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Foliated ρ-invariants

Doctoral thesis by

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Acknowledgements

It is with great pleasure that I take this opportunity to thank everyone who has given me the strength and support to complete this thesis. First and foremost I would like to express my deep gratitude towards my thesis advisors Prof. Moulay Benameur and Prof. Joachim Hilgert, without their inspiration and help, both professionally and personally, it is impossible for me to imagine completing this task. I consider myself extremely lucky to have found such inspiring teachers.

I also thank Prof. Paolo Piazza and Prof. Mathai Varghese for agreeing to be referees for the thesis, and the jury members, Prof. James Heitsch, Prof. Jean-Louis Tu and Prof. Bekka Bachir. I drew great strength and inspiration from discussions with Prof. Paolo Piazza, Prof. Jean-Louis Tu, Prof. James Heitsch, Prof. Nigel Higson, Prof. John Roe, Prof. Alexander Gorokhovsky, Prof. Thomas Schick, Prof. Hervé Oyono-Oyono, Prof. S. Rajeev, Dr. Alexander Alldridge and Dr. Troels Johansen. I sincerely thank Prof. Tilmann Wurzbacher and Prof. Angela Pasquale for helping me in various administrative and personal reasons. I would also like to thank my teachers in IIT Kharagpur, especially Prof. Pratima Panigrahi, Prof. S. Banerjee, Prof. A. R. Roy and Prof. S.S. Alam. I find it humbling to have had the chance to learn and gain inspiration from Prof. K. B. Sinha, Prof. V. S. Sundar, and Prof. Peter Fillmore. It was Prof. Fillmore who inspired me to take up research in Mathematics. I also cherish the many friends I made during conferences and workshops, especially Paolo Antonini, Paolo Carillo, Alessandro Fermi, Pierre Clare, Haija Mostafa, Uuye Otogonbayar and Maria Paula Gomez.

Among my colleagues from Paderborn, I can't thank enough Nikolay Dichev, Carsten Balleier, Anke Pohl, Jan Möllers and Stefan Wolf and Sameh Keliny for making me feel like home in the Department and for countless other ways in which they helped me, both professionally and personally. Among my friends in Paderborn, I must mention "Mahomies" Su, Martin, Cheng Yee, Erik, Dalimir, Melisa, Maria, Mariana, Fran, Timo and Luz-Marina for making it feel like a home away from home and for so many wonderful memories to cherish. No expression of thanks suffices for the wonderful people at PACE, especially Frau Canisius, who made it so much easier to deal with administrative problems in her magically efficient ways. I would also like to express my gratitude for Herr Das and Frau Hartmann who helped me in my neediest hour.

I take this opportunity to thank my colleagues from Metz, Mathieu, Sara, Mike, Fred, Elkaioum, Alexandre, Ivan, Wael, among others, who helped me innumerable times in administrative, personal, and professional matters. I especially thank Mathieu and Sara for helping me grasp many mathematical concepts with their sharp insights during our many enlightening discussions. I cannot express enough my gratitude for Fred and Elkaioum who helped me at crucial times when my poor French got in the way of things. Outside the professional circles, I express my deepest thanks to Jawad, Saba, Sundar, Praveena, Ananthi and Sajid for creating a homely environment in Metz and their generosity in lending a helping hand at the drop of a hat.

Last, but certainly not the least, it would be a futile attempt to express in words my gratitude to my family. This day would not have come without the love and support of my wife, Saumya. She has been a source of strength all throughout, especially when the chips were down. The influence of my parents and my elder brother to achieve my dreams can never be fathomed. Their constant encouragement and support saw me through many a difficult time. It is my pleasure to dedicate this thesis to all of them.

Résumé de la thèse en français

Dans cette thèse, nous avons introduit, pour un feuilletage mesuré donné, les " rho-invariants feuilleté " pour lesquels nous nous sommes attachés de prouver certaines de leurs propriétés de stabilité. En particulier, nous avons démontré que le rho-invariant associé à l'opérateur de signature est indépendant de la métrique considérée sur le feuilletage, ainsi que son invariance par rapport aux " difféomorphismes de feuilletage ", ce que généralise un résultat classique de Cheeger et Gromov.

Nous avons également obtenu une généralisation du théorme du Gamma-indice d'Atiyah pour les feuilletages. Ce résultat est déjà connu des experts, mais une preuve détaillée n'est pas disponible dans la littérature. De plus, nous avons étendu le formalisme des complexes de Hilbert-Poincaré (HP) aux cas des feuilletages, et avons construit une équivalence d'homotopie explicite pour les HP-complexes sur des feuilletages équivalents par homotopie feuilleté. Cela nous permet en particulier de donner une preuve directe d'un résultat déjà connu sur l'invariance par homotopie de la classe " signature d'indice " pour les feuilletages. Enfin, nous indiquons, comme application de ce formalisme, comment prolonger partiellement la preuve de l'invariance par homotopie sur les rho-invariants classiques de Cheeger et Gromov.

Zusammenfassung der Dissertation auf Deutsch

Wir führen in dieser Dissertation die foliated rho-Invarianten auf measured Blätterungen ein und beweisen einige Stabilittseigenschaften. Wir beweisen insbesondere, dass die "foliated rho-Invariante" metrisch unabhängig und invariant unter Diffeomorphismen ist. Dies ist eine Erweiterung eines klassischen Resultats von Cheeger und Gromov. Wir erreichen so eine Verallgemeinerung des Gamma-Index Theorems von Atiyah für Foliations, die Experten bekannt, aber nicht in der Literatur zu finden war. Wir erweitern den Hilbert-Poincar (HP) Komplex Formalismus fr den Fall von Blätterungen und konstruieren eine explizite Homotopieäquivalenz von HP-Komplexen auf leafwise Homotopie äquivalenten Blätterungen. Das liefert einen direkten Beweis des bereits bekannten Resultats über die Homotopieinvarianz der Signaturindexklasse für Blätterungen. Wir geben zuletzt eine Anwendung dieses Formalismus, um den Beweis der Homotopieinvarianz der klassischen Cheeger-Gromov rho-Invarianten teilweise auf den foliated Fall zu erweitern.

Contents

Chapter 1

Introduction

For a closed even-dimensional manifold M, the index theorem of Atiyah and Singer for Dirac operators on compact manifolds [AS:68] gave a deep connection between analysis, topology and geometry. The statement of the Atiyah-Singer index theorem equates the analytical index $\mathrm{Ind}_{an}(D)$ of a Dirac operator D with the topological index $\mathrm{Ind}_{top}(D)$. The Atiyah-Singer theorem spurred mathematicians of the era to further explore and understand the subject of index theory. One of the extensions of their results to study the index theory of compact manifolds with boundary was the Atiyah-Patodi-Singer index theorem. It states that under suitable global boundary conditions, for an even-dimensional compact manifold X with boundary ∂X, there is an extra term in the formula for the index:

$$\mathrm{Ind}_{an}(D) = \int_X AS + \frac{\eta(D_0) + h}{2}$$

where D is a Dirac-type operator on X and D_0 is the induced operator on ∂X, AS is the Atiyah-Singer characteristic class associated to the curvature form and the symbol of D, h is the dimension of $Ker(D_0)$, and $\eta(D_0)$ is the η-invariant of D_0, which is given by the value at zero of the so-called eta function given by:

$$\eta(s) = \sum_{\lambda \neq 0} sign(\lambda)|\lambda|^{-s}, \quad Re(s) >> 0$$

with λ varying over the spectrum of D_0. In [APS3:79] Atiyah, Patodi and Singer proved that $\eta(s)$ has a meromorphic continuation to the entire complex plane and it has a regular value at $s = 0$. Therefore the η-invariant turned out to be a well-defined spectral invariant on odd-dimensional manifolds. Using the spectral theorem, one can express the η-invariant of D_0 as

$$\eta(D_0) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} Tr(D_0 e^{-tD_0^2}) dt$$

where Tr denotes the trace of the operator. This quantity measures the 'spectral asymmetry' of the Dirac operator, in the sense that if the spectrum is symmetric about the origin then the η-invariant vanishes. The η-invariant then turned out to be an interesting secondary-invariant in its own right. In contrast to the index, it displays nonlocal behaviour; in particular it cannot be expressed as an integral. Moreover, it is unstable under perturbations of the operator. Atiyah, Patodi and Singer in their seminal paper [APS2:78] defined a quantity that was much more stable: they called it the "Relative eta-invariant", nowadays also called the APS ρ-invariant. This is defined as follows. Let M be a closed odd-dimensional Riemannian manifold with fundamental group Γ, and let α and β be two finite-dimensional representations of Γ of the same dimension.

Let D be a Dirac-type operator on M. Then one can define the twisted Dirac operators D_α and D_β obtained by twisting the operator D by the canonical flat bundles corresponding to the representations α and β, respectively. Then the APS ρ-invariant is defined as

$$\rho_{[\alpha-\beta]}(D) = \eta(D_\alpha) - \eta(D_\beta)$$

It has the following remarkable properties:

1. Let $D = D^{sign}$ be the signature operator on M. Then $\rho_{[\alpha-\beta]}(D^{sign})$ does not depend on the metric that is used to define D^{sign}. [APS2:78]

2. If M is spin and $D = \not{D}$ is the spin Dirac operator then $\rho_{[\alpha-\beta]}(\not{D})$ is constant on the path connected components of the space of metrics of positive scalar curvature on M. [APS2:78]

3. (a) If Γ is torsion-free and the maximal Baum-Connes assembly map

$$\mu_{max} : K_*(B\Gamma) \to K_*(C^*\Gamma)$$

is an isomorphism[1], then $\rho_{[\alpha-\beta]}(D^{sign})$ only depends on the oriented Γ-homotopy type of $(M, f : M \to B\Gamma)$. (see [Ne:79], [Ma:92], [We:88], [KeI:00])

(b) under the same assumptions as in 3(a), for the spin Dirac operator $\not{D}$ on a spin manifold with a metric of positive scalar curvature, $\rho_{[\alpha-\beta]}(\not{D})$ vanishes. [APS2:78]

The APS ρ-invariant was generalized by Cheeger and Gromov in [ChGr:85] to the case of coverings $\tilde{M} \xrightarrow{\Gamma} M$, using the L^2-trace defined by Atiyah in [At:76] to prove the famous L^2-index theorem. An equivalent theorem was proved by Singer [Si:77]. Atiyah used the von Neumann algebra of Γ-invariant operators on $L^2(\tilde{M})$ which carries a semifinite faithful normal trace Tr_Γ, to define the index of the lifted Dirac operator on $L^2(\tilde{M})$. Using this trace instead of the usual one in the integral formula for the η-invariant one obtains the L^2-η-invariant for the lifted operator $\tilde{D}$:

$$\eta_{(2)}(\tilde{D}) = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} Tr_\Gamma(\tilde{D} e^{-t\tilde{D}^2}) dt$$

Notice that the spectrum of $\tilde{D}$ is not discrete in general and hence the convergence of the integral at zero is a non-trivial matter, which follows from a deep estimate of Bismut and Freed (see [BiFr:86]). Then the Cheeger-Gromov ρ-invariant is defined as

$$\rho_{(2)}(D) := \eta_{(2)}(\tilde{D}) - \eta(D)$$

The Cheeger-Gromov ρ-invariant again turned out to be a nice invariant with many stability properties. Let us list a few of them. Let D^{sign} denote the signature operator on M and $\tilde{D}^{sign}$ its lift to $\tilde{M}$. We have

1. $\rho_{(2)}(D^{sign})$ does not depend on the metric on M used to define D^{sign} [ChGr:85].

2. If Γ is torsion-free and satisfies the maximal Baum-Connes conjecture, then $\rho_{(2)}(D^{sign})$ depends only on the oriented homotopy type of M ([Ke:00], [PiSch1:07]).

3. Let M be a compact oriented Riemannian manifold of dimension $4k+3, k > 0$. If $\pi_1(M)$ has torsion, then there are infinitely many manifolds that are homotopy equivalent to M but not diffeomorphic to it: they are distinguished by $\rho_{(2)}(D^{sign})$. (see [ChWe:03])

4. For a spin manifold M and $D = \not{D}$ the spin Dirac operator we have the following properties:

(a) $\rho(\not{D})$ is constant on the path connected components of the space of metrics of positive scalar curvature $\mathcal{R}^+(M)$ on M. [PiSch2:07]

[1] here $B\Gamma$ is the classyfying space for Γ and $C^*\Gamma$ is the maximal C^*-algebra of Γ, the left side of the arrow is K-homology for spaces while the right side is the K-theory for C^*-algebras

(b) If M is of dimension $4k+3$, $\mathcal{R}^+(M)$ is nonempty and Γ has torsion, then M has infinitely many different Γ-bordism classes of metrics with positive scalar curvature, the different classes are distinguished by $\rho(\not{D})$.[PiSch2:07]

(c) If Γ is a torsion-free group satisfying the maximal Baum-Connes conjecture then for a spin manifold with metric of positive scalar curvature g, $\rho(\not{D}_g)=0$.[PiSch1:07]

In the 1970's Alain Connes founded the subject of noncommutative geometry and one of its successful applications was in the theory of foliations. In [Co:79], he generalized the Atiyah-Singer index theorem to Dirac operators acting tangentially on the leaves of a measured foliation, proving the so called measured index theorem. Ramachandran [Ra:93] extended this work by proving an index theorem on foliated manifolds with boundary and defined the foliated η-invariant associated with the leafwise Dirac operator on the foliation. He showed that, like in the classical case, the foliated η-invariant appears as the error term in the formula for the index given by Connes. Independently, Peric [Pe:92] has defined the η-invariant of the Dirac operator lifted to the holonomy groupoid. As we shall see, the techniques of Peric and Ramachandran extend immediately to define the foliated η-invariant on the monodromy groupoid $\mathcal{G}$ of a foliation. Thus, we have a definition of the foliated ρ-invariant as the difference of two η-invariants on the monodromy groupoid of the foliation and on the leaves of the foliation.

In this thesis we have studied the η-invariant on the leaves of an oriented foliation[2] endowed with a holonomy-invariant transverse measure Λ, and on its monodromy groupoid $\mathcal{G}$. The ρ-invariant is then defined as the difference of these two quantities and we study in this thesis its stability properties extending the ones known for the classical cases, i.e. for the Atiyah-Patodi-Singer invariant and for the Cheeger-Gromov ρ-invariant. We now describe more precisely the results obtained in this thesis.

1.1 Part I: Foliated Atiyah's theorem and the Baum-Connes map

Using the pseudodifferential calculi developed by A. Connes [Co:79], and extended by Nistor-Weistein-Xu [NWX:99], and Monthubert-Pierrot [MoPi:97] to almost smooth groupoids, we have given a proof for a generalization of Atiyah's L^2-index theorem to foliations. More precisely, we prove the equality of different functionals, induced by traces, on the image of the Baum-Connes map. This latter equality is crucial in the proof of the homotopy invariance of the ρ-invariant. Although this theorem is known to experts, we couldn't find any published proof in the literature.

By using semifinite, faithful normal traces, τ^Λ and $\tau^\Lambda_{\mathcal{F}}$, associated to the invariant measure Λ on the corresponding groupoid von Neumann algebras with coefficients in a longitudinally smooth continuous vector bundle E, denoted by $W^*(\mathcal{G},E)$ and $W^*(M,\mathcal{F},E)$ [Co:79], one defines the measured indices $\mathrm{Ind}_\Lambda(D)$ and $\mathrm{Ind}_\Lambda(\tilde{D})$ of the Dirac operator acting on the leaves and of its lift on the monodromy groupoid, respectively. These are defined as follows

$$\mathrm{Ind}_\Lambda(\tilde{D}^+)=\tau^\Lambda(\tilde{\pi}^+)-\tau^\Lambda(\tilde{\pi}^-) \text{ and } \mathrm{Ind}_\Lambda(D^+)=\tau^\Lambda_{\mathcal{F}}(\pi^+)-\tau^\Lambda_{\mathcal{F}}(\pi^-)$$

where $D=\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ and $\tilde{D}=\begin{pmatrix} 0 & \tilde{D}^- \\ \tilde{D}^+ & 0 \end{pmatrix}$ with respect to the $\mathbb{Z}_2$-grading, and where $\pi^\pm$ are the projections onto the L^2-kernels of $D^\pm$, and similarly for $\tilde{D}$. The 'Foliated Atiyah's theorem' then states the equality of these two indices.

Theorem 1.1.1 (Foliated Atiyah's theorem)**.** $\mathrm{Ind}_\Lambda(D)=\mathrm{Ind}_\Lambda(\tilde{D})$

In the process, we also reprove the analogue of Calderon's formula in our geometric setting. To see the relation between this theorem and the Baum-Connes map for foliations, recall the now classical maximal Hilbert C^*-module $\mathcal{E}_m$ associated to a foliation $(M,\mathcal{F})$ (cf. [HiSk:84], [CoSk:84],[Co:94],[BePi:08]). The lifted Dirac

[2] in the sequel all foliated manifolds are assumed to be oriented

operator $\tilde{D}$ on the monodromy groupoid induces a self-adjoint, unbounded, regular Fredholm operator $\mathcal{D}_m$ on $\mathcal{E}_m$ whose K-theoretic index class $\mathrm{Ind}(\mathcal{D}_m)$ lies in the K-theory of the C^*-algebra of compact operators on the Hilbert C^*-module $\mathcal{K}(\mathcal{E}_m)$. By the Hilsum-Skandalis stability theorem, this index class corresponds to a class $\mathrm{ind}(\mathcal{D}_m)$ in the K-theory of the C^*-algebra of the monodromy groupoid $C^*(\mathcal{G})$. Then we have,

Proposition 1.1.2. *We have the following equalities:*

$$\tau_*^\Lambda \circ \pi_*^{reg}(\mathrm{ind}(\mathcal{D}_m)) = \mathrm{Ind}^\Lambda(\tilde{D}) \text{ and } \tau_{\mathcal{F},*}^\Lambda \circ \pi_*^{av}(\mathrm{ind}(\mathcal{D}_m)) = \mathrm{Ind}^\Lambda(D)$$

where π^{reg} and π^{av} are the regular and average representations of $C^*(\mathcal{G})$ in the two groupoid von Neumann algebras (see section 2.3 for the definitions).

Therefore, when the groupoid is torsion-free, we can reformulate our theorem as follows.

Theorem 1.1.3. *The functionals induced by the regular and average representations coincide on the image of the maximal Baum-Connes map*

$$\mu_{max} : K_0(B\mathcal{G}) \to K_0(C^*(\mathcal{G})).$$

1.2 Part II: Stability properties of the foliated ρ-invariant

Using the traces on the von Neumann algebras $W^*(\mathcal{G}, E)$ and $W^*(M, \mathcal{F}, E)$, the foliated η-invariants are defined as:

$$\eta_{\mathcal{F}}^\Lambda(D) = \frac{2}{\sqrt{\pi}} \int_0^\infty \tau_{\mathcal{F}}^\Lambda(D \exp(-t^2 D^2))dt \text{ and } \eta^\Lambda(\tilde{D}) = \frac{2}{\sqrt{\pi}} \int_0^\infty \tau^\Lambda(\tilde{D} \exp(-t^2 \tilde{D}^2))dt$$

That the integrals are well-defined follows from the following proposition which is a consequence of a foliated Bismut-Freed estimate.

Proposition 1.2.1. *The functions $t \longmapsto \tau_{\mathcal{F}}^\Lambda(D \exp(-t^2 D^2))$ and $t \longmapsto \tau^\Lambda(\tilde{D} \exp(-t^2 \tilde{D}^2))$ are Lebesgue integrable on $(0, \infty)$.*

Therefore the foliated η-invariants are well-defined.

Definition The foliated ρ-invariant associated to the longitudinal Dirac operator D on the foliated manifold $(M, \mathcal{F})$ is defined as

$$\rho^\Lambda(D; M, \mathcal{F}) = \eta^\Lambda(\tilde{D}) - \eta_{\mathcal{F}}^\Lambda(D) \tag{1.2.1}$$

Notice that we use the monodromy groupoid and hence $\rho^\Lambda(D; M, \mathcal{F})$ will not be trivial in general. For a foliation of maximal dimension, i.e. with one closed leaf M, the foliated ρ-invariant coincides with the Cheeger-Gromov ρ-invariant [ChGr:85]. For a fibration of closed manifolds $M \to B$ with typical fiber F, the foliated ρ-invariant with respect to a given measure Λ on the base B, is simply the integral over B of the ρ-function [Az:07]. Lastly, for foliations given by suspensions, the foliated ρ-invariant coincides with the one introduced and studied in [BePi:08].

Extending to foliations the proof given for the Cheeger-Gromov invariant in [ChGr:85], we have been able to prove that the ρ-invariant $\rho^\Lambda(M, \mathcal{F}, g)$ associated to the leafwise signature operator is independent of the leafwise metric g used to define it. So we have,

Theorem 1.2.2. *$\rho^\Lambda(M, \mathcal{F}, g) = \rho^\Lambda(M, \mathcal{F})$ does not depend on the leafwise metric g.*

As a corollary, we also establish the following generalization of a classical Cheeger-Gromov theorem.

Theorem 1.2.3 (Diffeomorphism invariance). *Let $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a leafwise diffeomorphism of foliated manifolds. Let $f_*\Lambda$ denote the holonomy-invariant transverse measure induced on $(M', \mathcal{F}')$ (see subsection 2.2.2). Then we have,*

$$\rho^{\Lambda}(M, \mathcal{F}) = \rho^{f_*\Lambda}(M', \mathcal{F}')$$

1.3 Part III: Hilbert-Poincaré complexes for foliations

In their work on mapping surgery to analysis, Nigel Higson and John Roe have given an appropriate framework of Hilbert-Poincaré (abbreviated HP) complexes (cf. [HiRoI:05],[HiRoII:05],[HiRoIII:05]). They define HP complexes as complexes of finitely-generated projective Hilbert C^*-modules on a C^*-algebra A with adjointable differentials, and an additional structure of adjointable Poincaré duality operators that induce isomorphism on cohomology from the original complex to its dual complex. Associated to an HP-complex there is a canonically defined class in $K_1(A)$, called the signature of the HP-complex. It is shown in [HiRoI:05] that a homotopy equivalence of such complexes leaves the signature class invariant. Moreover, an explicit path connecting the representatives of the two signature classes in $K_1(A)$ is constructed.

In this section our goal is to construct such an explicit homotopy equivalence in the case of HP-complexes associated to leafwise homotopy equivalent foliations. Although the homotopy invariance of the signature index class for foliations is well-known [KaMi:85], [HiSk:87], an explicit path connecting the two signatures has not yet appeared in the literature. Such a path will be crucial in the construction of the 'Large Time Path' in Chapter 6, which is an important step in the proof of the foliated homotopy invariance of $\rho^{\Lambda}(M, \mathcal{F})$, see [Ke:00; KeI:00; BePi:08].

Notice that even in the K-theory proof of the homotopy invariance of the Cheeger-Gromov invariant [Ke:00], it is important to extend the Higson-Roe formalism to deal with countably generated Hilbert modules and regular operators. Moreover, in the case of foliations, we needed to extend it further to cover homotopy equivalences of HP-complexes on Morita-equivalent C^*-algebras.

For a foliated manifold $(M, \mathcal{F}^{(p)})$ with a complete transversal X we associate the HP-complex

$$\mathcal{E}_X^0 \xrightarrow{d_X} \mathcal{E}_X^1 \xrightarrow{d_X} \cdots \xrightarrow{d_X} \mathcal{E}_X^p$$

where $\mathcal{E}_X^k$ is the completion of $C_c^{\infty}(\mathcal{G}_X, r^*(\bigwedge^k T^*\mathcal{F}))$ with respect to a $C^*(\mathcal{G}_X^X)$-valued inner product (see section 3.3.1). The Poincaré duality operator, denoted T_X, is induced on $\mathcal{E}_X^k$ by the lift of the Hodge $*$-operator on $\mathcal{G}_X$, and d_X is the regular operator induced by the lift of the de Rham differential to $\mathcal{G}_X$. So the HP-complex, denoted $(\mathcal{E}_X, d_X, T_X)$, consists of Hilbert modules on the maximal C^*-algebra $C^*(\mathcal{G}_X^X)$. Now consider a leafwise homotopy equivalence between two foliated manifolds $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$. Let X (resp. X') be a complete transversal on $(M, \mathcal{F})$ (resp. $(M', \mathcal{F}')$). Denote the maximal C^*-algebra $C^*(\mathcal{G}_X^X)$ (resp. $C^*(\mathcal{G}'^{X'}_{X'})$) as $\mathcal{A}_X^X$ (resp. $\mathcal{A}_{X'}^{X'}$). Then we can give a stepwise description of the results of this section.

