Uniformly summable multiplicative functions on additive arithmetical semigroups

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Abstract

We present the additive arithmetical semigroups and summarize the improvements on prime number theorems and mean-value theorems on additive arithmetical semigroups. We start with definitions and examples, then compare the approaches, which have been used to prove prime number theorems. Thereafter, we give a short outline of the convolution theory and generating functions.

Then we proceed with complex-valued multiplicative functions on additive arithmetical semigroups. First we summarize some results for multiplicative functions of modulus ≤ 1, and more generally for uniformly summable multiplicative functions. Afterwards, we prove new mean-value theorems for uniformly summable multiplicative functions on additive arithmetical semigroups. These theorems are more general than the previous results because our conditions on the additive arithmetical semigroups are weaker and we can prove our mean-value theorems for a larger class of functions. In the proof we use some tauberian theorems by Indlekofer, and some ideas of the proof of mean-value theorems for multiplicative functions in the classical number theory.

Finally, we give an application of our results by proving a characterization of finitely distributed functions on additive arithmetical functions and the Three-series theorem on additive arithmetical semigroups.

Zusammenfassung

Wir präsentieren die additiven arithmetischen Halbgruppen und fassen die wichtigsten Ergebnisse über Primzahl- und Mittelwertsätze auf additiven arithmetischen Halbgruppen zusammen. Wir beginnen mit Definitionen und Beispielen, danach vergleichen wir die Ansätze, die verwendet wurden, um Primzahlsätze zu beweisen. Anschließend geben wir einen kurzen Überblick über Faltung und erzeugende Funktionen.

Dann betrachten wir komplexwertige multiplikative Funktionen auf additiven arithmetischen Halbgruppen. Zuerst fassen wir einige Ergebnisse für multiplikative Funktionen vom Betrag ≤ 1 zusammen, dann allgemeiner für gleichgradig summierbare multiplikative Funktionen. Danach beweisen wir neue Mittelwertsätze für gleichgradig summierbare multiplikative Funktionen auf additiven arithmetischen Halbgruppen. Diese Sätze sind allgemeiner als die bisherigen, weil unsere Bedingungen an die additiven arithmetischen Halbgruppen schwächer sind und weil wir eine größere Klasse von Funktionen behandeln.
In dem Beweis benutzen wir eine Methode über Taubersätze von Indlekofer, und einige Ideen des Beweises der Mittelwertsätze für multiplikative Funktionen in der klassischen Zahlentheorie.

Schließlich geben wir als eine Anwendung unserer Ergebnisse eine Charakterisierung von endlich verteilten additiven Funktionen auf additiven arithmetischen Halbgruppen und einen Beweis für den Drei-Reihen-Satz auf additiven arithmetischen Halbgruppen.

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Contents

Notations

We use the following conventions: $\mathbb{N}$ is the set of the natural numbers; and $\mathbb{N}$ does not contain zero, while $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. The real- and imaginary parts of a complex number z are written $Re\ z$ and $Im\ z$. We use the the well-known Landau symbols little-o and big-O in this thesis, as well as the following notations to describe the limiting behaviour of a function: $f(n) \ll g(n)$ ($f(n) \gg g(n)$) as $n \to \infty$ means, that there exists a positive constant C and a natural number n_0 such that $|f(n)| \leq Cg(n)$ ($C|f(n)| \geq g(n)$) for all $n \geq n_0$. Summarizing the two assertions $f(n) \ll g(n)$ and $f(n) \gg g(n)$ as $n \to \infty$, we write $f(n) \asymp g(n)$ as $n \to \infty$.
The places where frequently used symbols are defined are indicated in the following table. Some symbols denote objects that depend on n, z, or a, where n represents an integer, z and y represent complex numbers, and a represents an element of an additive arithmetical semigroup. To emphasize the difference between functions defined on complex numbers or on additive arithmetical semigroups we shall in general use the extra " ˜ " sign over an alphabetic character (for example $\tilde{f}$) for functions defined on an additive arithmetical semigroup.
We use the notation $p^k||a$ for p^k divides exactly a when $p^k|a$ but $p^{k+1} \nmid a$.
Numbers in brackets like (1.1) indicate equation numbers, while numbers without brackets like 1.1 indicate numbers of definitions.

Symbol	Definition	page
$\partial(a)$	1.1	4
$G, G(n)$	1.1	4
P	1.1	4
$P(n)$	(1.1)	4
$\hat{Z}_G(z), \hat{Z}(z)$	(1.2)	5
q	(1.3)	5
A		6
$\hat{H}(z)$	(1.8)	6
$r(n)$	(1.12)	8
$\tilde{f}(a)$		11
$*$	(1.24)	11
$\mathbf{1}_B(a)$	(1.26)	12
$\tilde{\mu}(a)$	(1.27)	12
$\tilde{\omega}(a)$	(1.29)	12

Symbol	Definition	page
$\tilde{\Lambda}(a)$	(1.30)	13
$\hat{F}(z)$	(1.32)	13
$\hat{P}(z)$	(1.35)	14
$\Lambda(n)$	(1.37)	15
$\Lambda_f(n)$	(2.4)	17
$M(n, \tilde{f}), M(\tilde{f})$	2.1	17
L^α	2.7	21
L^*	2.8	22
$\gamma(n)$	2.21	27
$\lambda(n)$	(2.21)	27
$F(y)$	2.23	27
$Z(y)$	(2.22)	27

Introduction

Results from several different areas of mathematics have found a common generalization in the theory of arithmetical semigroups. An arithmetical semigroup is a free abelian semigroup generated by a countable set of prime elements with an associated mapping: a multiplicative norm, that leads to multiplicative arithmetical semigroups or an additive norm, that defines an additive arithmetical semigroup. In the case of multiplicative arithmetical semigroups one has to investigate the behavior of Dirichlet series, and for additive arithmetical semigroups we deal with power series.
Some of the research has focused on specific questions: e.g. polynomials over finite fields as composed of irreducibles or graphs as composed of connected components. This research example was the first instance in which many theorems were proven, and it also had a strong influence on the notation used in arithmetical semigroups. Others, such as Knopfmacher in his first books [30] and [31], began with given classical theorems and tried to prove them under the weakest possible conditions on the number of elements below a given size. It is this approach we follow in this thesis in the case of additive arithmetical semigroups.
There has been a considerable amount of research on the prime element theorem and therefore, a need to summarize the latest developments in this field. We focus on the other important question in the field, the mean-value theorems for multiplicative functions. The results in this thesis are improvements of the fundus presented by Knopfmacher and Zhang [32], the only recent book on the subject.

The idea of developing an arithmetical theory based mainly on the foundation of the axiom referred to here as Axiom $\mathcal{A}^{\#}$ seems to have first been pointed out in papers by Fogels [14] and [15]. However, in these papers, Fogels dealt only with some very special consequences of Axiom $\mathcal{A}^{\#}$, and referred only to polynomial rings and algebraic function fields over finite fields in order to motivate the axiom. These papers were followed by books written by J. Knopfmacher [30] and [31].

The zeta function of an additive arithmetical semigroup is the associated power series, where the nth coefficient gives the total number of elements in the additive arithmetical semigroup with degree n, which is by definition always finite. In the first investigations on additive arithmetical semigroups, the zeta function was regarded as having a pole on the boundary of its circle of convergence. Such an assumption is

Axiom $\mathcal{A}^{\#}$ (see [30]), which is a requirement on the coefficients of the zeta function. Knopfmacher stated that the zeta function has no zeros on its circle of convergence (see [31]), but Indlekofer, Manstavičius and Warlimont ([26]) have given examples of additive arithmetical semigroups satisfying Axiom $\mathcal{A}^{\#}$ with a zero on the boundary of the circle of convergence. This is an improvement of the prime number theorem on additive arithmetical semigroups. A major difference from the classical number theory is that the zeta function in the classical number theory has no zeros on the boundary; in the case of an additive arithmetical semigroup there can be one zero. Further, if there is a zero on the boundary then there are no further zeros in the circle of convergence. The third difference is that in contrast to the natural numbers on additive arithmetical semigroups, the Chebyshev lower estimate does not hold in general.

The argument has been developed in two different ways: some mathematicians, such as Knopfmacher and Zhang, have used requirements on the coefficients of the zeta function, where others, such as Indlekofer, have used the boundary behaviour of the zeta function and Axiom $\bar{\mathcal{A}}^{\#}$ introduced by Indlekofer [20]. If Axiom $\mathcal{A}^{\#}$ or Axiom $\bar{\mathcal{A}}^{\#}$ holds, then the zeta function is meromorphic on its circle of convergence with a simple pole.

Following this, the investigations have continued with the restriction that the circle of convergence is a natural boundary for the zeta function. Several mathematicians have made improvements in this case, particularly Indlekofer, Knopfmacher, Manstavičius, Warlimont and Zhang.

Another important area of investigation is complex-valued multiplicative functions on an additive arithmetical semigroup and mean-value theorems.
First, mean-value theorems for multiplicative functions on additive arithmetical semigroups were considered with functions of modulus ≤ 1. Several mathematicians, such as Barát, Indlekofer, Manstavičius, Warlimont, Wehmeier and Zhang, have contributed to this field. In this thesis we consider a larger class of functions, uniformly summable multiplicative functions, instead of the restriction on the modulus of the function. On the other hand, our condition on the additive arithmetical semigroup is more general.
The class of uniformly summable functions has been defined by Indlekofer (see [17]) for functions defined on $\mathbb{N}$ and correspond to integrable functions with respect to the asymptotic density.
In this thesis, we proceed as follows: Chapter 1 presents, just for reference, some well-known facts about the subject. We summarize the basic definitions and the most important conditions on additive arithmetical semigroups. The remaining part deals with multiplicative functions.
Chapter 2 contains the latest mean-value theorems for multiplicative functions on additive arithmetical semigroups for multiplicative functions of modulus ≤ 1. Then,

we formulate our new mean-value theorems for unifomly summable multiplicative functions with nonzero mean-value. The necessary assumptions on the semigroup are weaker than the conditions of the previous results. We proceed to give a characterization of uniformly summable functions on additive arithmetical semigroups. For the proof of our theorems we introduce the new method developed by Indlekofer that compares the coefficients of power series. Later, we prove some lemmas and give the proof of our theorems.
In Chapter 3 we give an application of our results by proving a characterization of finitely distributed additive functions defined on an additive arithmetical semigroup. Finally, we apply our results to prove the well-known Three-series theorem for additive arithmetical semigroups, and give an outline of our work with Indlekofer and Kaya on the more general Two-series theorem (see [3]) which was motivated among others by our previous work with Indlekofer and Wagner on Stone-Cech compactifications (see [4]).

Chapter 1

Basic definitions and facts about additive arithmetical semigroups

In this chapter, we introduce the additive arithmetical semigroups, and some functions related to them, such as the zeta function; and collate the most used conditions on additive arithmetical semigroups with the historical results on prime number theorems. We briefly expound the convolution on complex-valued functions on additive arithmetical semigroups and define the generating functions.

1.1 Definition and examples

Definition 1.1. We call (G, ∂) an *additive arithmetical semigroup*, if G is a commutative semigroup with identity element 1_G, generated by a countable set P of primes and ∂ is an integer valued degree mapping $\partial : G \to \mathbb{N}_0$, which satisfies

(i) $\partial(ab) = \partial(a) + \partial(b)$ for all $a, b \in G$,

(ii) the total number $G(n)$ of elements of degree n in G is finite for each $n \geq 0$.

Therefore, $\partial(1_G) = 0$, $\partial(p) > 0$ for all $p \in P$, $G(0) = 1$ and G is countable.

In this work (unless otherwise stated), G denotes an additive arithmetical semigroup related to an integer valued mapping ∂.
We write

$$P(n) := \#\{p \in P : \partial(p) = n\}. \tag{1.1}$$

In using these notations, we shall be particularly concerned with arithmetical consequences of assumptions on the total number $G(n)$ of elements of degree n in G, or on the total number $P(n)$ of primes of degree n in G.
J. Knopfmacher gave numerous natural examples (see [31]). We give here the most natural one.

Example 1.2. Galois polynomial rings. Let $F_q[X]$ denote a polynomial ring in an indeterminate X over the finite Galois field F_q with q elements (q a prime power). The subset $G_q = G(q, X)$ consisting of all monic polynomials in $F_q[X]$ forms a semigroup under multiplication. In particular, G_q together with the usual degree mapping on polynomials forms an additive arithmetical semigroup such that

$$G_q(n) = q^n \quad (n = 0, 1, 2, ...).$$

Example 1.3. Let (G, ∂) be an additive arithmetical semigroup such that there exists exactly one prime element of degree n for each $n \in \mathbb{N}$, i.e. $P(n) = 1$ for all $n \in \mathbb{N}$. In this case $G(n)$ is the number of ways to partition the number n into a sum of positive integers, so $G(1) = 1$, $G(2) = 2$, $G(3) = 3$, $G(4) = 5$, etc.

To investigate additive arithmetical semigroups, it is essential to work with the (generating) power series

$$\begin{aligned} \hat{Z}_G(z) := \hat{Z}(z) &= \sum_{n=0}^{\infty} G(n) z^n \\ &= \prod_{n=1}^{\infty} (1 - z^n)^{-P(n)} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} dP(d) z^n \right) \end{aligned} \tag{1.2}$$

which we call the *zeta function* $\hat{Z}_G$ *of* G. In the case of the Galois polynomial rings $G_q = G(q, X)$ (q a prime power), we obtain

$$\hat{Z}_{G_q}(z) = \sum_{n=0}^{\infty} G_q(n) z^n = \sum_{n=0}^{\infty} q^n z^n = \frac{1}{1 - qz}$$

that is convergent for $|z| < q^{-1}$. For the additive arithmetical semigroup described in Example 1.3, we obtain the fundamental identity ($|z| < 1$)

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-1}$$

the famous identity studied by Hardy and Ramanujan. Using their circle method in complex analysis, they found an asymptotic expression for $G(n)$, namely

$$G(n) \sim \frac{e^{\pi\sqrt{\frac{2}{3}n}}}{4n\sqrt{3}},$$

as well as efficient methods to actually compute $G(n)$.
Here we restrict ourselves to additive arithmetical semigroups satisfying

$$G(n) \ll q^n n^{\varrho} \tag{1.3}$$

with some $q > 1$ and $\varrho \in \mathbb{R}$ and generating zeta functions the circle of convergence of which is equal to $|z| < q^{-1}$ with radius q^{-1}.

1.2 Special conditions and their consequences

To get more precise results on additive arithmetical semigroups we need an assumption stronger than (1.3). Therefore Knopfmacher introduced the following condition.

Axiom $\mathcal{A}^{\#}$ (see J. Knopfmacher [30]). There exist constants $A > 0$, $q > 1$ and ν with $0 \leq \nu < 1$ (all depending on G), such that

$$G(n) = Aq^n + O(q^{\nu n}) \quad as \;\; n \to \infty. \tag{1.4}$$

If Axiom $\mathcal{A}^{\#}$ holds, we get

$$\hat{Z}(z) = \frac{A}{1-qz} + \hat{H}_1(z), \tag{1.5}$$

where

$$\hat{H}_1(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1.6}$$

with

$$c_n = G(n) - Aq^n = O(q^{\nu n}). \tag{1.7}$$

Thus $\hat{H}_1$ is holomorphic for $|z| < q^{-\nu}$. Put

$$\hat{H}(z) := A + (1-qz)\hat{H}_1(z). \tag{1.8}$$

Then the zeta function can be written as

$$\hat{Z}(z) = \frac{\hat{H}(z)}{1-qz}. \tag{1.9}$$

Obviously $\hat{H}(0) = 1$ and $\hat{H}(q^{-1}) = A$, furthermore $\hat{H}$ is holomorphic for $|z| < q^{-\nu}$.

Whereas Axiom $\mathcal{A}^{\#}$ gives conditions for the power series coefficients c_n of $\hat{H}_1$, Indlekofer formulated in [20] an assumption on the *boundary behaviour* of $\hat{H}$. This reads as

Axiom $\bar{\mathcal{A}}^{\#}$ (see Indlekofer [20]). There exist constants $q > 1$ and ν with $0 \leq \nu < 1$ (all depending on G) such that

(i) the function $\hat{H}$ defined by (1.8) is holomorphic in the open disc $|z| < q^{-\nu}$, and $\hat{H}(q^{-1}) > 0$,

(ii) the function $\bar{H}$ defined by $\bar{H}(y) := \hat{H}(q^{-\nu}y)$ is an element of the Nevanlinna class N.

Remark 1.4. If Axiom $\mathcal{A}^{\#}$ or Axiom $\bar{\mathcal{A}}^{\#}$ holds, then the zeta function $\hat{Z}(z)$ is meromorphic for $|z| < q^{-\nu}$ with a simple pole at $z = q^{-1}$ with residue A. Knopfmacher stated that $\hat{Z}(z) \neq 0$ for $|z| \leq q^{-1}$ (see [31]), but the proof was not correct for $z = -q^{-1}$, since Indlekofer, Manstavičius and Warlimont ([26]) have given examples of additive arithmetical semigroups G satisfying Axiom $\mathcal{A}^{\#}$ with $\nu = 1/2$ and $\hat{Z}(-q^{-1}) = 0$. On the other hand, these authors have also shown in [26] that, if G satisfies Axiom $\mathcal{A}^{\#}$ with $\nu < 1/2$, then $\hat{Z}(-q^{-1}) \neq 0$. Indlekofer, Manstavičius and Warlimont also proved that if $\hat{Z}(-q^{-1}) = 0$, then $\hat{Z}(z) \neq 0$ for all $|z| < q^{-\nu}, z \neq -q^{-1}$. As a result the authors showed the following *prime number theorem*:
Let $0 < \varepsilon < 1 - \nu$, if $\hat{Z}(-q^{-1}) \neq 0$, then there exist $l = l(\varepsilon) \in \mathbb{N}_0, 0 < \varepsilon' < \varepsilon$ and complex numbers β_i $(i = 1, \ldots, l)$ -the zeros of $\hat{Z}(z)$ in the disc $|z| \leq q^{-\nu-\varepsilon}$- such that:

$$\hat{\Lambda}(z) = \frac{z}{q^{-1} - z} - z \sum_{i=1}^{l} \frac{1}{\beta_i - z} + zR(z)$$

where $q^{-1} < \min\limits_{i=1,\ldots,l} |\beta_i| \leq \max\limits_{i=1,\ldots,l} |\beta_i| \leq q^{-\nu-\varepsilon}$ and $R(z)$ is holomorphic for $|z| \leq q^{-\nu-\varepsilon'}$, furthermore

$$P(n) = \frac{q^n}{n} - \frac{1}{n} \sum_{i=1}^{l} \beta_i^{-n} + O_\varepsilon \left(\frac{q^{n(\nu+\varepsilon')}}{n} \right). \tag{1.10}$$

as $n \to \infty$. If $\hat{Z}(-q^{-1}) = 0$, then

$$P(n) = \left(1 + (-1)^{n+1}\right) \frac{q^n}{n} + O_\varepsilon \left(\frac{q^{n(\nu+\varepsilon)}}{n} \right) \tag{1.11}$$

for all $0 < \varepsilon < 1 - \nu$ as $n \to \infty$.

