Representations and actions of general groups

In context of C^*-algebras the theory of dynamical systems and covariant representation is formulated without refering to a Hilbert space. Woronowicz and Napiórkowski [115] have presented the definition of unitary representations of groups in C^*-algebras.

Definition 12.2.58. *Let G be a topological group and $\mathfrak{A}$ be a C^*-algebra.*

*A mapping $U : G \to M(\mathfrak{A})$ define a **unitary representation of** G **in** $\mathfrak{A}$ if*

(i) $U(g)$ is unitary for any $g \in G$,

(ii) $U(g_1)U(g_2) = U(g_1 g_2)$ for all $g_1, g_2 \in G$ and

(iii) for every $A \in \mathfrak{A}$ the mapping $G \ni g \mapsto U(g)A \in \mathfrak{A}$ is norm-continuous.

The set of all representations is denoted by $\mathrm{Rep}(G, \mathfrak{A})$.

Remark that, each $U \in \mathrm{Rep}(G, \mathcal{K}(\mathcal{H}))$ is a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}$. Conversely each strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}$ is a unitary representation of G on the C^*-algebra $\mathcal{K}(\mathcal{H})$.

There is an action of a group on a C^*-algebra defined as follows.

Definition 12.2.59. *Let G be a locally compact group and let $\mathfrak{A}$ be a C^*-algebra. A map*

$$\alpha : G \to \mathrm{Aut}(\mathfrak{A}), g \mapsto \alpha_g$$

*is called an **action of a group** G **on the** C^***-algebra** $\mathfrak{A}$ if*

(i) for every $g, h \in G$ it is true that $\alpha_{gh} = \alpha_g \circ \alpha_h$ and

(ii) for every $A \in \mathfrak{A}$ the map $G \ni g \mapsto \alpha_g(A) \in \mathfrak{A}$ is norm-continuous.

The set of all such action is denoted by $\mathrm{Act}(G, \mathfrak{A})$.

Note that, in this dissertation additional conditions are required for an action α in $\mathrm{Act}(G, \mathfrak{A})$, the action has to be automorphic. Hence additionally the action α is assumed to satisfy

$$\begin{aligned} \alpha_g(AB) = \alpha_g(A)\alpha_g(B) \text{ for all } A, B \in \mathfrak{A}, \quad g \in G \\ \alpha_g(A^*) = \alpha_g(A)^* \text{ for all } A \in \mathfrak{A}, \quad g \in G \end{aligned}$$

In literature the object $g \mapsto \alpha_g$ is also called a strongly continuous one-parameter group of *-automorphisms.

Furthermore remark that, the C^*-algebra $\mathfrak{A}$ is not required to be commutative. In the framework of section 6 the transformation groups are introduced. There it is used that, there is a correspondence between transformations on the configuration space X and actions on C^*-algebra $C_0(X)$.

Definition 12.2.60. *A* ***C^*-dynamical system*** $(G, \mathfrak{A}, \alpha)$ *is a triple of a topological group G, a C^*-algebra and an action α of the group G on $\mathfrak{A}$.*

Definition 12.2.61. *A* ***covariant representation of a C^*-dynamical system*** $(G, \mathfrak{A}, \alpha)$ ***in a C^*-algbra*** $\mathfrak{B}$ *is a pair (M, U) consisting of a morphism $M \in \mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$ and a unitary representation of G in $\mathfrak{A}$ such that for any $A \in \mathfrak{A}$ and $g \in G$ it is true that*

$$M(\alpha_g(A)) = U_g M(A) U_g^*$$

Definition 12.2.62. *A* ***C^*-dynamical system*** $(H, \mathfrak{A}, \alpha)$ ***in a C^*-algebra*** $\mathfrak{B}$ *is given by a triple consisting of a locally compact group H, a C^*-algebra $\mathfrak{A}$ and an automorphism $H \ni h \mapsto \alpha_h \in \mathrm{Aut}(\mathfrak{A})$, which defines an action α of the group H on $\mathfrak{A}$.*

A ***covariant pair*** (Φ, V) ***of*** $(H, \mathfrak{A}, \alpha)$ ***in a C^*-algebra*** $\mathfrak{B}$ *is given by a morphism $\Phi \in \mathrm{Mor}(\mathfrak{A}, \mathfrak{B})$ and a unitary representation V of H in C^*-algebra $\mathfrak{B}$, $V \in \mathrm{Rep}(H, \mathfrak{B})$.*

Infinitesimal representations of Lie groups

In context of Lie groups the unitary representations can be related to infinitesimal representations of the Lie algebras in C^*-algebras. This is the generalisation of the infinitesimal representations of the universal enveloping $\mathcal{E}$ or of $\mathfrak{g}$ in Hilbert spaces, which has been presented by Schmüdgen [89, section 10].