Step I: Tensor product with Morita modules

Since the C^*-algebras $\mathcal{A}_X^X$ and $\mathcal{A}_{X'}^{X'}$ are rarely isomorphic we cannot use directly the Higson-Roe formalism, as their definition only considers HP-complexes on the same C^*-algebra. To overcome this problem we exploit the fact that since the foliations are leafwise homotopy equivalent the C^*-algebras $\mathcal{A}_X^X$ and $\mathcal{A}_{X'}^{X'}$ are nevertheless Morita-equivalent, and so there exists an explicit Morita bimodule which implements the Morita equivalence. In the first two subsections of Chapter 5 we extend some constructions from [CoSk:84] and [HiSk:84] and define a Hilbert C^*-module $\mathcal{E}_{X'}^X(f)$ which is an $\mathcal{A}_X^X$-$\mathcal{A}_{X'}^{X'}$ imprimitivity bimodule. Therefore the interior tensor product of the Hilbert C^*-modules $\mathcal{E}_X^k$ with $\mathcal{E}_{X'}^X(f)$ allows us to form a HP-complex $(\mathcal{E}_X \otimes \mathcal{E}_{X'}^X(f), d_X \otimes I, T_X \otimes I)$ on $\mathcal{A}_{X'}^{X'}$:

$$\mathcal{E}_X^0 \otimes \mathcal{E}_{X'}^X(f) \xrightarrow{d_X \otimes I} \mathcal{E}_X^1 \otimes \mathcal{E}_{X'}^X(f) \xrightarrow{d_X \otimes I} \cdots \xrightarrow{d_X \otimes I} \mathcal{E}_X^p \otimes \mathcal{E}_{X'}^X(f)$$

Consequently, we can now define out of a leafwise homotopy equivalence, a homotopy equivalence between the complex $(\mathcal{E}_X \otimes \mathcal{E}^X_{X'}(f), d_X \otimes I, T_X \otimes I)$ and the complex $(\mathcal{E}'_{X'}, d'_{X'}, T_{X'})$ associated to $(M', \mathcal{F}', X')$, since they are now on the same C^*-algebra. We note that $\mathcal{E}^k_X \otimes \mathcal{E}^X_{X'}(f)$ is isomorphic to a certain Hilbert $\mathcal{A}^{X'}_{X'}$-module $\mathcal{E}^{V,k}_{X'}(f)$ which implements the Morita equivalence between the C^*-algebras $C^*(\mathcal{G}, \bigwedge^k T^*\mathcal{F})$ and $\mathcal{A}^{X'}_{X'}$ and which is more convenient to work with. We denote by $(\mathcal{E}^V_{X'}(f), d_f, T_f)$ the complex

$$\mathcal{E}^{V,0}_{X'}(f) \xrightarrow{d_f} \mathcal{E}^{V,1}_{X'}(f) ... \xrightarrow{d_f} \mathcal{E}^{V,p}_{X'}(f)$$

where T_f correspond to $T_X \otimes I$ and d_f correspond to $d_X \otimes I$, under the isomorphism between $\mathcal{E}^k_X \otimes \mathcal{E}^X_{X'}(f)$ and $\mathcal{E}^{V,k}_{X'}(f)$.

Let the signatures of the complexes $(\mathcal{E}_X, d_X, T_X)$ and $(\mathcal{E}^V_{X'}(f), d_f, T_f)$ in $K_1(\mathcal{A}^X_X)$ and $K_1(\mathcal{A}^{X'}_{X'})$ be denoted as $\sigma(\mathcal{E}_X, d_X, T_X)$ and $\sigma(\mathcal{E}^V_{X'}(f), d_f, T_f)$, respectively. Then,

Proposition 1.3.1. *Let $\mathcal{M} : K_1(\mathcal{A}^X_X) \xrightarrow{\cong} K_1(\mathcal{A}^{X'}_{X'})$ be the isomorphism induced by the Morita equivalence between $\mathcal{A}^X_X$ and $\mathcal{A}^{X'}_{X'}$. Then we have*

$$\mathcal{M}(\sigma(\mathcal{E}_X, d_X, T_X)) = \sigma(\mathcal{E}^V_{X'}(f), d_f, T_f) \quad \text{in } K_1(\mathcal{A}^{X'}_{X'}).$$

Step II: Construction of the homotopy equivalence

We now proceed to explain how we construct an explicit homotopy equivalence between the complexes $(\mathcal{E}^V_{X'}(f), d_f, T_f)$ and $(\mathcal{E}'_{X'}, d'_{X'}, T_{X'})$. The leafwise homotopy equivalence f allows us to construct a chain map

$$\Xi_f : \mathcal{E}'^{\bullet}_{X'} \to \mathcal{E}^{\bullet}_X \otimes \mathcal{E}^X_{X'}(f),$$

which is our desired homotopy equivalence. We first use a Poincaré lemma adapted to this context and prove the following:

Proposition 1.3.2. *With the above notations, Ξ_f induces an isomorphism on unreduced cohomology between the complexes $(\mathcal{E}'_{X'}, d'_{X'}, T_{X'})$ and $(\mathcal{E}^V_{X'}(f), d_f, T_f)$.*

Moreover, if we construct in the same way chain maps Ξ_g and $\Xi_{f\circ g}$ for any leafwise homotopy inverse $g : M' \to M$ to f, then we prove that the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}'^{\bullet}_{X'} \arrow[r, "\Xi_f"] \arrow[d, "\Xi_{f\circ g}"] & \mathcal{E}^{\bullet}_X \otimes \mathcal{E}^X_{X'}(f) \arrow[d, "\Xi_g \otimes I"] \\
\mathcal{E}'^{\bullet}_{X'} \otimes \mathcal{E}^{X'}_{X'}(f\circ g) \arrow[r, "I\otimes\Omega"] & \mathcal{E}'^{\bullet}_{X'} \otimes \mathcal{E}^X_{X'}(f) \otimes \mathcal{E}^{X'}_X(g)
\end{tikzcd}$$

where $\Omega : \mathcal{E}^X_{X'}(f \circ g) \to \mathcal{E}^X_{X'}(f) \otimes \mathcal{E}^{X'}_X(g)$ is some explicit isomorphism. Now the main theorem of this section can be stated as:

Theorem 1.3.3. *As per the notations above, there is an explicit homotopy equivalence between the HP-complexes $(\mathcal{E}^V_{X'}(f), d_f, T_f)$ and $(\mathcal{E}'_{X'}, d'_{X'}, T'_{X'})$ which is associated to the leafwise homotopy equivalence f.*

As an immediate corollary we get the leafwise homotopy invariance of the index class of the leafwise signature operator on a foliation $(M, \mathcal{F})$ (cf. [HiSk:83], [KaMi:85]), but more importantly an explicit path connecting the signature representatives.

Corollary 1.3.4. *Let $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a leafwise homotopy equivalence of foliated manifolds. Then* $\mathrm{ind}(\mathcal{D}_{sign}) = \mathrm{ind}(\mathcal{D}'_{sign})$ *in $K_1(C^*(\mathcal{G}))$.*

1.4 Part IV: Application to homotopy invariance

Finally, as an important application of our results on HP-complexes on foliations, we would like to extend Keswani's proof [Ke:00] of the homotopy invariance for the foliated ρ-invariant when the monodromy groupoid is torsion-free. We recall here that the homotopy invariance of the ρ-invariant was first conjectured by Mathai. The first results in this direction for classical Atiyah-Patodi-Singer ρ-invariants were obtained by Neumann [Ne:79], who proved the homotopy invariance of the APS ρ-invariant when Γ is a free abelian group and Mathai in [Ma:92] who proved it for Bieberbach groups. When Γ is torsion-free and satisfies the Borel conjecture, the homotopy invariance of the ρ-invariant was proved by Weinberger [We:88]. The homotopy invariance of Cheeger-Gromov ρ-invariants have been studied by Chang[Ch:04], Chang and Weinberger [ChWe:03], Piazza and Schick [PiSch1:07], and Keswani [KeI:00], [Ke:00]. The results of Keswani and Piazza-Schick were improved by Chang [Ch:04] who used topological methods to prove the homotopy invariance of Cheeger-Gromov ρ-invariants under the condition that Γ is torsion-free and satisfies the rational Borel conjecture. For a recent reformulation of Keswani's results exploiting links with surgery theory, see Higson and Roe [HiRo:10]. The case of foliated bundles has been dealt with by Benameur and Piazza in [BePi:08], and the homotopy invariance is established for a special class of homotopy equivalences and conjectured for the general case. Their proof extends the techniques of Keswani for foliated bundles. For the sake of clarity, we recall the skeleton of the proof of the homotopy invariance of classical ρ-invariants given by Keswani in [Ke:00]. Let then $f : M \to M'$ be an oriented homotopy equivalence and assume that $\Gamma = \pi_1(M)$ is torsion-free and satisfies the maximal Baum-Connes conjecture. Let D and D' be the signature operators on M and M', respectively.

- Using functional calculus for the regular self-adjoint operator $\mathcal{D}_m$, which is induced by D on the maximal Mishchenko-Fomenko Hilbert module $\mathcal{E}_m$, Keswani constructed a path $V_\epsilon(\mathcal{D}_m) := (\psi_t(\mathcal{D}_m))_{\epsilon\leq t\leq 1/\epsilon}$ of unitaries acting on $\mathcal{E}_m$ such that

$$\lim_{\epsilon\to 0}((w_\Gamma \circ \pi_*^{reg} - w\circ \pi_*^{av})(V_\epsilon(\mathcal{D}_m))) = \frac{1}{2}\rho(D)$$

  where $\pi_*^{reg}(V_\epsilon(\mathcal{D}_m))$ and $\pi_*^{av}(V_\epsilon(\mathcal{D}_m))$ are the push-forward paths in the Atiyah von Neumann algebra $B(L^2(\tilde{M},\tilde{E}))^\Gamma$ and the algebra of bounded operators on the Hilbert space $L^2(M,E)$, respectively and w_Γ and w the Fuglede-Kadison determinants on $B(L^2(\tilde{M},\tilde{E}))^\Gamma$ and $B(L^2(M,E))$, respectively. Hence we get with obvious notations

$$\rho(D)-\rho(D') = 2\times \lim_{\epsilon\to 0}\left((w_\Gamma\circ\pi_*^{reg} - w\circ\pi_*^{av})\begin{pmatrix} V_\epsilon(\mathcal{D}_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}'_m)\end{pmatrix}\right)$$

- Using the Higson-Roe formalism of Hilbert-Poincaré complexes, Keswani constructed his 'Large Time Path' $LT_\epsilon = (LT_\epsilon(t))_{1/\epsilon\leq t\leq 2/\epsilon}$ composed of unitaries such that

$$LT_\epsilon(\frac{2}{\epsilon}) = Id_{\mathcal{E}_m\oplus\mathcal{E}'_m}, LT_\epsilon(\frac{1}{\epsilon}) = \begin{pmatrix} \psi_{1/\epsilon}(\mathcal{D}_m) & 0 \\ 0 & \psi_{1/\epsilon}(-\mathcal{D}'_m)\end{pmatrix} \text{ and } \lim_{\epsilon\to 0}(w_\Gamma\circ\pi_*^{reg} - w\circ\pi_*^{av})(LT_\epsilon) = 0.$$

- Using injectivity of the maximal Baum-Connes map, he then constructed his 'Small Time Path' $ST_\epsilon = (ST_\epsilon(t))_{\epsilon/2\leq t\leq\epsilon}$ of unitaries such that, up to stabilization,

$$ST_\epsilon(\frac{\epsilon}{2}) = Id_{\mathcal{E}_m\oplus\mathcal{E}'_m}, ST_\epsilon(\epsilon) = \begin{pmatrix} \psi_\epsilon(\mathcal{D}_m) & 0 \\ 0 & \psi_\epsilon(-\mathcal{D}'_m)\end{pmatrix} \text{ and } \lim_{\epsilon\to 0}(w_\Gamma\circ\pi_*^{reg} - w\circ\pi_*^{av})(ST_\epsilon) = 0.$$

Therefore Keswani ended up with a loop of unitaries which represents, through Morita equivalences, a class in $K_1(C^*_{max}\Gamma)$. Using Atiyah's theorem and surjectivity of the maximal Baum-Connes map, he was then able to deduce the theorem.

The goal of this section is to explain how our techniques can be applied to tackle, following the same lines, the leafwise homotopy invariance of the foliated ρ-invariant. We first interpret the foliated η as a generalized determinant (à la Fuglede-Kadison) of a path of operators on the maximal Connes-Skandalis Hilbert module. To do this, we prove the following

Proposition 1.4.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Then $\varphi(\mathcal{D}_m) \in \mathcal{K}(\mathcal{E}_m)$ and induces a trace class operator under both traces $\tau^\Lambda \circ \pi_{reg}$ and $\tau^\Lambda_{\mathcal{F}} \circ \pi_{av}$. Moreover, with simplified notations, compatibility of functional calculi allows to deduce the following identities:*

$$\tau^\Lambda \circ \pi^{reg}(\varphi(\mathcal{D}_m)) = \tau^\Lambda(\varphi(\tilde{D})) \text{ and } \tau^\Lambda_{\mathcal{F}} \circ \pi^{av}(\varphi(\mathcal{D}_m)) = \tau^\Lambda_{\mathcal{F}}(\varphi(D)).$$

Using this proposition, we deduced the first item, namely, again with simplified notations, that

$$\lim_{\epsilon \to 0}(w^\Lambda \circ \pi^{reg}_* - w^\Lambda_{\mathcal{F}} \circ \pi^{av}_*)(V_\epsilon(\mathcal{D}_m)) = \frac{1}{2}\rho^\Lambda(D),$$

where w^Λ and $w^\Lambda_{\mathcal{F}}$ are Fuglede-Kadison determinants in the regular von Neumann algebra and the foliation von Neumann algebra, respectively.

As per the second item of Keswani's proof, we apply our results on the HP-complexes for foliations and define the Large Time Path $LT_\epsilon = (LT_\epsilon(t))_{1/\epsilon \leq t \leq 2/\epsilon}$ which now acts on the direct sum of Hilbert modules $\mathcal{E}^{V,k}_{X'}(f) \oplus \mathcal{E}'^k$. In order to define and estimate the determinant of this Large Time Path, we were led to consider appropriate von Neumann algebras $W^*(f)$ and $W^*_{\mathcal{F}}(f)$ on which we have defined semi-finite normal positive traces $\tilde{\tau}_{\Lambda',f}$ and $\tilde{\tau}^{\mathcal{F}}_{\Lambda',f}$ and therefore determinants $\tilde{w}^{\Lambda',f}$ and $\tilde{w}^{\Lambda',f}_{\mathcal{F}}$, where Λ' is the holonomy-invariant measure on the foliation $(M', \mathcal{F}')$ which is the image under f of Λ. Using the Morita-equivalence induced by the leafwise homotopy equivalence, we replace the Hilbert module $\mathcal{E}_m$ by the Hilbert module $\mathcal{E}^V_{X'}(f)$ and consider the operator $\mathcal{D}_f$ on $\mathcal{E}^V_{X'}(f)$ corresponding to $\mathcal{D}_m$. We then prove that

$$(\tilde{w}^{\Lambda',f} \circ \pi^{f,reg}_*) \begin{pmatrix} V_\epsilon(\mathcal{D}_f) & 0 \\ 0 & V_\epsilon(-\mathcal{D}'_m) \end{pmatrix} = (w^\Lambda \circ \pi^{reg}_*)(V_\epsilon(\mathcal{D}_m)) - (w^{\Lambda'} \circ \pi^{reg}_*)(V_\epsilon(\mathcal{D}'_m))$$

where $\pi^{f,reg}_*$ is the push-forward to the von Neumann algebra $W^*(f)$. The same relation holds with the average representations and the von Neumann algebra $W^*_{\mathcal{F}}(f)$. Notice that we have the crucial relation

$$\rho_\Lambda(D) - \rho_{\Lambda'}(D') = 2 \times \lim_{\epsilon \to 0}(\tilde{w}^{\Lambda',f} \circ \pi^{f,reg}_* - \tilde{w}^{\Lambda',f}_{\mathcal{F}} \circ \pi^{f,av}_*) \begin{pmatrix} V_\epsilon(\mathcal{D}_f) & 0 \\ 0 & V_\epsilon(-\mathcal{D}'_m) \end{pmatrix}$$

where $\pi^{f,av}_*$ is the push-forward to the von Neumann algebra $W^*_{\mathcal{F}}(f)$. To end the proof of the second item, we then estimate

$$\lim_{\epsilon \to 0}(\tilde{w}^{\Lambda',f} \circ \pi^{f,reg}_* - \tilde{w}^{\Lambda',f}_{\mathcal{F}'} \circ \pi^{f,av}_*)(LT_\epsilon) = 0.$$

Finally, the last item is too long and tedious to be included in the present thesis and turned out to be of deep independent interest in its own. It will be treated in the work in progress [R].

Chapter 2

Background on foliations and Operator algebras

2.1 Foliated Charts and Foliated Atlases

We now give the formal definition of a foliation using foliated charts and foliated atlases. We refer the reader for instance to [CaCoI:99; MoSc:06; MkMr:03] for more details about the definitions and properties briefly reviewed in this section.

Definition Let M be a smooth compact manifold of dimension n without boundary. A *foliated chart* on M of codimension $q \leq n$ is a pair (U, ϕ) where $U \subseteq M$ is open and $\phi : U \xrightarrow{\cong} L \times T$ is a diffeomorphism, L and T being products of open intervals in $\mathbb{R}^{n-q}$ and $\mathbb{R}^q$, respectively.

The sets $P_y = \phi^{-1}(L \times \{y\})$ for $y \in T$ are called the *plaques* of the foliated chart (U, ϕ).

Definition A *smooth foliated atlas* of codimension q is an atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ of foliated charts such that the change of charts diffeomorphisms are locally of the form:

$$\phi_{\alpha\beta}(x, y) = \phi_\alpha \circ \phi_\beta^{-1}(x, y) = (g_{\alpha\beta}(x, y), h_{\alpha\beta}(y))$$

where $x \in \mathbb{R}^{n-q}, y \in \mathbb{R}^q$ and $g_{\alpha\beta}, h_{\alpha\beta}$ are smooth functions. We call such an atlas *coherently foliated*.

Definition Two foliated atlases $\mathcal{U}$ and $\mathcal{V}$ are called coherent if both $\mathcal{U}$ and $\mathcal{V}$ have the same codimension q, and $\mathcal{U} \cup \mathcal{V}$ is again a foliated atlas of codimension q. We denote this relation as $\mathcal{U} \approx \mathcal{V}$.

Lemma 2.1.1. *Coherence of foliated atlases is an equivalence relation.*

Proof. See [CaCoI:99, Lemma 1.2.9] for a proof.

□

Definition A foliated atlas is called *regular* if:

(i) for each $\alpha \in A$, $\overline{U_\alpha}$ is compact in a foliated chart (W_α, ψ_α) and $\psi_\alpha|_{U_\alpha} = \phi_\alpha$.

(ii) $\{U_\alpha\}_{\alpha \in A}$ is a locally finite cover.

(iii) The interior of each closed plaque $P \subset \overline{U_\alpha}$ meets at most one plaque in $\overline{U_\beta}$.

Lemma 2.1.2. *Every foliated atlas has a coherent refinement which is regular.*

Proof. We refer the reader to [CaCoI:99, Lemma 1.2.17] for a proof.

□

Definition A foliation $\mathcal{F}$ on M is a pair $(M,\mathcal{U})$ such that $\mathcal{U}$ is a maximal regular foliated atlas of M. The leaves of $\mathcal{F}$ are locally given by the plaques of a foliated chart (U, ϕ) of $\mathcal{U}$. We will denote by $(M, \mathcal{F})$ a foliation $\mathcal{F}$ on M.

Definition The set of points in $(M, \mathcal{F})$ such that for any two elements $x, y \in M$ belonging to this set there exists a sequence of foliated charts $U_1, U_2, \cdots, U_k$ and a sequence of points $x = p_1, p_2, ..., p_k = y$ such that p_{j-1} and p_j lie on the same plaque in U_j for $j = 2, \cdots, k$ is called a leaf of the foliation $(M, \mathcal{F})$.

Remark. *Each leaf is a topologically immersed submanifold of M.*

If $\mathcal{F}$ is a foliation of M, then the vector subspace of T_xM for $x \in M$ given by the vectors tangent to the leaves of the foliation forms a vector subbundle of TM, called the tangent bundle of the foliation, and is denoted $T\mathcal{F}$. Conversely, by the Frobenius theorem [CaCoI:99, Theorem 1.3.8], given a completely integrable subbundle E of TM, one can define a foliation M whose tangent bundle $T\mathcal{F}$ is exactly E.

2.1.1 Holonomy

Let $(M, \mathcal{F})$ be a foliated manifold. In informal terms, holonomy measures the magnitude of deviation of leaves close to each other, i.e. how they grow apart, wind around or come closer together as one "travels" along the leaves. The concept comes from the notion of the "first return map" given by Poincaré in his study of dynamical systems.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a regular foliated atlas of M. Consider a continuous leafwise path $\gamma : [0, 1] \to M$ from $\gamma(0) = x$ to $\gamma(1) = y$. Let $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ be a partition of $[0, 1]$ such that $\gamma([t_{i-1}, t_i])$ lies completely in a foliated chart. We denote the local transversals of the foliated chart around x and y as T_x and T_y respectively. Since $\mathcal{U}$ is a regular atlas, every plaque in a foliated chart U_i meets at most one plaque of any other foliated chart that intersects U_i. Shrinking, if necessary the foliated charts which intersect γ, we can ensure that each plaque of the foliated chart around $\gamma([t_{i-1}, t_i])$ meets *exactly* one plaque of the foliated chart around $\gamma([t_i, t_{i+1}])$. Therefore, if $U_{i_0}, U_{i_1}, ..., U_{i_n}$ are foliated charts covering $\gamma([t_0, t_1]), \gamma([t_1, t_2]), ..., \gamma([t_{n-1}, t_n])$, we get a one-to-one correspondence between plaques of U_{i_0} and plaques of U_{i_n}, thus inducing a diffeomorphism $H(\gamma)$ between the local transversals T_x and T_y (as long as the foliated charts are small enough).

Then the germ of $H(\gamma)$ at x is called the holonomy map from x to y associated to the leafwise path γ. The holonomy map does not depend on the choice of the partition of $[0, 1]$. Moreover, if γ' is another path from x to y which is fixed end-point homotopic to γ, then the germ of $H(\gamma)$ is equal to the germ of $H(\gamma')$. Thus the holonomy map only depends on the fixed end-point homotopy class of leafwise paths from x to y.

2.1.2 Groupoids associated to a foliation

We first recall the definition of a groupoid.

Definition A groupoid G is a couple $(G^{(1)}, G^{(0)})$, where $G^{(0)} = X$ is the space of units and $G^{(1)}$ is the space of arrow $\gamma : X \to X$ together with the following structure maps:

• the inclusion map $\Delta : X \to G^{(1)}$.

• the inverse map $i : G^{(1)} \to G^{(1)}$.

• the range map $r : G^{(1)} \to X$.

• the source map $s : G^{(1)} \to X$.

• the composition map $m : G_2^{(1)} \to X$, where $G_2^{(1)}$ is the set of pairs of composable elements in $G^{(1)}$, (γ, γ') such that $r(\gamma') = s(\gamma)$.

The above maps verify the following conditions:

1. $r(\Delta(x)) = s(\Delta(x))$, and $m(u, \Delta(s(u))) = u = m(\Delta(r(u)), u)$.

2. $r(i(u)) = s(u)$, and $m(u, i(u)) = \Delta(r(u))$, $m(i(u), u) = \Delta(s(u))$.

3. $s(m(u, v)) = s(v)$ and ,$r(m(u, v)) = r(u)$.

4. $m(u, m(v, w)) = m(m(u, v), w)$ if $r(w) = s(v)$ and $s(u) = r(v)$.

A groupoid is called topological (respectively differentiable of class C^k) if X and $G^{(1)}$ are topological spaces (resp. manifolds of class C^k), Δ is a continuous map (resp. of class C^k), m, r and s are continuous (resp. submersions of class C^k, and i is a homeomorphism (resp. diffeomorphism of class C^k).

If X and $G^{(1)}$ are smooth manifolds and all the structure maps above are smooth, the groupoid is called a Lie groupoid.

Let $(M, \mathcal{F})$ be a compact foliated manifold without boundary. There are various groupoids that one can associate to $(M, \mathcal{F})$. We give here the ones that are most important for us.

1. **Monodromy groupoid** : The monodromy groupoid $\mathcal{G}$ is the set of homotopy classes with fixed points of leafwise paths on $(M, \mathcal{F})$. The set of units $\mathcal{G}^{(0)}$ is the manifold M, the set of arrows $\mathcal{G}^{(1)}$ is given by the homotopy classes with fixed end-points $[\gamma]$ of leafwise paths (i.e. paths that are completely contained in a single leaf). The inclusion map Δ is given by the class of the constant path at a point, the inverse map is given by the homotopy class of the revsersed path, the source and range maps are the starting and ending points of a representative in the homotopy class, and composition is given by the homotopy class of the concatenated path.

2. **Holonomy groupoid**: It is very similar to the monodromy groupoid, we just replace 'homotopy' by 'holonomy' in the above description of the monodromy groupoid. Therefore, the holonomy groupoid is a quotient of the monodromy groupoid.

3. **Leafwise equivalence relation**: It is defined as the equivalence relation given by following relation on M: $x \sim y$ if and only if x and y belong to the same leaf. So it is the set of pairs $\{(x, y) | x, y \text{ in the same leaf}\}$. We have $\Delta(x) = (x, x), r(x, y) = x, s(x, y) = y, i(x, y) = (y, x)$ and the composition of two pairs (x, y) and (u, v) is given by the pair (x, v) if $y = u$. This is only a Borel groupoid in general (cf. [MoSc:06]).