Axiom $\bar{\mathcal{A}}^{\#}$ provides a better remainder term in the prime number theorem (see [20], Theorem 2, Theorem 5 and Corollary 3). For instance, if $\hat{Z}(-q^{-1}) = 0$, then Axiom $\mathcal{A}^{\#}$ yields (1.11) for some $(\varepsilon > 0)$, whereas Axiom $\bar{\mathcal{A}}^{\#}$ implies

$$P(n) = (1 - (-1)^n) \frac{q^n}{n} + 2a_n q^{n\nu}$$

with

$$a_n = \int_{-\pi}^{\pi} e^{-int} dm(t),$$

where m is the (real) measure occuring in the factorization of the Nevanlinna function $\bar{H}$ (cf. [20], Proposition). Furthermore, Axiom $\bar{\mathcal{A}}^{\#}$ follows from this asymptotic formula for $P(n)$ (cf. [20], Corollary 3).

Subsequently, the investigations continued with the restriction that $|z| = q^{-1}$ is a natural boundary for the zeta function. Zhang required assumptions on the coefficients of the zeta function and introduced

Axiom $\mathcal{A}$ (see Zhang [32]). There exist constants $A > 0$, $1 < q < \infty$ and a real-valued function r such that

$$G(n) = (A + r(n))q^n \tag{1.12}$$

with

$$\sum_{n=0}^{\infty} \sup_{m \geq n} |r(m)| < \infty.$$

Zhang and Warlimont proved the following prime number theorem (see [32], Theorem 5.4.1): if

$$G(n) = Aq^n + O\left(q^n n^{-\gamma}\right) \tag{1.13}$$

with $A > 0, q > 1$, and $\gamma > 2$, then either

$$P(n) = \frac{q^n}{n} + O\left(q^n n^{-\gamma}\right) \tag{1.14}$$

or

$$P(n) = (1 - (-1)^n)\,\frac{q^n}{n} + O(q^n n^{-\gamma+1}). \tag{1.15}$$

Indlekofer assumed a boundary behaviour on $\hat{H}$ using the notation of (1.8) (see [20]). This reads as

Axiom $\mathcal{A}_1$ (see Indlekofer [21]). There exists a constant $q > 1$ (depending on G), such that

(i) the function $\hat{H}$ is holomorphic in the disc $|z| < q^{-1}$, and continuous on $|z| \leq q^{-1}$ with $A := \hat{H}(q^{-1}) > 0$,

(ii) the derivative $\hat{H}'$ of $\hat{H}$ is bounded on $|z| < q^{-1}$.

Indlekofer showed if Axiom $\mathcal{A}_1$ holds and if $\hat{Z}(-q^{-1}) \neq 0$, then

$$P(n) = \frac{q^n}{n} + o\left(\frac{q^n}{n}\right) \tag{1.16}$$

and if $\hat{Z}(-q^{-1}) = 0$, then

$$\frac{nP(n)}{q^n} + \frac{(n-1)P(n-1)}{q^{n-1}} = 2 + o\,(1)\,. \tag{1.17}$$

Furthermore, the asymptotic formula

$$\sum_{m\le n} mP(m)q^{-m} = n + o(n^{1/2}), \quad n \to \infty \tag{1.18}$$

holds for the mean of the number of prime elements $P(n)$.

A small change of Axiom $\mathcal{A}_1$ leads to the abstract prime number theorem and to the asymptotic formula (1.18) with remainder term $o(1)$. This modification is contained in

Axiom $\mathcal{A}_2$ (see Indlekofer [21]). The conditions of Axiom $\mathcal{A}_1$ hold, and in addition, the power series of $\hat{H}'$ converges absolutely for $|z| \le q^{-1}$.

Axiom $\mathcal{A}_2$ yields the following prime number theorem (see [22], Theorem 1):

$$P(n) = \frac{q^n}{n} + O\left(q^n \max_{\frac{n}{4}\le m\le n} |h(m)|q^{-m}\right) \quad \text{as } n \to \infty \tag{1.19}$$

if $\hat{Z}(-q^{-1}) \neq 0$, and

$$P(n) = (1-(-1)^n)\,\frac{q^n}{n} + O\left(q^n \sum_{\frac{n}{8}\le m} |h(m)|q^{-m}\right) \quad \text{as } n \to \infty \tag{1.20}$$

if $\hat{Z}(-q^{-1}) = 0$.
It may be observed, that if

$$\sum_{n=1}^{\infty} n^k |h(n)|q^{-n} < \infty$$

then

$$\max_{\frac{n}{4}\le m\le n} |h(m)|q^{-m} = o(n^{-k})$$

and

$$\sum_{n\le m} |h(m)|q^{-m} = o(n^{-k}).$$

Remark 1.5. If the condition (1.13) holds with $\gamma > 2$, then (1.19) and (1.20) imply (1.14) and (1.15), respectively.

For the investigation of the mean-value of a multiplicative function it is sufficient to assume conditions weaker than the above assumptions, which were used to prove a prime number theorem on additive arithmetical semigroups.

There are several approaches to weaken the assumption on the additive arithmetical semigroup. Instead of (ii) in Axiom $\mathcal{A}_1$ Barát and Indlekofer assumed

$$\sum_{n\le N}\left(\sum_{d|n} dP(d)q^n\right)^2 = O(N), \qquad \text{as } N\to\infty. \tag{1.21}$$

This estimate is an assumption about the mean behaviour of the prime coefficients $P(n)$, which does not yield the *Chebyshev upper estimate*

$$P(n) = O\left(\frac{q^n}{n}\right). \tag{1.22}$$

It is natural to ask for the *Chebyshev lower estimate*:

$$\frac{q^n}{n} \ll P(n).$$

The asymptotic (1.11) or (1.20) show that the prime number theorem does not yield the Chebyshev lower estimate. This indicates a major divergence of the theory of additive arithmetical semigroups from the classical number theory. The question arises: what can we say about an additive arithmetical semigroup if the Chebyshev lower estimate is satisfied? Indlekofer investigated this case and considered additive arithmetical semigroups with the following condition (see [23], Example 4)

$$0 < c_1 \le nP(n)q^{-n} \le \sum_{d|n} dP(d)q^{-n} \le c_2 < \infty \qquad (n\in\mathbb{N}). \tag{1.23}$$

Beside (1.23) there are no further assumptions about the additive arithmetical semigroup; nevertheless, we can prove an estimate for $G(n)$. The zeta function can be written as (see (1.2))

$$\hat{Z}(z) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(\sum_{d|n} dP(d))z^n\right).$$

Therefore

$$\begin{aligned}
|\hat{Z}(z)| &= \hat{Z}(|z|)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(\sum_{d|n} dP(d))q^n|z|^n(\cos(nt)-1)\right)\\
&\le \hat{Z}(|z|)\exp\left(c_1\sum_{n=1}^{\infty}\frac{|z|^n}{n}q^n(\cos(nt)-1)\right)\\
&= \hat{Z}(|z|)\left|\frac{1-|qz|}{1-qz}\right|^{c_1}.
\end{aligned}$$

In addition, elementary estimates immediately yield (cf. [23])

$$G(n) \asymp \frac{q^n}{n} \exp\left(\sum_{m\le n}\sum_{d|m} dP(d)q^{-m}\right)$$

where the constants involved in $\asymp$ only depend on c_1 and c_2.

1.3 Convolution

In this section, we begin the study of general arithmetical functions, complex-valued functions on a given additive arithmetical semigroup G.

A complex-valued function $\tilde{f}(a)$ defined for all $a \in G$ is an *arithmetical function* on G. The set of all arithmetical functions on G will be denoted by $Dir(G)$. This set can be made into a complex vector space (of infinite dimension) by means of the point-wise operations

$$(\tilde{f}+\tilde{g})(a) = \tilde{f}(a)+\tilde{g}(a),$$
$$(\lambda\tilde{f})(a) = \lambda\tilde{f}(a)$$

for $\tilde{f},\tilde{g} \in Dir(G)$, $a \in G$ and $\lambda \in \mathbb{C}$. Further, this vector space becomes an associative algebra, which is the *Dirichlet-algebra* of G, under the *convolution* operation $*$ defined by

$$(\tilde{f}*\tilde{g})(a) = \sum_{d|a} \tilde{f}(d)\tilde{g}(a/d) \tag{1.24}$$

for $\tilde{f},\tilde{g} \in Dir(G)$ and $a \in G$. Unless otherwise stated $\tilde{f}$ denotes an arithmetical function on a given additive arithmetical semigroup G. It is easy to see that the convolution is commutative and associative. Also, the convolution and addition are distributive in the sense that

$$(\tilde{f}+\tilde{g})*\tilde{h} = (\tilde{f}*\tilde{h})+(\tilde{g}*\tilde{h}).$$

Moreover, for $\lambda \in \mathbb{C}$,

$$\lambda(\tilde{f}*\tilde{g}) = (\lambda\tilde{f})*\tilde{g} = \tilde{f}*(\lambda\tilde{g}).$$

The function

$$\tilde{\epsilon}(a) = \begin{cases} 1 & \text{for } a = 1_G \\ 0 & \text{otherwise} \end{cases} \tag{1.25}$$

is the neutral element for the convolution, that is

$$\tilde{\epsilon}*\tilde{f} = \tilde{f}*\epsilon = \tilde{f}$$

for the complex-valued function $\tilde{f}$ on G. Therefore, the arithmetical functions under the addition, scalar multiplication, and convolution form a commutative algebra. Although we shall make no use of the fact, it may be interesting to point out that $Dir(G)$ is actually a *unique factorization domain* (see [29]).

If there exists an arithmetical function $\tilde{f}^{-1}$ on G such that $\tilde{f} * \tilde{f}^{-1} = \tilde{f}^{-1} * \tilde{f} = \tilde{\epsilon}$, then $\tilde{f}$ is *invertible* and $\tilde{f}^{-1}$ is the *convolution inverse* of $\tilde{f}$. The equation $\tilde{f} * \tilde{f}^{-1} = \tilde{\epsilon}$ is equivalent to the system of equations

$$\begin{aligned} 1 &= \tilde{f}(1_G)\tilde{f}^{-1}(1_G), \\ 0 &= \sum_{d|a} \tilde{f}(d)\tilde{f}^{-1}(a/d), \quad (a \neq 1_G). \end{aligned}$$

Hence $\tilde{f}(1_G) \neq 0$ is necessary for the existence of an inverse $\tilde{f}^{-1}$. This condition is also sufficient. Actually, if $\tilde{f}(1_G) \neq 0$, then the first equation of the above system gives the value $\tilde{f}^{-1}(1_G)$. First, we obtain $\tilde{f}^{-1}(p) = 1/\tilde{f}(p)$ for all $p \in P$, and thereafter we get the values for $\tilde{f}^{-1}(a)$ for all $a \in G$.
Therefore, an arithmetical function $\tilde{f}$ on G is invertible if and only if $\tilde{f}(1_G) \neq 0$.

If B is a subset of G, then the *characteristic function of* B is given by

$$\mathbf{1}_B(a) = \begin{cases} 1 & \text{for } a \in B \\ 0 & \text{otherwise} \end{cases} \tag{1.26}$$

and the inverse of $\mathbf{1}_B$ exists if and only if $1_G \in B$. In particular, $\mathbf{1}_G$ is invertible and this leads to

$$\tilde{\mu} * \mathbf{1}_G = \tilde{\epsilon} \tag{1.27}$$

where $\tilde{\mu}$ is the *Möbius function on* G. This relation is sometimes also called the *Möbius inversion formula on* G.

As for the natural numbers, the Möbius function on G satisfies

$$\tilde{\mu}(a) = \begin{cases} 0 & \text{if } p^2|a \text{ for some } p \in P \\ (-1)^{\tilde{\omega}(a)} & \text{otherwise} \end{cases} \tag{1.28}$$

hereby $\tilde{\omega}(a)$ is

$$\tilde{\omega}(a) := \sum_{p|a} 1 \tag{1.29}$$

the prime divisor function on G.

Besides the Möbius function on G the *von Mangoldt's function* $\tilde{\Lambda}$ plays an important role in the investigations of additive arithmetical semigroups. We define $\tilde{\Lambda}$ by

$$\tilde{\Lambda}(a) = \begin{cases} \partial(p), & \text{if } a \text{ is a prime power } p^r \neq 1_G, \\ 0, & \text{otherwise.} \end{cases} \tag{1.30}$$

Let $a \in G$ has the prime factorization $a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, $(k \in \mathbb{N})$. Then the following holds

$$\begin{aligned} \sum_{d|a} \tilde{\Lambda}(d) =& \sum_{i=1}^{k} \sum_{d|p_i^{\alpha_i}} \tilde{\Lambda}(d) \\ =& \sum_{i=1}^{k} \partial(p_i)\alpha_i = \sum_{i=1}^{k} \partial(p_i^{\alpha_i}) \\ =& \partial(a). \end{aligned}$$

Therefore the von Mangoldt's function $\tilde{\Lambda}$ satisfies

$$\tilde{\Lambda} = \tilde{\mu} * \partial. \tag{1.31}$$

1.4 Generating functions

For each function $\tilde{f}$, we associate a power series $\hat{F}$, the *generating function* $\hat{F}$ *of* $\tilde{f}$. We define $\hat{F}$ by

$$\hat{F}(z) = \sum_{a \in G} \tilde{f}(a) z^{\partial(a)} = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a) \right) z^n \tag{1.32}$$

for $z \in \mathbb{C}$.
In particular, the generating function of $\mathbf{1}_G$, is just the zeta function $\hat{Z}_G$ of G, since

$$\hat{Z}_G(z) := \hat{Z}(z) = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \mathbf{1}_G(a) \right) z^n = \sum_{n=0}^{\infty} G(n) z^n. \tag{1.33}$$

For the special functions $\tilde{\epsilon}$, ∂ and $\mathbf{1}_P$ that we defined in the previous sections, we can give the associate generating functions as follows:

$$\hat{E}(z) := \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{\epsilon}(a) \right) z^n = 1,$$

$$\hat{D}(z) := \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \partial(a) \right) z^n = \sum_{n=0}^{\infty} nG(n)z^n = z\hat{Z}'(z) \tag{1.34}$$

and the *prime generating function of* G is

$$\hat{P}(z) := \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \mathbf{1}_P(a) \right) z^n = \sum_{n=0}^{\infty} P(n)z^n. \tag{1.35}$$

Since we assume $G(n) \ll q^n n^{\varrho}$ with some real constants δ and q, $q > 1$, the functions $\hat{Z}, \hat{E}, \hat{D}$ and $\hat{P}$ are holomorphic for $|z| < q^{-1}$.

The generating function of the convolution of the functions $\tilde{f}$ and $\tilde{g}$

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} (\tilde{f} * \tilde{g})(a) \right) z^n &= \sum_{n=0}^{\infty} \sum_{\substack{a \in G \\ \partial(a)=n}} \left(\sum_{d|a} \tilde{f}(d)\tilde{g}(a/d) \right) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{\substack{b,d \in G, \\ \partial(b)+\partial(d)=n}} \tilde{f}(d)\tilde{g}(b) \right) z^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{\substack{d \in G \\ \partial(d)=k}} \tilde{f}(d) \sum_{\substack{b \in G \\ \partial(b)=n-k}} \tilde{g}(b) \right) z^n \\
&= \left(\sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a) \right) z^n \right) \left(\sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{g}(a) \right) z^n \right)
\end{aligned}$$

is the pointwise product of their generating functions. If the inverse function $\tilde{f}^{-1}$ of $\tilde{f}$ exists then its generating function satisfies

$$\hat{F}(z)\hat{F}^{-1}(z) = 1,$$

which implies

$$\hat{F}^{-1}(z) = \frac{1}{\hat{F}(z)}.$$

For instance, the generating function $\hat{M}$ of the Möbius function of G has the following form

$$\hat{M}(z) = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{\mu}(a) \right) z^n = \frac{1}{\hat{Z}(z)}. \tag{1.36}$$

For the von Mangoldt's function $\tilde{\Lambda} = \tilde{\mu} * \partial$, we obtain

$$\begin{aligned} \hat{\Lambda}(z) := \sum_{n=0}^{\infty} \Lambda(n) z^n &= \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{\Lambda}(a) \right) z^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} (\mu_G * \partial)(a) \right) z^n \\ &= \frac{z\hat{Z}'(z)}{\hat{Z}(z)}, \end{aligned} \tag{1.37}$$

where the coefficients $\Lambda(n)$ of the power series $\hat{\Lambda}(z)$ are the *von Mangoldt's coefficients*.

We can write the zeta function of an additive arithmetical semigroup in several forms. We will summarize the most important ones, as we shall be making use of these representations in our proofs. The zeta function has an Euler product representation (cf. (1.2))

$$\hat{Z}(z) = \prod_{n=1}^{\infty} (1 - z^n)^{-P(n)}, \tag{1.38}$$

in the disc $\{z : |z| < q^{-1}\}$, using (1.2) and (1.37) that can be written in the form

$$\hat{Z}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{d|n} dP(d) z^n \right) = \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} z^n \right). \tag{1.39}$$

In particular, $\hat{Z}(z) \neq 0$ for $|z| < 1/q$.
Using (1.39) we obtain that the coefficients $\Lambda(n)$ and $P(n)$ are related by

$$\Lambda(n) = \sum_{d|n} dP(d)$$

and by the Möbius inversion formula on the positive integers $\mathbb{N}$

$$nP(n) = \sum_{d|n} \Lambda(d) \mu\left(\frac{n}{d}\right). \tag{1.40}$$

Chapter 2

Investigation of arithmetical functions

This chapter contains some basic definitions followed by the latest mean-value theorems for multiplicative functions on additive arithmetical semigroups. We begin by summarizing mean-value theorems for multiplicative functions of modulus ≤ 1, and proceed to introduce our new mean-value theorems for unifomly summable multiplicative functions with nonzero mean-value. The necessary assumptions in our mean-value theorems on the semigroup are weaker than the conditions of the previous results. Afterwards, we give a characterization of uniformly summable functions on additive arithmetical semigroups. For the proof of our theorems we introduce the new method developed by Indlekofer, that compares the coefficients of power series. Later, we prove some lemmas and give the proof of our theorems.