Definition 12.2.63. *Let U be a unitary representation of a n-dimensional Lie group G in a C^*-algebra $\mathfrak{A}$.*

The operators $\mathrm{d}\,U(X)$ for every $X \in \mathcal{E}$ are defined on the invariant dense domain

$$D^\infty(U) := \left\{ A \in \mathfrak{A} : \begin{array}{c} \textit{the mapping } G \ni g \mapsto U(g)A \in \mathfrak{A} \\ \textit{is of } C^\infty\textit{-class in the sense} \\ \textit{of norm topology in } \mathfrak{A} \end{array} \right\}$$

and for every $X \in \mathcal{E}$ and $A \in D^\infty(U)$

$$\mathrm{d}\,U(X)A := XU(g)A|_{g=e_G}$$

The ***infinitesimal representation of the universal enveloping $\mathcal{E}$ in*** $\mathfrak{A}$ *is denoted by $\mathrm{d}\,U$.*

Let $(X_1, ..., X_N)$ be a basis of the Lie algebra $\mathfrak{g}$. Then the infinitesimal operator $\mathrm{d}\,U(X_i)$ is skew-adjoint and affiliated with $\mathfrak{A}$. Moreover, the Nelson operator $\triangle = \sum X_l^+ X_l$ is self-adjoint and positive.

12.2.4.8 C^*-bialgebras

A compact quantum group $(\mathfrak{A}, \triangle)$ is a pair of a unital C^*-algebra and a unital *-homomophism. In a more general case, it is necessary to consider an arbitrary C^*-algebra $\mathfrak{A}$ and a comultiplication $\triangle$.

Definition 12.2.64. *Let $\mathfrak{A}$ be a C^*-algebra.*

Then a comultiplication $\triangle$ is an element of $\mathrm{Mor}(\mathfrak{A}, \mathfrak{A} \otimes \mathfrak{A})$ such that

- $(\triangle \otimes \iota)\triangle = (\iota \otimes \triangle)\triangle$

· $\{\triangle(A)(B \otimes \mathbb{1}) : A, B \in \mathfrak{A}\}$ *is a dense subspace of* $\mathfrak{A} \otimes \mathfrak{A}$

· $\{\triangle(A)(\mathbb{1} \otimes B) : A, B \in \mathfrak{A}\}$ *is a dense subspace of* $\mathfrak{A} \otimes \mathfrak{A}$

*An C^*-algebra $\mathfrak{A}$ with a comultiplication $\triangle$ is called a C^*-**bialgebra**.*

Since $(\triangle \otimes \iota)\triangle$ and $(\iota \otimes \triangle)\triangle$ are products of morphisms it is necessary to extend $\triangle \otimes \iota$ and $\iota \otimes \triangle$ to maps from $M(\mathfrak{A} \otimes \mathfrak{A})$ to $M(\mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A})$.

12.2.4.9 Derivations of C^*-algebras

Refer to Bratteli and Robinson [22].

Definition 12.2.65. *A (symmetric) ***-derivation** δ of a C^*-algebra $\mathfrak{A}$ is a linear operator from a *-subalgebra $D(\delta)$, the domain of δ, into $\mathfrak{A}$ with the properties*

(i) $\delta(A)^* = \delta(A^*)$

(ii) $\delta(AB) = \delta(A)B + A\delta(B)$

for $A, B \in D(\delta)$.

Set $\mathrm{Der}(D(\delta), \mathfrak{A}) := \{\delta : \delta \text{ derivation on } D(\delta) \text{ into } \mathfrak{A}\}$

Definition 12.2.66. *Let $\mathfrak{A}$ be a unital C^*-algebra.*

*A *-derivation $-\delta$ is called **dissipative** if*

(i) $D(\delta)$ is a dense *-subalgebra of $\mathfrak{A}$

(ii) $\delta(A^*A) = \delta(A^*)A + A^*\delta(A) \quad \forall A \in D(\delta)$ *and*

(iii) $\mathbb{1} \in D(\delta)$, *if* $A \geq 0$ *and* $A \in D(\delta)$, *then* $A^{\frac{1}{2}} \in D(\delta)$

Let $\mathbb{R} \ni t \mapsto \alpha_t$ be a strongly continuous one-parameter group of *-automorphisms, then the infinitesimal generator of α is defined by

$$\begin{aligned} D(\delta) &:= \left\{ A \in \mathfrak{A} : \lim_{t \to 0} \frac{1}{t}\Big(\alpha_t(A) - A\Big) \quad \exists \text{ in norm } \right\} \\ \delta(A) &:= \lim_{t \to 0} \frac{1}{t}\Big(\alpha_t(A) - A\Big) \end{aligned} \tag{12.51}$$

Conversely let δ be a symmetric *-derivation on a unital C^*-algebra $\mathfrak{A}$, which is as an operator norm-densely defined and norm-closed on $\mathfrak{A}$. Then δ is a generator of a strongly continuous one-parameter group of *-automorphisms of $\mathfrak{A}$ iff δ has a dense set of analytic elements and δ and $-\delta$ are dissipative. A derivation δ is dissipative iff $\|(I + \delta)(A)\| \geq \|A\|$ for all $A \in D(\delta)$. The idea behind this property is that for an element A of the set of analytic elements in $D(\delta)$ the automorphism is represented by

$$\alpha_t(A) = \sum_{n \geq 0} \frac{t^n}{n!}\delta^n(A) = \lim_{n \to \infty} \left(I + \frac{t}{n}\delta\right)^n (A) \text{ for } |t| < t_A \tag{12.52}$$

where t_A is the convergence radius of the series $\sum_{n \geq 0} \frac{t^n}{n!}\|\delta^n(A)\|$.