2.2 Noncommutative integration theory on foliations

Let $(M, \mathcal{F})$ be a foliated manifold. The theory of noncommutative integration given by Connes (cf. [Co:79]) provides a notion of integration on a foliated manifold which takes into account the local product structure of the foliation. So locally one can 'put together' a measure in the direction of leaves and a transverse measure satisfying some invariance conditions in a Fubini decomposition inside a foliated chart and then sum over all charts. In this section we describe this process of integration.

2.2.1 Tangential measures

Definition Let $\{\lambda_x\}_{x\in M}$ be a family of measures with λ_x a σ-finite measure on $\mathcal{G}_x$. Such a system is called right $\mathcal{G}$-equivariant if the following condition holds:

for all $\phi \in C_c(\mathcal{G})$, $\delta \in \mathcal{G}_y^x$, we have,

$$\int_{\beta\in\mathcal{G}_x} \phi(\beta)d\lambda_x(\beta) = \int_{\gamma\in\mathcal{G}_y} \phi(\gamma\delta^{-1})d\lambda_y(\gamma)$$

The definition implies in particular that each λ_x is $\mathcal{G}_x^x$-invariant. So each measure λ_x on $\mathcal{G}_x$ descends to a well-defined measure on L_x, the leaf through x, through the identification $\mathcal{G}_x/\mathcal{G}_x^x \cong L_x$.

Definition A tangential measure is a right $\mathcal{G}$-equivariant family of measures such that for $\phi \in C_c(\mathcal{G})$ the function from M to $\mathbb{C}$ given by $x \mapsto \lambda_x(\phi) = \int_{\mathcal{G}_x} \phi(\gamma)d\lambda_x(\gamma)$ is Borel measurable.

In the above definition one can ask for C^k-continuity rather than just Borel measurability. See [Re:80] for more details.

Definition (Haar System) A (right) Haar system on $\mathcal{G}$ is a family of measures $\{\lambda_x\}_{x\in M}$, satisfying the following conditions:

(i) $supp(\lambda_x) = \mathcal{G}_x$

(ii) (Continuity) for $f \in C_c(\mathcal{G})$, $x \mapsto \int_{\mathcal{G}_x} f(\gamma)d\lambda_x(\gamma)$ is continuous.

(iii) (Right invariance) for all $\phi \in C_c(\mathcal{G})$, $\delta \in \mathcal{G}_y^x$, we have,

$$\int_{\beta\in\mathcal{G}_x} \phi(\beta)d\lambda_x(\beta) = \int_{\gamma\in\mathcal{G}_y} \phi(\gamma\delta^{-1})d\lambda_y(\gamma)$$

One similarly defines a *smooth* Haar system by replacing "continuous" by "smooth" in condition (ii) of the definition above.

2.2.2 Transverse measures

Definition ([Co:81]) A Borel transversal to a foliation is a Borel subset T of M such that T intersects each leaf at most countably many times.

Definition ([Co:81]) A transverse measure on the foliation $(M, \mathcal{F})$ is a countably additive Radon measure on the σ-ring of all Borel transverals.

Definition ([Co:81]) A holonomy-invariant transverse measure is a transverse measure Λ such that for any leaf-preserving Borel bijection between Borel transversal T_1 and T_2, $\psi : T_1 \to T_2$ (i.e. $x \in T_1$ and $\psi(x) \in T_2$ are on the same leaf) we have $\Lambda(T_1) = \Lambda(T_2)$.

Definition Consider foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$.

- Two leafwise maps f,g: $(M, \mathcal{F}) \to (M', \mathcal{F}')$ are called leafwise homotopic if there exists a leafwise map $h : (M \times [0,1], \mathcal{F} \times [0,1]) \to (M', \mathcal{F}')$ such that $h(.,0) = f$ and $h(.,1) = g$.
- A leafwise map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is called a leafwise homotopy equivalence if there exists a leafwise map $g : (M', \mathcal{F}') \to (M, \mathcal{F})$ such that $f \circ g$ is leafwise homotopic to the identity map in M' and $g \circ f$ is leafwise homotopic to the identity map in M.
- We will call $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ leafwise homotopy equivalent if there exists a leafwise homotopy equivalence from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$.

Proposition 2.2.1. *Let $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a $C^{\infty,0}$ leafwise homotopy equivalence between foliated manifolds. Let Λ be a holonomy-invariant transverse measure on M. Then f induces a transverse measure $f_*\Lambda$ on M' which is also holonomy-invariant.*

Proof. Let $(U'_\alpha)_{\alpha \in A}$ be a distinguished open cover on $(M', \mathcal{F}')$. Let X'_α denote the local transversal of U'_α. Without loss of generality one can assume that $\overline{X'}_\alpha \cap \overline{X'}_\beta = \emptyset$ for $\alpha \neq \beta$ (cf. [HiSk:84]). Then we can choose a distinguished open cover $(U_i)_{i \in I}$ of $(M, \mathcal{F})$ such that for $i \in I$ there exists $\alpha(i) \in A$ such that $f(U_i) \subseteq U'_{\alpha(i)}$. Let $U_i \cong W_i \times X_i$, where X_i is transversal to the plaques W_i. One can also assume without loss of generality that the induced map on the transversal $\hat{f} : X_i \to \hat{f}(X_i)$ is a homeomorphism onto its image (cf. [CoSk:84], [BePi:08]). Let $\pi_{\alpha(i)} : \hat{f}(X_i) \to X'_{\alpha(i)}$ be the map which projects to the local transversal. Denote $X'_i := \pi_{\alpha(i)}(\hat{f}(X_i))$. Then it can be easily seen that $X' := \bigcup_{i \in I} X'_i$ is a complete transversal for $(M', \mathcal{F}')$.

Now let T' be a Borel transversal on M'. Then locally on $U'_{\alpha(i)}$, T' is homeomorphic to a Borel subset $T''_{\alpha(i)}$ of X'_i, which is in turn homeomorphic to a Borel subset $T'_{\alpha(i)}$ of $\hat{f}(X_i)$. Since $\hat{f}$ is a homeomorphism onto its image on X_i, $\hat{f}^{-1}(T'_{\alpha_i})$ is a Borel subset of X_i and we set

$$f_*\Lambda(T'_i) := \Lambda(\hat{f}^{-1}(T'_{\alpha_i}))$$

Since T' is a disjoint union of such subsets we define $f_*\Lambda(T')$ as the sum $\sum_{\alpha_i} f_*\Lambda(T'_{\alpha_i})$ where the index runs over all α_i such that $T \cap U_{\alpha(i)} \neq \emptyset$ and $T'_{\alpha(i)}$ is homeomorphic to a Borel subset of $\hat{f}(X_i)$. Then from the properties of Λ we see that $f_*\Lambda$ is a countably additive Radon measure on the σ-ring of Borel transversals on M'.

Now let $\psi' : T'_1 \to T'_2$ be a Borel bijection between Borel transversals T'_1 and T'_2 in M'. We need to show that $f_*\Lambda(T'_1) = f_*\Lambda(T'_2)$. Assume that T'_1 lies completely in some $U'_{\alpha(i_1)}$, T'_2 lies completely in some $U'_{\alpha(i_2)}$, so that there exist subsets Y'_{i_k} of X'_{i_k} such that $T'_k \cong Y'_{i_k}$ for $k = 1, 2$. Then there exist subsets Y_{i_k} of $\hat{f}(X_{i_k})$ such that $Y_{i_k} \cong Y'_{i_k}$ for $k = 1, 2$. The Borel bijection ψ' induces a Borel bijection ψ between $\hat{f}^{-1}(Y_{i_1})$ and $\hat{f}^{-1}(Y_{i_2})$. Now, since Λ is holonomy-invariant, we have $\Lambda(\hat{f}^{-1}(Y_{i_1})) = \Lambda(\hat{f}^{-1}(Y_{i_2}))$. Since by definition we have $f_*\Lambda(T'_k) = \Lambda(\hat{f}^{-1}(Y_{i_k}))$ for $k = 1, 2$, and a general Borel transversal is the disjoint union of such 'local' Borel transversals, the result follows.

□

The following corollary is immediate.

Corollary 2.2.2. *Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be closed smooth foliated manifolds and $f : M \to M'$ a leafwise diffeomorphism. If Λ is a holonomy invariant transverse measure on M then $f_*\Lambda$ is a holonomy invariant transverse measure on M'.*

2.2.3 Integrating a tangential measure against a holonomy invariant measure

It turns out that holonomy-invariant transverse measure are the right objects of integration against tangential measures, by exploiting the local product structure of the foliation. The integration process is described as follows.

Let $(\nu_x)_{x\in M}$ be a tangential measure and Λ a holonomy-invariant transverse measure on $(M,\mathcal{F})$. Let $\{(U_i,\phi_i)\}_{i\in I}\}$ be a regular cover on $(M,\mathcal{F})$ with each $U_i \cong L_i \times T_i$ having a local transversal T_i and plaques L_i. Consider the complete transerval $T = \cup_{i\in I} T_i$. Let $(\psi_i)_{i\in I}$ be a partition of unity subordinate to the cover U_i. Then for any function $f \in C(M)$, we define a functional on $C(M)$ by the formula:

$$\Lambda_\nu(f) = \sum_{i\in I} \int_{t_i\in T_i} \int_{l_i\in L_i} \psi_i(l_i,t_i) f(l_i,t_i) d\nu^L_{t_i}(l_i) d\Lambda_i(t_i)$$

where $\nu^L_{t_i}$ is the restriction of $\nu_{(l_i,t_i)}$ to the plaque through t_i and Λ_i is the restriction of Λ to T_i.

The above formula is well-defined due to the holonomy-invariance of Λ and is independent of the choices of the regular cover and the partition of unity. Therefore this functional defines a Borel measure μ on M, which is expressed through notation as $\mu = \int \nu d\Lambda$.

2.3 Operator algebras on foliations

2.3.1 The convolution algebra on a groupoid

Let $(M,\mathcal{F})$ be a foliation. Consider the monodromy groupoid $\mathcal{G}$ and the space of compactly supported continuous functions $\mathcal{B}_c = C_c(\mathcal{G})$ on $\mathcal{G}$.

We fix a Haar system $\{\lambda_x\}_{x\in M}$ on $\mathcal{G}$. We define the multiplication and involution on $\mathcal{B}_c$ by the following formulae:

$$(f * g)(u) = \int_{v\in\mathcal{G}_{r(u)}} f(v^{-1}) g(vu) d\lambda_{r(u)}(v), \text{ and } f^*(u) = \overline{f(u^{-1})} \text{ for any } u \in \mathcal{G} \tag{2.3.1}$$

The Haar system is $\mathcal{G}$-equivariant on the right, i.e. for all $\phi \in \mathcal{B}_c$, $\delta \in \mathcal{G}^x_y$, we have,

$$\int_{\beta\in\mathcal{G}_x} \phi(\beta) d\lambda_x(\beta) = \int_{\gamma\in\mathcal{G}_y} \phi(\gamma\delta^{-1}) d\lambda_y(\gamma) \tag{2.3.2}$$

With this the convolution formula 2.3.10 becomes:

$$(f * g)(u) = \int_{v\in\mathcal{G}_{s(u)}} f(uv^{-1}) g(v) d\lambda_{s(u)}(v), \quad u \in \mathcal{G} \tag{2.3.3}$$

The L^1 norm on $\mathcal{B}_c$ is given by

$$||f||_1 = \sup \{\sup_{x\in M} \int_{\mathcal{G}_x} |f(\alpha)| d\lambda_x(\alpha), \sup_{x\in M} \int_{\mathcal{G}_x} |f(\alpha^{-1})| d\lambda_x(\alpha)\} \tag{2.3.4}$$

2.3.2 Representations of $\mathcal{B}_c$

Let $x \in M$. We have a representation π_x^{reg} of $\mathcal{B}$ on the Hilbert space of square integrable sections $\mathcal{H}_x = L^2(\mathcal{G}_x, \lambda_x)$ defined as

$$\pi_x^{reg}(f)(\xi)(\alpha) = \int_{\beta \in \mathcal{G}_x} f(\alpha\beta^{-1})\xi(\beta)d\lambda_x(\beta) \text{ for } f \in \mathcal{B}_c, \xi \in \mathcal{H}_x \text{ and } \alpha \in \mathcal{G}_x \tag{2.3.5}$$

It is easy to check that π_x^{reg} is a $*$-representation. Indeed for $f, g \in \mathcal{B}_c$, we have,

$$\pi_x^{reg}(f * g) = \pi_x^{reg}(f) \circ \pi_x^{reg}(g) \text{ and } \pi_x^{reg}(f^*) = (\pi_x^{reg}(f))^* \tag{2.3.6}$$

We also consider another representation π_x^{av} of $\mathcal{B}_c$ on $H_x := L^2(L_x, \lambda^L) \cong L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^L)$ given by the following formula:

$$\pi_x^{av}(f)(\psi)([\alpha]) = \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} f(\alpha\beta\theta^{-1})\psi[\theta]d\tilde{\lambda}^l([\theta]) \tag{2.3.7}$$

Here, as before, L_x is the leaf through x and λ^L is the leafwise Lebesgue measure induced by the Haar system.

Lemma 2.3.1. *For every $x \in M$, π_x^{av} is a $*$-representation.*

Proof. We identify $L^2(L_x, \lambda^L)$ with $L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^L)$. Then, for $\xi \in L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^L), [\alpha] \in \mathcal{G}_x/\mathcal{G}_x^x$ we have,

$$(\pi_x^{av}(f) \circ \pi_x^{av}(g))(\xi)([\alpha]) = \int_{\mathcal{G}_x/\mathcal{G}_x^x} \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x/\mathcal{G}_x^x} \sum_{\delta \in \mathcal{G}_x/\mathcal{G}_x^x} f(\alpha\beta\theta^{-1})g(\theta\delta\gamma^{-1})\xi([\gamma])d\tilde{\lambda}^L([\gamma])d\tilde{\lambda}^L([\theta]) \tag{2.3.8}$$

We also have,

$$\begin{aligned}
\pi_x^{av}(f * g)(\xi)([\alpha]) &= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} (f * g)(\alpha\beta\theta^{-1})\xi([\theta])d\tilde{\lambda}^L([\theta]) \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} \{\int_{\mathcal{G}_{r(\alpha\beta\theta^{-1})}} f(\delta^{-1})g(\delta\alpha\beta\theta^{-1})d\lambda_{r(\alpha\beta\theta^{-1})}(\delta)\}\xi([\theta])d\tilde{\lambda}^L([\theta]) \\
&\quad \text{Using the property 2.3.2 of the Haar system,} \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} \{\int_{\mathcal{G}_x} f(\alpha\eta^{-1})g(\eta\beta\theta^{-1})d\lambda_x(\eta)\}\xi([\theta])d\tilde{\lambda}^L([\theta]) \\
&\quad \text{Choosing } \gamma \in \mathcal{G}_x \text{ and } \delta \in \mathcal{G}_x^x \text{ such that } \eta = \gamma\delta, \text{ we have} \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} \{\int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\delta \in \mathcal{G}_x^x} f(\alpha\delta^{-1}\gamma^{-1})g(\gamma\delta\beta\theta^{-1})d\tilde{\lambda}^L([\gamma])\}\xi([\theta])d\tilde{\lambda}^L([\theta]) \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} \sum_{\delta \in \mathcal{G}_x^x} f(\alpha\delta^{-1}\gamma^{-1})g(\gamma\delta\beta\theta^{-1})\xi([\theta])d\tilde{\lambda}^L([\gamma])d\tilde{\lambda}^L([\theta]) \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta' \in \mathcal{G}_x^x} \sum_{\delta' \in \mathcal{G}_x^x} f(\alpha\delta'^{-1}\gamma^{-1})g(\gamma\beta'\theta^{-1})\xi([\theta])d\tilde{\lambda}^L([\gamma])d\tilde{\lambda}^L([\theta]) \qquad (2.3.9)
\end{aligned}$$

Which is the same as (2.3.8). We can also check similarly that it satisfies the $*$-relation.

∎

2.3.3 C^*-algebra of a foliation with coefficients in a vector bundle

Let $E \to M$ be a Hermitian vector bundle. Let $C_c(\mathcal{G}, E)$ be the space of smooth sections of $(s^*(E))^* \otimes r^*(E)$. So, for $\gamma \in \mathcal{G}, f \in C_c(\mathcal{G}, E)$, we have $f(\gamma) \in \mathrm{Hom}(E_{s(\gamma)}, E_{r(\gamma)})$.

We have a $*$-algebra structure on $C_c(\mathcal{G}, E)$. Let $f, g \in C_c(\mathcal{G}, E)$. Then, we define the convolution and involution as

$$(f * g)(\gamma) = \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} f(\gamma'^{-1}) \circ g(\gamma'\gamma) d\lambda_{r(\gamma)}(\gamma') \text{ and } f^*(\gamma) = \overline{f(\gamma^{-1})} \text{ for } \gamma \in \mathcal{G} \tag{2.3.10}$$

As in the trivial case ($E = M \times \mathbb{C}$), we have a representation of $\mathcal{B}_c^E := C_c^\infty(\mathcal{G}, E)$ on $\mathcal{H}_x := L^2(\mathcal{G}_x, r^*(E), \lambda_x)$ given by

$$\pi_x^{reg}(f)(\xi)(\alpha) = \int_{\beta \in \mathcal{G}_x} f(\alpha\beta^{-1})\xi(\beta) d\lambda_x(\beta) \text{ for } f \in \mathcal{B}_c^E, \xi \in \mathcal{H}_x \text{ and } \alpha \in \mathcal{G}_x \tag{2.3.11}$$

The regular norm on $\mathcal{B}_c^E$ is thus given by

$$||f||_{reg} = \sup_x ||\pi_x^{reg}(f)|| \tag{2.3.12}$$

We define the reduced C^*-algebra of the foliation with coefficients in E as the completion of $\mathcal{B}_c^E$ in the regular norm, denoted by $\mathcal{B}_r^E$, and the maximal C^*-algebra $\mathcal{B}_m^E$ its completion in the maximal norm given by

$$||f||_{max} = \sup_\pi ||\pi(f)||$$

where the supremum is taken over all L^1-continuous $*$-representations of $\mathcal{B}_c$. Recall that a $*$-representation is called L^1-continuous if it is continuous with respect to the L^1 norm on $\mathcal{B}_c$.

2.3.4 Von Neumann Algebras for foliations

For each desingularization of the foliation $(M, \mathcal{F})$ that associates a groupoid to the foliation, one can define von Neumann algebras on the groupoids which reflect many geometrical properties of the foliation. More importantly, operators arising in geometric settings that act on sections on the groupoids will be naturally associated with these von Neumann algebras using the "affiliation" relation. We will work with von Neumann algebras associated to the leafwise equivalence relation $\mathcal{R}$ and the monodromy groupoid of the foliation. We define them as follows:

The regular von Neumann algebra

Definition The regular von Neumann algebra $W^*(\mathcal{G})$ can be described as the space of measurable families of operators $T = \{T_x\}_{x \in M}$ such that for each x, $T_x \in B(L^2(\mathcal{G}_x, \lambda_x))$ and the following conditions hold:

- The mapping $x \mapsto ||T_x||$ is measurable and Λ-essentially bounded, i.e. Ess $\sup_x ||T_x|| < \infty$.
- We have for $x, y \in M, \gamma \in \mathcal{G}_x^y$:$T_y = R_\gamma \circ T_x \circ R_{\gamma^{-1}}$ where $R_\gamma : L^2(\mathcal{G}_x, \lambda_x) \to L^2(\mathcal{G}_y, \lambda_y)$ is given by

$$(R_\gamma \xi)(\alpha) = \xi(\alpha\gamma) \qquad \forall \xi \in L^2(\mathcal{G}_x, \lambda_x), \alpha \in \mathcal{G}_y \tag{2.3.13}$$

Remark. *For the definition of measurable families of operators we refer the reader to [Di:57, Chapitre 2, Sections 1 and 2].*

Lemma 2.3.2. *The image of the representation π^{reg} lies in $W^*(\mathcal{G})$.*

Proof. For $f \in C_c(\mathcal{G})$ the family of operators $(\pi_x(f))_{x\in M}$ is measurable, hence from [Di:57, Proposition 1, p.156] we get that $x \mapsto ||\pi_x(f)||$ is measurable. Now for $\alpha \in \mathcal{G}_y, \gamma \in \mathcal{G}_x^y$ and $\xi_y \in L^2(\mathcal{G}_y)$, we have

$$\begin{aligned}
(R_\gamma \circ \pi_x^{reg}(f) \circ R_{\gamma^{-1}})(\xi_y)(\alpha) &= (\pi_x^{reg}(f) \circ R_{\gamma^{-1}})(\xi_y)(\alpha\gamma) \\
&= \int_{\mathcal{G}_x} f(\alpha\gamma\theta^{-1})\xi_y(\theta\gamma^{-1})d\lambda_x(\theta) \\
&\quad \text{Using the invariance of the measure, we get} \\
&= \int_{\mathcal{G}_y} f(\alpha\eta^{-1})\xi_y(\eta)d\lambda_y(\eta) \\
&= \pi_y^{reg}(f)(\xi_y)(\alpha)
\end{aligned}$$

Therefore the above computation together with the L^1-continuity of π_x^{reg} shows that the image of π^{reg} is in $W^*(\mathcal{G})$.

□

Remark: As can be checked easily, the regular von Neumann algebra of $\mathcal{G}$ $W^*(\mathcal{G})$, actually coincides with the weak closure of the image of $L^1(\mathcal{G})$ under π^{reg} in $B(L^2(\mathcal{G}))$. See [Co:79].

The foliation von Neumann algebra

The foliation von Neumann algebra of $\mathcal{R}$ is defined as the space of measurable families of operators $T = \{T_x\}_{x\in M}$ such that for each x, $T_x \in B(L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^L))$ and the following conditions hold:

- The mapping $x \mapsto ||T_x||$ is measurable and Λ-essentially bounded.
- We have for $x, y \in M, \gamma \in \mathcal{G}_x^y$

$$T_y = R_\gamma \circ T_x \circ R_{\gamma^{-1}}, \tag{2.3.14}$$

where $R_\gamma : L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^l) \to L^2(\mathcal{G}_y/\mathcal{G}_y^y, \tilde{\lambda}^L)$ is given by

$$(R_\gamma\xi)([\alpha]) = \xi([\alpha\gamma]) \qquad \forall\xi \in L^2(\mathcal{G}_x/\mathcal{G}_x^x, \tilde{\lambda}^L), \alpha \in \mathcal{G}_y \tag{2.3.15}$$

Let $L^2([\mathcal{G}])$ be the field of Hilbert spaces $(L^2(\mathcal{G}_x/\mathcal{G}_x^x))_{x\in M}$.

Lemma 2.3.3. *The image of the representation π^{av} lies in the von Neumann algebra $W^*(M, \mathcal{F})$.*

Proof. For $\alpha \in \mathcal{G}_y, \gamma \in \mathcal{G}_x^y$ and $\xi_y \in L^2(\mathcal{G}_y/\mathcal{G}_y^y)$, we have

$$
\begin{aligned}
(R_\gamma \circ \pi_x^{av}(f) \circ R_{\gamma^{-1}})(\xi_y)([\alpha]) &= (\pi_x^{av}(f) \circ R_{\gamma^{-1}})(\xi_y)([\alpha\gamma]) \\
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta\in\mathcal{G}_x^x} f(\alpha\gamma\beta\theta^{-1})\xi_y([\theta\gamma^{-1}])d\tilde{\lambda^l}([\theta])
\end{aligned}
$$

Using the invariance of the measure, we get

$$
\begin{aligned}
&= \int_{\mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta\in\mathcal{G}_x^x} f(\alpha\gamma\beta\gamma^{-1}\eta^{-1})\xi_y([\eta])d\tilde{\lambda^l}([\eta]) \qquad (2.3.16)\\
&= \int_{\mathcal{G}_y/\mathcal{G}_y^y} \sum_{\beta'\in\mathcal{G}_y^y} f(\alpha\beta'\eta^{-1})\xi_y([\eta])d\tilde{\lambda^l}([\eta]) \\
&= \pi_y^{av}(f)(\xi_y)([\alpha])
\end{aligned}
$$

□

We define similarly the regular and foliation von Neumann algebras associated to a vector bundle E on M, and we denote them by $W^*(\mathcal{G}, E)$ and $W^*(M, \mathcal{F}; E)$.

2.3.5 Traces on foliations

Let $U = (U_i)_{i\in I}$ be a regular covering of M (i.e. composed of distinguished charts) and $(\phi_i)_{i\in I}$ be a partition of unity subordinate to U on M. Let T_i be the local transversals at $U_i \simeq L_i \times T_i$. We also assume that there is a holonomy invariant transverse measure on M which we call Λ. We will now define traces on the von Neumann algebras $W^*(\mathcal{G}, E)$ and $W^*(M, \mathcal{F}, E)$. To this end, let $T = (T_L)_{L\in M/\mathcal{F}}$ be a positive element of $W^*(M, \mathcal{F}, E)$. Then, we define the traces as follows

Definition We define the trace $\tau_{\mathcal{F}}^\Lambda$ on $W^*(M, \mathcal{F}, E)$ as

$$
\tau_{\mathcal{F}}^\Lambda(T) = \sum_{i\in I} \int Tr_x(M_{\phi_i^{1/2}} T_x M_{\phi_i^{1/2}})d\Lambda \text{ for } T \in W^*(M, \mathcal{F}, E)^+ \qquad (2.3.17)
$$

where $M_{\phi_i^{1/2}}$ denotes the multiplication operator by $\phi_i^{1/2}$, Tr_x is the usual trace on the Hilbert space $B(L^2(L_x, E|_{L_x}))$, and the integration procedure is done according to [MoSc:06, Ch. IV, p.90], since $(Tr_x(\phi_i^{1/2} T_x \phi_i^{1/2}))_{x\in M}$ is a family of tangential measures.