Let us start with some basic definitions in this topic. An arithmetical function $\tilde{f}$ on G is *additive* if

$$\tilde{f}(ab) = \tilde{f}(a) + \tilde{f}(b) \quad \text{for all coprime } a, b \in G. \tag{2.1}$$

An arithmetical function $\tilde{f}$ on G is *multiplicative* if

$$\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b) \quad \text{for all coprime } a, b \in G. \tag{2.2}$$

An arithmetical function $\tilde{f}$ on G is *completely multiplicative* if

$$\tilde{f}(ab) = \tilde{f}(a)\tilde{f}(b) \quad \text{for all } a, b \in G. \tag{2.3}$$

If $\tilde{f}$ is a multiplicative function on G, then $\sum_{\substack{a \in G \\ \partial(a)=0}} \tilde{f}(a) = 1 \ (\neq 0)$, therefore its

generating function $\hat{F}$ satisfies

$$\hat{F}(z) = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a) \right) z^n \tag{2.4}$$
$$= \prod_{p} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)} \right)$$
$$=: \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda_f(m)}{m} z^m \right).$$

In this thesis we investigate the mean-value of an arithmetical function on G.

Definition 2.1. Let $\tilde{f} : G \mapsto \mathbb{C}$. We define the average value of $\tilde{f}$ on elements of degree n by

$$M(n, \tilde{f}) := \begin{cases} \frac{1}{G(n)} \sum\limits_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a), & \text{if } G(n) \neq 0, \\ 0, & \text{if } G(n) = 0. \end{cases}$$

If $\lim\limits_{n\to\infty} M(n, \tilde{f})$ exists, it is called the *mean-value* of $\tilde{f}$ and is denoted by $M(\tilde{f})$.

The mean-value $M(\mathbf{1}_S)$ of a characteristic function $\mathbf{1}_B$ of a set $B \subseteq G$ is called the *asymptotic density* of S.

2.1 Mean-value theorems for multiplicative functions of modulus ≤ 1

The first mean-value theorems for multiplicative functions on additive arithmetical semigroups considered functions $\tilde{f}$ of modulus ≤ 1 (i.e. with $|\tilde{f}| \leq 1$). Several mathematicians, such as Barát, Indlekofer, Manstavičius, Warlimont, Wehmeier and Zhang, made contributions to this subject. In the results of Indlekofer and Manstavičius (see [24] and [25]) the authors required Axiom $\mathcal{A}^{\#}$ and proved analogues of the results of Delange, Wirsing and Halász on $\mathbb{N}$, which describe the mean-value of a multiplicative function of modulus ≤ 1 (cf. chapter 6 in [9]).
In this section, we compare two typical mean-value theorems for multiplicative functions of modulus ≤ 1 on additive arithmetical semigroups, where the circle of convergence of the zeta function is a natural boundary. The first result is a theorem by Zhang and the second theorem was proven by Barát and Indlekofer. These results are typical in the sense that Zhang made assumption on the coefficients $G(n)$ of the zeta function, whereas Barát and Indlekofer used analytical conditions on the zeta function.
Zhang proved the following result for additive arithmetical semigroups satisfying Axiom $\mathcal{A}$ (see [32], Theorem 6.3.1).

Proposition 2.2. *Suppose that either*

(i) G is an additive arithmetical semigroup satisfying the Chebyshev upper estimate and

$$\sum_{n=1}^{\infty} |G(n)q^{-n} - A| < \infty$$

or

(ii)

$$\sum_{n=1}^{\infty} \sup_{n \le m} |G(m)q^{-m} - A| < \infty,$$

holds.

Let $\tilde{f}$ be a multiplicative function with $|\tilde{f}(a)| \le 1$ for all $a \in G$. If there exists a real number τ_0 such that the series

$$\sum_{p \in P} q^{-\partial(p)} \left(1 - Re(\tilde{f}(p)e^{-i\tau\partial(p)})\right) \tag{2.5}$$

converges for $\tau = \tau_0$, then

$$M(n, \tilde{f}) = Aq^{n(1+i\tau_0)} \prod_{\partial(p) \le n} \left(1 - q^{-\partial(p)}\right) \left(1 + \sum_{k=1}^{\infty} q^{-k\partial(p)(1+i\tau_0)} \tilde{f}(p^k)\right) + o(1)$$

as $n \to \infty$. On the other hand, if there exists no such τ, then

$$M(n, \tilde{f}) = o(1).$$

Remark 2.3. We note, from Theorem 3.2.1 in [32], that an additive arithmetical semigroup satisfying condition (ii) of Proposition 2.2 satisfies the Chebyshev upper estimate.

Barát and Indlekofer formulated conditions on $\hat{H}$ which leads to a proof of the results in [24] (see Theorem 2 in [2]). These conditions essentially imply the estimate

$$\sum_{n \le N} \left(\Lambda(n)q^{-n}\right)^2 = O(N) \quad \text{as } N \to \infty \tag{2.6}$$

which is weaker than the Chebyshev upper estimate $\Lambda(n)q^{-n} = O(1)$ as $n \to \infty$. Putting $z = q^{-1}y$ in (1.8) we define $Z(y) := \hat{Z}(q^{-1}y)$ and $H(y) := \hat{H}(q^{-1}y)$ and obtain

$$Z(y) = \frac{H(y)}{1-y} \quad \text{for } |y| < 1, \tag{2.7}$$

and assume that $H(y)$ is bounded in the disc $|y| < 1$ satisfying

$$\lim_{y \to 1^-} H(y) = A > 0. \tag{2.8}$$

Put $H(y) = \sum_{n=0}^{\infty} h(n) y^n$. Then the following holds (see [2], Theorem 1).

Theorem 2.4. *Let $H(y)$ be continuous for $|y| \leq 1$ and satisfy (2.8). If*

$$\sum_{n=1}^{\infty} n^2 h^2(n) r^{2n} = O\left(\frac{1}{1-r}\right) \quad \text{as } 0 < r < 1,\ r \to 1 \tag{2.9}$$

then (2.6) holds.

The main result of Barát and Indlekofer ([2], Theorem 2) is the following mean-value theorem

Theorem 2.5. *Let G be an additive arithmetical semigroup satisfying (2.6) and let $H \in \mathbb{H}^{\infty}$ (i.e. H is bounded in $|y| < 1$) satisfy (2.8). Further, let $\tilde{f}$ be a completely multiplicative function, $|\tilde{f}| \leq 1$. Then the following two assertions hold.*

(i) If the series (2.5) diverges for each $\tau \in (-\pi, \pi]$, then

$$M(n, \tilde{f}) = o(1)$$

as $n \to \infty$.

(ii) If the series (2.5) converges for some $\tau = \tau_0 \in (-\pi, \pi]$, then

$$\begin{aligned} M(n, \tilde{f}) =& A q^{n(1+i\tau_0)} \prod_{\partial(p) \leq n} \left(1 - q^{-\partial(p)}\right) \left(1 + \sum_{k=1}^{\infty} q^{-k\partial(p)(1+i\tau_0)} \tilde{f}(p^k)\right) + o(1) \\ =& cL(n) + o(1) \end{aligned}$$

as $n \to \infty$, where c is an appropriate real constant, and $L(y)$ is a slowly oscillating function.

Theorem 2.5 supersedes Proposition 2.2, the corresponding result of Zhang. His assumption

$$\sum_{n=1}^{\infty} \left| G(n) q^{-n} - A \right| < \infty \tag{2.10}$$

implies, since $h(n) = G(n)q^{-n} - G(n-1)q^{-n+1}$ the absolute convergence

$$\sum_{n=0}^{\infty} |h(n)| < \infty,$$

and thus H is continuous on the closed disc $\bar{D} = \{y : |y| \leq 1\}$. Similarly, the condition (ii) of Proposition 2.2

$$\sum_{n=0}^{\infty} \sup_{n \leq m} |G(m)q^{-m} - A| < \infty$$

yields

$$\sum_{n=0}^{\infty} |h(n)| < \infty \tag{2.11}$$

and $h(n) = o(n^{-1})$. This can easily be seen from (1.8) in section 1.2. From our Remark 1.2 we obtain also, that (ii) of Proposition 2.2 yields the Chebyshev upper estimate.
In Theorem 2.5, we have assumed only $H \in \mathbb{H}^{\infty}$ and (2.6), that follow from the condition (2.11) and the Chebyshev upper estimate, but conversely $H \in \mathbb{H}^{\infty}$ and (2.6) do not yield (2.11) and the Chebyshev upper estimate in general.

In [2], Barát and Indlekofer also gave an example (Example 1), which satisfies the assumption (2.6) but does not satisfy the Chebyshev upper estimate $\Lambda(N) \ll q^N$.

The mean-value theorems so far assumed that the zeta function has the form $\hat{Z}(z) = \hat{H}(z)(1-qz)^{-\delta}$ with $\delta \geq 1$ (see for example [44]). In this thesis we deal with a more general case:

$$\hat{Z}(z) = \hat{H}(z)(1-qz)^{-\delta} \tag{2.12}$$

with $\delta > 0$. We use a method different from the one that have been used to prove the results which we have described in this section.
In his recent work Indlekofer also formulated a general mean-value theorem for multiplicative functions of modulus ≤ 1 on additive arithmetical semigroups under the above mentioned general condition (2.12) (see Theorem 4, [23]).

Proposition 2.6. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n)z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\bar{\Lambda}(m)}{m} z^m\right) = \frac{\hat{H}(z)}{(1-qz)^{\delta}}$$

where $\hat{H}(z) = O(1)$ for $|z| < q^{-1}$, $\lim_{z \to \frac{1}{q}-} \hat{H}(z) = A > 0$ and $\delta > 0$. Assume that $\bar{\Lambda}(m) = O(q^m)$ and $G(n) \asymp q^n n^{\delta-1}$. Suppose $|\tilde{f}(g)| \leq 1$ for all $g \in G$ and either

(i) $\tilde{f}$ is a completely multiplicative function on G, or

(ii) $\tilde{f}$ is a multiplicative function such that $\tilde{f}(p^k) = 0$ for each prime power p^k with $\partial(p) \leq \frac{\log 2}{\log q}$.

If there exists a real number a such that the series (2.5) *converges for $\tau = a$, then*

$$\sum_{\substack{g\in G\\ \partial(g)=n}} \tilde{f}(g) = q^{ina} \prod_{\partial(p)\leq n} (1-q^{-\partial(p)}) \left(1+\sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)(1+ia)}\right) G(n) + o(G(n)).$$

If (2.5) *diverges for all $\tau \in \mathbb{R}$ then*

$$\sum_{\substack{g\in G\\ \partial(g)=n}} \tilde{f}(g) = o(G(n)).$$

2.2 New mean-value theorems for uniformly summable multiplicative functions

We begin by introducing some definitions, and so on to present our main theorems about mean-values of uniformly summmable multiplicative functions on additive arithmetical semigroups.
The class of uniformly summable multiplicative functions has been defined by Indlekofer (see [17]) for functions defined on $\mathbb{N}$. Indlekofer proved mean-value theorems for the class of functions L^α and for uniformly summable functions.
In the case of additive arithmetical semigroups, the main contributions, to date, are by Wehmeier (see [39]) and Zhang (see for example [44]). Therefore we restrict ourselves to comparing their most recent results with our new results. Our new mean-value theorems are more general than the ones proposed before. We prove our results for a larger class of functions and under a weaker condition on the additive arithmetical semigroups.

In the previous results in this field the class of arithmetical functions L^α was of of great interest. We introduce this class of functions as follows

Definition 2.7. Let $\tilde{f} : G \mapsto \mathbb{C}$. If $1 \leq \alpha < \infty$, then $\tilde{f}$ is said to be in L^α if

$$||\tilde{f}||_\alpha := (\limsup_{n\to\infty} M(n, |\tilde{f}|^\alpha))^{1/\alpha}$$

is finite.

After the classes L^α for $1 \leq \alpha < \infty$ the development continued and motivated by Indlekofer's results in the classical number theory (see [17]), the class of uniformly summable functions became very important for additive arithmetical semigroups. Wehmeier initiated the investigations in this direction for additive arithmetical semigroups. In our thesis we extend the previous results. Therefore we introduce the following:

Definition 2.8. Let $\tilde{f} : G \mapsto \mathbb{C}$. Let $K \in \mathbb{R}$; then we define $\tilde{f}_K$ by

$$\tilde{f}_K(a) = \begin{cases} \tilde{f}(a), & \text{if } |\tilde{f}(a)| \geq K, \\ 0, & \text{otherwise.} \end{cases}$$

$\tilde{f}$ is called *uniformly summable* if

$$\lim_{K\to\infty} \sup_{n\geq 1} M(n, |\tilde{f}_K|) = 0.$$

We denote the set of all uniformly summable functions by L^*.

Remark 2.9. To emphasize the importance of the class L^* we remark, that for $\alpha > 1$

$$L^\alpha \subsetneqq L^* \subsetneqq L^1$$

holds.

Now, we introduce our results. Let $\tilde{f}$ be a multiplicative function on G. The idea is to use Indlekofer's latest approach which we summarize in section 2.4. His method differs from the methods used before since he does not prove an asymptotic formula for $\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a)n$ via Cauchy's theorem

$$n \sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) = \frac{1}{2\pi} \int_{|z|=r<q^{-1}} \frac{\hat{F}'(z)}{z^{n+1}}\, dz$$

but compare $\sum_{a\in G, \partial(a)=n} \tilde{f}(a)n$ with $nG(n)$

$$|n \sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) - A_n nG(n)| = \Big| \sum_{m\leq n} \Lambda_f(m) \{ \sum_{\substack{a\in G\\ \partial(a)=n-m}} \tilde{f}(a) - A_n G(n-m)\}$$

$$+ A_n \sum_{m\leq n} (\Lambda_f(m) - \Lambda(m)) G(n-m) \Big| \tag{2.13}$$

$$\ll \sum_{m\leq n} | \sum_{a\in G, \partial(a)=m} \tilde{f}(a) - A_n G(m)| \tag{2.14}$$

$$+ |A_n| \sum_{m\leq n} |\Lambda_f(m) - \Lambda(m)| G(n-m)$$

This leads via Parseval's equality to an estimate of the distance between $\sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a)n$ and $A_n nG(n)$, a procedure which is also effective for quantitative investigations of occuring remainder terms.

Motivated by the above mentioned method and the corresponding papers of Indlekofer about mean-value theorems for multiplicative functions defined on $\mathbb{N}$ (see Indlekofer [17], [18] and [19]) we formulate our main theorems.

Theorem 2.10. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$\hat{Z}(z) = \sum_{m=0}^{\infty} G(m) z^m = \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} z^n\right) = \frac{\hat{H}(z)}{(1-qz)^{\delta}}$$

where $\hat{H}(z) = O(1)$ for $|z| < q^{-1}$, $\lim\limits_{z \to \frac{1}{q}^-} \hat{H}(z) = A > 0$ and $\delta > 0$. Assume that $\Lambda(m) = O(q^m)$. Let $\tilde{f}$ be a multiplicative function and $\alpha \geq 1$. If $G(n) \asymp q^n n^{\delta-1}$ and $\tilde{f} \in L^* \cap L^{\alpha}$ and if $M(\tilde{f})$ exists and is nonzero, then the following series

$$\sum_{p \in P} \frac{\tilde{f}(p) - 1}{q^{\partial(p)}} \tag{2.15}$$

$$\sum_{\substack{p \in P \\ |\tilde{f}(p)| \leq 3/2}} \frac{|\tilde{f}(p) - 1|^2}{q^{\partial(p)}} \tag{2.16}$$

$$\sum_{p \in P; n \geq 2} \frac{|\tilde{f}(p^n)|^{\lambda}}{(q^{\partial(p)})^n}. \tag{2.17}$$

$$\sum_{\substack{p \in P \\ ||\tilde{f}(p)| - 1| > 1/2}} \frac{|\tilde{f}(p)|^{\lambda}}{q^{\partial(p)}} \tag{2.18}$$

converge for $1 \leq \lambda \leq \alpha$, and for each prime p

$$\sum_{n=1}^{\infty} \frac{\tilde{f}(p^n)}{q^{n\partial(p)}} + 1 \neq 0. \tag{2.19}$$

In the converse direction we have two cases: $1 \leq \delta$ and $0 < \delta < 1$. If the above mentioned series of Theorem 2.10 converge, then we can prove the following:

Theorem 2.11. *Let an additive arithmetical semigroup fulfill the conditions of Theorem 2.10, with $G(n) \asymp q^n n^{\delta-1}$ for $1 \leq \delta$. Further we assume that $\frac{G(n-1)}{G(n)} = q^{-1} + o(1)$ as $n \to \infty$.*
Let $\tilde{f}$ be a multiplicative function and $\alpha \geq 1$. If the series (2.15)-(2.18) converge, then $M(\tilde{f})$ exists, $\tilde{f} \in L^* \cap L^{\alpha}$ and $M(|\tilde{f}|^{\lambda})$ exists for $1 \leq \lambda \leq \alpha$. If in addition (2.19) holds then $M(\tilde{f}) \neq 0$ and $M(|\tilde{f}|^{\lambda}) \neq 0$ for $1 \leq \lambda \leq \alpha$.

For $0 < \delta < 1$ we need a further assumption on our multiplicative function $\tilde{f}$ in order to prove the existence of the mean-value of $\tilde{f}$.