Definition 12.2.67. *A *-derivation is called **inner**, if there exists an element H of $M(\mathfrak{A})$ such that*

$$\delta(A) = [H, A] \quad \text{for } A \in D(\delta) \tag{12.53}$$

12.2.5 Banach *-algebras

12.2.5.1 Commutative Banach *-algebras

Let $\mathfrak{A}$ be a commutative Banach *-algebra.

Definition 12.2.68. *The structure space $\Delta(\mathfrak{A})$ of a commutative Banach *-algebra $\mathfrak{A}$ is the set of all nonzero linear maps $\omega : \mathfrak{A} \to \mathbb{C}$ for which*

$$\omega(AB) = \omega(A)\omega(B) \tag{12.54}$$

for all $A, B \in \mathfrak{A}$. In other words,

$$\Delta(\mathfrak{A}) := \{\omega \in \mathrm{Hom}(\mathfrak{A}, \mathbb{C}) : \omega \textit{ non-zero}\} \tag{12.55}$$

Since

$$|\omega(A)| \leq \|A\| \quad \forall A \in \mathfrak{A} \tag{12.56}$$

the structure space $\Delta(\mathfrak{A})$ is a subset of the dual $\mathfrak{A}^*$ of the $*$-Banach space $\mathfrak{A}$. The weak*- topology on the dual $\mathfrak{A}^*$ is defined by the convergence $\omega_n \to \omega$ iff $\omega_n(A) \to \omega(A)$ for all $A \in \mathfrak{A}$. The **Gel'fand topology** on $\Delta(\mathfrak{A})$ is the relative weak*- topology.

Proposition 12.2.69. *The structure space $\Delta(\mathfrak{A})$ of a unital commutative *-Banach algebra is compact and Hausdorff in the Gel'fand topology.*

There is a map $A \mapsto \hat{A} := \omega(A)$, which maps elements of the Banach *-algebra $\mathfrak{A}$ to $\mathcal{B}(\Delta(\mathfrak{A}))$ and is called the Gel'fand transform. The commutative Banach *-algebra $\mathcal{B}(X)$ is the algebra of all continuous bounded complex-valued functions on a set X with a pointwise multiplication on X and supremum norm.

Proposition 12.2.70. *The Gelfand transform $A \mapsto \hat{A}$ is a norm-decreasing algebra-homomorphism of $\mathfrak{A}$ into $\mathcal{B}(\Delta(\mathfrak{A}))$.*

Symbols

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Danksagung

Ich danke meine Eltern, die mir bei allen schwierigen Problemen geholfen haben und mich weiter begleiten werden. Herzlichen Dank an alle meine Freunde, die mit Geduld und Zuspruch diese Arbeit ermöglicht haben. Diese Dissertation widme ich allen, die mich auf meinem Weg des Lebens begleitet haben. Ich trage die außerwählten Momente des Lebens, welche ich mit Euch verbracht habe, für immer in meinem Herzen. In Gedanken werden mir auch die Menschen bleiben, die unsere Welt bereits verlassen haben.

Ich danke Christian Fleischhack für die Finanzierung während der Erarbeitung dieser Dissertation im Rahmen des Emmy-Noether-Projektes der DFG. Darüberhinaus habe ich in unserer Arbeitsgruppe gern mit Johannes Brunnemann diskutiert. Des Weiteren danke ich allen Wissenschaftlern mit denen ich mathematisch-physikalische Probleme erörtert konnte. Insbesondere möchte ich an dieser Stelle Johannes Aastrup und Jesper Grimstrup für die Gespräche über KMS- und Operator-Theorie erwähnen. Die Fragen zur Theorie der Wege - Zusammenhänge a là Mackenzie hat mir Iakovos Androulidakis beantwortet. Ebenso danke ich Rainer Verch, Klaus Fredenhagen, Stanisław Lech Woronowicz, Thomans Thiemann und Jerzy Lewandowski für persönliche Einladungen zu Gesprächen über meine Arbeit.

Zuletzt möchte ich die Institutionen erwähnen die mich finanziell bzw. durch die Bereitstellung von Büroräumen, Büchern, Reisemittel oder ähnlichem unterstützt haben. Diese sind die Deutsche Forschungsgemeinschaft im Rahmen des Emmy-Noether-Progammes "Quantengeometrie: Mathematische Physik auf dem Weg zur Quantengravitation" (grant FL 622/1-1), die Universität Hamburg, das Desy, die Universität Paderborn, das Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig und das Netzwerk "Quantum Geometry and Quantum Gravity" im Rahmen der European Science Foundation.

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne unerlaubte fremde Hilfe angefertigt habe. Alle dargestellten Zusammenhänge und Rechnungen habe ich selbst erarbeitet. Abschnitte, in denen ich Literatur herangezogen habe, sind als solche kenntlich gemacht.

Leipzig, den 19. August 2011