We also have a trace τ^Λ on $W^*(\mathcal{G}, E)$

Definition Let $T = (T_x)_{x\in M}$ be a positive element in $W^*(\mathcal{G}, E)$. Then define

$$
\tau^\Lambda(T) = \sum_{i\in I} \int Tr_x(\tilde{\phi_i}^{1/2} T_x \tilde{\phi_i}^{1/2})d\Lambda \qquad (2.3.18)
$$

where $\tilde{\phi}_i \in C_c^\infty(\mathcal{G})$ is supported in a compact neighbourhood W_i very close to the diagonal $\mathcal{G}^{(0)}$ in $\mathcal{G}$ such that $\tilde{\phi}_i(\gamma) = \phi_i(r(\gamma))$ on $W_i \subset \mathcal{G}$, Tr_x is the usual trace on $B(L^2(\mathcal{G}_x, r^*E))$ and as in the previous case $Tr_x(\tilde{\phi}_i^{1/2} T_x \tilde{\phi}_i^{1/2})$ is a well defined tangential measure on the leaf through x and is independent of the choice of x (cf. [MoSc:06, Chapter VI, p.149]). The compact neighbourhood W_i of $\mathcal{G}^{(0)}$ above is chosen such that for $x \in U_i$, $\tilde{\phi}_i(\gamma) = 0$ for $\gamma \in \mathcal{G}_x^x, \gamma \neq e$.

Proposition 2.3.4. *(1) $\tau_{\mathcal{F}}^{\Lambda}$ is a positive faithful normal semi-finite trace on $W^*(M,\mathcal{F},E)$.*

(2) τ^{Λ} is a positive faithful normal semi-finite trace on $W^*(\mathcal{G},E)$.

Proof. We prove the assertions in (1).

• Positivity: Let $T \in W^*(M,\mathcal{F};E)^+$. Then $T = S^*S$ for some $S \in W^*(M,\mathcal{F};E)$. So we have $T_L = S_L^*S_L$, Λ-a.e. Then, $M_{\phi_i^{1/2}}T_LM_{\phi_i^{1/2}} = M_{\phi_i^{1/2}}S_L^*S_LM_{\phi_i^{1/2}} = (S_LM_{\phi_i^{1/2}})^*(S_LM_{\phi_i^{1/2}}) \geq 0$. So by the positivity of the trace Tr we get positivity of $\tau_{\mathcal{F}}^{\Lambda}$.

• Faithfulness: Let $T \in W^*(M,\mathcal{F},E)^+$ be such that $\tau^{\Lambda}(T) = 0$. Then we have $Tr_L(M_{\phi_i^{1/2}}T_LM_{\phi_i^{1/2}}) = Tr_L(T_L^{1/2}M_{\phi_i}T_L^{1/2}) = 0$, $\Lambda - a.e.$. Then $T_L^{1/2}M_{\phi_i}T_L^{1/2} = 0$, Λ-$a.e.$ follows from the faithfulness of Tr. Therefore, $0 = \sum_{i\in I} T_L^{1/2}M_{\phi_i}T_L^{1/2} = T_L^{1/2}\sum_{i\in I} M_{\phi_i}T_L^{1/2} = T_L$. Hence $T = 0$.

• Traciality: Let $T_1, T_2 \in W^*(M,\mathcal{F},E)^+$

$$\begin{aligned}
\tau_{\mathcal{F}}^{\Lambda}(T_1T_2) &= \sum_{i\in I}\int Tr_L(M_{\phi_i^{1/2}}T_{1,L}T_{2,L}M_{\phi_i^{1/2}})d\Lambda \\
&= \sum_{i\in I}\int Tr_L(M_{\phi_i^{1/2}}T_{1,L}\sum_{j\in I}M_{\phi_j}T_{2,L}M_{\phi_i^{1/2}})d\Lambda \\
&= \sum_{i\in I}\sum_{j\in I}\int Tr_L((M_{\phi_i^{1/2}}T_{1,L}M_{\phi_j^{1/2}})(M_{\phi_j^{1/2}}T_{2,L}M_{\phi_i^{1/2}}))d\Lambda \\
&= \sum_{i\in I}\sum_{j\in I}\int Tr_L((M_{\phi_j^{1/2}}T_{2,L}M_{\phi_i^{1/2}})(M_{\phi_i^{1/2}}T_{1,L}M_{\phi_j^{1/2}}))d\Lambda \text{ (by the traciality of Tr)} \\
&= \sum_{i\in I}\sum_{j\in I}\int Tr_L(M_{\phi_j^{1/2}}T_{2,L}M_{\phi_i}T_{1,L}M_{\phi_j^{1/2}})d\Lambda \\
&= \sum_{j\in I}\int Tr_L(M_{\phi_j^{1/2}}T_{2,L}\sum_{i\in I}M_{\phi_i}T_{1,L}M_{\phi_j^{1/2}})d\Lambda \\
&= \sum_{j\in I}\int Tr_L(M_{\phi_j^{1/2}}T_{2,L}T_{1,L}M_{\phi_j^{1/2}})d\Lambda \\
&= \tau_{\mathcal{F}}^{\Lambda}(T_2T_1)
\end{aligned} \tag{2.3.19}$$

• Normality: Let $A_\gamma \nearrow A$ be a net of positive operators in $W^*(M,\mathcal{F},E)^+$. Then from the normality of the trace Tr, we know that $Tr(M_{\phi_i^{1/2}}A_\gamma M_{\phi_i^{1/2}}) \nearrow Tr(M_{\phi_i^{1/2}}AM_{\phi_i^{1/2}})$. So we can use the 'convergence from below' theorem of Lebesgue[1] to show that $\tau_{\mathcal{F}}^{\Lambda}(A_\gamma) \nearrow \tau_{\mathcal{F}}^{\Lambda}(A)$.

• Semifiniteness: This again follows from the semifiniteness of Tr as for any $T \in W^*(M,\mathcal{F},E)^+$ we can find a net $(M_{\phi_i^{1/2}}T_\gamma M_{\phi_i^{1/2}})_{\gamma\in A}$ converging to $M_{\phi_i^{1/2}}TM_{\phi_i^{1/2}}$ with $Tr(M_{\phi_i^{1/2}}T_\gamma M_{\phi_i^{1/2}}) < \infty$. Since one can choose the complete transversal to be pre-compact, we have $\tau_{\mathcal{F}}^{\Lambda}(T_\gamma) < \infty$. ☐

Let there be a $\mathbb{Z}_2$-grading on E, so that we can write $E = E^+ \oplus E^-$. Then we have two von Neumann algebras $W^*(\mathcal{G},E^{\pm})$ of bounded operators acting on $L^2(\mathcal{G},E^{\pm})$. Denote $\tau_{\pm}^{\Lambda}$ the traces on $W^*(\mathcal{G},E^{\pm})$. We have the following

[1]which states that for a measure space (M,μ) if we have a sequence of measurable functions $f_n : M \to [0,\infty]$ converging pointwise to f and $f_n(x) \leq f(x)\forall n \in \mathbb{N}, x \in M$, then $\int f d\mu = \lim_{n\to\infty}\int f_n d\mu$

Proposition 2.3.5. *Let $A^+ : L^2(\mathcal{G}, E^+) \to L^2(\mathcal{G}, E^-)$, $B^- : L^2(\mathcal{G}, E^-) \to L^2(\mathcal{G}, E^+)$ be bounded positive operators, and A^+B^- be τ_-^Λ-trace class and B^-A^+ be τ_+^Λ-trace class in the respective Von Neumann algebras. Then,*

$$\tau_+^\Lambda(B^-A^+) = \tau_-^\Lambda(A^+B^-) \tag{2.3.20}$$

Proof. To prove 2.3.20 we note that $W^*(\mathcal{G}, E^\pm) = e^\pm W^*(\mathcal{G}, E^+ \oplus E^-)e^\pm$, with $e^\pm$ projections in $W^*(\mathcal{G}, E^+ \oplus E^-)$, and $\tau_\pm^\Lambda$ are the restrictions of the trace τ^Λ on $W^*(\mathcal{G}, E^+ \oplus E^-)$. We write $\hat{A} = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$, $\hat{B} = \begin{pmatrix} 0 & 0 \\ B^- & 0 \end{pmatrix}$, $\hat{A}, \hat{B} \in W^*(\mathcal{G}, E^+ \oplus E^-)$. Then,

$$\hat{A}\hat{B} = \begin{pmatrix} A^+B^- & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{B}\hat{A} = \begin{pmatrix} 0 & 0 \\ 0 & B^-A^+ \end{pmatrix},$$

But for the trace τ^Λ on $W^*(\mathcal{G}, E)$, we know that $\tau^\Lambda(\hat{A}\hat{B}) = \tau^\Lambda(\hat{B}\hat{A})$, and we have,

$$\tau^\Lambda(\hat{A}\hat{B}) = \tau^\Lambda(e^- A^+B^- e^-) = \tau_-^\Lambda(A^+B^-).$$

Similarly,

$\tau^\Lambda(\hat{B}\hat{A}) = \tau_+^\Lambda(B^-A^+)$, thus proving 2.3.20.

□

We now state dominated convergence theorems for traces, following [Sh:, 2.2.4 Theorem 1, p. 54].

Theorem 2.3.6. *: Let H be a Hilbert space, S a trace class operator on H,$\{A_\gamma\ \gamma \in \Gamma\}$ a net of bounded linear operators on H such that*

(i) there exists $C > 0$ such that $||A_\gamma|| \leq C, \forall \gamma \in \Gamma$,

(ii) w-$\lim_\gamma A_\gamma = A$, where w-$\lim_\gamma A_\gamma$ denotes the weak limit of the net A_γ.

Then, $\lim_\gamma Tr(SA_\gamma) = Tr(SA)$.

Proof. Choose an orthonormal basis $(e_j)_{j\in J}$ of H. Then,

$$Tr(SA_\gamma) = \sum < SA_\gamma e_j, e_j >= \sum < A_\gamma e_j, S^* e_j > \tag{2.3.21}$$

To prove the result it would suffice to show the uniform convergence of the sum in the last part of 2.3.21 with respect to $\gamma \in \Gamma$. To this end, let $S = S_1S_2$ where $S_i, i = 1, 2$ are Hilbert Schmidt operators on H. Then we have,

$$| < SA_\gamma e_j, e_j > | = | < S_2A_\gamma e_j, S_1^* e_j > | \leq ||S_2A_\gamma e_j||||S_1^* e_j|| \tag{2.3.22}$$

Using Holder's inequality for sums, we get for an arbitrary finite subset J_1 of J,

$$\begin{aligned} \sum_{j\in J_1} | < SA_\gamma e_j, e_j > | &\leq (\sum_{j\in J_1} ||S_2A_\gamma e_j||^2)^{1/2}(\sum_{j\in J_1} ||S_1^* e_j||^2)^{1/2} \\ &\leq ||S_2A_\gamma||_{HS}(\sum_{j\in J_1} ||S_1^* e_j||^2)^{1/2} \\ &\leq C||S_2||_{HS}(\sum_{j\in J_1} ||S_1^* e_j||^2)^{1/2} \end{aligned} \tag{2.3.23}$$

where $||.||_{HS}$ is the Hilbert-Schmidt norm and we have used the inequality $||SA||_{HS} \leq ||S||_{HS}||A||$. Since the series $(\sum_{j\in J}||S_1^* e_j||^2)^{1/2}$ converges as S_1 is Hilbert-Schmidt (therefore so is S_1^*) the uniform convergence of the series in 2.3.21 is established. □

Corollary 2.3.7. *: If S is a trace class operator on H and $\{A_k, k = 1, 2, ...\}$ is a sequence of operators in $B(H)$ such that the weak limit $w\text{-}\lim_{k\to\infty} A_k = A$, then*

$$\lim_{k\to\infty} Tr(SA_k) = Tr(SA)$$

Proof. This follows from the Banach-Steinhaus principle in functional analysis which guarantees $||A_k|| < C$, and the application of Theorem (2.3.6). □

We give next a proposition which we will need later in the proof of Theorem (3.2.2).

Proposition 2.3.8. *If S is a τ^Λ-trace class operator in $W^*(\mathcal{G}, E)$ and $\{A_n, n = 1, 2, ...\}$ is an increasing sequence of positive operators in $W^*(\mathcal{G}, E)$ such that the strong limit of A_n $s\text{-}\lim(A_n)_x = A_x$ for each $x \in M$ for some $A \in W^*(\mathcal{G}, E)$, then*

$$\lim_{n\to\infty} \tau^\Lambda(SA_n) = \tau^\Lambda(SA)$$

Proof. Since $s\text{-}\lim(A_n)_x = A_x \Rightarrow w\text{-}\lim_{n\to\infty}(A_n)_x = A_x$, so we can apply Corollary(2.3.7) to get

$$\lim_{n\to\infty} Tr_x(S_x(A_n)_x) = Tr_x(S_xA_x) \Rightarrow \lim_{n\to\infty} Tr_x(\tilde{\phi_i}^{1/2}S_x(A_n)_x\tilde{\phi_i}^{1/2}) = Tr_x(\tilde{\phi_i}^{1/2}S_xA_x\tilde{\phi_i}^{1/2}),$$

and due to the normality of the trace Tr_x we have $Tr_x(\tilde{\phi_i}^{1/2}S_x(A_n)_x\tilde{\phi_i}^{1/2}) \leq Tr_x(\tilde{\phi_i}^{1/2}S_xA_x\tilde{\phi_i}^{1/2})$ Then, we have,

$$\lim_{n\to\infty} \tau^\Lambda(SA_n) = \lim_{n\to\infty} \sum_{i\in I} \int Tr_x(\tilde{\phi_i}^{1/2}(SA_n)_x\tilde{\phi_i}^{1/2})d\Lambda \tag{2.3.24}$$

We use Lebesgue's 'convergence from below' theorem[2] then to infer that

$$\begin{aligned}\lim_{n\to\infty} \tau^\Lambda(SA_n) &= \sum_{i\in I} \int \lim_{n\to\infty} Tr_x(\tilde{\phi_i}^{1/2}(SA_n)_x\tilde{\phi_i}^{1/2})d\Lambda \\ &= \sum_{i\in I} \int Tr_x(\tilde{\phi_i}^{1/2}S_xA_x\tilde{\phi_i}^{1/2})d\Lambda \\ &= \tau^\Lambda(SA)\end{aligned} \tag{2.3.25}$$

□

We define two functionals τ^Λ_{reg} and τ^Λ_{av} on $\mathcal{B}_c$ with the help of the measure $\mu = \int \lambda d\Lambda$ on M as follows:

$$\tau^\Lambda_{reg}(f) = \int_M f(1_x)d\mu(x) \tag{2.3.26}$$

and

$$\tau^\Lambda_{av}(f) = \int_M \sum_{\beta\in\mathcal{G}^x_x} f(\beta)d\mu(x) \tag{2.3.27}$$

[2]which states that for a measure space (M, μ) if we have a sequence of measurable functions $f_n : M \to [0, \infty]$ converging pointwise to f and $f_n(x) \leq f(x)\forall n \in \mathbb{N}, x \in M$, then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$

Proposition 2.3.9. *On $\mathcal{B}_c$, we have*

(i) $\tau^{\Lambda}_{reg} = \tau^{\Lambda} \circ \pi^{reg}$ *, and*

(ii) $\tau^{\Lambda}_{av} = \tau^{\Lambda}_{\mathcal{F}} \circ \pi^{av}$

Proof. As $K_{\pi^{reg}_x(f)}(\alpha, \beta) = f(\alpha\beta^{-1})$ and $K_{\pi^{av}_x(f)}([\alpha], [\beta], t) = \sum_{[\gamma]=[\beta]} f(\alpha\gamma^{-1})$, so for $f \in \mathcal{B}_c$, $\pi^{av}_x(f)$ has a compactly supported Schwartz kernel $K_{\pi^{av}_x(f)}$ and $\pi^{reg}_x(f)$ has a compactly supported continuous Schwartz kernel $K_{\pi^{reg}_x(f)}$ and so they are trace-class operators.

We prove (ii), the proof of (i) is similar.

$$
\begin{aligned}
\tau^{\Lambda}_{\mathcal{F}} \circ \pi^{av}(f) &= \sum_{i \in I} \int Tr_x(M_{\phi_i} \pi^{av}_x(f) M_{\phi_i}) d\Lambda \\
&= \sum_{i \in I} \int_{t \in T_i} \int_{l \in L_i} \phi_i(l,t) K_{\pi^{av}_x(f)}(l,l,t) d\Lambda_i(t) \\
&= \sum_{i \in I} \int_{t \in T_i} \int_{l \in L_i} \phi_i(l,t) \sum_{\beta \in \mathcal{G}^{(l,t)}_{(l,t)}} f(\beta) d\Lambda_i(t) \\
&= \int_M \sum_{\beta \in \mathcal{G}^x_x} f(\beta) d\mu(x) \\
&= \tau^{\Lambda}_{av}(f)
\end{aligned}
$$

□

Corollary 2.3.10. *1.* τ^{Λ}_{reg} *extends to a positive, faithful, normal trace on* $\mathcal{B}_c$*.*

2. τ^{Λ}_{av} *extends to a positive, normal trace on* $\mathcal{B}_c$*.*

Proof. We only show that τ^{Λ}_{reg} is faithful, as all other properties follow from the previous proposition. Let $f \in \mathcal{B}_c$ and $g = f^* * f$. Then we have

$$g(1_x) = \int_{v \in \mathcal{G}_x} |f(v)|^2 d\lambda_x(v)$$

Now

$$\tau^{\Lambda}_{reg}(g) = 0 \Rightarrow \int_M \int_{v \in \mathcal{G}_x} |f(v)|^2 d\lambda_x(v) d\mu(x) = 0$$

So for μ a.e. everywhere x, $f(v) = 0$ λ_x a.e. for $v \in \mathcal{G}_x$. The continuity of f on $\mathcal{G}$ then implies that $f = 0$ on $\mathcal{G}$, hence $g = 0$. Thus τ^{Λ}_{reg} is faithful.

□

Chapter 3

Foliated Atiyah's theorem

3.1 Pseudodifferential operators on Groupoids

3.1.1 Longitudinal Pseudodifferential operators on Foliations and its monodromy groupoid

Let W and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Definition We denote by $C_c^{\infty,0}(W \times V)$ the space of maps $f : W \times V \to \mathbb{R}$ such that $f(.,y) : W \to \mathbb{R}$ is smooth for all $y \in V$ and $f(x,.) : V \to \mathbb{R}$ is continuous for all $x \in W$.

Definition An operator $P = (P_v)_{v \in V} : C_c^{\infty,0}(W \times V) \to C^{\infty,0}(W \times V)$, such that for $v \in V$, $P_v : C_c^{\infty}(W) \to C^{\infty}(W)$ is a classical pseudodifferential operator is called a continuous family indexed by V of pseudodifferential operators on W , if it satisfies the following relation: for $f \in C_c^{\infty,0}(W \times V), w \in W$,

$$P_v(f_v)(w) = (Pf)(w,v)$$

We note that in the above definition the Schwartz kernel K_P of P can be viewed as a distribution on $W \times W \times V$.

Now let X and Y be smooth manifolds with dimensions q and $p+q$, respectively, and let $s : Y \to X$ be a submersion. For all $x \in X$, one can find an open neighbourhood V_x of x in X and an open subset $\tilde{V}$ of Y, such that there are diffeomorphisms $\psi : V_x \to V$, $\phi : \tilde{V} \to W \times V$, where V is an open subset of $\mathbb{R}^q$ and W is an open subset of $\mathbb{R}^p$ and we have $s|_{\tilde{V}} = \psi^{-1} \circ \mathrm{pr}_2 \circ \phi$, where $\mathrm{pr}_2 : W \times V \to V$ is the projection onto the second factor. Let $C^{\infty,0}(Y)$ be the space of functions $f : Y \to \mathbb{R}$ such that for each such trivialisation given above, the function $f \circ \phi^{-1} : W \times V \to \mathbb{R}$ is in $C^{\infty,0}(W \times V)$.

Definition A family $P = (P_x)_{x \in X}$ is called a continuous family of pseudodifferential operators indexed by the submersion $s : Y \to X$ if each P_x is a classical pseudodifferential operator on $s^{-1}(x)$ such that for each local trivialisation of the submersion, with the same notations as above, the family $(P_v)_{v \in V}$ associated via ψ with the family $(P_{\psi^{-1}(v)})_{\psi^{-1}(v) \in V_x}$, is a continuous family indexed by V of pseudodifferential operators on W, as in the previous definition. One can write P as an operator $P : C_c^{\infty,0}(Y) \to C^{\infty,0}(Y)$.

Let $(M, \mathcal{F})$ be a compact foliated manifold without boundary, and $\mathcal{G}$ be its monodromy groupoid. We denote by $C^{\infty,0}(\mathcal{G})$ the space associated with the submersion $s : \mathcal{G} \to M$, as given above.

Definition The space $\mathcal{P}_c^m(U,\mathcal{F})$ of longitudinal pseudodifferential operators of order m on a foliated chart $U \simeq L \times T$ is defined as the set of continuous families $P = (P_t)_{t\in T}$ of operators indexed by T, with a compactly supported Schwratz kernel K_P viewed as a distribution on $L \times L \times T$, such that each P_t is a classical pseudodifferential operator of order m on $C_c^\infty(L \times \{t\})$.

Definition A uniformly-supported $\mathcal{G}$-operator is a family of operators indexed by M, written $P = (P_x)_{x\in M}$, where $P : C_c^{\infty,0}(\mathcal{G}) \to C_c^{\infty,0}(\mathcal{G})$ is a linear operator, and each $P_x : C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{G}_x)$ is a linear operator satisfying the following relations:

$$Pf(\gamma) = P_{s(\gamma)}\left(f|_{\mathcal{G}_{s(\gamma)}}\right)(\gamma)\ \forall f \in C_c^{\infty,0}(\mathcal{G}), \gamma \in \mathcal{G}, \tag{3.1.1}$$

for $x, y \in M, \gamma' \in \mathcal{G}_x^y$,

$$P_y R_{\gamma'} = R_{\gamma'} P_x \qquad (\mathcal{G}\text{-equivariance}) \tag{3.1.2}$$

where $R_{\gamma'} : C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{G}_y)$ is given by

$$(R_{\gamma'}f)(\alpha) = f(\alpha\gamma')$$

We note that due to the condition 3.1.2, the kernel K_P of a $\mathcal{G}$-operator P can be viewed as a distribution on $\mathcal{G}$ by the formula

$$k_P(\gamma) = K_P(1_{s(\gamma)}, \gamma), \forall \gamma \in \mathcal{G}$$

Definition : A $\mathcal{G}$-pseudodifferential operator of order m with compact support is a $\mathcal{G}$-operator $P = (P_x)_{x\in M}$ such that each P_x is a pseudodifferential operator $P_x : C_c^\infty(\mathcal{G}_x) \to C_c^\infty(\mathcal{G}_x)$ of order m, whose kernel as a distribution on $\mathcal{G}$ has compact support in $\mathcal{G}$. The set of all such operators is denoted $\Psi_c^m(\mathcal{G})$.

We call a $\mathcal{G}$-pseudodifferential operator *compactly smoothing* if its kernel viewed on $\mathcal{G}$ is in $C_c^{\infty,0}(\mathcal{G})$. The set of compactly smoothing operators are denoted $\Psi_c^{-\infty}(\mathcal{G})$.

The following proposition summarizes the properties of operators in $\mathcal{P}_c^m(U,\mathcal{F})$ and $\Psi_c^m(\mathcal{G})$. It is proved in [Va:01].

Proposition 3.1.1. *(i) A family of operators $P \in \mathcal{P}_c^m(U,\mathcal{F})$ induces a $\mathcal{G}$-pseudodifferential operator $\tilde{P}$ of order m with compact support i.e. $\tilde{P} \in \Psi_c^m(\mathcal{G})$. There is an injective map $i_U : \mathcal{P}_c^m(U,\mathcal{F}) \to \Psi_c^m(\mathcal{G})$.*

(ii) Let $(U_j)_{j\in J}$ be a regular covering of M with foliated charts. Then, an operator in $\Psi_c^m(\mathcal{G})$ can be written as a finite linear combination of elements in $i_{U_j}(\mathcal{P}_c^m(U_j,\mathcal{F}))$ and a compactly smoothing operator,i.e. $\Psi_c^m(\mathcal{G}) \subseteq \sum_{j\in S} i_{U_j}(\mathcal{P}_c^m(U_j,\mathcal{F})) + \Psi_c^{-\infty}(\mathcal{G})$, where S is a finite subset of J.

(iii) $\Psi_c^\infty(\mathcal{G}) := \bigcup_{m\in\mathbb{Z}} \Psi_c^m(\mathcal{G})$ is an involutive filtered algebra.