Theorem 2.12. *Let an additive arithmetical semigroup fulfill the conditions of Theorem 2.10, with $G(n) \asymp q^n n^{\delta-1}$ for $0 < \delta < 1$. Let $\tilde{f}$ be a multiplicative function and $\alpha \geq 1$. Further we assume that $\frac{G(n-1)}{G(n)} = q^{-1} + o(1)$ as $n \to \infty$ and*

$$\forall \varepsilon > 0 : \exists K > 0 : \forall n \in \mathbb{N} : \forall S \subseteq G : \tag{2.20}$$

$$S = \{a \in G : p^k || a, p \in P; |\tilde{f}(p^k)|^\alpha > K\} \Rightarrow M(n, \mathbf{1}_S |\tilde{f}|^\alpha) < \varepsilon$$

holds. If the series (2.15)-(2.17) *converge, then $M(\tilde{f})$ exists, $\tilde{f} \in L^* \cap L^\alpha$ and $M(|\tilde{f}|^\lambda)$ exists for $1 \leq \lambda \leq \alpha$. If in addition* (2.19) *holds then $M(\tilde{f}) \neq 0$ and $M(|\tilde{f}|^\lambda) \neq 0$ for $1 \leq \lambda \leq \alpha$.*

The class of uniformly summable functions was defined by Indlekofer (see [17]) for functions defined on $\mathbb{N}$, and he has proved mean-value theorems for uniformly summable multiplicative functions.
In the case of additive arithmetical semigroups, as mentioned before, it was mainly Wehmeier (see [39]) and Zhang (see for example [44]) who have made contributions to this subject before this thesis. Zhang's most recent results in this topic appeared in 2008 [44]. In his paper he has presented mean-value theorems for functions of the class L^α ($\alpha > 1$) and for the case $1 \leq \delta$ applying the methods which have been used in the proofs in the classical case also.
Wehmeier's most recent results in this field appeared in his PhD thesis in 2005 (see [39], chapter 6). Wehmeier has proven his mean-value theorems for uniformly summable multiplicative functions applying the methods of Indlekofer's proof for the natural numbers $\mathbb{N}$ but only for the case $\delta = 1$.
In this thesis we prove our mean-value theorems in the case $0 < \delta$ and for $L^* \cap L^\alpha$ ($\alpha \geq 1$), which is a larger class of functions using other methods.

2.3 Characterization of uniformly summable functions

In this section we give a characterization of uniformly summable functions on G. We summarize the equivalent properties of L^* in the following lemma, that corresponds to Lemma 1 in [17]. For additive arithmetical semigroups Wehmeier also proved the equivalence of the assertions 1.-3. of the subsequent lemma (see Lemma 6.4 in [39]). We shall apply in particular the first property of this lemma in our proof.

Lemma 2.13. *The following statements are equivalent:*

1. *$\tilde{f} \in L^1$ and*

$$\forall \varepsilon > 0 : \exists \gamma > 0 : \forall n \in \mathbb{N} : \forall S \subseteq G : (M(n, \mathbf{1}_S) < \gamma \Rightarrow M(n, \mathbf{1}_S |\tilde{f}|) < \varepsilon);$$

2. *$\tilde{f} \in L^*$;*

3.
$$\forall \varepsilon > 0\, \exists \alpha \in \mathbb{R} :\ (||(|\tilde{f}| - \alpha)^+||_1 < \varepsilon)$$
where $(|\tilde{f}| - \alpha)^+ = \max((|\tilde{f}| - \alpha), 0)$ *is the positive part of* $|\tilde{f}| - \alpha$.

4. There exists a monotonic function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ *such that*
(i) $\varphi \geq 0$,
(ii) $\varphi(t)/t \to \infty$ *as* $t \to \infty$,
(iii) $\varphi \circ \tilde{f} \in L^1$.

Proof. 1. $\Rightarrow$ *2.:* Let $\varepsilon > 0$. To prove that $\tilde{f} \in L^*$, we have to find K_0 such that $M(n, |\tilde{f}_K|) < \varepsilon$ for all $K \geq K_0$, $n \in \mathbb{N}$.
Apply the assumption *1.* to obtain γ such that $M(n, \mathbf{1}_S|\tilde{f}|) < \varepsilon$ if $M(n, \mathbf{1}_S) < \gamma$. Choose $K_0 := 2||\tilde{f}||_1/\gamma$. Since $\tilde{f} \in L^1$ we know that $M(n, |\tilde{f}|) < 2||\tilde{f}||_1 < \infty$ for all $n \in \mathbb{N}$. Let
$$S := \{a \in G; |\tilde{f}(a)| \geq K\},$$
then $\tilde{f}_K = \mathbf{1}_S \tilde{f}$. For $K \geq K_0$ it yields

$$M(n, \mathbf{1}_S) \leq M(n, \mathbf{1}_S|\tilde{f}|)/K \leq \frac{M(n, |\tilde{f}|)}{K_0} \leq \frac{\gamma M(n, |\tilde{f}|)}{2||\tilde{f}||_1} < \gamma.$$

Hence $M(n, \mathbf{1}_S|\tilde{f}|) = M(n, |\tilde{f}_K|) < \varepsilon$ for all $K \geq K_0$, $n \in \mathbb{N}$.

2. $\Rightarrow$ *3.:* Let $\varepsilon > 0$. There exists K such that $||\tilde{f}_K||_1 < \varepsilon$. Since $\max(|\tilde{f}| - \alpha, 0) < |\tilde{f}|_\alpha$ it suffices to set $\alpha := K$.

3. $\Rightarrow$ *4.:* There exist real numbers $n_k \nearrow \infty$ such that

$$\sup_{n \geq 1} \frac{1}{G(n)} \sum_{\partial(a) = n} (|\tilde{f}(a)| - n_k)^+ < 2^{-k}.$$

Define $\varphi : \mathbb{R} \mapsto \mathbb{R}$ by $\varphi(t) := \sum_{k=1}^{\infty} (m - n_k)^+$ if $m \leq t < m + 1$ where $m \in \mathbb{Z}$. Then $\varphi \geq 0$ and φ is monotonic. Further, $\varphi(m)/m = \sum_{k=1}^{\infty} (1 - n_k/m)^+ \to \infty$ as $m \to \infty$

and therefore (ii) of *4.* holds. Now,

$$\begin{aligned}
\frac{1}{G(n)}\sum_{\partial(a)=n}\varphi(|\tilde{f}(a)|) &= \frac{1}{G(n)}\sum_{m\in\mathbb{Z}}\sum_{k=1}^{\infty}(m-n_k)^+ \sum_{\substack{\partial(a)=n\\ m\le|\tilde{f}(a)|<m+1}} 1\\
&= \frac{1}{G(n)}\sum_{k=1}^{\infty}\sum_{m\in\mathbb{Z}}(m-n_k)^+ \sum_{\substack{\partial(a)=n\\ m\le|\tilde{f}(a)|<m+1}} 1\\
&\le \frac{1}{G(n)}\sum_{k=1}^{\infty}\sum_{\substack{\partial(a)=n\\ |\tilde{f}(a)|\ge n_k}}(m-n_k)\\
&\le \frac{1}{G(n)}\sum_{k=1}^{\infty}\sum_{\partial(a)=n}(|\tilde{f}(a)|-n_k)^+\\
&\le 1.
\end{aligned}$$

Therefore $\varphi\circ\tilde{f}\in L^1$, which shows the assertion *4.*

4. $\Rightarrow$ *3.:* Let $\varepsilon>0$ and put $c:=\sup_{n\ge1}\frac{1}{G(n)}\sum_{\partial(a)=n}\varphi(|\tilde{f}(a)|)$. Choosing $\alpha>0$ such that $\varphi(t)/t\ge c/\varepsilon$ for all $t\ge\alpha$ we have

$$\begin{aligned}
\frac{1}{G(n)}\sum_{\partial(a)=n}(|\tilde{f}(a)|-\alpha)^+ &\le \frac{1}{G(n)}\sum_{\substack{\partial(a)=n\\ |\tilde{f}(a)|>\alpha}}|\tilde{f}(a)|\\
&\le \frac{\varepsilon}{c}\frac{1}{G(n)}\sum_{\substack{\partial(a)=n\\ |\tilde{f}(a)|>\alpha}}\varphi(|\tilde{f}(a)|)\\
&\le \varepsilon.
\end{aligned}$$

3. $\Rightarrow$ *1.:* For $\varepsilon>0$ the assumption *3.* yields that there exists a real number α_1 such that

$$||\tilde{f}||_1=||(|\tilde{f}|-\alpha_1)+\alpha_1||_1\le\varepsilon+2\alpha_1$$

since $||\tilde{f}|-\alpha_1|(a)\le\max((|\tilde{f}|-\alpha_1)^+(a),\alpha_1)$. Hence $\tilde{f}\in L^1$.
Apply now the assumption *3.* to $\varepsilon/2$. It yields that there are $\alpha_2\in\mathbb{R}$ and $N\in\mathbb{N}$ such that

$$n\ge N\Rightarrow M(n,(|\tilde{f}|-\alpha_2)^+)<\varepsilon/2.$$

Choose γ so small that $G(n)>1/\gamma$ implies $n\ge N$, and that $\gamma<\frac{\varepsilon}{2\alpha_2}$.
Let $S\subseteq G$ and $n\in\mathbb{N}$ such that $M(n,\mathbf{1}_S)<\gamma$. We have to show that

$$M(n,\mathbf{1}_S|\tilde{f}|)<\varepsilon.$$

If $G_n \cap S = \emptyset$, then $M(n, \mathbf{1}_S|\tilde{f}|) = 0$. Otherwise it follows that $n \geq N$. The inequality $|\tilde{f}(a)| \leq (|\tilde{f}| - \alpha_2)^+(a) + \alpha_2$ holds for all $a \in G$. Hence

$$M(n, \mathbf{1}_S|\tilde{f}|) \leq M(n, (|\tilde{f}| - \alpha_2)^+) + M(n, \mathbf{1}_S\alpha_2) \leq \varepsilon/2 + \alpha_2\gamma < \varepsilon.$$

This ends the proof of our lemma. □

2.4 Indlekofer's method

In this section we summarize the ideas and main results of [23].
To ease notational difficulties we make a variable transformation and put

$$y = qz, \lambda(m) = q^{-m}\Lambda(m) \quad \text{and} \quad \gamma(n) = q^{-n}G(n) \tag{2.21}$$

which leads to

$$Z(y) := \hat{Z}(yq^{-1}) = \sum_{n=0}^{\infty} \gamma(n)y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda(m)}{m} y^m\right). \tag{2.22}$$

Then $Z(y)$ is holomorphic for $|y| < 1$.
The same transformation yields for the generating function $\hat{F}$ of an arithmetical function $\tilde{f}$ the following

$$F(y) := \hat{F}(yq^{-1}) = \sum_{n=0}^{\infty} \left(\sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a)\right) q^{-n}y^n. \tag{2.23}$$

For an arithmetical function $\tilde{f}$ on G we define $f : \mathbb{N}_0 \mapsto \mathbb{C}$ by

$$f(n) := q^{-n} \sum_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a) \tag{2.24}$$

and call it the *summatory function* of $\tilde{f}$.
It may be observed, that

$$M(n, \tilde{f}) = \frac{\sum\limits_{\substack{a \in G \\ \partial(a)=n}} \tilde{f}(a)}{G(n)} = \frac{f(n)}{\gamma(n)}.$$

For example, the investigation of the mean-value of $\tilde{f} = \tilde{\Lambda}$ leads to defining the asymptotic behaviour of $\frac{\Lambda(n)}{\gamma(n)}$, which corresponds to the prime number theorem. In this section we consider functions $f : \mathbb{N}_0 \to \mathbb{C}$ with $f(0) = 1$ and $\gamma(n) \geq 0$ for $n \in \mathbb{N}$ and $\gamma(0) = 1$.

The transformed generating function F of an arithmetical function $\tilde{f}$ with summatory function f satisfying $f(0) = 1$ can be written as

$$F(y) := \sum_{n=0}^{\infty} f(n)y^n = \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m} y^m\right). \tag{2.25}$$

The basic conditions in this section will be

$$0 \leq \lambda(m) = O(1) \quad (m \in \mathbb{N}) \tag{2.26}$$

and

$$|Z(y)| \ll Z(|y|) \left|\frac{1-|y|}{1-y}\right|^{\varepsilon} \qquad (|y| < 1) \tag{2.27}$$

for some $\varepsilon > 0$. Then we assume that

$$n\gamma(n) \asymp \exp\left(\sum_{m \leq n} \frac{\lambda(m)}{m}\right) \tag{2.28}$$

and

$$\exp\left(\sum_{k \leq m} \frac{\lambda(k)}{k}\right) = o\left(\exp\left(\sum_{k \leq n} \frac{\lambda(k)}{k}\right)\right) \quad \text{if } m = o(n)\ (n \to \infty). \tag{2.29}$$

Definition 2.14. We say that the function Z given in (2.22) belongs to the $\exp - \log$ class $\mathcal{F}$ in case (2.26)-(2.29) hold.

We notice that the definition of the functions $Z \in \mathcal{F}$ does not require any analytic continuation of $Z(y)$ over the boundary $|y| = 1$.
We assume that λ_f splits into

$$\lambda_f = \lambda_{f,1} + \lambda_{f,2} \tag{2.30}$$

such that

$$|\lambda_{f,1}(m)| \leq \lambda(m) \quad (m \leq n) \qquad \text{and} \qquad \sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} = c_1 < \infty. \tag{2.31}$$

We may assume that $\lambda_{f,1}(m) = 0$ if $m > n$ since these values do not influence $f(n)$.
We can formulate the following (Theorem 2 in [23])

Theorem 2.15. *Let Z be an element of the exp-log class $\mathcal{F}$ and let $F(y)$ in (2.25) satisfy (2.30) and (2.31) with*

$$\lambda_f(m) = O(1),\ |\lambda_{f,1}(m)| \leq \lambda(m) \qquad \textit{for all } m \in \mathbb{N}$$

and

$$\sum_{m=1}^{\infty} \frac{|\lambda_{f,2}(m)|}{m} < \infty.$$

If

$$F(y) = F_I(y)F_{II}(y)$$

where

$$F_I(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,1}(m)}{m} y^m\right), \qquad F_{II}(y) := \exp\left(\sum_{m=1}^{\infty} \frac{\lambda_{f,2}(m)}{m} y^m\right)$$

for $|y| < 1$, then the following two assertions hold.

(i) Let

$$\sum_{m=1}^{\infty} \frac{\lambda(m) - Re\,\lambda_{f,1}(m)e^{ima}}{m} \tag{2.32}$$

converge for some $a \in \mathbb{R}$. Put

$$A_n = \exp\left(-ina + \sum_{m \le n} \frac{\lambda_{f,1}(m)e^{ima} - \lambda(m)}{m}\right) F_{II}(1).$$

It yields

$$f(n) = A_n\gamma(n) + o(\gamma(n)) \qquad as \quad n \to \infty.$$

(ii) Let (2.32) diverge for all $a \in \mathbb{R}$. Then

$$f(n) = o(\gamma(n)) \qquad as \quad n \to \infty. \tag{2.33}$$

It may be observed, that we do not require analytic continuation of the generating functions outside the disk of convergence.

2.5 Lemmata

We prove some lemmata that we shall use in the proofs of our theorems. The proof of the first lemma follows the lines of Elliott [9], the second and the third lemma were proven by Indlekofer in [23], therefore we omit their proofs here. Afterwards, we present a mean-value theorem for multiplicative functions which are bounded on the prime powers. Thereafter, we prove the fifth lemma, that is a tauberian theorem, of which the first part was a problem proposed by Schur ([37]).
We shall use the following definition for our results. Put $G_n := \{a \in G : \partial(a) = n\}$ then

Definition 2.16. A function $\tilde{h} : G \mapsto \mathbb{R}$ is called *finitely distributed* if there exists a sequence of integers $(n_1, n_2, ...)$ and a subset $H \subseteq G$ such that for every n_l, $\#(H \cap G_{n_l}) \geq cG(n_l)$ and $|\tilde{h}(a_1) - \tilde{h}(a_2)| < C$ for all $a_1, a_2 \in H \cap G_{n_l}$ with some parameters $c > 0$, $C > 0$.

We can describe a finitely distributed additive function on G as follows

Lemma 2.17. *If an additive function $\tilde{g}$ on G is finitely distributed, then there is an additive function $\tilde{h}$ on G and a constant c, $c \in \mathbb{R}$ so that*

$$\tilde{g}(a) = c\partial(a) + \tilde{h}(a)$$

where both the series

$$\sum_{\substack{p \\ |\tilde{h}(p)|>1}} \frac{1}{q^{\partial(p)}} \qquad \sum_{\substack{p \\ |\tilde{h}(p)|<1}} \frac{\tilde{h}(p)^2}{q^{\partial(p)}} \tag{2.34}$$

converge.