Let $C^{\infty,0}(M,\mathcal{F})$ be the space of functions $f : M \to \mathbb{R}$ such that on a foliated chart (U,θ) of M with $\theta : U \xrightarrow{\cong} L \times T$, where L (resp. T) is an open subset of $\mathbb{R}^p$ (resp. $\mathbb{R}^q$), the map $f_{|_U} \circ \theta^{-1} : L \times T \to \mathbb{R}$ is in $C^{\infty,0}(L \times T)$.

Definition For $P \in \Psi_c^m(\mathcal{G})$ define the operator $r_*(P) : C^{\infty,0}(M,\mathcal{F}) \to C^{\infty,0}(M,\mathcal{F})$ by the formula

$$r_*P(f)(x) := P_{s(\gamma)}(f \circ r)(\gamma), \text{ for any } \gamma \in \mathcal{G}^x$$

The above definition is independent of the choice of γ due to the property 3.1.2 of P.

Let $r_*(\Psi_c^{-\infty}(\mathcal{G}))$ be denoted as $\mathcal{P}_c^{-\infty}(M,\mathcal{F})$ and the vector space generated by linear combinations of elements in $\mathcal{P}_c^m(U,\mathcal{F})$ and $\mathcal{P}_c^{-\infty}(M,\mathcal{F})$ be denoted as $\mathcal{P}_c^m(M,\mathcal{F})$. Let $C_c^{\infty,0}(M\times M,\mathcal{F})$ be the space of functions k on $M\times M$ such that $k(x,y)=0$ if x and y are not on the same leaf and which is locally of the form $k\in C^{\infty,0}(L_i\times L_i\times T_i)$, for regular foliated charts $(U_i)_{i\in S}$ such that $U_i\cong L_i\times T_i$, with compact support in $L\times L$ for each leaf L in $(M,\mathcal{F})$.

Proposition 3.1.2. *(i) If $P\in\mathcal{P}_c^{-\infty}(M,\mathcal{F})$ then $k_P\in C_c^{\infty,0}(M\times M,\mathcal{F})$.*

(ii) $\mathcal{P}_c^m(M,\mathcal{F})\circ\mathcal{P}_c^n(M,\mathcal{F})\subset\mathcal{P}_c^{m+n}(M,\mathcal{F})$.

Proof. (i) Since $P\in\mathcal{P}_c^{-\infty}(M,\mathcal{F})\Leftrightarrow P\in r_*(\Psi_c^{-\infty}(\mathcal{G}))$ by definition, $P=r_*(\tilde{P})$ for some $\tilde{P}\in\Psi_c^{-\infty}(\mathcal{G})$. Therefore the kernel $k_{\tilde{P}}$ (seen as a distribution on $\mathcal{G}$) is in $C_c^{\infty,0}(\mathcal{G})$. Now the kernel of P as a distribution on $M\times M$ is given by the following formula: for x,y in the same leaf,

$$k_P(x,y)=\sum_{\gamma\in\mathcal{G}_y^x}k_{\tilde{P}}(\gamma)$$

Indeed, we have,

$$\begin{aligned}
r_*\tilde{P}(f)(x) &= \tilde{P}_{s(\gamma)}(f\circ r)(\gamma)\\
&= \int_{\mathcal{G}_{s(\gamma)}} K_{\tilde{P}}(\gamma,\gamma_1)(f\circ r)(\gamma_1)d\lambda_{s(\gamma)}(\gamma_1)\\
&= \int_{v\in L_{s(\gamma)}}\sum_{\gamma_1\in\mathcal{G}^v_{s(\gamma)}} K_{\tilde{P}}(\gamma,\gamma_1)(f\circ r)(\gamma_1)d\lambda^{L_{s(\gamma)}}(v)\\
&= \int_{v\in L_{s(\gamma)}}\sum_{\gamma_1\in\mathcal{G}^v_{s(\gamma)}} k_{\tilde{P}}(\gamma\gamma_1^{-1})(f\circ r)(\gamma_1)d\lambda^{L_{s(\gamma)}}(v)\\
&= \int_{v\in L_x}\left(\sum_{\gamma_1\in\mathcal{G}^v_{s(\gamma)}} k_{\tilde{P}}(\gamma\gamma_1^{-1})\right)f(v)d\lambda^{L_x}(v)\\
&= \int_{v\in L_x}\left(\sum_{\alpha\in\mathcal{G}^x_v} k_{\tilde{P}}(\alpha)\right)f(v)d\lambda^{L_x}(v)\ \text{(putting } \alpha=\gamma\gamma_1^{-1})\\
&= \int_{v\in L_x} k_P(x,v)f(v)d\lambda^{L_x}(v)
\end{aligned}$$

hence we have the desired equality. Since $k_{\tilde{P}}$ has compact support C_x in each $\mathcal{G}_x$ for $x\in M$, k_P has support inside $r(C_x)\times s(C_x)$ in $L_x\times L_x$. Moreover, since $k_{\tilde{P}}$ is in $C_c^{\infty,0}(\mathcal{G})$ it is locally of the form $C^{\infty,0}(L_i\times L_i\times T_i)$ and hence k_P is also locally of the same form.

(ii) This follows from Proposition 3.2.4 of [Va:01] and the fact that $\Psi_c^{-\infty}(\mathcal{G})$ is an algebra , thus so is $r_*(\Psi_c^{-\infty}(\mathcal{G}))$ because $r_*(P\circ Q)=r_*(P)\circ r_*(Q)$ for $P\in\Psi_c^m(\mathcal{G}),Q\in\Psi_c^l(\mathcal{G})$. Indeed, we have,

$$\begin{aligned}
r_*(P\circ Q)(f)(x) &= (P\circ Q)_{s(\gamma)}(f\circ r)(\gamma)\ \text{for } \gamma\in\mathcal{G}^x\\
&= (P_{s(\gamma)}\circ Q_{s(\gamma)})(f\circ r)(\gamma)\\
&= P_{s(\gamma)}(Q_{s(\gamma)}(f\circ r))(\gamma)\\
&= P_{s(\gamma)}(r_*Q(f)\circ r)(\gamma)\\
&= (r_*P)(r_*Q(f))(x)\\
&= (r_*P\circ r_*Q)(f)(x)
\end{aligned}$$

where we have used $r_*Q(f)(r(\gamma))=Q_{s(\gamma)}(f\circ r)(\gamma)$ in the fourth equality above. $\square$

3.1.2 Almost local pseudodifferential operators on foliations

Let U be a regular foliated chart and $\theta : U \to L \times T$ be a diffeomorphism, with L (resp. T) open intervals in $\mathbb{R}^p$ (resp. $\mathbb{R}^q$). Let $g_{\mathcal{F}}$ be a leafwise smooth metric on $(M, \mathcal{F})$ whose restriction to U is a continuous family of metrics $(g_t)_{t \in T}$, where g_t is a metric on the plaque $\theta^{-1}(L \times \{t\})$. The associated distance functions are denoted $d_{\mathcal{F}}$ and d_t.

Definition An operator $P \in \mathcal{P}^m_c(M, \mathcal{F})$ is called a c-almost local operator if there exists a constant c, $c \geq 0$ such that the Schwartz kernel of P vanishes outside the set

$$\{(x, y) \in M \times M |\ x, y \text{ are in the same leaf}, d_{\mathcal{F}}(x, y) \leq c\}$$

Remark. *An operator $P \in \mathcal{P}^m_c(U, \mathcal{F})$ is c-almost local for some $c \geq 0$.*

Assume that $\mathcal{G}$ is Hausdorff. With the help of $d_{\mathcal{F}}$ on $(M, \mathcal{F})$, we can define a length function on the groupoid. For a leafwise path a in M (i.e. a lies completely in a leaf) starting at x and ending at y, we denote by $l(a)$ the length of a. Since elements of $\mathcal{G}$ are homotopy classes of leafwise paths in M, we can define for $\gamma \in \mathcal{G}$, $l(\gamma) := \inf\{l(a) | a \in \gamma\}$. Now we define the almost local property for a pseudodifferential operator on the groupoid.

Definition An operator $P \in \Psi^m_c(\mathcal{G})$ is called a c-almost local operator if there exists a constant c, $c \geq 0$ such that the Schwartz kernel of P viewed on $\mathcal{G}$ vanishes outside the set $\{\gamma \in \mathcal{G} | l(\gamma) \leq c\}$.

We call an operator almost local if it is c-almost local for some $c \geq 0$.

Proposition 3.1.3. *A c-almost local operator $P \in \mathcal{P}^m_c(U, \mathcal{F})$ lifts to a c-almost local operator $\tilde{P} \in \Psi^m_c(\mathcal{G})$.*

Proof. Let P be c-almost local. The Schwartz kernel K_P of P has compact support in $\mathcal{G}(U) := L \times L \times T$, and can be extended to all of $\mathcal{G}$ by setting it zero outside $\mathcal{G}(U)$, which gives us an operator $\tilde{P} \in \Psi^m_c(\mathcal{G})$. Then, for $\gamma \in \mathcal{G}$, $l(\gamma) > c \Rightarrow d_t(\theta_1(s(\gamma)), \theta_1(r(\gamma))) > c$, θ_1 being the projection onto the first component under the image of θ. This implies that $K_{\tilde{P}}(\gamma) = 0$ for $l(\gamma) > c$. Thus $\tilde{P}$ is c-almost local. □

Proposition 3.1.4. *Let $k \in C^{\infty,0}_c(M \times M, \mathcal{F})$ be such that it vanishes outside the set $\Delta_c := \{(x, y) \in M \times M | x, y$ are in the same leaf and $d_{\mathcal{F}}(x, y) \leq c\}$. Then k induces $\tilde{k} \in C^{\infty,0}_c(\mathcal{G})$ which defines a c-almost local operator $P \in \Psi^{-\infty}_c(\mathcal{G})$ such that $r_* P$ is c-almost local with Schwartz kernel k.*

Proof. Let $\{U_i\}_{i \in I}$ be a regular foliated covering of $(M, \mathcal{F})$. Let c be small enough so that $d_{\mathcal{F}}(x, y) \leq c$ implies that there exist a regular foliated chart $U \cong L \times T$ for which x and y belong to the some plaque $L \times \{t\}, t \in T$. Define $\tilde{k}(\gamma) = k(s(\gamma), r(\gamma))$ for $\gamma \in \mathcal{G}$. Then $l(\gamma) > c \Rightarrow d_{\mathcal{F}}(x, y) > c$ and therefore $\tilde{k}$ vanishes outside the set $\{\gamma \in \mathcal{G} | l(\gamma) \leq c\}$. Since $k \in C^{\infty,0}_c(M \times M, \mathcal{F})$, [Tu:99, Proposition 1.8] implies that $\tilde{k} \in C^{\infty,0}_c(\mathcal{G})$. Therefore, by definition, $\tilde{k}$ defines a c-almost local operator $P \in \Psi^{-\infty}_c(\mathcal{G})$. Now, by Proposition 3.1.2, the Schwartz kernel $k_{r_* P}$ of $r_*(P)$ is in $C^{\infty,0}_c(M \times M, \mathcal{F})$ and is given by

$$k_{r_* P}(x, y) = \sum_{\mathcal{G}^x_y} \tilde{k}(\gamma)$$

But due to the assumption on c, $\tilde{k}(\gamma)$ is non-zero only for a unique $\gamma \in \mathcal{G}^x_y$ for which there exists a $\delta > 0$ such that all paths a in the class of γ for which $l(a) - l(\gamma) < \delta$ lie completely inside the plaque containing x and y. Therefore if $\tilde{k}(\gamma)$ is non-zero γ can be identified with $(l, l', t) \in L \times L \times T$ where $x = (l, t), y = (l', t)$ in a foliated chart $U \cong L \times T$. Therefore we have,

$$k_{r_* P}(x, y) = \tilde{k}(l, l', t) = k(x, y)$$

Thus the Schwartz kernel of $r_* P$ is k and is c-almost local by hypothesis.

□

3.2 Measured Index of Dirac Operators

3.2.1 Statement of foliated Atiyah's theorem

Let $(M, \mathcal{F})$ be a smooth even-dimensional foliation on a closed (i.e. compact without boundary) manifold M, and Λ be a holonomy invariant transverse measure on M. Let $D = (D_L)_{L \in M/\mathcal{F}}$ be a family of leafwise Dirac-type operators on M. Let $\tilde{D} = (\tilde{D}_x)_{x \in M}$ be the lift of D to the monodromy groupoid, i.e. the pullback r^*D by the range map $r : \mathcal{G} \to M$. For each $x \in M$, $\tilde{D}_x$ is a first order elliptic differential operator acting on $C_c^\infty(\mathcal{G}_x)$. $\tilde{D}$ is then the unique operator satisfying $r_*\tilde{D} = D$, where $r_*\tilde{D}$ is given by

$$r_*\tilde{D}(f)(x) = \tilde{D}_{s(\gamma)}(f \circ r)(\gamma) \text{ for } \gamma \in \mathcal{G}^x$$

Let $E \to M$ be a longitudinally smooth continuous vector bundle over M having a $\mathbb{Z}_2$ grading $E = E^+ \oplus E^-$. Let $C^{\infty,0}(M, E^\pm)$ denote the space of tangentially smooth sections on M which are locally of the form $C^{\infty,0}(L \times T, E^\pm)$. Then we have an induced grading on the pullback bundle r^*E over $\mathcal{G}$. Then we can define D as acting on sections over M in a 2x2 matrix:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \text{ where } D^\pm : C^{\infty,0}(M, E^\pm) \to C^{\infty,0}(M, E^\mp),$$

and similarly for $\tilde{D}$. The operators $D^\pm$, $\tilde{D}^\pm$ are closable and extend to unbounded, densely-defined operators acting on L^2 sections. Let $\pi^\pm$ be the projection onto the space of L^2-solutions of $D^\pm$. We define similarly $\tilde{\pi}^\pm$ for the operator $\tilde{D}$ on the groupoid.

We have the following well-known result. Recall the traces τ^Λ and $\tau^\Lambda_{\mathcal{F}}$ defined in section 2.3.5.

Proposition 3.2.1. *:*

- $\pi^\pm$ *are positive self-adjoint* $\tau^\Lambda_{\mathcal{F}}$*-trace class elements of* $W^*(M, \mathcal{F}, E^\pm)$.
- $\tilde{\pi}^\pm$ *are positive self-adjoint* τ^Λ*-trace class elements of* $W^*(\mathcal{G}, E^\pm)$.

Proof. We only prove the second statement. By [Theorem 7.6, [Ro:88]], we know that for any Schwartz function $f \in \mathcal{S}(\mathbb{R})$, $f(D) \in W^*(\mathcal{G}, E)$ and is a tangentially smoothing operator with uniformly bounded kernel. By approximating χ_0(the characteristic function at 0) by rapidly decreasing functions with pointwise convergence and using the spectral theorem, we get a sequence of operators $A^\pm_\lambda \in W^*(\mathcal{G}, E)^+$ which converge strongly to $\tilde{\pi}^\pm$. Since $W^*(\mathcal{G}, E)$ is strongly closed, we get that $\tilde{\pi}^\pm \in W^*(\mathcal{G}, E)^+$. Now consider the function $\exp(-tx^2)$ on $\mathbb{R}$. By [Prop 7.37, [Ro:88]], $\exp(-tD^2)$ is of τ^Λ-trace class. Since $\chi_0(x)\exp(-tx^2) = \chi_0(x)$, we get by the spectral theorem that

$$\tilde{\pi} = \tilde{\pi}\exp(-tD^2)$$

Since $\exp(-tD^2)$ is τ^Λ-trace class, we get that $\tilde{\pi}$ is τ^Λ-trace class. ∎

We can now define the measured indices of D^+ and $\tilde{D}^+$:

Definition The measured index of D^+ is defined as $\mathrm{Ind}^\Lambda(D^+) = \tau^\Lambda_{\mathcal{F}}(\pi^+) - \tau^\Lambda_{\mathcal{F}}(\pi^-)$.

The measured index of $\tilde{D}^+$ is defined as $\mathrm{Ind}^\Lambda(\tilde{D}^+) = \tau^\Lambda(\tilde{\pi}^+) - \tau^\Lambda(\tilde{\pi}^-)$.

Theorem 3.2.2 (Foliated Atiyah's Theorem)**.** $\mathrm{Ind}^\Lambda(\tilde{D}^+) = \mathrm{Ind}^\Lambda(D^+)$

We will extend Atiyah's theorem and show the following steps:

1. There exists an almost-local parametrix $Q^- : C^{\infty,0}(M, E^-) \to C^{\infty,0}(M, E^+)$ for D^+, i.e. $Q^- \in \mathcal{P}^{-1}(M, \mathcal{F})$ such that the kernel of Q^- has support as localized near the diagonal $\Delta = \{(x, x)|x \in M\}$ as wanted, and $S^+ = 1 - Q^- D^+ \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$ and $S^- = 1 - D^+ Q^- \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$, with both S^+ and S^- $\tau_{\mathcal{F}}^{\Lambda}$-trace class.

2. If Q^- is localized near Δ, then it lifts to a parametrix $\tilde{Q}^-$ of $\tilde{D}^+$, such that we have $\tilde{Q}^- \in \Psi_c^{-1}(\mathcal{G})$, with $\tilde{S}^+ = 1 - \tilde{Q}^- \tilde{D}^+ \in \Psi_c^{-\infty}(\mathcal{G})$ and $\tilde{S}^- = 1 - \tilde{D}^+ \tilde{Q}^- \in \Psi_c^{-\infty}(\mathcal{G})$. Moreover, $\tilde{Q}^-, \tilde{S}^+, \tilde{S}^-$ are τ^{Λ}-trace class, and satisfy (cf. 3.2.9)

$$\tau^{\Lambda}(\tilde{S}^{\pm}) = \tau_{\mathcal{F}}^{\Lambda}(S^{\pm}) \tag{3.2.1}$$

3. For any such almost-local parametrix $Q^- : E^- \to E^+$ of D^+ as in step 1 above , we have the Atiyah-Bott formula $\tau_{\mathcal{F}}^{\Lambda}(\pi^+) - \tau_{\mathcal{F}}^{\Lambda}(\pi^-) = \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-)$.

4. For the parametrix $\tilde{Q}^-$ of $\tilde{D}^+$ given by the lift of Q^- in step 2 above, we have $\tau^{\Lambda}(\tilde{\pi}^+) - \tau^{\Lambda}(\tilde{\pi}^-) = \tau^{\Lambda}(\tilde{S}^+) - \tau^{\Lambda}(\tilde{S}^-)$.

Then we would have:

$$\begin{aligned}
\mathrm{Ind}^{\Lambda}(D^+) &= \tau_{\mathcal{F}}^{\Lambda}(\pi^+) - \tau_{\mathcal{F}}^{\Lambda}(\pi^-) \\
&= \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-) \\
&= \tau^{\Lambda}(\tilde{S}^+) - \tau^{\Lambda}(\tilde{S}^-) \\
&= \tau^{\Lambda}(\tilde{\pi}^+) - \tau^{\Lambda}(\tilde{\pi}^-) \\
&= \mathrm{Ind}^{\Lambda}(\tilde{D}^+)
\end{aligned} \tag{3.2.2}$$

3.2.2 Construction of the parametrix

For an elliptic operator $P \in \mathcal{P}_c^m(M, \mathcal{F})$, we want to construct an almost-local parametrix Q. To do this, we will follow the general construction of a parametrix for an elliptic operator to get a parametrix Q' (which need not be almost-local), and then show that there exists an almost-local operator of the same order as Q' such that $Q - Q' \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$. It will follow that Q is a parametrix for P which can be chosen as almost-local as wanted.

Let us begin by solving the analogous local problem, i.e. to construct an almost-local parametrix for an elliptic operator $P \in \mathcal{P}_c^m(U, \mathcal{F})$, where $U \simeq L \times T$ is a regular foliated chart of M. Recall that an operator $P \in \mathcal{P}_c^m(U, \mathcal{F})$ is called elliptic if the principal symbol $\sigma_m(P)(x, t, \xi)$ is invertible for $(x, t, \xi) \in L \times T \times \mathbb{R}^p \backslash \{0\}$. The following proposition is a consequence of the proofs in [Co:79, Page 128], [MoSc:06, Page179 Proposition 7.10].

Proposition 3.2.3. *For an elliptic operator $P \in \mathcal{P}_c^m(U, \mathcal{F})$, there exists $Q \in \mathcal{P}_c^{-m}(U, \mathcal{F})$ such that $S_0 = 1 - QP \in \mathcal{P}_c^{-\infty}(U, \mathcal{F}), S_1 = 1 - PQ \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$.*

We now construct a c-almost local parametrix for $P \in \mathcal{P}_c^m(U, \mathcal{F})$ with arbitrarily small $c > 0$.

Proposition 3.2.4. *For a given parametrix Q' of an elliptic operator $P \in \mathcal{P}_c^m(U, \mathcal{F})$, there exists for any $\epsilon > 0$ an ϵ-almost local operator $Q \in \mathcal{P}_c^{-m}(U, \mathcal{F})$ for P, such that $Q' - Q \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$.*

Proof. : Let the Schwartz Kernel of Q' be denoted by K, which is a distribution over $\mathcal{G}(U) := L \times L \times T$. We know that K is smooth outside a compact neighbourhood W of the diagonal where $d_{\mathcal{F}}(x, y) > \epsilon$ outside W. We choose a smooth cutoff function χ on $\mathcal{G}(U)$ with the following properties:

(i) $supp(\chi) \subseteq W$.

(ii) $\chi = 1$ on an open neighbourhood W' of $\Delta \times T$ with $\overline{W'} \subset \mathrm{int}(W)$.

Then, letting Q denote the operator given by $M_\chi Q'$, where M_χ is the multiplication operator with the function χ, we find that $Q' - Q$ has kernel which vanishes on W'. This implies, by the pseudolocal property, that it is smooth on $\mathcal{G}(U)$. Hence we have $Q' - Q \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$ and Q is ϵ-almost local. $\square$

Remark. *If $P_1 \in \mathcal{P}_c^m(M, \mathcal{F})$ is c_1-almost local and $P_2 \in \mathcal{P}_c^n(M, \mathcal{F})$ is c_2-almost local then $P_1 P_2 \in \mathcal{P}_c^{m+n}(M, \mathcal{F})$ is $(c_1 + c_2)$-almost local.*

Corollary 3.2.5. *: For an ϵ_0-almost local elliptic operator $P \in \mathcal{P}_c^m(U, \mathcal{F})$, there exists for every $\epsilon > 0$ an ϵ-almost local parametrix $Q \in \mathcal{P}_c^{-m}(U, \mathcal{F})$ such that $S_0 = I - QP \in \mathcal{P}_c^{-\infty}(U, \mathcal{F}), S_1 = I - PQ \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$, with S_0 and S_1 $(\epsilon + \epsilon_0)$-almost local.*

Proof. : Let Q' be a parametrix for P. Then from proposition 3.2.4, there exists an almost local operator $Q \in \mathcal{P}_c^{-m}(U, \mathcal{F})$ such that $R := Q' - Q \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$. Then, we have

$$\begin{aligned} QP &= (Q - Q' + Q')P = RP + Q'P \\ &= RP + I - S_0' = I - S_0 \end{aligned} \tag{3.2.3}$$

where $I - QP = S_0 = RP - S_0' \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$. Similarly we can find an $S_1 = I - PQ \in \mathcal{P}_c^{-\infty}(U, \mathcal{F})$. The almost local property of S_0 and S_1 follows from the convolution formula of kernels together with the remark above, and that I is 0-almost local. $\square$

We now patch together our local parametrices to get

Proposition 3.2.6. *: For an elliptic operator $P \in \mathcal{P}_c^m(M, \mathcal{F})$, there exists an almost local parametrix $Q \in \mathcal{P}_c^{-m}(M, \mathcal{F})$ such that $R = 1 - PQ \in \mathcal{P}_c^{-\infty}(M, \mathcal{F}), R' = 1 - QP \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$, with R and R' almost-local.*

Proof. : Let $\{\phi_i\}_{i=1}^N$ be a partition of unity subordinate to a regular covering $\{U_i\}_{i=1}^N$ for $(M, \mathcal{F})$. For each i, let $\phi_i' \in C_c^\infty(U_i)$ be such that $\phi_i' = 1$ on $\mathrm{supp}(\phi_i)$. Let C be a compact neighbourhood of the diagonal in $M \times M$ such that the set $\{(x, y) | x \in \mathit{supp}(\phi_i')\} \subseteq C$ for all i. Since P is pseudolocal, we can assume that its Schwartz kernel has support inside C. Let $M_{\phi_i'}$ be the multiplication with the function ϕ_i', and put $P_i = PM_{\phi_i'}$. Then $P_i \in \mathcal{P}_c^m(U_i, \mathcal{F})$. Let $Q = \sum_{i=1}^N M_{\phi_i'} Q_i M_{\phi_i}$, where Q_i is the almost local parametrix for P_i, such that $P_i Q_i = M_{\phi_i'} - R_i$. Then Q is the required almost local parametrix for P. To check this, we calculate:

$$\begin{aligned} PQ &= \textstyle\sum_{i=1}^N PM_{\phi_i'} Q_i M_{\phi_i} \\ &= \textstyle\sum_{i=1}^N (P_i Q_i M_{\phi_i}) \\ &= \textstyle\sum_{i=1}^N (M_{\phi_i'} M_{\phi_i} - R_i M_{\phi_i}) \\ &= I - R \end{aligned} \tag{3.2.4}$$

with $R \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$. It remains to see that Q and R are almost local. Indeed, since Q is a sum of almost local operators, it is almost local. The same argument holds for R as each R_i is almost local. $\square$

Definition An operator $P \in \Psi_c^m(\mathcal{G})$ is called elliptic if the operator $r_* P \in \mathcal{P}_c^m(M, \mathcal{F})$ is elliptic.

The following proposition gives the existence of parametrices of elliptic operators in $\Psi_c^m(\mathcal{G})$.

Proposition 3.2.7. *Let $P \in \Psi_c^m(\mathcal{G})$ be an elliptic operator. Then there exists an operator $Q \in \Psi_c^{-m}(\mathcal{G})$ such that*

$$PQ = I - R \text{ and } QP = I - R'$$

where $R, R' \in \Psi_c^{-\infty}(\mathcal{G})$.

Proof. See [Co:79, Page 128] or [MoSc:06, Page179 Proposition 7.10] for a detailed proof. □

Let us keep the notations from the statement of Theorem 3.2.2.

Corollary 3.2.8. *: Let $Q^- \in \mathcal{P}_c^{-1}(M, \mathcal{F})$ be a parametrix for D^+ which is c-almost local for c sufficiently small, with $D^+Q^- = I - S^-$ and $Q^-D^+ = I - S^+$ for $S^\pm \in \mathcal{P}_c^{-\infty}(M, \mathcal{F})$. Let $\tilde{Q}^- \in \Psi_c^{-1}(\mathcal{G})$ be the lift of Q^- and $\tilde{S}^\pm \in \Psi_c^{-\infty}(\mathcal{G})$ be the lifts of $S^\pm$. Then we have*

$$\tilde{D}^+\tilde{Q}^- = I - \tilde{S}^- \text{ and } \tilde{Q}^-\tilde{D}^+ = I - \tilde{S}^+$$

Proof. : This follows from the construction of the parametrix Q^- in the proof of Proposition 4.1.1 and the construction of the parametrix for an elliptic operator in $\Psi_c^m(\mathcal{G})$ (see [MoSc:06, Page 179 Proposition 7.10]) , choosing the parametrix Q^- to be as almost-local as we want. □

Proposition 3.2.9. *: Let us take the notations given after the statement of Theorem 3.2.2. Then we have,*

$$\tau^\Lambda(\tilde{S}^\pm) = \tau^\Lambda_{\mathcal{F}}(S^\pm) \tag{3.2.5}$$

Proof. Let $(U_i)_{i\in I}$ be a regular foliated cover of $(M, \mathcal{F})$. The operators $\phi_i^{1/2} S \phi_i^{1/2} \in \mathcal{P}_c^{-\infty}(U_i, \mathcal{F})$ for each $i \in I$. Then as in Proposition 3.1.3 and Proposition 3.1.4, we let $c > 0$ to be small enough such that the Schwartz kernel $k_{S^\pm} \in C_c^{\infty,0}(M \times M, \mathcal{F})$ of $S^\pm$ coincides with the Schwartz kernel of $\tilde{S}^\pm$, since $r_*\tilde{S}^\pm = S^\pm$. Then choosing S to be c-almost local, the operators $\phi_i^{1/2} S \phi_i^{1/2}$ lift to the operator $\tilde{\phi}_i^{1/2} \tilde{S} \tilde{\phi}_i^{1/2} \in \Psi_c^{-\infty}(\mathcal{G})$ which are also c-almost local and the Schwartz kernels of $\phi_i^{1/2} S \phi_i^{1/2}$ and $\tilde{\phi}_i^{1/2} \tilde{S} \tilde{\phi}_i^{1/2}$ coincide by the proof of Proposition 3.1.4. Hence, by the definition of the traces τ^Λ and $\tau^\Lambda_{\mathcal{F}}$, we get the desired result. □

3.2.3 Atiyah-Bott formula for the measured index

In the following proposition and proof we denote the trace on $W^*(M, \mathcal{F}, E)$ as τ^Λ for convenience of notation.

Proposition 3.2.10. *: Let Q^- be a parametrix for D^+, put $S^+ = I - Q^-D^+$, $S^- = I - D^+Q^-$. Then , we have*

$$\tau^\Lambda(\pi^+) - \tau^\Lambda(\pi^-) = \tau^\Lambda(S^+) - \tau^\Lambda(S^-) \tag{3.2.6}$$

Proof. : We will follow the method of Atiyah [At:76] to prove (3.2.6). We have the following relations:

- $D^+S^+ = D^+ - D^+Q^-D^+ = S^-D^-$
- $S^+Q^- = Q^- - Q^-D^+Q^- = Q^-S^-$ (*)
- $S^+\pi^+ = \pi^+ - Q^-D^+\pi^+ = \pi^+$
- $\pi^-S^- = \pi^- - \pi^-D^+Q^- = \pi^-$

The last relation uses the fact that $\pi^- = P|_{kerD^-} = P|_{ImD^{+\perp}}$, as D^+ is the adjoint of D^-. Indeed, we have, for $u, v \in L^2(L(x), E^+)$,

$< \pi^- D^+ u, v >=< D^+ u, \pi^- v >=< u, D^- \pi^- v >= 0.$

Now, let us define even operators T^+ and T^- as follows:

$$T^+ = (1 - \pi^+) S^+ (1 - \pi^+) \tag{3.2.7}$$

$$T^- = (1 - \pi^-) S^- (1 - \pi^-) \tag{3.2.8}$$

Then,

$$\begin{aligned} \tau^\Lambda(T^+) &= \tau^\Lambda((1 - \pi^+) S^+ (1 - \pi^+)) \\ &= \tau^\Lambda((S^+ - \pi^+ S^+)(1 - \pi^+)) \\ &= \tau^\Lambda(S^+) - \tau^\Lambda(\pi^+) + \tau^\Lambda(\pi^+ Q^- D^+) \end{aligned}$$

Similarly we get

$$\begin{aligned} \tau^\Lambda(T^-) &= \tau^\Lambda((1 - \pi^-) S^- (1 - \pi^-)) \\ &= \tau^\Lambda((S^- - \pi^- S^-)(1 - \pi^-)) \\ &= \tau^\Lambda(S^-) - \tau^\Lambda(\pi^-) + \tau^\Lambda(D^+ Q^- \pi^-) \end{aligned}$$

But from the trace property, we know $\tau^\Lambda(AB) = \tau^\Lambda(BA)$, so $\tau^\Lambda(\pi^+ Q^- D^+) = \tau^\Lambda(Q^- D^+ \pi^+) = 0$. Similary, $\tau^\Lambda(D^+ Q^- \pi^-) = \tau^\Lambda(\pi^- D^+ Q^-) = 0$

Therefore, proving equation(3.2.6) is equivalent to showing that

$$\tau^\Lambda(T^+) = \tau^\Lambda(T^-) \tag{3.2.9}$$

Using the previous relations (*), We also have the following relations:

- $D^+ T^+ = D^+ (I - \pi^+) S^+ (I - \pi^+) = D^+ S^+ (I - \pi^+) = S^- D^+ - D^+ \pi^+ = S^- D^+$
- $T^- D^+ = (I - \pi^-) S^- (I - \pi^-) D^+ = (I - \pi^-) S^- D^+ = S^- D^+ - \pi^- D^+ = S^- D^+$ So, we have,

$$D^+ T^+ = T^- D^+ \tag{3.2.10}$$

Now D^+ has a polar decomposition $D^+ = U^+ A^+$, where U^+ is a partial isometry and A^+ is a positive self-adjoint operator, this is well defined from the functional calculus of measurable families, with both U^+ and A^+ being measurable family of operators [Di:57] Page169. Define

$R^+ := (U^+)^* T^- U^+$ Since U^+ is a partial isometry,by the unitary equivalence of the usual trace tr, we have

$$\tau^\Lambda(R^+) = \tau^\Lambda(T^-), \tag{3.2.11}$$

while (3.2.10) gives

$$A^+ T^+ = T^- A^+ \tag{3.2.12}$$

Now, let P_n^+ be the spectral projection of A^+ corresponding to the closed interval $[\frac{1}{n}, n]$, and we put

$$T_n^+ = P_n^+ T^+ P_n^+, \qquad R_n^+ = P_n^+ R^+ P_n^+, \qquad A_n^+ = P_n^+ A^+ P_n^+ + (I - P_n^+)$$

We claim that A_n^+ is bounded and invertible.Since $P_n^+ A^+ P_n^+$ is bounded, so is A_n^+. To prove invertibility, it suffices to prove that 0 is not in the spectrum of $(A_n^+)_L$ for almost every $L \in M/\mathcal{F}$. We apply the spectral mapping theorem to see that $\sigma(A_n^+) = f(\sigma(A^+))$, where

$$f(\lambda) = \chi_{[\frac{1}{n}, n]}(\lambda) \lambda \chi_{[\frac{1}{n}, n]}(\lambda) + 1 - \chi_{[\frac{1}{n}, n]}(\lambda)$$

Since $f(\lambda) = \lambda$ if $\lambda \in [\frac{1}{n}, n]$ and takes the value 1 otherwise, we see that $\sigma(A_n^+) \subseteq [\frac{1}{n}, n] \cup \{1\}$. Therefore A_n^+ is invertible as it is invertible for $L \in M/\mathcal{F}$(cf. [Di:57] Page159).

Hence, from (3.2.12),we have,$\Rightarrow A_n^+ T^+ (A_n^+)^{-1} = R_n^+$. Taking the trace on both sides gives, due to the unitary equivalence of the trace,

$$\tau^\Lambda(T_n^+) = \tau^\Lambda(R_n^+) \tag{3.2.13}$$

Note that P_n^+ commutes with both T^+ and R^+, so we get

$$\tau^\Lambda(T_n^+) = \tau^\Lambda(T^+ P_n^+), \quad \text{and} \quad \tau^\Lambda(R_n^+) = \tau^\Lambda(R^+ P_n^+) \tag{3.2.14}$$

Since $\chi_{[\frac{1}{n},n]}$ is a uniformly bounded sequence($|\chi_{[\frac{1}{n},n]}(\lambda)| \leq 1$) and $\chi_{[\frac{1}{n},n]} \to \chi_{(0,\infty)}$ converges pointwise, by Lebesgue's dominated convergence theorem we have $\chi_{[\frac{1}{n},n]}(A_L^+) \to \chi_{(0,\infty)}(A_L^+)$ strongly with $\sup_L ||\chi_{[\frac{1}{n},n]}(A_L^+)|| = 1 < \infty$ for each $L \in M/\mathcal{F}$. Proposition 4(ii) page 160 of [Di:57] then guarantees the strong convergence of $P_n^+ = \chi_{[\frac{1}{n},n]}(A^+)$ to $\chi_{(0,\infty)}(A^+) = I - \chi_{(-\infty,0]}(A^+) = I - \chi_{\{0\}}(A^+) = I - \pi^+$.

So, by Proposition(2.3.8) for the trace τ^Λ, we get

$$\lim_{n\to\infty} \tau^\Lambda(T^+ P_n^+) = \tau^\Lambda(T^+(I - \pi^+)) = \tau^\Lambda(T^+),$$

where the last equality comes from the definition of T^+. Similarly we have

$$\lim_{n\to\infty} \tau^\Lambda(R^+ P_n^+) = \tau^\Lambda(R^+(I - \pi^+)) = \tau^\Lambda(R^+),$$

as $KerU^+ = KerD^+ \Rightarrow U^+\pi^+ = 0$

Therefore by equations (3.2.13) and (3.2.14) we get $\tau^\Lambda(T^+) = \tau^\Lambda(R^+)$, which together with (3.2.11) proves (3.2.9):

$$\tau^\Lambda(T^+) = \tau^\Lambda(T^-), \tag{3.2.15}$$

thus completing the proof of (3.2.6). □

We can repeat the same arguments given above to prove that

Proposition 3.2.11. *: Let $\tilde{Q}^-$ be a parametrix for $\tilde{D}^+$, put $\tilde{S}^+ = I - \tilde{Q}^-\tilde{D}^+$, $\tilde{S}^- = I - \tilde{D}^+\tilde{Q}^-$. Then , we have*

$$\tau^\Lambda(\tilde{\pi}^+) - \tau^\Lambda(\tilde{\pi}^-) = \tau^\Lambda(\tilde{S}^+) - \tau^\Lambda(\tilde{S}^-) \tag{3.2.16}$$

Equations (3.2.6) and (3.2.16) thus complete the proof of Theorem(3.2.2).

For completeness, we add the following important corollary.

Proposition 3.2.12 (Calderon's formula)**.** *For any $n \geq 1$, and with notations as in the previous section, we have*

$$\mathrm{Ind}^\Lambda(\tilde{D}) = \tau^\Lambda((\tilde{S}^+)^n) - \tau^\Lambda((\tilde{S}^-)^n) \text{ and } \mathrm{Ind}^\Lambda(D) = \tau^\Lambda_{\mathcal{F}}((S^+)^n) - \tau^\Lambda_{\mathcal{F}}((S^-)^n) \tag{3.2.17}$$

Proof. We will give the proof of the second equation, the proof of the first one is similar. We first note that since S^+ and S^- are leafwise smoothing operators, D^+S^+ is a bounded operator which lies in the von Neumann algebra $W^*(M, \mathcal{F}; E)$. As in the proof of Theorem 3.2.2, we recall the following relations between S^+, S^-, Q^+, Q^-, D^+ and D^-:

- $S^+ = I - Q^-D^+$

- $S^- = I - D^+Q^-$
- $D^+S^+ = D^+ - D^+Q^-D^+ = S^-D^+$
- $S^+Q^- = Q^- - Q^-D^+Q^- = Q^-S^-$

We now calculate:

$$\begin{aligned}(S^+)^2 &= (I - Q^-D^+)^2 \\ &= I - 2Q^-D^+ + Q^-D^+Q^-D^+ \\ &= I - 2Q^-D^+ + Q^-(I - S^-)D^+ \\ &= I - Q^-D^+ - Q^-S^-D^+ \\ &= S^+ - Q^-S^-D^+\end{aligned} \tag{3.2.18}$$

Similarly, $(S^-)^2 = S^- - D^+S^+Q^-$. Since S^+ and S^- are $\tau_{\mathcal{F}}^{\Lambda}$-trace class, so are the operators $(S^+)^2, (S^-)^2$. The operators $Q^-S^-D^+$ and $D^+S^+Q^-$ are also $\tau_{\mathcal{F}}^{\Lambda}$-trace class, since we have $Q^-S^-D^+ = Q^-D^+S^+$ and $D^+S^+Q^- = S^-D^+Q^-$. Now, taking traces on both sides of the above equations and taking their difference, we get,

$$\begin{aligned}\tau_{\mathcal{F}}^{\Lambda}((S^+)^2) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^2) &= \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-) + \tau_{\mathcal{F}}^{\Lambda}(Q^-S^-D^+) - \tau_{\mathcal{F}}^{\Lambda}(D^+S^+Q^-) \\ &= \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-) + \tau_{\mathcal{F}}^{\Lambda}(Q^-D^+S^+) - \tau_{\mathcal{F}}^{\Lambda}(D^+S^+Q^-) \\ &= \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-)\end{aligned} \tag{3.2.19}$$

where we have used the relations given above, the fact that Q^- and D^+S^+ are bounded operators in the von Neumann algebra $W^*(M, \mathcal{F}, E)$ and $Q^-D^+S^+$ (resp. $D^+S^+Q^-$) lies in the von Neumann algebra $W^*(M, \mathcal{F}, E^+)$ (resp. $W^*(M, \mathcal{F}, E^-)$ and the tracial property for $\tau_{\mathcal{F}}^{\Lambda}$ (cf. 2.3.20).

Now let us assume that (**??**) is satisfied for $n = m$. We will prove it for $n = m + 1$. To this end, we calculate:

$$\begin{aligned}\tau_{\mathcal{F}}^{\Lambda}((S^+)^{m+1}) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^{m+1}) &= \tau_{\mathcal{F}}^{\Lambda}(S^+(S^+)^m) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^mS^-) \\ &= \tau_{\mathcal{F}}^{\Lambda}((I - Q^-D^+)(S^+)^m) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^m(I - D^+Q^-)) \\ &= \tau_{\mathcal{F}}^{\Lambda}((S^+)^m - Q^-D^+(S^+)^m) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^m - (S^-)^mD^+Q^-) \\ &= \tau_{\mathcal{F}}^{\Lambda}((S^+)^m) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^m) - \tau_{\mathcal{F}}^{\Lambda}(Q^-D^+(S^+)^m) + \tau_{\mathcal{F}}^{\Lambda}((S^-)^mD^+Q^-)\end{aligned}$$

However, $D^+(S^+)^m$ is a bounded $\tau_{\mathcal{F}}^{\Lambda}$-trace class operator in $W^*(M, \mathcal{F}, E)$, and we also have $D^+(S^+)^m = (S^-)^mD^+$. So $\tau_{\mathcal{F}}^{\Lambda}(Q^-D^+(S^+)^m) = \tau_{\mathcal{F}}^{\Lambda}((S^-)^mD^+Q^-)$. Therefore we get,

$$\begin{aligned}\tau_{\mathcal{F}}^{\Lambda}((S^+)^{m+1}) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^{m+1}) &= \tau_{\mathcal{F}}^{\Lambda}((S^+)^m) - \tau_{\mathcal{F}}^{\Lambda}((S^-)^m) \\ &= \tau_{\mathcal{F}}^{\Lambda}(S^+) - \tau_{\mathcal{F}}^{\Lambda}(S^-) \qquad \text{(by the induction hypothesis)} \\ &= \mathrm{Ind}^{\Lambda}(D) \qquad \text{(by the Atiyah-Bott formula)}\end{aligned}$$

□

Remark. *We also have the following formulae which are easily proved by induction:*

$$\begin{aligned}&\bullet (S^+)^m = S^+ - Q^-(S^- + (S^-)^2 + ... + (S^-)^{m-1})D^+ \\ &\bullet (S^-)^m = S^- - D^+(S^+ + (S^+)^2 + ... + (S^+)^{m-1})Q^-\end{aligned} \tag{3.2.20}$$

3.3 K-theoretic Index of Dirac operators

3.3.1 Hilbert C^*-modules on foliations

Let T be a complete transversal for the foliation (M, F). Then we have a subgroupoid $\mathcal{G}_T^T$ of $\mathcal{G}$ which consists of arrows in $\mathcal{G}$ starting and ending in T, and $\mathcal{A}_c^T := C_c(\mathcal{G}_T^T)$ naturally inherits a $*$-algebra structure by convolution and involution defined in a similar way as in the previous section.

Let $f \in \mathcal{A}_c^T, \xi \in l^2(\mathcal{G}_x^T)$. The regular representation $\rho_x^{reg} : \mathcal{A}_c^T \to B(l^2(\mathcal{G}_x^T))$ is defined as

$$[\rho_x^{reg}(f)](\xi)(\gamma) = \sum_{\gamma' \in \mathcal{G}_x^T} \xi(\gamma') f(\gamma\gamma'^{-1}), \tag{3.3.1}$$

So in a similar manner as in section 1 we get a regular norm $||.||_{reg} := \sup_{x\in T} ||\rho_x^{reg}(f)||$, and the completion of $\mathcal{A}_c^T$ in this norm is the reduced C^*-algebra $\mathcal{A}_r^T$. Similarly, the maximal norm, obtained by taking the *sup* over all representations of $\mathcal{A}_c^T$ which are l^1-continuous, gives the maximal C^*-algebra $\mathcal{A}_m^T$ on completion.

Similarly, we have an average representation ρ_x^{av} of $\mathcal{A}_c^T$ on $l^2(\mathcal{G}_x^T/\mathcal{G}_x^x)$ given by

$$\begin{aligned} [\rho_x^{av}(f)](\phi)([\gamma]) &= \textstyle\sum_{\gamma' \in \mathcal{G}_x^T} \xi([\gamma']) f(\gamma\gamma'^{-1}) \\ &= \textstyle\sum_{[\gamma'] \in \mathcal{G}_x^T/\mathcal{G}_x^x} \sum_{[\eta]=[\gamma']} \xi([\gamma']) f(\gamma\eta^{-1}) \end{aligned} \tag{3.3.2}$$

Let $\mathcal{G}_T := s^{-1}(T)$. A right action of $\mathcal{A}_c^T$ on $\mathcal{E}_c := C_c(\mathcal{G}_T, r^*E)$ is defined as follows:

$$(\xi f)(\gamma) = \sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} f(\gamma'\gamma)\xi(\gamma'^{-1}) \text{ for } f \in \mathcal{A}_c^T, \xi \in \mathcal{E}_c, \gamma \in \mathcal{G}_T \tag{3.3.3}$$

Proposition 3.3.1. *We have $(\xi f)g = \xi(fg)$ for $f, g \in \mathcal{A}_c^T, \xi \in \mathcal{E}_c$.*

Proof. We have for $\gamma \in \mathcal{G}_T$,

$$\begin{aligned} [(\xi f)g](\gamma) &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} g(\gamma'\gamma)(\xi f)(\gamma'^{-1}) \\ &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} g(\gamma'\gamma) \sum_{\beta \in \mathcal{G}_{s(\gamma')}^T} f(\beta\gamma'^{-1})\xi(\beta^{-1}) \\ &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} \sum_{\beta \in \mathcal{G}_{s(\gamma')}^T} g(\gamma'\gamma) f(\beta\gamma'^{-1})\xi(\beta^{-1}) \end{aligned} \tag{3.3.4}$$

On the other hand , we have,

$$\begin{aligned} [\xi(fg)](\gamma) &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} (f * g)(\gamma'\gamma)\xi(\gamma'^{-1}) \\ &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} \sum_{\alpha \in \mathcal{G}_{r(\gamma')}^T} g(\alpha\gamma'\gamma) f(\alpha^{-1})\xi(\beta^{-1}) \\ &= \textstyle\sum_{\gamma' \in \mathcal{G}_{r(\gamma)}^T} \sum_{\alpha \in \mathcal{G}_{s(\gamma')=r(\gamma)}^T} f(\gamma'\alpha^{-1}) g(\alpha\gamma)\xi(\gamma'^{-1}) \end{aligned} \tag{3.3.5}$$

where in the last equality we have used the $\mathcal{G}$-equivariance of the Haar system. As f, g are of compact support we can interchange the order of summations to see that the last lines of 3.3.4 and 3.3.5 are the same. Thus we have a well-defined right-$C_c(\mathcal{G}_T^T)$ module structure on $\mathcal{E}_c := C_c(\mathcal{G}_T, r^*E)$. □

We also define an $\mathcal{A}_c^T$-valued inner product on $\mathcal{E}_c$:

$$< \xi_1, \xi_2 > (u) = \int_{v \in \mathcal{G}_{r(u)}} < \xi_1(v), \xi_2(vu) >_{E_{r(v)}} d\lambda_{r(u)}(v) \text{ for } \xi_1, \xi_2 \in \mathcal{E}_c, u \in \mathcal{G}_T^T. \tag{3.3.6}$$

We will prove the following linearity property for the inner product defined above[1]:

Proposition 3.3.2. *Let $\xi_1, \xi_2 \in \mathcal{E}_c, \gamma \in \mathcal{G}_T^T$, and $f \in C_c(\mathcal{G}_T^T$. Then*

$$< \xi_1, \xi_2 f >=< \xi_1, \xi_2 > *f, \text{ where } * \text{ denotes convolution in } \mathcal{A}_c^T.$$

Proof. We compute first the LHS in Proposition(3.3.2):

$$\begin{aligned}
< \xi_1, \xi_2 f > (\gamma) &= \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} < \xi_1(\gamma'), (\xi_2 f)(\gamma'\gamma) >_{E_{r(\gamma')}} d\lambda_{r(\gamma)}(\gamma') \\
&= \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} < \xi_1(\gamma'), \textstyle\sum_{\alpha \in \mathcal{G}^T_{r(\gamma')}} f(\alpha\gamma'\gamma)\xi(\alpha^{-1}) >_{E_{r(\gamma')}} d\lambda_{r(\gamma)}(\gamma') \\
&= \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \textstyle\sum_{\alpha \in \mathcal{G}^T_{r(\gamma')}} f(\alpha\gamma'\gamma) < \xi_1(\gamma'), \xi_2(\alpha^{-1}) >_{E_r(\gamma')} d\lambda_{r(\gamma)}(\gamma') \\
&= \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \textstyle\sum_{\alpha \in \mathcal{G}^T_{s(\gamma')=r(\gamma)}} f(\alpha\gamma) < \xi_1(\gamma'), \xi_2(\gamma'\alpha^{-1}) >_{E_r(\gamma')} d\lambda_{r(\gamma)}(\gamma')
\end{aligned} \tag{3.3.7}$$

Computing now the RHS, we have:

$$\begin{aligned}
< \xi_1, \xi_2 > *f(\gamma) &= \textstyle\sum_{\gamma' \in \mathcal{G}^T_{r\gamma}} < \xi_1, (\xi_2) > (\gamma'^{-1}) f(\gamma'\gamma) \\
&= \textstyle\sum_{\gamma' \in \mathcal{G}^T_{r(\gamma)}} \int_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\gamma'\gamma) < \xi_1(\alpha), \xi(\alpha\gamma'^{-1}) >_{E_{r(\alpha)}} d\lambda_{r(\gamma)}(\alpha)
\end{aligned} \tag{3.3.8}$$

From 3.3.7 and 3.3.8 we thus get $< \xi_1, \xi_2 f >=< \xi_1, \xi_2 > *f$. □

Definition We define the Connes-Skandalis Hilbert C^*-modules $\mathcal{E}_r$ and $\mathcal{E}_m$ as the completion of the pre-Hilbert $\mathcal{A}_c^T$-module $\mathcal{E}_c$ in this inner product with respect to the reduced and maximal completions of $\mathcal{A}_c^T$, respectively.