Proof. Since $\tilde{g}$ on G is an additive function, therefore the function $\exp(it\tilde{g})$ is a multiplicative function of modulus 1 on G for any real number t. Define the real-valued function l as follows

$$l(t) = \lim_{n\to\infty} \frac{1}{G(n)} \left| \sum_{\partial(a)=n} \exp(it\tilde{g}(a)) \right|.$$

The function l is well-defined, since the existence of this limit is guaranteed by Proposition 2.6.
Further, we define the function

$$D(\Theta) = \begin{cases} \left(\frac{\sin \pi\Theta}{\pi\Theta}\right)^2, & \text{if } \Theta \neq 0, \\ 1, & \text{if } \Theta = 0. \end{cases}$$

Then, for each real number y,

$$\int_{-\infty}^{\infty} e^{2\pi i\Theta y} D(\Theta)\, d\Theta = \begin{cases} 1 - |y|, & \text{if } |y| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Interchanging summation and integration shows that for a positive α

$$\int_{-\infty}^{\infty} \alpha \left| \sum_{\partial(a)=n} \exp(it\tilde{g}(a)) \right|^2 D(\alpha t)dt = \sum_{\substack{a_1,a_2\in G_n \\ |\tilde{g}(a_1)-\tilde{g}(a_2)|\leq\alpha}} (1 - \alpha^{-1}|\tilde{g}(a_1) - \tilde{g}(a_2)|).$$

We divide by $G(n)$, and let $n \to \infty$, and apply Lebesgue's theorem on dominated convergence. Since $\tilde{g}$ is a finitely distributed additive function and if α is sufficiently large, then

$$\int_{-\infty}^{\infty} \alpha l(t)^2 D(\alpha t) dt > 0.$$

To put it more precisely, if $\tilde{g}$ satisfies the condition given in the definition of a finitely distributed additive function, and if $\alpha \geq 2C$, then the value of this integral is at least as large as $c^2/2$.
As a consequence, there is a set E, of positive Lebesgue measure, on which $l(t) > 0$. If, for some value of t, we have $l(t) > 0$, then according to Indlekofer's theorem (see Proposition 2.6) on the multiplicative function $\exp(it\tilde{g})$ of modulus 1 it yields that there is a *unique* real number $\tau = \tau(t)$, so that the series

$$\sum_{p \in P} q^{-\partial(p)}(1 - Re\, e^{it\tilde{g}(p)} q^{-i\tau\partial(p)}) \tag{2.35}$$

converges. The convergence of this series is equivalent to that of the series

$$L(t,\tau) = \sum_{p \in P} q^{-\partial(p)} \sin^2\left(\frac{1}{2} t\tilde{g}(p) - \frac{1}{2}\tau\partial(p)\right).$$

Such a number τ may be found for each member t of E. Indeed, there is a number K, and a subset F of E, of positive measure, so that whenever t belongs to F the inequality

$$L(t,\tau) \leq K \tag{2.36}$$

is satisfied. Steinhaus proved the following: *The differences generated by a set of a real numbers of positive Lebesgue measure, cover an open interval around the origin* (for the proof see [9]). Therefore there is a proper interval around the origin, $(-2\delta, 2\delta)$ say, each point w of which has a representation $w = t_1 - t_2$, with both t_1 and t_2 belonging to the set F. In view of the inequality

$$\sin^2(x \pm y) \leq 2\sin^2 x + 2\sin^2 y,$$

which is valid for all real numbers x and y, by (2.36) we see that

$$L(w, \tau(t_1) - \tau(t_2)) \leq 2L(t_1, \tau(t_1)) + 2L(t_2, \tau(t_2)) \leq 4K.$$

In particular, $\tau(w)$ exists and has the value

$$\tau(w) = \tau(t_1 - t_2) = \tau(t_1) - \tau(t_2).$$

A simple extension of this argument shows that $L(t,\tau)$ is defined (and finite) for every real number t, and that for every rational number r, the relation $\tau(rt) = r\tau(t)$

holds. For $w = \frac{j}{k}\delta$ $(k \in \mathbb{N}, j = 1, \ldots, k)$ it yields $\tau(w) = \tau(\frac{j}{k}\delta) = \frac{j}{k}\tau(\delta)$. Since $\frac{j}{k}\delta \in (-2\delta, 2\delta)$ we also get the inequality

$$L\left(\frac{j}{k}\delta, \tau\left(\frac{j}{k}\delta\right)\right) \leq 4K$$

which holds uniformly for each positive integer k and $j = 1, 2, ..., k$.
For our next step we shall need the inequality

$$\frac{1}{k}\sum_{j=1}^{k}(1 - \cos jy) \geq \frac{1}{2}, \tag{2.37}$$

that is certainly valid when k is an integer, $k \geq 2$, and y is a real number in the range $\pi/k \leq |y| \leq \pi$. This inequality may be deduced from the identity

$$\frac{1}{k}\sum_{j=1}^{k}(1 - \cos jy) = 1 + \frac{1}{2k} - \frac{\sin((2k+1)y/2)}{2k\sin(y/2)},$$

by means of the inequality

$$|2k\sin(y/2)| \geq 2k|y|/\pi.$$

We set $c := \tau(\delta)/\delta$, define $\tilde{h}(p) = \tilde{g}(p) - c\partial(p)$, and deduce that for the argument of the sinus function in the series $L\left(\frac{j}{k}\delta, \tau\left(\frac{j}{k}\delta\right)\right)$ yields

$$\frac{1}{2}\frac{j}{k}\delta\tilde{g}(p) - \frac{1}{2}\tau\left(\frac{j}{k}\delta\right)\partial(p) = \frac{j\delta}{k}\left(\tilde{g}(p) - c\partial(p)\right) = \frac{j\delta}{k}\tilde{h}(p).$$

Using the cosinus addition formula it follows

$$2sin^2\left(\frac{1}{2}\frac{j}{k}\delta\tilde{g}(p) - \frac{1}{2}\tau\left(\frac{j}{k}\delta\right)\partial(p)\right) = 1 - \cos\left(j\delta h(p)/k\right))$$

By the inequality (2.36) we get

$$\sum_{p}{}' q^{-\partial(p)}(1 - \cos(j\delta\tilde{h}(p)/k)) \leq 8K \tag{2.38}$$

where ' indicates that the summation runs over those primes for which

$$\pi\delta^{-1} \leq |\tilde{h}(p)| \leq \pi k\delta^{-1}.$$

Utilising the inequality (2.37) and (2.38) we deduce that

$$\begin{aligned}\sum_p{}' \frac{1}{q^{\partial(p)}} &\le 2\sum_p{}' \frac{1}{q^{\partial(p)}}\frac{1}{k}\sum_{j=1}^{k}(1-\cos(j\delta\tilde{h}(p)/k))\\ &\le 2\frac{1}{k}\sum_{j=1}^{k}\sum_p{}' \frac{1}{q^{\partial(p)}}(1-\cos(j\delta\tilde{h}(p)/k))\\ &\le \frac{2}{k}\sum_{j=1}^{k} 8K\\ &\le 16K,\end{aligned}$$

and, since k may be chosen arbitrarily large,

$$\sum_{|\tilde{h}(p)|\ge\pi\delta^{-1}} \frac{1}{q^{\partial(p)}} \le 16K.$$

It yields the convergence of the first series in (2.34). Also,

$$\begin{aligned}\frac{\delta^2}{\pi^2}\sum_{|\tilde{h}(p)|\le\pi\delta^{-1}} q^{-\partial(p)}\tilde{h}^2(p) &\le \sum_{|\tilde{h}(p)|\le\pi\delta^{-1}} q^{-\partial(p)}\sin^2(\delta\tilde{h}(p)/2)\\ &\le L(\delta,\tau(\delta)) \le 4K.\end{aligned}$$

This completes the proof of Lemma 2.17. □

Indlekofer's method can be used to deal with multiplicative functions which are bounded on prime powers. For this let $\tilde{f} : G \mapsto \mathbb{C}$ be multiplicative such that, for some constant $c > 1$,

$$|\tilde{f}(p^k)| \le c \qquad \text{for all prime powers } p^k. \tag{2.39}$$

Referring to the proof in [23] we recall

Lemma 2.18. *Let (G,∂) be an additive arithmetical semigroup satisfying $G(n) \ll n^\rho q^n$ where $q > 1$ and $\rho \in \mathbb{R}$. If $\tilde{f}$ is multiplicative satisfying (2.39), then there exists $m_0 \in \mathbb{N}$ such that*

$$\begin{aligned}\Pi(y) &= \prod_{\substack{p\\ \partial(p)\ge m_0}} \left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}y^{k\partial(p)}\right)\\ &= \exp\left(\sum_{m=m_0}^{\infty}\frac{\lambda_f(m)}{m}y^m\right) \qquad (|y|<1)\end{aligned}$$

where

$$\frac{\lambda_f(m)}{m} = \sum_{\substack{p\\ \partial(p)=m}} \tilde{f}(p)q^{-m} + O\left(q^{-\frac{m}{4}}\right)$$

as $m \to \infty$.

Motivated by the results for multiplicative functions on $\mathbb{N}$ (see [17]) we shall assume that

$$\sum_p \frac{(|\tilde{f}(p)|-1)^2}{q^{\partial(p)}} < \infty \tag{2.40}$$

and $\tilde{f} \in \mathcal{L}^1$, i.e.

$$M(n, |\tilde{f}|) \ll 1. \tag{2.41}$$

We define the multiplicative function $\tilde{f}_1$ by

$$\tilde{f}_1(p^k) = \begin{cases} \tilde{f}(p^k), & \text{if } \partial(p) \geq m_0, \\ 0, & \text{if } \partial(p) < m_0. \end{cases} \tag{2.42}$$

Clearly $M(n, |\tilde{f}_1|) \leq M(n, |\tilde{f}|)$, and (2.41) implies

$$\sum_{n=0}^{\infty} \sum_{\substack{g \in G \\ \partial(g)=n}} |\tilde{f}_1(g)| q^{-n} |y|^n \ll Z(|y|)$$

Since $P(d) \leq G(d) \ll q^d d^{\delta-1}$,

(2.43)

$$\Lambda(m) = mP(m) + O\left(mG\left(\frac{m}{2}\right) \sum_{r \leq m} \frac{1}{r}\right) = mP(m) + O\left(mq^{\frac{m}{2}} \left(\frac{m}{2}\right)^{\delta-1} \log m\right).$$

By Lemma 2.18 we get

$$\sum_p \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}} r^{\partial(p)} \leq c_1 \qquad \text{with some } c_1 > 0 \tag{2.44}$$

uniformly as $r \to 1^-$.
Under these conditions the following holds

Lemma 2.19. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} z^m\right) = \frac{\hat{H}(z)}{(1-qz)^{\delta}}$$

where $\hat{H}(z) = O(1)$ for $|z| < q^{-1}$, $\lim_{z \to \frac{1}{q}-} \hat{H}(z) = A > 0$ and $\delta > 0$. Assume that $\Lambda(m) = O(q^m)$ and $G(n) \asymp q^n n^{\delta-1}$. Let $\tilde{f}$ be multiplicative and assume (2.40) *and* (2.41). *If $\tilde{f}_1$ satisfy* (2.42) *and* (2.44), *then, as $n \to \infty$,*

$$M(n, |\tilde{f}_1|) = c_2 \exp\left(\sum_{m_0 \leq \partial(p) \leq n} \frac{|\tilde{f}(p)|-1}{q^{\partial(p)}}\right) + o(1)$$

with some positive constant c_2.

Proof. For further details see Theorem 6 in [23]. □

Our next lemma can be proven similarly as Proposition 2.6 (cf. Theorem 4 in [23]).

Lemma 2.20. *Let (G, ∂) be an additive arithmetical semigroup such that*

$$\hat{Z}(z) = \sum_{n=0}^{\infty} G(n) z^n = \exp\left(\sum_{m=1}^{\infty} \frac{\bar{\Lambda}(m)}{m} z^m \right) = \frac{\hat{H}(z)}{(1-qz)^{\delta}}$$

where $\hat{H}(z) = O(1)$ for $|z| < q^{-1}$, $\lim_{z \to \frac{1}{q}^-} \hat{H}(z) = A > 0$ and $\delta > 0$. Assume that $\bar{\Lambda}(m) = O(q^m)$ and $G(n) \asymp q^n n^{\delta - 1}$. Suppose $\tilde{f}$ is a multiplicative function such that $|\tilde{f}(p^k)| < K$ for each prime power p^k with $\partial(p) \leq \frac{\log 2}{\log q}$, and the series $\sum_{p \in P} (|\tilde{f}(p)| - 1) q^{-\partial(p)}$ converges. If there exists a real number a such that the series (2.5) converges for $\tau = a$, then

$$\sum_{\substack{g \in G \\ \partial(g) = n}} \tilde{f}(g) = q^{ina} \prod_{\partial(p) \leq n} (1 - q^{-\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)(1+ia)} \right) G(n) + o(G(n)).$$

If (2.5) diverges for all $\tau \in \mathbb{R}$ then

$$\sum_{\substack{g \in G \\ \partial(g) = n}} \tilde{f}(g) = o(G(n)).$$

In the proof of our mean-value theorems we often compute the coefficients of power series, that can be written as a product of two other power series. Under certain conditions we can compute the required coefficients. We describe our result in the following lemma.

Lemma 2.21. *Let $C(z) = A(z)B(z)$ where the power series are defined as $A(z) = \sum_{n=1}^{\infty} a_n z^n$, $B(z) = \sum_{n=1}^{\infty} b_n z^n$, $C(z) = \sum_{n=1}^{\infty} c_n z^n$. Assume that for a real number ρ the relation $b_{n-1}/b_n = \rho + o(1)$ holds as $n \to \infty$ and assume that $\sum_{n=1}^{\infty} |a_n||\rho|^n < \infty$. Let the radius r of converge of $A(z)$ satisfy $r \geq |\rho|$. If*

(i) $|\rho| < r$, or

(ii) $|\rho| = r$ and $b_m = O(|b_n||\rho|^{n-m})$ for all $m \leq n$

then

$$c_n \sim A(\rho) b_n \quad (n \to \infty).$$

Proof. Let $\varepsilon > 0$ be so small that $|\rho| + \varepsilon < r$. Then there exists a constant A independent of n and ν such that

$$\left| \frac{b_{n-\nu}}{b_n} \right| = \left| \frac{b_{n-\nu}}{b_{n-\nu+1}} \right| \cdot \ldots \cdot \left| \frac{b_{n-1}}{b_n} \right| < A(|\rho| + \varepsilon)^{\nu}, \qquad \nu = 0, 1, \ldots, n;\ n = 0, 1, 2, \ldots$$

For $n > m$ we obtain

$$\frac{c_n}{b_n} - A(\rho) = \sum_{\nu=0}^{m} a_\nu \left(\frac{b_{n-\nu}}{b_n} - \rho^\nu \right) + \sum_{\nu=m+1}^{n} a_\nu \frac{b_{n-\nu}}{b_n} - \sum_{\nu=m+1}^{\infty} a_\nu \rho^\nu =: \Sigma_1 + \Sigma_2 - \Sigma_3. \tag{2.45}$$

In the case (i) there exists a positive integer m_0 such that

$$\Sigma_2 \ll \sum_{\nu=m+1}^{\infty} |a_\nu| (|\rho| + \varepsilon)^\nu \leq \varepsilon$$

for $m \geq m_0$, and

$$\Sigma_3 \leq \sum_{\nu=m+1}^{\infty} |a_\nu| |\rho|^\nu \leq \varepsilon$$

for $m \geq m_0$.
In the case (ii), we obtain

$$\Sigma_2 \ll \sum_{\nu=m+1}^{n} |a_\nu| |\rho|^\nu \leq \varepsilon$$

since $\frac{|b_{n-\nu}|}{|b_n|} = O(|\rho|^\nu)$ holds by our assumption in (ii). Further, we know that

$$\Sigma_3 \leq \sum_{\nu=m+1}^{\infty} |a_\nu| |\rho|^\nu \leq \varepsilon.$$

Thus in both cases the sum Σ_2 and Σ_3 are absolutely smaller than ε for $m \geq m_0$. Choose m so large that these two terms are smaller than ε. For a fixed m we can choose n in both cases (i) and (ii) so that the first sum Σ_1 in (2.45) becomes absolutely smaller than ε. It yields

$$c_n \sim A(\rho) b_n$$

as $(n \to \infty)$. This ends the proof of Lemma 2.21. □

2.6 Proof of the new mean-value theorems

First we prove Theorem 2.10.

Proof. We assume that $M(\tilde{f}) \neq 0$ and $\tilde{f} \in L^1$. Then there exists a natural number n_0 and constants $0 < c_1, c_2 < \infty$ such that

$$0 < c_1 \leq \frac{1}{G(n)} \sum_{\substack{a \in G \\ \partial(a) = n}} |\tilde{f}(a)| \leq c_2 < \infty \tag{2.46}$$

for all $n \geq n_0$. Then, for $\varepsilon > 0$ that is small enough,

$$\frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ |\tilde{f}(a)| \leq \varepsilon}} |\tilde{f}(a)| \leq \frac{c_1}{4}$$

and with suitable $K > 0$,

$$\frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ |\tilde{f}(a)| > K}} |\tilde{f}(a)| \leq \frac{c_1}{4}$$

because $\tilde{f}$ is uniformly summable. Thus

$$c_2 \geq \frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ \varepsilon < |\tilde{f}(a)| \leq K}} |\tilde{f}(a)| \geq c_1 - \frac{c_1}{4} - \frac{c_1}{4} = \frac{c_1}{2} > 0.$$

It follows also, that

$$\frac{c_2}{\varepsilon} \geq \frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ \varepsilon < |\tilde{f}(a)| \leq K}} 1 \geq \frac{c_1}{2K} > 0,$$

i.e.

$$\frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ \varepsilon < |\tilde{f}(a)| \leq K}} 1 \asymp 1.$$

We define an additive function $\tilde{g}$ by

$$\tilde{g}(p^k) = \begin{cases} \log |\tilde{f}(p^k)|, & \text{if } \tilde{f}(p^k) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

It yields

$$\frac{1}{G(n)} \sum_{\substack{a \in G, \partial(a)=n \\ \log \varepsilon < \tilde{g}(a) \leq \log K}} 1 \asymp 1, \tag{2.47}$$

therefore $\tilde{g}$ is finitely distributed. Then by Lemma 2.17 we deduce

$$\tilde{g}(a) = c\partial(a) + \tilde{h}(a) \tag{2.48}$$

where $\tilde{h}$ satisfies (2.34).
Our next step is to prove $c = 0$. With (2.48) we derive

$$0 \neq |\tilde{f}(p)| = e^{c\partial(p)} e^{\tilde{h}(p)}. \tag{2.49}$$

By Lemma 2.17

$$\sum_{\substack{p\in P\\ |\tilde{h}(p)|>C}} \frac{1}{q^{\partial(p)}} < \infty. \tag{2.50}$$

for all $C > 0$. If $|e^{\tilde{h}(p)} - 1| > \eta_1$ for some $\eta_1, 0 < \eta_1 < 3/4$, then $|\tilde{h}(p)| \geq \min\{\log(1+\eta_1), -\log(1-\eta_1), 1\}$ hence with (2.50)

$$\sum_{\substack{p\in P\\ |e^{\tilde{h}(p)}-1|>\eta_1}} \frac{1}{q^{\partial(p)}} < \infty. \tag{2.51}$$

We define

$$P_1 := \{p \in P; e^{\tilde{h}(p)} < 1 - \eta_1\} \tag{2.52}$$

and

$$P_2 := \{p \in P; e^{\tilde{h}(p)} > 1 + \eta_1\} \tag{2.53}$$

with $0 < \eta_1 < 3/4$. Let

$$S_1 := \{a \in G; \exists p \in P_1 \cup P_2 : p|a, \partial(p) \geq n_0\}$$

$$S_2 := \{a \in G; \exists p \in P : p^2|a, \partial(p) \geq n_0\}$$

and

$$S_3 := \{a \in G; \exists p \in P : p^k|a, k \geq k_0, \partial(p) \leq n_0\}.$$

Put

$$S := S_1 \cup S_2 \cup S_3.$$

We apply Proposition 2.6 on the multiplicative function $\mathbf{1}_{G\setminus S}$. Therefore we can choose n_0 and k_0 such that for all $n_0 \leq n$

$$M(n, \mathbf{1}_S) < \gamma. \tag{2.54}$$

Then S has a density less than γ, hence $\limsup_{n\to\infty} M(n, \mathbf{1}_S\tilde{f}) < \varepsilon$ and therefore

$$\liminf_{n\to\infty} M(n, \mathbf{1}_{G\setminus S}\tilde{f}) > \varepsilon,$$

by our choice of ε.