Now, there is a representation $\chi : \mathcal{B}_c^E \to \mathcal{L}(\mathcal{E}_c)$ of $\mathcal{B}_c^E := C_c(\mathcal{G}, E)$ on $\mathcal{E}_c$, given by the following formula:

$$[\chi(\phi)](\xi)(\gamma) = \int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \phi(\gamma'^{-1})\xi(\gamma'\gamma) d\lambda_{r(\gamma)}(\gamma') \text{ for } \phi \in \mathcal{B}_c^E, \xi \in \mathcal{E}_c, \gamma \in \mathcal{G}_T. \tag{3.3.9}$$

We have the following:

Proposition 3.3.3. *With the notations above, we have:*

(i) $\chi(\phi) : \mathcal{E}_c \to \mathcal{E}_c$ is $\mathcal{A}_c^T-$linear.

(ii) χ is faithful and extends to C^*-algebra isomorphisms $\chi_r : \mathcal{B}_r^E \to \mathcal{K}_{\mathcal{A}_r^T}(\mathcal{E}_r)$ and $\chi_m : \mathcal{B}_m^E \to \mathcal{K}_{\mathcal{A}_m^T}(\mathcal{E}_m)$.

[1]Note that we prove linearity in the second variable. We assume the same for the fiberwise inner product on E

Proof. (i) Let $f \in \mathcal{A}_c^T, \gamma \in \mathcal{G}_T, \xi \in \mathcal{E}_c$. We need to prove $[\chi(\phi)(\xi)]f(\gamma) = \chi(\phi)(\xi f)(\gamma)$. Computing the LHS, we have,

$$\begin{aligned}
[\chi(\phi)(\xi)]f(\gamma) &= \textstyle\sum_{\gamma' \in \mathcal{G}^T_{r(\gamma)}} f(\gamma'\gamma)[\chi(\phi)(\xi)](\gamma'^{-1}) \\
&= \textstyle\sum_{\gamma' \in \mathcal{G}^T_{r(\gamma)}} f(\gamma'\gamma) \int_{\alpha \in \mathcal{G}_{s(\gamma')=r(\gamma)}} \phi(\alpha^{-1})\xi(\alpha\gamma'^{-1}) d\lambda_{s(\gamma')}(\alpha) \\
&= \textstyle\sum_{\gamma' \in \mathcal{G}^T_{r(\gamma)}} \int_{\alpha \in \mathcal{G}_{s(\gamma')=r(\gamma)}} f(\gamma'\gamma)\phi(\alpha^{-1})\xi(\alpha\gamma'^{-1}) d\lambda_{s(\gamma')}(\alpha)
\end{aligned} \tag{3.3.10}$$

Similary we compute the RHS,

$$\begin{aligned}
[\chi(\phi)(\xi f)(\gamma) &= \textstyle\int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \phi(\gamma'^{-1})(\xi f)(\gamma'\gamma) d\lambda_{r(\gamma)}(\gamma') \\
&= \textstyle\int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \phi(\gamma'^{-1}) \sum_{\alpha \in \mathcal{G}^T_{r(\gamma')}} f(\alpha\gamma'\gamma)\xi(\alpha^{-1}) d\lambda_{r(\gamma)}(\gamma') \\
&= \textstyle\int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \sum_{\alpha \in \mathcal{G}^T_{s(\gamma')=r(\gamma)}} \phi(\gamma'^{-1}) f(\alpha\gamma)\xi(\gamma'\alpha^{-1}) d\lambda_{r(\gamma)}(\gamma') \\
&= \textstyle\int_{\gamma' \in \mathcal{G}_{r(\gamma)}} \sum_{\alpha \in \mathcal{G}^T_{s(\gamma')=r(\gamma)}} f(\alpha\gamma)\phi(\gamma'^{-1})\xi(\gamma'\alpha^{-1}) d\lambda_{r(\gamma)}(\gamma')
\end{aligned} \tag{3.3.11}$$

which is the same as the LHS computed above. The last item is proved in [HiSk:84]. ☐

With the representation (**??**) and the Hilbert module $\mathcal{E}_m$ we can define, for any $x \in M$, a Hilbert space $H_x := \mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T)$ with the inner product given by

$$< \zeta_1 \otimes \xi_1, \zeta_2 \otimes \xi_2 > = < \xi_1, \rho_x^{reg}(< \zeta_1, \zeta_2 >)\xi_2 >_{l^2(\mathcal{G}_x^T)} .$$

Let $\Psi_{x,reg} : H_x \to L^2(\mathcal{G}_x, r^*E)$ be the map induced by the formula

$$[\Psi_{x,reg}(\zeta \otimes \xi)](u) = \sum_{v \in \mathcal{G}_x^T} \xi(v)\zeta(uv^{-1}), \tag{3.3.12}$$

where $\zeta \in \mathcal{E}_c, \xi \in l^2(\mathcal{G}_x^T), u \in \mathcal{G}_x$.

Proposition 3.3.4. *With the above notations, we have:*

(i) $\Psi_{x,reg}$ is an isometry.

(ii) $\Psi_{x,reg}(\zeta f \otimes \xi) = \Psi_{x,reg}(\zeta \otimes [\rho_x^{reg}(f)]\xi)$ for $f \in \mathcal{A}_c^T$ and ζ, ξ as above.

(iii) We have,

$$\pi_x^{reg}(S) = \Psi_{x,reg} \circ [\chi_m(S) \otimes Id_{B(l^2(\mathcal{G}_x^T))}] \circ \Psi_{x,reg}^{-1} \textit{ for } S \in \mathcal{B}_m^E \tag{3.3.13}$$

(iv) $\Psi_{x,reg}$ is surjective when the $\mathcal{G}$ is Hausdorff.

Proof. (i) First, we compute for $\zeta \in C_c^\infty(\mathcal{G}_T, r^*E), \xi \in l^2(\mathcal{G}_x^T)$,

$$\begin{aligned} < \zeta \otimes \xi, \zeta \otimes \xi > &=< [\rho_x^r eg(< \zeta, \zeta >_{\mathcal{E}_c})]\xi, \xi >_{l^2(\mathcal{G}_x^T)} \\ &= \textstyle\sum_{u \in \mathcal{G}_x^T} [\rho_x^r eg(< \zeta, \zeta >_{\mathcal{E}_c})]\xi(u)\overline{\xi(u)} \\ &= \textstyle\sum_{u \in \mathcal{G}_x^T} \sum_{v \in \mathcal{G}_x^T} \xi(v)(< \zeta, \zeta >)(uv^{-1})\overline{\xi(u)} \\ &= \textstyle\sum_{u \in \mathcal{G}_x^T} \sum_{v \in \mathcal{G}_x^T} \xi(v)\overline{\xi(u)} \int_{\alpha \in \mathcal{G}_{r(u)}} < \zeta(\alpha), \zeta(\alpha u v^{-1}) >_{E_{r(\alpha)}} d\lambda_{r(u)}(\alpha) \\ &= \textstyle\sum_{u \in \mathcal{G}_x^T} \sum_{v \in \mathcal{G}_x^T} \int_{\alpha \in \mathcal{G}_x} \xi(v)\overline{\xi(u)} < \zeta(\alpha u^{-1}), \zeta(\alpha v^{-1}) >_{E_{r(\alpha)}} d\lambda_x(\alpha) \end{aligned} \tag{3.3.14}$$

where in the last line we have used $\mathcal{G}$-equivariance in the variable α. Next, we have,

$$< \Psi_{x,reg}(\zeta \otimes \xi), \Psi_{x,reg}(\zeta \otimes \xi) >_{L^2(\mathcal{G}_x, r^*E)} = \int_{u \in \mathcal{G}_x} < \Psi_{x,reg}(\zeta \otimes \xi)(u), \Psi_{x,reg}(\zeta \otimes \xi(u)) >_{E_{r(u)}} d\lambda_x(u).$$

Computing the integrand, we get

$$\begin{aligned} < \Psi_{x,reg}(\zeta \otimes \xi)(u), \Psi_{x,reg}(\zeta \otimes \xi)(u) >_{E_{r(u)}} &=< \textstyle\sum_{v \in \mathcal{G}_x^T} \xi(v)\zeta(uv^{-1}), \sum_{\beta \in \mathcal{G}_x^T} \xi(\beta)\zeta(u\beta^{-1}) >_{E_{r(u)}} \\ &= \textstyle\sum_{v \in \mathcal{G}_x^T} \sum_{\beta \in \mathcal{G}_x^T} \overline{\xi(v)}\xi(\beta) < \zeta(uv^{-1}), \zeta(u\beta^{-1}) >_{E_{r(u)}} \end{aligned} \tag{3.3.15}$$

Thus,

$$< \Psi_{x,reg}(\zeta \otimes \xi), \Psi_{x,reg}(\zeta \otimes \xi) >_{L^2(\mathcal{G}_x, r^*E)} = \int_{u \in \mathcal{G}_x} \sum_{v \in \mathcal{G}_x^T} \sum_{\beta \in \mathcal{G}_x^T} \overline{\xi(v)}\xi(\beta) < \zeta(uv^{-1}), \zeta(u\beta^{-1}) >_{E_{r(u)}} d\lambda_x(u).$$

which is the same as the last line of 3.3.14, since $< \zeta \otimes \xi, \zeta \otimes \xi > = \overline{< \zeta \otimes \xi, \zeta \otimes \xi >}$.

(ii) Let $u \in \mathcal{G}_x$. Then we have,

$$\begin{aligned} \Psi_x(\zeta f \otimes \xi)(u) &= \sum_{v \in \mathcal{G}_x^T} \xi(v)(\zeta f)(uv^{-1}) \\ &= \sum_{v \in \mathcal{G}_x^T} \sum_{w \in \mathcal{G}_{r(u)}^T} \xi(u) f(wuv^{-1})\zeta(w^{-1}) \\ &= \sum_{v \in \mathcal{G}_x^T} \sum_{w \in \mathcal{G}_{s(u)=x}^T} \xi(u) f(wv^{-1})\zeta(uw^{-1}) \qquad \text{(by } \mathcal{G}\text{-equivariance)} \end{aligned} \tag{3.3.16}$$

Computing the RHS, we have,

$$\begin{aligned} \Psi_x(\zeta \otimes [\rho_x^{reg}(f)]\xi)(u) &= \sum_{v \in \mathcal{G}_x^T} [\rho_x^{reg}(f)]\xi(v)(\zeta)(uv^{-1}) \\ &= \sum_{v \in \mathcal{G}_x^T} \sum_{w \in \mathcal{G}_x^T} \xi(w) f(vw^{-1})\zeta(uv^{-1}) \end{aligned} \tag{3.3.17}$$

which is equal to 3.3.16.

(iii) We shall prove the equation first for an element $\phi \in C_c^\infty(\mathcal{G}, E)$. We have for ζ, ξ, u as above,

$$\begin{aligned}
[\Psi_{x,reg} \circ (\chi_m(\phi) \otimes Id_{B(l^2(\mathcal{G}_x^T))})](\zeta \otimes \xi)(u) &= \Psi_{x,reg}(\chi_m(\phi)(\zeta) \otimes \xi)(u) \\
&= \sum_{v \in \mathcal{G}_x^T} \xi(v)[\chi_m(\phi)\zeta](uv^{-1}) \\
&= \sum_{v \in \mathcal{G}_x^T} \xi(v) \int_{w \in \mathcal{G}_{r(u)}} \phi(w^{-1})\zeta(wuv^{-1}) d\lambda_{r(u)}(w) \\
&= \sum_{v \in \mathcal{G}_x^T} \xi(v) \int_{w \in \mathcal{G}_x} \phi(uw^{-1})\zeta(wv^{-1}) d\lambda_x(w) \qquad (3.3.18)
\end{aligned}$$

where in the last equality we have used the $\mathcal{G}$-equivariance. Also, we have,

$$\begin{aligned}
[\pi_x^{reg}(\phi)](\Psi_{x,reg}(\zeta \otimes \xi))(u) &= \int_{v \in \mathcal{G}_x} \phi(uv^{-1})(\Psi_{x,reg}(\zeta \otimes \xi))(v) d\lambda_x(v) \\
&= \int_{v \in \mathcal{G}_x} \phi(uv^{-1}) \sum_{w \in \mathcal{G}_x^T} \xi(w)\zeta(vw^{-1}) d\lambda_x(v) \\
&= \int_{v \in \mathcal{G}_x} \sum_{w \in \mathcal{G}_x^T} \phi(uv^{-1})\xi(w)\zeta(vw^{-1}) d\lambda_x(v) \qquad (3.3.19)
\end{aligned}$$

which is the same as 3.3.18, hence proving the desired equation.

Now for a general element $S \in \mathcal{B}_m^E$, there exists a sequence $\phi_n \in C_c^\infty(\mathcal{G}, E)$ such that $\phi_n \xrightarrow{n\to\infty} S$ in the maximal norm. Then, using the continuity of the representations π_x^{reg} and χ_m with respect to the maximal norm on $\mathcal{B}_m^E$ and the isometry property of $\Psi_{x,reg}$, and the fact that $||\chi_m(S) \otimes I|| \leq ||\chi_m(S)||$ (cf. [La:95, pg. 42]), we have $\pi_x^{reg}(\phi_n) \xrightarrow{n\to\infty} \pi_x^{reg}(S)$, and $\Psi_{x,reg} \circ [\chi_m(\phi_n) \otimes I] \circ \Psi_{x,reg}^{-1} \xrightarrow{n\to\infty} \Psi_{x,reg} \circ [\chi_m(S) \otimes I] \circ \Psi_{x,reg}^{-1}$. Since we have $\pi_x^{reg}(\phi_n) = \Psi_{x,reg} \circ [\chi_m(\phi_n) \otimes I] \circ \Psi_{x,reg}^{-1}$ for all $n \geq 0$, we get the desired result.

(iv) To see this, first let $x \in T$. Then if $\eta \in C_c(\mathcal{G}_x, r^*E)$, we can extend η to $\tilde{\eta} \in C_c(\mathcal{G}_T, r^*E)$. Now take the delta function δ_x at the identity element in $\mathcal{G}_x^x$, and extend it by zero to $\mathcal{G}_x^T$. Then $\delta_x \in l^2(\mathcal{G}_x^T)$ and η is the image under $\Psi_{x,reg}$ of $\tilde{\eta} \otimes \delta_x$. Indeed, we have,

$$\begin{aligned}
\Psi_{x,reg}(\tilde{\eta} \otimes \delta_x)(u) &= \sum_{v \in \mathcal{G}_x^T} \delta_x(v)\tilde{\eta}(uv^{-1}) \\
&= \tilde{\eta}(u1_x^{-1}) \\
&= \tilde{\eta}(u) \\
&= \eta(u)
\end{aligned}$$

since $u \in \mathcal{G}_x$. If $x \in M \setminus T$, as T is a complete transversal, there exists a $t(x) \in T$ and $\gamma \in \mathcal{G}_x^{t(x)} \subseteq \mathcal{G}_x^T$ which induces unitary isomorphisms $R_\gamma : L^2(\mathcal{G}_x, r^*E) \to L^2(\mathcal{G}_{t(x)}, r^*E)$ and $R_\gamma : l^2(\mathcal{G}_x^T) \to l^2(\mathcal{G}_{t(x)}^T)$. Then, we have, for $\xi_1, \xi_2 \in l^2(\mathcal{G}_x^T), \xi_1', \xi_2' \in l^2(\mathcal{G}_{t(x)}^T)$ and $\zeta_1, \zeta_2 \in \mathcal{E}_c$,

$$\begin{aligned}
< \xi_1, \rho_x^{reg}(< \zeta_1, \zeta_2 >)\xi_2 > &= < R_{\gamma^{-1}}\xi_1', \pi_x^{reg}(< \zeta_1, \zeta_2 >)R_{\gamma^{-1}}\xi_2' > \\
&= < R_{\gamma^{-1}}\xi_1', R_{\gamma^{-1}}\pi_{t(x)}^{reg}(< \zeta_1, \zeta_2 >)\xi_2' > \\
&= < \xi_1', \rho_{t(x)}^{reg}(< \zeta_1, \zeta_2 >)\xi_2' >
\end{aligned}$$

where we have used the $\mathcal{G}$-equivariance of the representation ρ^{reg} in the second line. Therefore the map $\Phi : H_x \to H_{t(x)}$ given by $\Phi(\zeta \otimes \xi) = \zeta \otimes R_\gamma \xi$ is an isometric isomorphism. $\square$

We have proven:

Proposition 3.3.5. *$\forall x \in M, \Psi_{x,reg} : H_x \to L^2(\mathcal{G}_x, r^*E)$ is an isometric isomorphism.*

Similarly using the average representation ρ_x^{av} we define a map $\Psi_{x,av} : \mathcal{E}_m \otimes_{\rho_x^{av}} l^2(\mathcal{G}_x^T/\mathcal{G}_x^x) \to L^2(\mathcal{G}_x/\mathcal{G}_x^x, E|_L)$ induced by

$$\Psi_{x,av}(\zeta \otimes \phi)([u]) = \sum_{v \in \mathcal{G}_x^T} \phi([v])\zeta(uv^{-1}) \text{ for } \zeta \in \mathcal{E}_c, \phi \in l^2(\mathcal{G}_x^T/\mathcal{G}_x^x).$$

We have the following

Proposition 3.3.6. *$\forall x \in M, \Psi_{x,av} : \mathcal{E}_m \otimes_{\rho_x^{av}} l^2(\mathcal{G}_x^T/\mathcal{G}_x^x) \to L^2(\mathcal{G}_x/\mathcal{G}_x^x, E|_L)$ is an isometric isomorphism of hilbert spaces. Moreover, we have the following properties of $\Psi_{x,av}$:*

(i)$\Psi_{x,av}(\zeta f \otimes \phi) = \Psi_{x,av}(\zeta \otimes [\rho_x^{av}(f)]\phi)$ for $f \in \mathcal{A}_c^T$ and ζ, ϕ as above.

(ii)$\pi_x^{av}(T) = \Psi_{x,av} \circ [\chi_m(T) \otimes Id_{B(l^2(\mathcal{G}_x^T/\mathcal{G}_x^x))}] \circ \Psi_{x,av}^{-1}$, for $T \in \mathcal{B}_m^E$

Proof. : In this proof we drop the subscripts x, av from the notation $\Psi_{x,av}$ and simply write Ψ for simplicity. It is easy to see that the definition does not depend on the choice of representative u for the class $[u] \in \mathcal{G}_x^T/\mathcal{G}_x^x$. So we have

$$\Psi(\zeta \otimes \phi)([u']) = \Psi(\zeta \otimes \phi)([u]) \text{ for } u' = ug, g \in \mathcal{G}_x^x.$$

Now we prove the properties of Ψ:

• Ψ is an isometry:

We have

$$\begin{aligned} \Psi(\zeta \otimes \phi)([u]) &= \sum_{[v] \in \mathcal{G}_x^T/\mathcal{G}_x^x} \sum_{[w]=[v]} \zeta(uw^{-1})\phi([v]) \\ &= \sum_{v \in \mathcal{G}_x^T} \zeta(uv^{-1})\phi([v]) \end{aligned} \tag{3.3.20}$$

So,

$$\begin{aligned} & < \Psi(\zeta \otimes \phi), \Psi(\zeta \otimes \phi) >_{L^2(L(x),E|_{L(x)})} \qquad (3.3.21) \\ = & \int_{z=[\alpha] \in L(x)} < \Psi(\zeta \otimes \phi)(z), \Psi(\zeta \otimes \phi)(z) >_{E_z} d\lambda_x^l(z) \\ = & \int_{z=[\alpha] \in L(x)} < \sum_{v \in \mathcal{G}_x^T} \zeta(\alpha v^{-1})\phi([v]), \sum_{w \in \mathcal{G}_x^T} \zeta(\alpha w^{-1})\phi([w]) >_{E_z} d\lambda_x^l(z) \\ = & \int_{z=[\alpha] \in L(x)} \sum_{v \in \mathcal{G}_x^T} \sum_{w \in \mathcal{G}_x^T} \overline{\phi([v])}\phi([w]) < \zeta(\alpha v^{-1}), \zeta(\alpha w^{-1}) >_{E_{r(\alpha)}} d\lambda_x^l(z) \\ = & \int_{z=[\alpha] \in L(x)} \sum_{[v] \in [\mathcal{G}_x^T]} \sum_{\beta \in \mathcal{G}_x^x} \sum_{w \in \mathcal{G}_x^T} \overline{\phi([v])}\phi([w]) < \zeta(\alpha\beta v^{-1}), \zeta(\alpha w^{-1}) >_{E_{r(\alpha)}} d\lambda_x^l(z) \\ = & \sum_{[v] \in [\mathcal{G}_x^T]} \overline{\phi([v])} \sum_{w \in \mathcal{G}_x^T} \int_{[\alpha] \in \mathcal{G}_x/\mathcal{G}_x^x} \sum_{\beta \in \mathcal{G}_x^x} \phi([w]) < \zeta(\alpha\beta v^{-1}), \zeta(\alpha w^{-1}) >_{E_{r(\alpha)}} d\lambda_x^l(z) \\ = & \sum_{[v] \in [\mathcal{G}_x^T]} \overline{\phi([v])} \sum_{w \in \mathcal{G}_x^T} \int_{\alpha \in \mathcal{G}_x} \phi([w]) < \zeta(\alpha v^{-1}), \zeta(\alpha w^{-1}) >_{E_{r(\alpha)}} d\lambda_x(\alpha) \qquad (3.3.22) \end{aligned}$$

where the last equality is clear since the measure λ_x is $\mathcal{G}_x^x$-invariant.

$$\begin{aligned}
<\zeta\otimes\phi,\zeta\otimes\phi> &= <\phi,\rho_x^{av}(<\zeta,\zeta>)\phi>_{l^2(\mathcal{G}_x^T/\mathcal{G}_x^x)} \\
&= \sum_{z=[u]\in[\mathcal{G}_x^T]} \overline{\phi(z)}[\rho_x^{av}(<\zeta,\zeta>](\phi)([v]) \\
&= \sum_{z=[u]\in[\mathcal{G}_x^T]} \overline{\phi(z)} \sum_{v\in\mathcal{G}_x^T} <\zeta,\zeta>(uv^{-1})\phi([v]) \\
&= \sum_{z=[u]\in[\mathcal{G}_x^T]} \overline{\phi(z)} \sum_{v\in\mathcal{G}_x^T} \left(\int_{\alpha\in\mathcal{G}_{r(u)}} <\zeta(\alpha),\zeta(\alpha uv^{-1})>_{E_{r(\alpha)}} d\lambda_{r(u)}\right)\phi([v]) \\
&= \sum_{[u]\in[\mathcal{G}_x^T]} \overline{\phi([u])} \sum_{v\in\mathcal{G}_x^T} \left(\int_{\alpha\in\mathcal{G}_{s(u)=x}} <\zeta(\alpha u^{-1}),\zeta(\alpha v^{-1})>_{E_{r(\alpha)}} d\lambda_x\right)\phi([v]) \quad (3.3.23)
\end{aligned}$$

Thus 6.3.3 and 3.3.23 give that

$$<\Psi(\zeta\otimes\phi),\Psi(\zeta\otimes\phi)>_{L^2(L(x),E|_{L(x)})}=<\zeta\otimes\phi,\zeta\otimes\phi>_{\mathcal{E}_m\otimes_{\rho_x^{av}}l^2(\mathcal{G}_x^T/\mathcal{G}_x^x)}$$

proving that Ψ is an isometry.