Since $G(n) \ll q^n n^{\delta-1}$, we have

$$\begin{aligned}
\sum_{\partial(a)=n} \tilde{\omega}(a) &= \sum_{\substack{p \\ \partial(p)=n}} 1 + \sum_{\substack{p \\ \partial(p)\le n-1}} G(n-\partial(p)) \\
&\ll P(n) + \sum_{\substack{p \\ \partial(p)\le n-1}} q^{n-\partial(p)}(n-\partial(p))^{\delta-1} \\
&\ll \frac{q^n}{n} + q^n \sum_{m=1}^{n-1} q^{-m} P(m)(n-m)^{\delta-1} \\
&= \frac{q^n}{n} + q^n \sum_{\frac{n}{2}\le m\le n-1} q^{-m} P(m)(n-m)^{\delta-1} + q^n \sum_{1\le m<\frac{n}{2}} q^{-m} P(m)(n-m)^{\delta-1} \\
&\ll \frac{q^n}{n} + \frac{q^n}{n} \sum_{1\le m\le n} m^{\delta-1} + q^n n^{\delta-1} \sum_{1\le m\le \frac{n}{2}} \frac{1}{m} \\
&\ll \frac{q^n}{n} + \frac{q^n}{n} n^{\delta} + q^n n^{\delta-1} \log n
\end{aligned}$$

and therefore

$$M(n, \tilde{\omega}) = O(\log n). \tag{2.55}$$

By our assumption $\log(|\tilde{f}(a)|) = c\partial(a) + \tilde{h}(a)$. Hence

$$|\tilde{f}(a)| = e^{c\partial(a)} e^{\tilde{h}(a)}.$$

Since $\tilde{f} \in L^*$

$$1 \asymp M(n, |\tilde{f}|\mathbf{1}_{G\setminus S}) = e^{cn} \frac{1}{G(n)} \sum_{\substack{a\in G\setminus S \\ \partial(a)=n}} e^{\tilde{h}(a)}. \tag{2.56}$$

We show that c has to be zero. For this we estimate

$$\frac{1}{G(n)} \sum_{\substack{a\in G \\ \partial(a)=n}} e^{\tilde{h}(a)}.$$

By the definition of the set S we have

$$a \in G \setminus S, p|a \Rightarrow \tilde{h}(p^k) = O(1), \tag{2.57}$$

and it yields

$$|\tilde{h}(a)| \le C\tilde{\omega}(a) \quad \text{for } a \in G \setminus S. \tag{2.58}$$

Therefore

$$\frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{\tilde{h}(a)} \leq \frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{|\tilde{h}(a)|}. \tag{2.59}$$

Put

$$\tilde{g}_{\tilde{h}}(a) := e^{|\tilde{h}(a)|}. \tag{2.60}$$

Then $\tilde{g}_{\tilde{h}}(a)$ is a multiplicative function with $e^{|\tilde{h}(p^k)|} = O(1)$ by (2.57). Thus, there exists a real constant c_3 such that

$$\sum_{n=0}^{\infty} g_{\tilde{h}}(n) z^n \ll \exp\left(\sum_{m=1}^{\infty} \frac{c_3 \Lambda(m)}{m} z^m \right). \tag{2.61}$$

It yields for $0 < |z| = r < q^{-1}$ there exist positive constants c_4 and c_5 so that

$$\frac{g_{\tilde{h}}(n) r^n}{q^n} \ll \exp\left(\sum_{m \leq n} \frac{c_4 \Lambda(m)}{m} r^m \right) \ll \exp(c_5 \log n). \tag{2.62}$$

Hence there exists a constant c_6 with

$$\frac{g_{\tilde{h}}(n)}{q^n n^{\delta - 1}} \ll \exp(c_6 \log n), \tag{2.63}$$

it means

$$\frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{\tilde{h}(a)} \ll \exp(c_6 \log n). \tag{2.64}$$

On the other hand, the equations (2.55) and (2.58) with the Jensen-inequality (see [36]) yields

$$\begin{aligned}
\frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{\tilde{h}(a)} &\geq \frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{-|\tilde{h}(a)|} \\
&\geq \frac{1}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} e^{-C\tilde{\omega}(a)} \\
&\gg \exp\left(-\frac{c_4}{G(n)} \sum_{\substack{a \in G \setminus S \\ \partial(a)=n}} \tilde{\omega}(a) \right) \\
&\gg \exp(-c_3 \log n)
\end{aligned}$$

Collating our last results we arrive at the conclusion that there exists a real constant c_7 such that

$$1 \asymp M(n, |\tilde{f}|\mathbf{1}_{G\setminus S}) \asymp e^{cn} e^{\pm c_7 \log n} \qquad \text{as } n \to \infty. \tag{2.65}$$

Hence c must be zero, and it follows that $|\tilde{f}(a)| = e^{\tilde{h}(a)}$, for all $a \in G$, for which $|\tilde{f}(a)| \neq 0$.
By Lemma 2.17 the series

$$\sum_{\substack{p\in P\\ |\tilde{g}(p)|<1}} \frac{(\tilde{g}(p))^2}{q^{\partial(p)}} \quad \text{and} \quad \sum_{\substack{p\in P\\ |\tilde{g}(p)|>1}} \frac{1}{q^{\partial(p)}} \tag{2.66}$$

converge. If $||\tilde{f}(p)| - 1| \leq \eta_1$, then the series expansion of the logarithm yields

$$\log|\tilde{f}(p)| = \log(1 + (|\tilde{f}(p)| - 1)) = |\tilde{f}(p)| - 1 + O((|\tilde{f}(p)| - 1)^2)$$

so that for $\eta_1 = 1/2$

$$||\tilde{f}(p)| - 1| \leq 2|\log|\tilde{f}(p)|| = 2|\tilde{g}(p)|$$

and

$$|\tilde{g}(p)| \leq 2||\tilde{f}(p)| - 1| \leq 1.$$

Therefore

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|<1/2}} \frac{(|\tilde{f}(p)| - 1)^2}{q^{\partial(p)}} \ll \sum_{\substack{p\in P\\ |\tilde{g}(p)|>\log(1/2)}} \frac{1}{q^{\partial(p)}} < \infty$$

and

$$\sum_{\substack{p\in P\\ 1/2\leq|\tilde{f}(p)|\leq 3/2}} \frac{(|\tilde{f}(p)| - 1)^2}{q^{\partial(p)}} \ll \sum_{\substack{p\in P\\ |\tilde{g}(p)|\leq 1}} \frac{(\tilde{g}(p))^2}{q^{\partial(p)}} < \infty.$$

Thus the series

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|\leq 3/2}} \frac{(|\tilde{f}(p)| - 1)^2}{q^{\partial(p)}}$$

converges. Furthermore

$$|\tilde{f}(p) - 1|^2 = (|\tilde{f}(p)| - 1)^2 + 2(|\tilde{f}(p)| - 1) - 2(Re\,(\tilde{f}(p)) - 1) \tag{2.67}$$

where the series over the first term on the right hand side converges (see above). Choose $K > 0$ large enough and let k_0, n_0 be the parameters, which we have chosen such that $M(n, \mathbf{1}_S) < \gamma$ holds.

We show that the series over the second term on the right hand side of (2.67) is bounded. Let the multiplicative function $\tilde{f}^*$ be defined as

$$\tilde{f}^* := \tilde{f}\mathbf{1}_{G\setminus S}. \tag{2.68}$$

Then the function $\tilde{f}^*$ is bounded on the set of the prime powers. Since $M(\tilde{f})$ exists and is nonzero there exists a natural number n_1, $n_1 \geq n_0$ such that

$$|M(n, \tilde{f})| \asymp 1 \quad \text{for all}\, n \geq n_1. \tag{2.69}$$

Our assumption $\tilde{f} \in L^1$ yields

$$M(n, |\tilde{f}|) \asymp 1 \quad \text{for all}\, n \geq n_1. \tag{2.70}$$

Since $\tilde{f} \in L^*$ we obtain also

$$|M(n, |\tilde{f}^*|)| \asymp 1 \quad \text{for all}\, n \geq n_1. \tag{2.71}$$

For the moment put $0 < |z| = tq^{-1} < 1$ with $0 < t < 1$. The assertion (2.71) and the definition of $\tilde{f}^*$ yield

$$1 \asymp \frac{\sum\limits_{n=1}^{\infty} \sum\limits_{\substack{a\in G \\ \partial(a)=n}} |\tilde{f}^*(a)| t^n q^{-n}}{\hat{Z}(tq^{-1})} \asymp \exp\left(\sum_{n\geq n_0} \sum_{\substack{p \\ \partial(p)=n}} \frac{|\tilde{f}^*(p)| - 1}{q^{\partial(p)}} t^{\partial(p)} \right). \tag{2.72}$$

Put

$$a_n = \sum_{\substack{p \\ n_0 \leq \partial(p)=n}} \frac{|\tilde{f}^*(p)| - 1}{q^n}.$$

From our assumption $\Lambda(n) = O(q^n)$ follows $P(n) = O(\frac{q^n}{n})$, therefore we obtain

$$a_n = O\left(\frac{1}{n}\right).$$

We have

$$\sum_{n=0}^{\infty} a_n t^n = O(1) \qquad \text{for } t \to 1.$$

We show that

$$\sum_{n\leq N} a_n = O(1). \tag{2.73}$$

If we put $t = 1 - \frac{1}{N}$, then

$$\begin{aligned}\left|\sum_{n=0}^{\infty} a_n t^n - \sum_{n\le N} a_n\right| &\le \left|\sum_{n\le N} a_n(t^n-1)\right| + \left|\sum_{n>N} a_n t^n\right| \\ &\le \sum_{n\le N} |a_n| \left|\exp\left(n\log\left(1-\frac{1}{N}\right)\right) - 1\right| + O\left(\frac{1}{N}\right)\sum_{n=0}^{\infty} t^n \\ &\le \sum_{n\le N} O\left(\frac{1}{n}\right) O\left(\frac{n}{N}\right) + O\left(\frac{1}{N}\right) N \\ &= O(1).\end{aligned}$$

Hence (2.73) is satisfied and thus

$$\sum_{n\le N} \sum_{\substack{p \\ n_0 \le \partial(p) = n}} \frac{|\tilde{f}^*(p)| - 1}{q^n} = O(1).$$

There exist only finitely many terms with $\partial(p) < n_0$, therefore

$$\sum_{n\le N} \sum_{\substack{p, \partial(p)=n \\ |\tilde{f}(p)|\le K}} \frac{|\tilde{f}(p)| - 1}{q^{\partial(p)}} = O(1). \tag{2.74}$$

Next, we prove that the sum over the third term on the right hand side of (2.67) is bounded, and with this we find that (2.16) converges, as has been claimed.

Let ε be an arbitrary positive number. By the definition of $\tilde{f}^*$ and the formula (2.68) we deduce

$$AG(n) \sim G(n)M(n,\tilde{f}) = G(n)M(n,\tilde{f}^*) + G(n)\vartheta\varepsilon \quad (\text{as } n \to \infty)$$

with $0 \le |\vartheta| \le 1$. Therefore

$$A\hat{Z}(z) \sim \hat{F}(z) = \hat{F}^*(z) + \vartheta\varepsilon\hat{Z}(z)$$

where $\hat{F}^*$ denotes the generating function of $\tilde{f}^*$. Dividing by $\hat{Z}(z)$ and utilising the formula (2.69) it follows

$$\left|\frac{\hat{F}^*(z)}{\hat{Z}(z)}\right| \asymp 1.$$

Using here, further, the notation $0 < |z| = tq^{-1} < 1$ with $0 < t < 1$, we obtain as by (2.72) that

$$1 \asymp \exp\left(\sum_{n\ge n_0} \sum_{\substack{p \\ \partial(p)=n}} \frac{Re\,\tilde{f}^*(p) - 1}{q^{\partial(p)}} r^{\partial(p)}\right).$$

Then with analogue tauberian arguments as above we obtain

$$\sum_{n\leq N}\sum_{\substack{p\in P\\ \partial(p)\geq n_0,|\tilde{f}(p)|\leq 3/2}}\frac{Re\,(\tilde{f}(p))-1}{q^{\partial(p)}}=O(1),$$

i.e that the partial sums of the series over the third term on the right hand side of (2.67) are bounded. Putting our results together in (2.67) we obtain the convergence of the series

$$\sum_{\substack{p\in P\\ |\tilde{f}(p)|\leq 3/2}}\frac{|\tilde{f}(p)-1|^2}{q^{\partial(p)}},$$

i.e the convergence of (2.16).
Next we prove the convergence of the series (2.18). Let

$$S_4:=\{a\in G;\exists p\in P: p|a;||\tilde{f}(p)|-1|>1/2,\partial(p)\geq n_0\}.$$

Since $M(\tilde{f})$ exists and is nonzero and $\tilde{f}\in L^1$ there exists a natural number n_1, $n_1\geq n_0$ such that (2.69) and (2.70) hold. By our assumption $\tilde{f}\in L^*$, thus

$$M(n,|\tilde{f}|\mathbf{1}_{G\setminus S_4})\asymp 1\quad\text{for all } n\geq n_1. \tag{2.75}$$

Put in what follows $1<\lambda\leq\alpha$ and $\beta\in\mathbb{R}$ with $\frac{1}{\lambda}+\frac{1}{\beta}=1$.
Then Hölder's inequality yields

$$\begin{aligned}
1\ll\frac{1}{G(n)}\sum_{\substack{a\in G\\ \partial(a)=n}}|\tilde{f}(a)|&\leq\frac{1}{G(n)}\left(\sum_{\substack{a\in G\\ \partial(a)=n}}|\tilde{f}(a)|^\lambda\right)^{\frac{1}{\lambda}}G(n)^{\frac{1}{\beta}}\\
&=\frac{G(n)^{1-\frac{1}{\lambda}}}{G(n)}\left(\sum_{\substack{a\in G\\ \partial(a)=n}}|\tilde{f}(a)|^\lambda\right)^{\frac{1}{\lambda}}\\
&=\left(\frac{1}{G(n)}\sum_{\substack{a\in G\\ \partial(a)=n}}|\tilde{f}(a)|^\lambda\right)^{\frac{1}{\lambda}}\\
&=M(n,|\tilde{f}|^\lambda)^{\frac{1}{\lambda}}\\
&\ll 1
\end{aligned}$$

since $\tilde{f}\in L^\alpha$. Hence

$$M(n,|\tilde{f}|^\lambda)\asymp 1\quad\text{for all } n\geq n_1.$$

By the formula (2.75) we obtain similarly

$$M(n, |\tilde{f}|^\lambda \mathbf{1}_{G\setminus S_4}) \asymp 1 \quad \text{for all } n \geq n_1.$$

For $0 < r = |z| < 1/q$ we obtain

$$1 \asymp \frac{\hat{Z}(r) \sum_{n=0}^{\infty} \left(\sum_{\substack{a\in G\setminus S_4 \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n}{\hat{Z}(r) \sum_{n=0}^{\infty} \left(\sum_{\substack{a\in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n} = \prod_{\substack{p\in P, \partial(p)\geq n_0 \\ ||\tilde{f}(p)|-1|>1/2}} \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda r^{k\partial(p)} \right)^{-1}. \tag{2.76}$$

Remark 2.22. Consider an infinite product $\prod_{n=1}^{\infty}(1+b_n)$, where $b_n \geq 0$ is satisfied. Then

$$\prod_{n=1}^{\infty}(1+b_n) = \lim_{N\to\infty} \prod_{n\leq N}(1+b_n) \leq \lim_{N\to\infty} \exp\left(\sum_{n\leq N} b_n \right),$$

where we have made use of the inequality $1 + x \leq \exp(x)$, which is valid for $x \geq 0$. On the other hand, we know

$$\lim_{N\to\infty} \sum_{n\leq N} b_n \leq \lim_{N\to\infty} \prod_{n\leq N}(1+b_n).$$

Thus the product $\prod_{n=1}^{\infty}(1+b_n)$ is convergent if and only if $\sum_{n=1}^{\infty} b_n$ is convergent.

The last product in (2.76) has the form $\prod_{n=1}^{\infty}(1+b_n)$, where $b_n \geq 0$. Therefore Remark 2.22 yields that there exists a real constant c_8 such that for all $r \in \mathbb{R}$

$$\sum_{p;||\tilde{f}(p)|-1|>1/2} |\tilde{f}(p)|^\lambda r^{\partial(p)} \leq c_8 < \infty.$$

Thus for $r \to 1/q$

$$\sum_{p;||\tilde{f}(p)|-1|>1/2} \frac{|\tilde{f}(p)|^\lambda}{q^{\partial(p)}} < \infty.$$

which yields the convergence of the series (2.18) for all $1 \leq \lambda \leq \alpha$.

Next, we prove the convergence of the series (2.17). Put

$$S_5 := \{a \in G; \exists p \in P : p^k | a; k \geq 2, \partial(p) \geq n_0\}.$$

Then, analogous to what we have seen above, there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$

$$M(n, |\tilde{f}|^\lambda) \asymp 1 \qquad \text{and} \qquad M(n, |\tilde{f}|^\lambda \mathbf{1}_{G\setminus S_5}) \asymp 1.$$

For $0 < r = |z| < 1/q$ the following holds

$$1 \asymp \frac{\sum\limits_{n=0}^{\infty} \left(\sum\limits_{\substack{a\in G\setminus S_5 \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n}{\sum\limits_{n=0}^{\infty} \left(\sum\limits_{\substack{a\in G \\ \partial(a)=n}} |\tilde{f}(a)|^\lambda \right) r^n} = \prod_{\substack{p\in P, k\geq 2 \\ \partial(p)\geq n_0}} \left(1 + \sum_{k=1}^{\infty} |\tilde{f}(p^k)|^\lambda r^{k\partial(p)} \right)^{-1}.$$

Using Remark 2.22 it follows that there exists a real constant c_9 such that for all $r \in \mathbb{R}$

$$\sum_{\substack{p\in P, k\geq 2 \\ \partial(p)\geq n_0}} |\tilde{f}(p^k)|^\lambda r^{k\partial(p)} \leq c_9 < \infty.$$

Thus for $r \to 1/q$

$$\sum_{p\in P; k\geq 2} \frac{|\tilde{f}(p^k)|^\lambda}{q^{k\partial(p)}} < \infty.$$

holds, and therefore the series (2.17) converges for all $1 \leq \lambda \leq \alpha$.
Next, we show the validity of (2.19) for every $p \in P$. By the convergence of the series (2.16) and (2.18) for $|z| = r < q^{-1}$ we have

$$\begin{aligned} \left| \prod_{p;\partial(p)\geq n_0} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)} \right) \right| &\leq \prod_{p;\partial(p)\geq n_0} \left(1 + \left| \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)} \right| \right) \\ &\ll \exp\left(\sum_{p;\partial(p)\geq n_0} (|\tilde{f}(p)| - 1) r^{k\partial(p)} \right) \hat{Z}(r) \\ &\ll \hat{Z}(r). \end{aligned}$$

Suppose now, that for some p_1 with $\partial(p_1) < n_0$ we have

$$1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)} = 0.$$

Hence $1 + \sum\limits_{k=1}^{\infty} \tilde{f}(p_1^k) z^{k\partial(p_1)} = o(1)$ as $r \to q^{-1}$. Thus, as $r \to q^{-1}$

$$\prod_{p\in P} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)} \right) = o(1) O(\hat{Z}(r)) = o(\hat{Z}(r))$$

and we achieve a contradiction to $\hat{F}(z) \sim c\hat{Z}(z)$ for $z \to q^{-1}$ with $c \neq 0$.

$$1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-\partial(p)} \neq 0,$$

which is then true for all primes $p \in P$.
Finally, we prove the convergence of the series (2.15). By the convergence of (2.18) and the condition (2.19), there exists some number m_0 sufficiently large such that $|\tilde{f}(p)q^{-\partial(p)}| < \frac{1}{4}$ and

$$\left| 1 + \sum_{k=1}^{\infty} \tilde{f}(p^k)(q^{-1}e^{i\Theta})^{k\partial(p)} \right| > \frac{1}{2} \tag{2.77}$$

for all p with $\partial(p) \geq m_0$ and all real Θ with $|\Theta| \leq \pi$. We write

$$\begin{aligned}\frac{\hat{F}(r)}{\hat{Z}(r)} &= \prod_{p,\partial(p)<m_0} (1 - r^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) r^{k\partial(p)} \right) \prod_{p,\partial(p)\geq m_0} (1 - r^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) r^{k\partial(p)} \right) \\ &=: \Pi_1(r)\Pi_2(r),\end{aligned}$$

where the first product $\Pi_1(r)$ is continuous by (2.18). We now estimate the second product $\Pi_2(r)$:

$$\begin{aligned}\Pi_2(r) &= \prod_{p,\partial(p)\geq m_0} (1 - r^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) r^{k\partial(p)} \right) \\ &= \prod_{p,\partial(p)\geq m_0} \left(1 + (\tilde{f}(p) - 1) r^{\partial(p)} + \sum_{k=2}^{\infty} (\tilde{f}(p^k) - \tilde{f}(p^{k-1})) r^{k\partial(p)} \right) \\ &= \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)-1|\leq \frac{1}{2}}} (1 - r^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) r^{k\partial(p)} \right) \prod_{\substack{p,\partial(p)\geq m_0 \\ |\tilde{f}(p)-1|> \frac{1}{2}}} (1 - r^{\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) r^{k\partial(p)} \right) \\ &=: \Pi_3(r)\Pi_4(r).\end{aligned}$$

By the convergence of the serie (2.18) the product $\Pi_4(r)$ of the last line is continuous for $r \leq q^{-1}$.

We can write the first product $\Pi_3(r)$ of the last line as follows

$$\begin{aligned}\Pi_3(r) &= \prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}} (1-r^{\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)r^{k\partial(p)}\right)\prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}}\frac{1-\tilde{f}(p)r^{\partial(p)}}{1-\tilde{f}(p)r^{\partial(p)}}\\ &= \prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}} (1-\tilde{f}(p)r^{\partial(p)})^{-1}\\ &\times \prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}} (1-r^{\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)r^{k\partial(p)}\right)(1-\tilde{f}(p)r^{\partial(p)})\\ &= \prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}} (1-\tilde{f}(p)r^{\partial(p)})^{-1}\times \prod_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}}\Big(1-r^{\partial(p)}-(\tilde{f}(p)r^{\partial(p)})^2+\tilde{f}(p)r^{2\partial(p)}\\ &+\sum_{k=2}^{\infty}(\tilde{f}(p^k)-\tilde{f}(p^{k-1}))r^{k\partial(p)}(1-\tilde{f}(p)r^{\partial(p)})\Big)\\ &=:\Pi_5(r)\Pi_6(r).\end{aligned}$$

The convergence of the series (2.16) yields that the second product $\Pi_6(r)$ of the above line is continuous for $r\leq q^{-1}$. Also

$$\Pi_5(r)=\exp\left(\sum_{\substack{p,\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}}\log(1-(\tilde{f}(p)-1)r^{\partial(p)})\right).$$

After the power series expansion of the logarithm we can summarize our last results and write

$$\frac{\hat{F}(r)}{\hat{Z}(r)}=F_1(r)\exp\left(-\sum_{\substack{\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}}(1-\tilde{f}(p))r^{\partial(p)}\right). \tag{2.78}$$

Here $F_1(r)$ is continuous for $r\leq q^{-1}$. If now the condition (2.19) is satisfied, then $F_1(q^{-1})\neq 0$. It follows from (2.78) that

$$\lim_{r\to q^{-1}}\sum_{\substack{\partial(p)\geq m_0\\ |\tilde{f}(p)-1|\leq\frac{1}{2}}}(1-\tilde{f}(p))r^{\partial(p)}$$

exists. Appealing directly to our condition $P(m) = O(\frac{q^m}{m})$ and to the well-known tauberian theorem of Ingham (see Theorem 106 in [16]),

$$\sum_{m \geq m_0} q^{-m} \sum_{\substack{p, \partial(p)=m \\ |\tilde{f}(p)-1| \leq \frac{1}{2}}} (1 - \tilde{f}(p))$$

converges and then the series

$$\sum_{p \in P} \frac{\tilde{f}(p) - 1}{q^{\partial(p)}}$$

converges, too. Hence (2.15) converges, as it has been claimed. This ends the proof of Theorem 2.10. □

Next, we prove Theorem 2.11

Proof. First we prove that $M(\tilde{f})$ exists. By the convergence of (2.18) and the condition (2.19), there exists some number m_0 sufficiently large such that $|\tilde{f}(p)q^{-\partial(p)}| < \frac{1}{4}$ and (2.77) holds for all p with $\partial(p) \geq m_0$ and all real Θ with $|\Theta| \leq \pi$. We write

$$\begin{aligned}
\hat{F}(z) = & \prod_{p, \partial(p) < m_0} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)}\right) \prod_{\substack{p, \partial(p) \geq m_0 \\ |\tilde{f}(p)| < K}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)}\right) \\
& \times \prod_{\substack{p, \partial(p) \geq m_0 \\ |\tilde{f}(p)| \geq K}} \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) z^{k\partial(p)}\right) \\
& =: \Pi_1(z)\Pi_2(z)\Pi_3(z),
\end{aligned}$$

where the first product $\Pi_1(z)$ is absolutely convergent for $|z| \leq q^{-1}$, since each factor of the finite product $\Pi_1(z)$ is convergent by (2.18). The third product $\Pi_3(z)$ is also absolutely convergent for $|z| \leq q^{-1}$. We now estimate the second product $\Pi_2(z)$:

$$\begin{aligned}
\Pi_2(z) = & \prod_{\substack{p, \partial(p) \geq m_0 \\ |\tilde{f}(p)| < K}} \left(1 + \sum_{k=2}^{\infty} \tilde{f}(p^k) z^{k\partial(p)}\right) \frac{1 - \tilde{f}(p) z^{\partial(p)}}{1 - \tilde{f}(p) z^{\partial(p)}} \\
= & \prod_{\substack{p, \partial(p) \geq m_0 \\ |\tilde{f}(p)| < K}} (1 - \tilde{f}(p) z^{\partial(p)})^{-1} \prod_{\substack{p, \partial(p) \geq m_0 \\ |\tilde{f}(p)| < K}} \left(1 + \sum_{k=2}^{\infty} \tilde{f}(p)(\tilde{f}(p^k) - \tilde{f}(p^{k-1})) z^{k\partial(p)}\right) \\
& =: \Pi_4(z)\Pi_5(z).
\end{aligned}$$

By the convergence of the series (2.18) the second product $\Pi_5(z)$ of the last line is absolutely convergent for $|z| \leq q^{-1}$. We apply Lemma 2.20 to the product $\Pi_4(z)$, that is

a generating function of a completely multiplicative function $\tilde{f}_1$, where $\tilde{f}_1(p) = \tilde{f}(p)$ for $\partial(p) \geq m_0$ and $|\tilde{f}(p)| < K$, and $\tilde{f}_1(p) = 0$ otherwise. We obtain

$$\sum_{a\in G,\partial(a)=n} \tilde{f}_1(a) = \prod_{p\in P}(1-q^{\partial(p)})(1-\tilde{f}(p)q^{-\partial(p)})^{-1}G(n) + o(G(n)).$$

Thus we can write

$$\hat{F}(z) = \Pi_4(z)(\Pi_1(z)\Pi_5(z)\Pi_3(z)) =: \Pi_4(z)A(z) \tag{2.79}$$

where $A(z)$ is absolutely convergent for $|z| = q^{-1}$. Applying Lemma 2.21 it follows

$$M(\tilde{f}) = M(\tilde{f}_1)A(q^{-1}).$$

Hence $M(\tilde{f})$ exists and has the above form. If in addition (2.19) holds, then $M(\tilde{f}) \neq 0$ as it has been claimed.

If $\alpha > 1$ and $||\tilde{f}(p)| - 1| < 1/2$, then

$$|\tilde{f}(p)|^\alpha - 1 = \alpha(|\tilde{f}(p)| - 1) + O((|\tilde{f}(p)| - 1)^2)$$

and

$$(|\tilde{f}(p)|^\alpha - 1)^2 = O((|\tilde{f}(p)| - 1)^2) = O(|\tilde{f}(p) - 1|^2),$$

and the corresponding series converge (cf (2.67)).
Therefore, in the same way as above we deduce that $M(|\tilde{f}|^\lambda)$ exists for $1 \leq \lambda \leq \alpha$ and $\tilde{f} \in L^\alpha$. If in addition (2.19) holds, then $M(|\tilde{f}|^\lambda) \neq 0$ for $1 \leq \lambda \leq \alpha$ as it has been claimed.

Next, we prove that $\tilde{f} \in L^*$. Using the equation (2.79) we can write the multiplicative function $\tilde{f}$ as the convolution

$$\tilde{f} = \tilde{f}_1 * \tilde{f}_2, \tag{2.80}$$

where $\tilde{f}_1$ is the completely multiplicative function defined above; and $\tilde{f}_2$ is a multiplicative function, such that its generating function $A(z)$ is absolutely convergent for $|z| \leq q^{-1}$. It yields

$$\Sigma_* := \sum_{m\in\mathbb{N}} \sum_{b\in G,\partial(b)=m} |\tilde{f}_2(b)|q^{-\partial(b)} < \infty. \tag{2.81}$$

Hence for an arbitrary ε_* there exists a natural number m_0 such that

$$\sum_{m\geq m_0} \sum_{b\in G,\partial(b)=m} |\tilde{f}_2(b)|q^{-\partial(b)} < \frac{\varepsilon_*}{2}.$$

Using the convergence of the series (2.15)-(2.18) we deduce by Lemma 2.19 that $M(|\tilde{f}_1|)$ and $M(|\tilde{f}_1|^2)$ exist.
Let $\varepsilon > 0$ be arbitrary and fixed. We prove that there exists K_0 such that

$$\sum_{a\in G,\partial(a)=n} |\tilde{f}_{K_0}(a)| < \varepsilon G(n)$$

holds for all $n \in \mathbb{N}$.

$$\begin{aligned}\sum_{a\in G,\partial(a)=n} |\tilde{f}_{K_0}(a)| &= \sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\geq K_0\\ \partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| \\ &= \sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\geq K_0\\ |\tilde{f}_2(b)|\geq K_1,\partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| + \sum_{\substack{a,b\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\geq K_0\\ |\tilde{f}_2(b)|<K_1,\partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| \\ &=: \Sigma_1 + \Sigma_2,\end{aligned}$$

where the parameter K_1 is chosen such that $\partial(b) \geq m_0$ as $|\tilde{f}_2(b)| \geq K_1$. Let us now estimate Σ_1. By our assumption $G(n) \asymp q^n n^{\delta-1}$ $(1 \leq \delta)$ we obtain

$$\begin{aligned}\Sigma_1 &= \sum_{\substack{b\in G\\ |\tilde{f}_2(b)|\geq K_1\\ \partial(b)\leq n}} |\tilde{f}_2(b)| \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde{f}_1(a)| \\ &\leq \sum_{\substack{b\in G\\ m_0\leq\partial(b)\leq n}} |\tilde{f}_2(b)| \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde{f}_1(a)| \ll \sum_{\substack{b\in G\\ m_0\leq\partial(b)\leq n}} |\tilde{f}_2(b)| q^{-\partial(b)} G(n) \\ &< \frac{\varepsilon}{2} G(n),\end{aligned}$$

whereby we have used the following

$$G(n-\partial(b)) \asymp q^{n-\partial(b)}(n-\partial(b))^{\delta-1} = q^n n^{\delta-1}(1-\partial(b)/n)^{\delta-1} q^{-\partial(b)} \ll q^{-\partial(b)} G(n).$$

Afterwards, we estimate Σ_2. We use (2.81) and our assumption $G(n) \asymp q^n n^{\delta-1}$ to obtain the following

$$\begin{aligned}\Sigma_2 &= \sum_{\substack{a,b\in G\\ |\tilde{f}_2(b)|<K_1\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\geq K_0, \partial(a)+\partial(b)=n}} |\tilde{f}_1(a)||\tilde{f}_2(b)| \\ &= \sum_{b\in G, |\tilde{f}_2(b)|<K_1} \sum_{\substack{a\in G\\ |\tilde{f}_1(a)||\tilde{f}_2(b)|\geq K_0\\ \partial(a)=n-\partial(b)}} \frac{|\tilde{f}_1(a)|^2}{|\tilde{f}_1(a)|} \\ &\leq \sum_{b\in G, |\tilde{f}_2(b)|<K_1} |\tilde{f}_2(b)| \frac{|\tilde{f}_2(b)|}{K_0} \sum_{\substack{a\in G\\ \partial(a)=n-\partial(b)}} |\tilde{f}_1(a)|^2 \\ &\ll \frac{K_1}{K_0} \sum_{b\in G} |\tilde{f}_2(b)| G(n-\partial(b)) \leq \frac{\varepsilon}{2} G(n),\end{aligned}$$

since $M(|\tilde{f}_1|^2)$ exists.
Therefore $\tilde{f} \in L^*$. This ends the proof of Theorem 2.11. □

Now we prove Theorem 2.12

Proof. Let $\varepsilon > 0$ be arbitrary and fixed. Then by (2.20) there exists $K > 0$ with

$$S = \{a \in G : p^k | a, p \in P, k \geq 1, |\tilde{f}(p^k)| > K\}$$

such that

$$M(n, |\tilde{f}|\mathbf{1}_S) < \varepsilon.$$

Let such a K be fixed. It yields

$$\left| \frac{1}{G(n)} \sum_{\substack{a\in G\\ \partial(a)=n}} \tilde{f}(a) - \frac{1}{G(n)} \sum_{\substack{a\in G\setminus S\\ \partial(a)=n}} \tilde{f}(a) \right| < \varepsilon.$$

By Lemma 2.20 we obtain

$$\frac{1}{G(n)} \sum_{\substack{a\in G\setminus S\\ \partial(a)=n}} \tilde{f}(a) \to M(\mathbf{1}_{G\setminus S}\tilde{f}) = \prod_{\substack{p\in P\\ |\tilde{f}(p^k)|\leq K}} (1-q^{-\partial(p)}) \left(1 + \sum_{k=1}^{\infty} \tilde{f}(p^k) q^{-k\partial(p)}\right)$$

as $n \to \infty$. Therefore

$$M(\mathbf{1}_{G\setminus S}\tilde{f}) = \prod_{\substack{p\in P\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K, K=2,3,\dots}} (1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right)$$
$$\times \prod_{\substack{p\in P\\ K\ge|\tilde{f}(p)|>K_2\\ |\tilde{f}(p^k)|\le K}} (1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right)$$
$$=:\Pi_{1,K}\Pi_{2,K}.$$

The product $\Pi_{2,K}$ is absolutely convergent for $|z| \le q^{-1}$ and

$$\lim_{K\to\infty}\Pi_{2,K} = \prod_{\substack{p\in P\\ |\tilde{f}(p)|>K_2}} (1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right).$$

Estimating the product $\Pi_{1,K}$ we deduce

$$\Pi_{1,K} = \prod_{\substack{p\in P\\ |\tilde{f}(p)|\le K_2}} (1-q^{-\partial(p)})(1+\tilde{f}(p)q^{-\partial(p)}) \prod_{\substack{p\in P\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K}} (1+\tilde{f}(p)q^{-\partial(p)})^{-1}\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right)$$
$$=:\Pi_1\Pi_{3,K}.$$

We derive

$$\Pi_{1,K} = \prod_{\substack{p\in P,\partial(p)\le m_0\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K}} (1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right)$$
$$\times \prod_{\substack{p,\partial(p)>m_0\\ |\tilde{f}(p)|\le K_2}} (1-q^{-\partial(p)})(1+\tilde{f}(p)q^{-\partial(p)}) \prod_{\substack{p\in P,\partial(p)>m_0\\ |\tilde{f}(p)|\le K_2\\ |\tilde{f}(p^k)|\le K}} (1+\tilde{f}(p)q^{-\partial(p)})^{-1}\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right)$$
$$=:\Pi_{4,K}\Pi_5\Pi_{6,K}.$$

Therefore

$$\lim_{K\to\infty}\Pi_{4,K} = \prod_{\substack{p\in P,\partial(p)\le m_0\\ |\tilde{f}(p)|\le K_2}} (1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right),$$

since $\Pi_{4,K}$ is a finite product. Since

$$(1-q^{-\partial(p)})(1+\tilde{f}(p)q^{-\partial(p)}) = 1-q^{-\partial(p)}+\tilde{f}(p)q^{-\partial(p)}-\tilde{f}(p)q^{-2\partial(p)}$$

the product Π_5 is convergent because of the convergence of the series (2.15) and (2.16). Hence for a given positive real number ε_1 there exists a natural number n_0 such that

$$\left|\frac{1}{G(n)}\sum_{\substack{a\in G\setminus S\\ \partial(a)=n}}\tilde{f}(a) - M(\mathbf{1}_{G\setminus S}\tilde{f})\right| < \varepsilon_1$$

holds for all $n \geq n_0$. Thus

$$\limsup_{n\to\infty}\left|\frac{1}{G(n)}\sum_{a\in G,\partial(a)=n}\tilde{f}(a) - M(\mathbf{1}_{G\setminus S}\tilde{f})\right| < \varepsilon+\varepsilon_1. \tag{2.82}$$

Considering the limit for $K\to\infty$ we obtain

$$\lim_{K\to\infty} M(\mathbf{1}_{G\setminus S}\tilde{f}) = \prod_{\partial(p)\in P}(1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right).$$

where the last product does not depend on K. Therefore (2.82) yields

$$M(\tilde{f}) = \prod_{\partial(p)\in P}(1-q^{-\partial(p)})\left(1+\sum_{k=1}^{\infty}\tilde{f}(p^k)q^{-k\partial(p)}\right).$$

It means that the mean-value $M(\tilde{f})$ exists and has the above form. If (2.19) holds, we obtain also, that $M(\tilde{f})$ is nonzero.

The existence of the mean-value $M(|\tilde{f}|^\lambda)$ for $1\leq\lambda\leq\alpha$ follows in the same way, since the series corresponding to (2.15)-(2.18) for $|\tilde{f}|^\lambda$ are convergent, then $\tilde{f}\in L^\alpha$ holds.

Finally, we prove that $\tilde{f}\in L^*$. For a real number K, $K>0$ it yields

$$\sum_{\substack{a\in G\\ |\tilde{f}(a)|>K\\ \partial(a)=n}}|\tilde{f}(a)| = \sum_{\substack{a\in G\setminus S\\ |\tilde{f}(a)|>K\\ \partial(a)=n}}|\tilde{f}(a)| + \sum_{\substack{a\in S\\ |\tilde{f}(a)|>K\\ \partial(a)=n}}|\tilde{f}(a)| \tag{2.83}$$

where the second sum on the right hand side is $<\varepsilon/2$ and tends to zero as $K\to\infty$. Put $\tilde{f}_3 = \tilde{f}\mathbf{1}_{G\setminus S}$. Then $\tilde{f}_3$ is a multiplicative function with $|\tilde{f}_3(p^k)|\leq K$ and Lemma

2.18 yields $M(n, \tilde{f}_3) = O(1)$. Therefore

$$\sum_{\substack{a\in G\setminus S\\ |\tilde{f}(a)|>K\\ \partial(a)=n}} |\tilde{f}(a)| \le \sum_{\substack{a\in G\setminus S\\ |\tilde{f}(a)|>K\\ \partial(a)=n}} |\tilde{f}(a)|\frac{|\tilde{f}(a)|}{K}$$

$$= \frac{1}{K}\sum_{\substack{a\in G\\ |\tilde{f}_3(a)|>K\\ \partial(a)=n}} |\tilde{f}_3(a)|^2 < G(n)\varepsilon/2$$

if K is large enough. By (2.83) it follows that $\tilde{f} \in L^*$.
This ends the proof of Theorem 2.12. □

Chapter 3

Applications

In this chapter we give two probabilistic applications of our main results. Let now $\tilde{g} : G \to \mathbb{R}$ be a (real-valued) additive function defined on G. Then, by the continuity theorem of Lévy, the distribution functions

$$\mathcal{G}_n(x) := \frac{1}{G(n)} \#\{a \in G : \partial(a) = n, \tilde{g}(a) \leq x\} \tag{3.1}$$

tend to a limit distribution $\mathcal{G}(x)$,

$$\mathcal{G}_n \Rightarrow \mathcal{G}, \tag{3.2}$$

if and only if there exists a function $\varphi(t)$ which is continuous at $t = 0$ such that

$$\frac{1}{G(n)} \sum_{\substack{a \in G \\ \partial(a)=n}} e^{it\tilde{g}(a)} \to \varphi(t)$$

as $n \to \infty$ for $t \in \mathbb{R}$. Moreover, $\varphi(t)$ is the characteristic function of $\mathcal{G}(x)$. We note that the function $\tilde{f}(a) := e^{it\tilde{g}(a)}$ is multiplicative and $|\tilde{f}(a)| = 1$ since $\tilde{g}$ is real-valued and additive.

3.1 Finitely distributed additive functions

In this section we characterize all additive functions $\tilde{g}$ on G which, after a suitable translation, possess a limiting distribution. In order that there exists a sequence $\{\alpha(n)\}$, $n \in \mathbb{N}$, for which the frequencies

$$\frac{1}{G(n)} \#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

converge to a weak limit as $n \to \infty$ we give here the necessary and sufficient conditions.

Theorem 3.1. *Suppose that an additive arithmetical semigroup (G, ∂) fulfills the assumptions of Theorem 2.11. Let $\tilde{g}$ be a real-valued additive function on G. Then the following assertions hold*

(i) If, for some $\alpha(n)$ the frequencies

$$\frac{1}{G(n)}\#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

converge to a weak limit as $n \to \infty$, then $\tilde{g}$ is finitely distributed.

(ii) If $\tilde{g}$ is finitely distributed, then it has a decomposition $\tilde{g}(a) = c\partial(a) + \tilde{h}(a)$ with a real constant c and an additive function $\tilde{h}$ where both the series

$$\sum_{\substack{p \\ |\tilde{h}(p)|>1}} \frac{1}{q^{\partial(p)}} \qquad \sum_{\substack{p \\ |\tilde{h}(p)|<1}} \frac{\tilde{h}(p)^2}{q^{\partial(p)}} \tag{3.3}$$

converge.

(iii) If $\tilde{g}$ has a representation $c\partial + \tilde{h}$, where the series (3.3) both converge, and if we define

$$\alpha(n) = cn + \sum_{\partial(p)\leq n, |\tilde{h}(p)|\leq 1} \frac{\tilde{h}(p)}{q^{\partial(p)}} \qquad (n \geq 1),$$

then the frequencies

$$\frac{1}{G(n)}\#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

converge to a weak limit as $n \to \infty$.

Proof of (i). If the number w is chosen sufficiently large, and such that $\pm w$ are continuity points of the limiting distribution of $\tilde{g}(a) - \alpha(n)$, then

$$\lim_{n\to\infty} \frac{1}{G(n)}\#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq w\} > \frac{1}{2}. \tag{3.4}$$

Moreover, if a_1 and a_2 are any two elements in G which are counted in a typical frequency,

$$|\tilde{g}(a_1) - \tilde{g}(a_2)| \leq |\tilde{g}(a_1) - \alpha(n)| + |\alpha(n) - \tilde{g}(a_2)| \leq 2w, \tag{3.5}$$

from which it is clear that $\tilde{g}$ is finitely distributed.
The assertion (ii) is contained in Lemma 2.17.
Proof of (iii). Consider the characteristic function

$$\psi(n, t) = \frac{1}{G(n)} \exp(-it\alpha(n)) \sum_{a\in G, \partial(a)=n} \exp(it\tilde{g}(a)).$$

In the same way as above, we can show, that $\psi(n,t)$ converges for $n \to \infty$ using Proposition 2.6 and taking in account the property $\frac{G(n-1)}{G(n)} = q^{-1} + o(1)$ as $n \to \infty$. Therefore the frequencies

$$\frac{1}{G(n)} \#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

converge to a weak limit as $n \to \infty$.
This ends the proof of Theorem 3.1.

3.2 Three-series theorem

In this section we present our version of the well-known Three-series theorem under weak conditions about the additive arithmetical semigroups. We remark here, that in a paper of Barát, Indlekofer and Kaya (see [3]), the authors prove the Two-series theorem in additive arithmetical semigroups and pull together the properties of finitely distributed functions and the characterisation of essentially convergent series in the Stone-Cech compactification of G. Some ideas of the construction of the described Stone-Cech compactification in [3] were motivated by the construction described by Barát and Indlekofer (see [2]).

Theorem 3.2. *(Three-series theorem).*
Suppose that an additive arithmetical semigroup (G, ∂) fulfills the assumptions of Theorem 2.11. A real-valued additive function $\tilde{g}$ on G has a limit distribution function $\mathcal{G}(x)$ if and only if the three series

$$\sum_{|\tilde{g}(p)| \geq 1} q^{-\partial(p)}, \qquad \sum_{|\tilde{g}(p)|<1} \tilde{g}(p) q^{-\partial(p)}, \qquad \sum_{|\tilde{g}(p)|<1} \tilde{g}^2(p) q^{-\partial(p)} \tag{3.6}$$

all converge. Moreover, the limit distribution function $\mathcal{G}(x)$ has the characteristic function

$$\phi(t) = \prod_p \left(1 - q^{-\partial(p)}\right) \left(1 + \sum_{k=1}^{\infty} q^{-k\partial(p)} e^{it\tilde{g}(p^k)}\right), \tag{3.7}$$

where the infinite product is taken over all $p \in P$ in ascending order of $\partial(p)$.

Proof. Assume that real-valued additive function $\tilde{g}$ on G has a limit distribution function $\mathcal{G}(x)$. Assertions (i) and (ii) of Theorem 3.1 yields that the first and the third series of (3.6)

$$\sum_{|\tilde{g}(p)| \geq 1} q^{-\partial(p)}, \qquad \sum_{|\tilde{g}(p)|<1} \tilde{g}(p) q^{-\partial(p)}$$

converge. By our assumption $\tilde{g}$ has a limit distribution function $\mathcal{G}(x)$, on the other hand, (iii) yields that $\frac{1}{G(n)}\#\{a \in G, \partial(a) = n : \tilde{a} - \alpha(n) \leq x\}$ tends to a weak limit, too. Then:

$$\alpha(n) = cn + \sum_{\partial(p)\leq n, |\tilde{g}(p)|\leq 1} \frac{\tilde{g}(p)}{q^{\partial(p)}}$$

converges and, since

$$\sum_{\partial(p)\leq n, |\tilde{g}(p)|\leq 1} \frac{\tilde{g}(p)}{q^{\partial(p)}} = O(\log n)$$

c must be zero, which implies the convergence of

$$\sum_{p, |\tilde{g}(p)|<1} \tilde{g}(p) q^{-\partial(p)}.$$

Put

$$\alpha(n) = \sum_{\partial(p)\leq n, |\tilde{g}(p)|\leq 1} \frac{\tilde{g}(p)}{q^{\partial(p)}}.$$

Then (iii) of Theorem 3.1 yields, that the frequencies

$$\frac{1}{G(n)}\#\{a \in G, \partial(a) = n : \tilde{g}(a) - \alpha(n) \leq x\}$$

has a limit distribution $\mathcal{D}(x)$ as $n \to \infty$; and $\lim_{n\to\infty} \alpha(n) = \alpha$ exists, then $\tilde{g}$ also has a limit distribution $\mathcal{D}(x - \alpha)$.
This ends the proof of Theorem 3.2 □

Bibliography

[1] Arratia R., Barbour A., Tavare S. Logarithmic combinatorial structures: a Probabilistic Approach, EMS Monographs in Mathemitics, (2003), Zürich

[2] Barát, A., Indlekofer, K.-H. On mean-value theorems for multiplicative functions in additive arithmetical semigroups, Annales Univ. Sci. Budapest, Sect. Comp. **33**, (2010), 49-72.

[3] Barát, A., Indlekofer, K.-H., Kaya, E. Two-Series Theorem in Additive Arithmetical Semigroups. Preprint.

[4] Barát, A., Indlekofer, K.-H., Wagner, R. On some compactifications of $\mathbb{N}$, Analytic and Probabilistic Methods in Number Theory, TEV, Vilnius, (2007), 6-16.

[5] Beurling, A. Analyse de la loi asymptotique de la distribution de nombres premiers généralisés. I. Acta Mathematica, **68**, (1974), 255-291. Mémoire dédié á M. Holmgren.

[6] Burris, S. N. Number theoretic density and logical limit laws, Vol. **86**, Mathematical surveys and monographs. American Mathematical Society, (2001).

[7] Daboussi, H. Sur le Theoreme des Nombres Premiers, C. R. Acad. Sci., Paris Ser. I, **298**, (1984), 161-164.

[8] Daboussi, H. and Indlekofer, K.-H. Two elementary proofs of Halász's theorem, Math. Z., **209**, (1992), 43-52.

[9] Elliott, P.D.T.A. Probabilistic Number Theory I, Mean-Value Theorems. Springer-Verlag, New York (1979).

[10] Flajolet, P., Odlyzko, A. Singularity analysis of generating functions. SIAM J. Discrete Math. **3(2)**, (1990), 216-240.

[11] Flajolet, P., Sedgewick R. Analytic Combinatorics. Cambridge University Press. (2009).

[12] Flajolet, P., Soria, M. Gaussian limiting distributions for the number of components in combinatorial structures. J. Combin. Theory Ser. A **53**, (1990), 165-182.

[13] Flajolet, P., Soria, M. General combinatorial schemas: Gaussian limiting distributions and exponential tails. Discrete Math. **114**, (1993), 159-180.

[14] Fogels, E. On the distribution of analogues of primes. Dokl. Akad. Nauk SSSR, **146**, (1962), 318-321.

[15] Fogels, E. On the abstract theory of primes. I-III, Acta. Arith., **10**, (1964), 137-182.

[16] Hardy, G. H. Divergent series. Oxford University Press, (1949).

[17] Indlekofer, K.-H. A Mean-Value Theorem for Multiplicative Functions. Mathematische Zeitschrift **172**, (1980), 255-271.

[18] Indlekofer, K.-H. On multiplicative arithmetical functions. Topics in classical number theory, Vol. I,II (Budapest 1981), 731-748, Colloq. Math. Soc. Janos Bolyai, **34** North-Holland, Amsterdam.

[19] Indlekofer, K.-H. Properties of uniformly summable multiplicative functions. Period. Math. Hungar. **17(2)**, (1986), 143-161.

[20] Indlekofer, K.-H. The abstract prime number theorem for function fields, Acta. Math. Hungar., **62**, (1993), 137-148.

[21] Indlekofer, K.-H. Some remarks on additive arithmetical semigroups. Lietuvos Matematikos Rinkenys, **42**(2), (2002), 185-204.

[22] Indlekofer, K.-H. Some remarks on additive arithmetical semigroups II. Šiauliai Math. Seminar **4(12)**, (2009), 83-104.

[23] Indlekofer, K.-H. Tauberian theorems with applications to arithmetical semigroups and probabilistic combinatorics. Annales Univ. Sci. Budapest., Sect. Comp. **34**, (2011), 135-177.

[24] Indlekofer. K.-H. and Manstavičius, E. Additive and multiplicative functions on additive and arithmetical semigroups, Publicationes Math. Debrecen, **45**, (1994), 1-17.

[25] Indlekofer. K.-H. and Manstavičius, E. New approach to multiplicative functions on arithmetical semigroups, Lithuanian Math. Journal, **34 (4)**, (1994), 356-363.

[26] Indlekofer, K.-H., Manstavičius, E. and Warlimont, R. On a certain class of infinite products with an application to aritmetical semigroups, Arch. Math., **56**, (1991), 446-453.

[27] Indlekofer, K.-H., Wehmeier, S. Mean behaviour of multiplicative function on additive arithmetical semigroups, Comp. and Math. with Appl., **52**, (2006), 577-592.

[28] Knofmacher, A. and Knopfmacher, J. Arithmetical semigroups related to trees and polyhedra Journal of Combinatorial Theory, Series A, **86**, (1999), 85-102.

[29] Knopfmacher, J. Arithmetical properties of finite rings and algebras, and analytic number theory, V: Categories and relative analytic number theory. J. Reine Angew. Math., **271**, (1974), 95-121.

[30] Knopfmacher, J. Abstract Analytic Number Theory. Noth-Holland Publ. Co., Amsterdam, 1975; Second Edition, Dover Publ., New York, (1996).

[31] Knofmacher, J. Analytic arithmetic of algebraic funcition fields. Lecture Notes in Pure and Applied Mathematics, Vol. **50**, Marcel Dekker, New York-Basel, (1979).

[32] Knopfmacher, J., Zhang, W.-B. Number theory arising from finite fields, Analytic and probabilistic theory, **241** Pure and Appl. Math., Marcel Decker, New York, (2001).

[33] Manstavičius, E. An analytic method in probabilistic combinatorics. Osaka J. Math. **46** (2009), 273-290.

[34] Panario, D., Richmond, B. Smallest Components in Decomposable Structures: Exp-Log Class. Algorithmica **29** (2001), 205-226.

[35] Pólya, G., Szegö, G. Problems and Theorems in Analysis. Die Grundlehren der math. Wissenschaften in Einzeldarstellungen, Band **193**, Berlin, (1972).

[36] Rudin, W. Real and Complex Analysis. McGraw-Hill, (1987).

[37] Schur, I. Problem: Arch. Math. Phys. Ser. 3, Vol. **27** (1918), p. 162 . Cf. Szász, O. Sber. Berlin Math. ges. Vol. **21** (1922), 25-29.

[38] Warlimont, R. Arithmetic semigroups, II: Sieving by large and small prime elements. Sets of multiples. Manuscr. Math. **71** (1991), 197-221.

[39] Wehmeier, S. Arithmetical semigroups. PhD Thesis, Paderborn, (2005).

[40] Zhang, W.-B. Mean-value theorems of multiplicative functions on additive arithmetic semigroups. Math. Z. **229** (1998), 195-233.

[41] Zhang, W.-B. The prime element theorem in additive arithmetic semigroups, II. Illinois J. Math. **42** (1998), 198-229.

[42] Zhang, W.-B. Mean-value theorems of multiplicative functions on additive arithmetic semigroups via Halasz's method. Monatsh. Math. **138** (2003), 319-353.

[43] Zhang, W.-B. A Chebyshev type upper estimate for prime elements in additive aritmetic semigroups, Monatsh. Math., **129** (2000), 227-260.

[44] Zhang, W.-B. Mean-value theorems and extensions of the Elliott-Daboussi theorem on additive arithmetic semigroups. Ramanujan J. **15** (2008), 47-75.

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