- Ψ is surjective when $\mathcal{G}$ is Hausdorff:

To see this, first let $x\in T$. Then if $\eta\in C_c^\infty(\mathcal{G}_x/\mathcal{G}_x^x, r^*E)$, we can lift η to a $\mathcal{G}_x^x$-invariant section $\tilde\eta\in C^\infty(\mathcal{G}_x, r^*E)$. Now extend $\tilde\eta$ to $C^\infty(\mathcal{G}_T, r^*E)$, and denote this extension by $\hat\eta$. We take a function κ on $C_c^\infty(\mathcal{G}_x)$ such that for $u\in\mathcal{G}_x$, we have

$$\sum_{\alpha\in\mathcal{G}_x^x}\kappa(u\alpha)=1.$$

Then η is the image under Ψ of $\kappa\cdot\hat\eta\otimes\delta_x$, where $\cdot$ is pointwise multiplication of functions and $\delta_x\in l^2(\mathcal{G}_x^T/\mathcal{G}_x^x)$ is the delta function which is equal to 1 on the subset $\mathcal{G}_x^x\subseteq\mathcal{G}_x^T$ and 0 elsewhere. Indeed, we have,

$$\begin{aligned}
\Psi(\kappa\cdot\hat\eta\otimes\delta_x)([u]) &= \sum_{v\in\mathcal{G}_x^T}\kappa\cdot\hat\eta(uv^{-1})\delta_x([v]) \\
&= \sum_{v\in\mathcal{G}_x^x}\kappa\cdot\tilde\eta(uv^{-1}) \\
&= \sum_{v\in\mathcal{G}_x^x}\kappa(uv^{-1})\tilde\eta(uv^{-1}) \\
&= \sum_{v\in\mathcal{G}_x^x}\kappa(uv^{-1})\eta(\pi(uv^{-1})) \\
&= \sum_{v\in\mathcal{G}_x^x}\kappa(uv^{-1})\eta([u]) \\
&= \eta([u])\sum_{v\in\mathcal{G}_x^x}\kappa(uv^{-1}) \\
&= \eta([u]) \qquad (3.3.24)
\end{aligned}$$

(ii) Let $\psi\in\mathcal{B}_c$, ζ,ϕ as above. Then we have,

$$\begin{aligned}\pi_x^{av}(\psi)(\Psi(\zeta\otimes\phi))([\alpha]) &= \int_{\mathcal{G}_x/\mathcal{G}_x^x}\sum_{\beta\in\mathcal{G}_x^x}\psi(\alpha\beta\theta^{-1})\Psi(\zeta\otimes\phi)([\theta])d\tilde{\lambda}^l([\theta])\\ &= \int_{\mathcal{G}_x/\mathcal{G}_x^x}\sum_{\beta\in\mathcal{G}_x^x}\psi(\alpha\beta\theta^{-1})\sum_{\delta\in\mathcal{G}_x^T}\phi([\delta])\zeta(\theta\delta^{-1})d\tilde{\lambda}^l([\theta])\end{aligned}$$

On the other hand, we have,

$$\begin{aligned}[\Psi(\chi_m(\psi)\otimes I)](\zeta\otimes\phi)([\alpha]) &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])[\chi_m(\psi)\zeta](\alpha\kappa^{-1})\\ &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])\int_{\gamma\in\mathcal{G}_{r(\alpha)}}\psi(\gamma^{-1})\zeta(\gamma\alpha\kappa^{-1})d\lambda_{r(\alpha)}(\gamma)\\ &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])\int_{v\in L_{r(\alpha)}}\sum_{\gamma_1\in\mathcal{G}_{r(\alpha)}^v}\psi(\gamma_1^{-1})\zeta(\gamma_1\alpha\kappa^{-1})d\lambda^{L_{r(\alpha)}}(v)\\ &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])\int_{v\in L_{r(\alpha)}}\sum_{\gamma_2\in\mathcal{G}_x^v}\psi(\alpha\gamma_2^{-1})\zeta(\gamma_2\kappa^{-1})d\lambda^{L_{r(\alpha)}}(v)\\ &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])\int_{v\in L_x}\sum_{\beta\in\mathcal{G}_x^x}\psi(\alpha\beta^{-1}\tilde{\gamma}_2^{-1})\zeta(\tilde{\gamma}_2\beta\kappa^{-1})d\lambda^{L_{r(\alpha)}}(v)\\ &\quad (\text{ fixing } \tilde{\gamma}_2\in\mathcal{G}_x^v \text{ for each } v\in L_x, \gamma_2=\tilde{\gamma}_2\beta)\\ &= \sum_{\kappa\in\mathcal{G}_x^T}\phi([\kappa])\int_{\tilde{\gamma}_2\in\mathcal{G}_x/\mathcal{G}_x^x}\sum_{\beta\in\mathcal{G}_x^x}\psi(\alpha\beta^{-1}\tilde{\gamma}_2^{-1})\zeta(\tilde{\gamma}_2\beta\kappa^{-1})d\tilde{\lambda}^l([\tilde{\gamma}_2])\\ &= \sum_{\kappa'\in\mathcal{G}_x^T}\phi([\kappa'])\int_{\tilde{\gamma}_2\in\mathcal{G}_x/\mathcal{G}_x^x}\sum_{\beta'\in\mathcal{G}_x^x}\psi(\alpha\beta'\tilde{\gamma}_2^{-1})\zeta(\tilde{\gamma}_2\kappa'^{-1})d\tilde{\lambda}^l([\tilde{\gamma}_2])\end{aligned}$$

Comparing the last lines of the above computations gives the result. ☐

3.3.2 Dirac Operators on Hilbert C^*-modules

Let $\{D_L\}_{L\in\mathcal{F}}$ be a family of leafwise Dirac operators on M, and $\{\tilde{D}_x\}_{x\in M}$ its lift to the monodromy groupoid via the covering map $r:\mathcal{G}_x\to L(x)$. Thus the family $\tilde{D}=\{\tilde{D}_x\}_{x\in M}$ is a $\mathcal{G}$-operator. Then, we define a densely defined $\mathcal{A}_c^T$-linear operator on the Hilbert module $\mathcal{E}_r$ as:

$$\mathcal{D}\xi(\gamma)=\tilde{D}_{s(\gamma)}(\xi|_{\mathcal{G}_{s(\gamma)}})(\gamma) \text{ for } \xi\in C_c^\infty(\mathcal{G}_T,r^*E), \gamma\in\mathcal{G}_T. \tag{3.3.25}$$

Then [Va:01, Proposition 3.4.9] applies to our case to give:

Proposition 3.3.7. *The operator $\mathcal{D}$ is a closable operator and extends to closed self-adjoint unbounded regular operators $\mathcal{D}_r$ and $\mathcal{D}_m$ on $\mathcal{E}_r$ and $\mathcal{E}_m$, respectively.*

Proof. We recall the proof given in [Va:01]. Since $\tilde{D}_x$ is a formally self-adjoint operator, $\mathcal{D}^*$ is densely defined and $\mathcal{D}^*=\mathcal{D}$ on the dense subspace $\mathcal{E}_c$. Since $\mathcal{D}^*$ is a closed operator, so is $(\mathcal{D}^*)^*$, and $G(\mathcal{D})\subseteq G((\mathcal{D}^*)^*)$ implies $\mathcal{D}$ is a closable operator. To show $G(\mathcal{D})\subseteq G((\mathcal{D}^*)^*)$, we use the definition of the graphs $G(\mathcal{D})=\{(x,y)\in\mathcal{E}^2|x\in Dom\mathcal{D}, y=\mathcal{D}x\}$ and $G(\mathcal{D}^*)=\{(y,x)\in\mathcal{E}^2|\forall z\in Dom\mathcal{D}, <x,z>=<y,\mathcal{D}z>\}$. Therefore, $G(\mathcal{D})\subseteq G((\mathcal{D}^*)^*)=\{(y,x)\in\mathcal{E}^2|\forall z\in Dom\mathcal{D}^*, <x,z>=<y,\mathcal{D}^*z>\}$

$\tilde{D}_x$ has a parametrix $\tilde{Q}_x$ which is a pseudodifferential operator of order -1, and satisfies $\tilde{D}_x\tilde{Q}_x = 1-R$, where R is a compactly smoothing tangential operator on the monodromy groupoid. So for large enough n, $(\tilde{Q}^*\tilde{Q})^n$ has a compactly supported continuous kernel on $\mathcal{G}$,i.e. an element of $\mathcal{B}_c^E$, and thus extends to an element in the C^*-algebra $\mathcal{B}_r^E$ and is therefore a compact operator on $\mathcal{E}_\pi$, by Proposition 3.3.3. But $(\tilde{Q}^*\tilde{Q})^n$ extending to a bounded operator implies the same is true for $\tilde{Q}$, let it be denoted by $\mathcal{Q}$.

We want to show that the extension of the operator $\tilde{D}\tilde{Q}$ coincides with $\mathcal{D}\mathcal{Q}$. To see this, let $(u,v) \in G(\overline{\tilde{D}\tilde{Q}})$. Then, there exists a sequence $\{u_n\}$ in $\mathcal{E}$ such that $||u_n - u|| \to 0$ as $n \to \infty$, for which $||\tilde{D}\tilde{Q}u_n - v|| \to 0$ as $n \to \infty$. But by the definition of $\mathcal{Q}$, there exists a $w \in \mathcal{E}$ such that $||\tilde{Q}u_n - w|| \to 0$ as $n \to \infty$. Thus $(w,v) \in G(\mathcal{D})$ and so $(u,v) \in G(\mathcal{D}\mathcal{Q})$. Thus we have shown $G(\overline{\tilde{D}\tilde{Q}}) \subseteq G(\mathcal{D}\mathcal{Q})$.

Now, $\tilde{D}\tilde{Q} = 1 - R$ is a zero-th order operator and also extends to a bounded operator on $\mathcal{E}$, and thus has full domain $\mathcal{E}$. So $G(\overline{\tilde{D}\tilde{Q}}) = G(\mathcal{D}\mathcal{Q}) \Leftrightarrow \overline{\tilde{D}\tilde{Q}} = \mathcal{D}\mathcal{Q}$.

But then we have the relations $\mathcal{D}\mathcal{Q} = I - \mathcal{R}$, $\mathcal{Q}\mathcal{D} = I - \mathcal{R}'$and $\mathcal{Q}^*\mathcal{D}^* \subseteq (\mathcal{D}\mathcal{Q})^* = I - \mathcal{R}^*$. Let $u_n \in Dom(\tilde{D})$ be a sequence converging to u, such that $\tilde{D}u_n \to \mathcal{D}u$. As $\tilde{Q}$ and R are continuous,we get $\mathcal{Q}\mathcal{D}u = u - \mathcal{R}'u$. This implies $Dom(\mathcal{D}) \subseteq Im(\mathcal{Q}) + Im(\mathcal{R}')$. As $\mathcal{Q}^*\mathcal{D}^* \subseteq (\mathcal{D}\mathcal{Q})^* = I - \mathcal{R}^*$, we get similarly as above $Dom(\mathcal{D}^*) \subseteq Im(\mathcal{Q}^*) + Im(\mathcal{R}^*)$. As $\mathcal{Q}$ is bounded and formally self-adjoint,it is self-adjoint by continuity.So $Dom(\mathcal{D}^*) \subseteq Im(\mathcal{Q}) + Im(\mathcal{R}^*) \subseteq Dom(\mathcal{D}) = Dom(\mathcal{D}^*)$, as $\mathcal{R}^*$ is the closure of a compactly smoothing operator.

Therefore, $G(\mathcal{D}) = \{(\mathcal{Q}x+\mathcal{R}y, \mathcal{D}\mathcal{Q}x+\mathcal{D}\mathcal{R}y); (x,y) \in \mathcal{E}^2\}$, which is an orthocomplemented submodule because it is the image of the bounded closed operator given by

$$\mathcal{U} = \begin{pmatrix} \mathcal{Q} & \mathcal{R}' \\ \mathcal{D}\mathcal{Q} & \mathcal{D}\mathcal{R}' \end{pmatrix}$$

The complement submodule of $Im(\mathcal{U})$ is $Ker(\mathcal{U}^*) = hG(\mathcal{D}^*)^\perp, h : (x,y) \mapsto (y,-x)$. Therefore $\mathcal{D}$ is regular by [La:95, Proposition 9.5]. ☐

Definition We define the index class $\text{ind}(\mathcal{D}_m^+) \in K_0(\mathcal{K}_{\mathcal{A}_m^T}(\mathcal{E}_m))$ as the class $[e] - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, where e is the idempotent

$$e = \begin{pmatrix} (\mathcal{S}^+)^2 & -\mathcal{Q}(\mathcal{S}^- + (\mathcal{S}^-)^2) \\ -\mathcal{S}^-\mathcal{D} & \mathcal{I} - (\mathcal{S}^-)^2 \end{pmatrix},$$

where $\mathcal{S}^+ = \mathcal{I} - \mathcal{Q}^-\mathcal{D}^+, \mathcal{S}^- = \mathcal{I} - \mathcal{D}^+\mathcal{Q}^-$ are the operators on the Hilbert module $\mathcal{E}_m$ associated to the remainders $\tilde{S}^\pm$ for the Dirac operator $\tilde{D}^+$.

We also recall that $\mathcal{S}$ is the extension of the operator $\tilde{S}$ and is defined by

$$\mathcal{S}\xi(\gamma) = \tilde{S}_{s(\gamma)}\xi|_{\mathcal{G}_{s(\gamma)}}(\gamma), \tag{3.3.26}$$

where $\xi \in C_c(\mathcal{G}_T, r^*E), \gamma \in \mathcal{G}_T$, and we have the isomorphism $\Psi_{x,reg} : \mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T) \to L^2(\mathcal{G}_x, r^*E)$ defined in 6.3.1.

Proposition 3.3.8. $\Psi_{x,reg} \circ [\mathcal{S} \otimes Id_{B(l^2(\mathcal{G}_x^T))}] \circ \Psi_{x,reg}^{-1} = \tilde{S}$

Proof. Since by definition,

$$[\tilde{S}_x(\Psi_{x,reg}(\zeta \otimes \xi))](u) = \sum_{v \in \mathcal{G}_x^T} \tilde{S}_x[R_{v^{-1}}\zeta|_{\mathcal{G}_{r(v)}}](u)\xi(v) \tag{3.3.27}$$

we have,

$$\begin{aligned}\Psi_{x,reg}\circ[\mathcal{S}_m\otimes Id](\zeta\otimes\xi)(u) &= \Psi_{x,reg}(\mathcal{S}_m\zeta\otimes\xi)(u)\\ &= \sum_{v\in\mathcal{G}_x^T}\mathcal{S}_m\zeta(uv^{-1})\xi(v)\\ &= \sum_{v\in\mathcal{G}_x^T}\tilde{S}_{r(v)}\zeta|_{\mathcal{G}_{r(v)}}(uv^{-1})\xi(v)\\ &= \sum_{v\in\mathcal{G}_x^T}R_{v^{-1}}[\tilde{S}_{r(v)}\zeta|_{\mathcal{G}_{r(v)}}](u)\xi(v)\\ &= \sum_{v\in\mathcal{G}_x^T}\tilde{S}_x[R_{v^{-1}}\zeta|_{\mathcal{G}_{r(v)}}](u)\xi(v) \qquad \text{(using } \mathcal{G}\text{-equivariance)}\\ &= [\tilde{S}_x(\Psi_{x,reg}(\zeta\otimes\xi))](u)\end{aligned} \tag{3.3.28}$$

□

Recall that we have an isomorphism $\chi_m:\mathcal{B}_m^E\to\mathcal{K}_{\mathcal{A}_m^T}(\mathcal{E}_m)$. We define another index class of $\mathcal{D}_m$ in $K_0(B_m^E)$, where $B_m^E\equiv C_m^*(\mathcal{G},E))$, as the image of $\operatorname{ind}(\mathcal{D}_m)$ under the map $\chi_{m,*}^{-1}:K_0(\mathcal{K}_{\mathcal{A}_m^T}(\mathcal{E}_m))\to K_0(B_m^E)$ induced by χ_m^{-1}. Let us denote this index class by $\operatorname{ind}(\mathcal{D}_m)$.

Let also $\tau_{reg,*}^\Lambda:=\tau_*^\Lambda\circ\pi_*^{reg}:K_0(B_m^E)\to\mathbb{C}$ and $\tau_{\mathcal{F},*}^\Lambda:=\tau_{\mathcal{F},*}^\Lambda\circ\pi_*^{av}:K_0(B_m^E)\to\mathbb{C}$ be the maps induced by the traces τ_{reg}^Λ and $\tau_{\mathcal{F}}^\Lambda$ on B_m^E. Then we have the following

Proposition 3.3.9. *We have the following equalities:*

$$\tau_{reg,*}^\Lambda(\operatorname{ind}(\mathcal{D}_m))=\operatorname{Ind}^\Lambda(\tilde{D})\ \textit{and}\ \tau_{\mathcal{F},*}^\Lambda(\operatorname{ind}(\mathcal{D}_m))=\operatorname{Ind}^\Lambda(D)$$

Proof. To show the first equality we will show that $\tau_{reg}^\Lambda(\chi_m^{-1}(\mathcal{S}))=\tau^\Lambda(\tilde{S})$. Recall that

$\tau_{reg}^\Lambda=\tau^\Lambda\circ\pi^{reg}$ and $\pi_x^{reg}(T)=\Psi_{x,reg}\circ[\chi_m(T)\otimes Id_{B(l^2(\mathcal{G}_x^T))}]\circ\Psi_{x,reg}^{-1}$, for $T\in\mathcal{B}_m^E$.

So letting $T=\chi_m^{-1}(\mathcal{S})$ in the last equation above, we get,

$$\pi_x^{reg}(\chi_m^{-1}(\mathcal{S}))=\Psi_{x,reg}\circ[\mathcal{S}\otimes Id_{B(l^2(\mathcal{G}_x^T))}]\circ\Psi_{x,reg}^{-1}=\tilde{S} \qquad \text{(by Proposition(3.3.8))} \tag{3.3.29}$$

Since $\tilde{S}$ is τ^Λ-trace class, we have the required identity $\tau_{reg}^\Lambda(\chi_m^{-1}(\mathcal{S}))=\tau^\Lambda(\tilde{S})$. Therefore, we have,

$$\begin{aligned}\tau_{reg,*}^\Lambda(\operatorname{ind}(\mathcal{D}_m)) &= \tau_*^\Lambda\circ(\pi_*^{reg}\circ\chi_{m,*}^{-1})(\operatorname{ind}(\mathcal{D}_m))=\tau_*^\Lambda\circ(\pi^{reg}\circ\chi_m^{-1})_*(IND(\mathcal{D}_m))\\ &= \tau^\Lambda((\tilde{S}^+)^2)-\tau^\Lambda((\tilde{S}^-)^2)\end{aligned} \tag{3.3.30}$$

But then the Calderon's formula for $N=2$ gives

$\tau^\Lambda((\tilde{S}^+)^2)-\tau^\Lambda((\tilde{S}^-)^2)=\operatorname{Ind}^\Lambda(\tilde{D})$, thus establishing the first equality. The second equality is similarly established. □

Corollary 3.3.10. *As per the above notation, we have* $\tau_{reg,*}^\Lambda(\operatorname{ind}(\mathcal{D}_m)))=\tau_{\mathcal{F},*}^\Lambda(\operatorname{ind}(\mathcal{D}_m))$.

Proof. This follows directly from Theorem 3.2.2 and Proposition 3.3.9. □

3.3.3 Remarks on the Baum-Connes map

Let $(V,\mathcal{F})$ be a compact foliated manifold without boundary. Let $\mathcal{G}$ be its monodromy groupoid, which we assume to be torsion-free, i.e. all isotropy groups $\mathcal{G}_x^x$ are torsion-free. Let $B\mathcal{G}$ be the classifying space of the groupoid $\mathcal{G}$ (cf. [Co:94]). We recall the definition of geometric K-homology for $B\mathcal{G}$ as given in [Co:94]. Recall that a generalized morphism $f:M\to V/\mathcal{F}$ (in the sense of [Co:94]) is called K-oriented if the bundle $T^*M\oplus f^*\tau$ admits a $Spin^c$- structure, where τ is the normal bundle to the foliation $\mathcal{F}$, i.e. $\tau_x=T_xV/T_x\mathcal{F}$.

Definition The geometric K-homology group for $B\mathcal{G}$, denoted $K_*(B\mathcal{G})$ is the set of triples (M, E, f), called K-cycles, where M is a closed Riemannian $Spin^c$-manifold, E is a complex vector bundle over M, and f is a K-oriented map from M to the space of leaves $V/\mathcal{F}$ of $(V, \mathcal{F})$, modulo the equivalence relation generated by the following relations:

(i) **Direct sum:** Let (M, E, f) be a K-cycle, and $E = E_1 \oplus E_2$. Then

$$(M, E, f) \sim (M, E_1, f) \sqcup (M, E_2, f)$$

where $\sqcup$ denotes the disjoint union operation given by $(M_1, E_1, f_1) \sqcup (M_2, E_2, f_2) = (M_1 \sqcup M_2, E_1 \sqcup E_2, f_1 \sqcup f_2)$.

(ii) **Bordism:** Let (M_1, E_1, f_1) and (M_2, E_2, f_2) be K-cycles. Then $(M_1, E_1, f_1) \sim (M_2, E_2, f_2)$ if there exists a smooth compact Riemannian $Spin^c$-manifold with boundary, W with a complex Hermitian vector bundle E on W and a K-oriented map $W \to V/\mathcal{F}$ such that the cycle $(\partial W, E|_{\partial W}, f_{\partial W})$ is isomorphic to the disjoint union $(M_1, E_1, f_1) \sqcup (-M_2, E_2, f_2)$. Here $-M_2$ denotes M_2 with the $Spin^c$-structure reversed, and we call two K-cycles isomorphic if there exists a diffeomorphism $h : M_1 \to M_2$ such that h preserves the Riemannian and $Spin^c$-structures, $h^* E_2 \cong E_1$ and we have $f_2 \circ h = f_1$.

(iii) **Vector bundle modification:** Let (M, E, f) be a K-cycle. Let H be a Hermitian vector-bundle on M with even-dimensional fibers. Let 1 denote the trivial line bundle on M, $1 = M \times \mathbb{R}$. Let $\hat{M} = S(H \oplus 1)$ denote the unit sphere bundle of $H \oplus 1$ corresponding to the inner product on H. The $Spin^c$-structures on TM and H yield a $Spin^c$-structure on $T\hat{M}$, so that $\hat{M}$ is a $Spin^c$-manifold. Let $\rho : \hat{M} \to M$ denote the projection to the zero section.

Since H has a $Spin^c$-structure, there is an associated bundle S_H of Clifford modules over TM such that $Cl(H) \otimes \mathbb{C} \cong End(S_H)$. The Clifford multiplication by the volume element induces a $\mathbb{Z}_2$ grading on S_H, which we write as $S_H = S_H^+ \oplus S_H^-$. Denote the pull-backs of S_H^+ and S_H^- to H by H_0 and H_1, respectively. Now, $\hat{M}$ can be seen as two copies of the unit ball bundle of H glued together via the identity map on S_H. Form a new bundle $\hat{H}$ on $\hat{M}$ by putting S_H^+ and S_H^- on the two copies of the unit ball bundle of H respectively and then gluing them together along S_H by a clutching map $\sigma : H_0 \to H_1$, where σ is given by the Clifford action of H on H_0 and H_1.

Then the vector bundle modification relation is defined as $(M, E, f) \sim (\hat{M}, \hat{H} \otimes \rho^* E, f \circ \rho)$.

The set $K_*(B\mathcal{G})$ is an abelian group with respect to the operations of disjoint union and reversal of the $Spin^c$-structure. The subgroup $K_0(B\mathcal{G})$ (resp. $K_1(B\mathcal{G})$) are the subgroups of $K_*(B\mathcal{G})$ given by K-cycles (M, E, f) such that each connected components of M are even (resp. odd)-dimensional.

Definition The maximal Baum-Connes map $\mu_{max} : K_*(B\mathcal{G}) \to K_*(C^*_{max}(\mathcal{G}))$ is given by the map $[(M, E, f)] \mapsto f_!([E])$, where $f_!$ denotes the shriek map implementing the wrong-way functoriality in K-theory (see [Co:94, Section 2.6, page 111], [Co:81]).

We have the following theorem which ensures that the map μ_{max} is well-defined.

Theorem 3.3.11 (Corollary 8.6, [Co:94])**.** *Let $x \in K_*(B\mathcal{G})$ and (M, E, f) be a K-cycle representing x. Then the element $f_!([E]) \in K_*(C^*_{max}(\mathcal{G}))$ only depends on the class of (M, E, f) under the equivalence relation generated by direct sum, bordism and vector-bundle modification, and μ_{max} is an additive map of abelian groups.*

Recall the functionals $\tau_{reg,*}$ and $\tau_{av,*}$ from the previous section.

Proposition 3.3.12. *We have $\tau_{reg,*} = \tau_{av,*}$ on the image of μ_{max} in $K_*(C^*_{max}(\mathcal{G}))$.*

Proof. We give a brief outline of the proof and refer the reader to [BeRo:10] for more details. By [Co:94, Proposition 8.4] and [Co:94, Theorem 8.5], one can restrict to K-cycles (M, E, f) such that $f : M \to V/\mathcal{F}$ is a smooth K-oriented submersion. Then there is a well-defined pull-back foliation on M, which we denote by $\mathcal{F}_M$. Then the maximal Baum-Connes map for the foliation $(V, \mathcal{F})$ is given by the index of an operator $\mathcal{D}_f$ on a certain Hilbert $C^*(\mathcal{G})$-module $\mathcal{E}_{f,E}$ induced by a family of order zero Dirac operators and lies in $K_*(C^*(\mathcal{G}))$ (for notations and details see [CoSk:84, Lemma 4.7, Definition 4.8, Theorem 4.14]). Then we can apply the same techniques as in the proof of Proposition 3.3.9 and Corollary 3.3.10 for the operator $\mathcal{D}_f$ to get the result. $\square$

3.4 Functional calculus of Dirac operators

3.4.1 Functional calculus of normal regular operators on Hilbert C^*-modules

The functional calculus of normal regular operators on Hilbert C^*-modules has been treated by [Wo:91] and [Ku:97]. We state some results which we will require for the functional calculus of the operator $\mathcal{D}$. We begin with the following definitions:

Definition ([Wo:91]) Consider a C^*-algebra A and let T be a linear mapping acting on A defined on a linear dense domain $Dom(T) \subset A$. T is said to be affiliated with A if and only if there exists $z \in M(A)$ such that $||z|| \leq 1$ and for any $x, y \in A$ we have

$x \in Dom(T), y = Tx \iff$ There exists $a \in A$ such that $x = (I - z^*z)^{1/2}a$ and $y = za$.

Definition ([La:95]) Let $\mathcal{E}, \mathcal{E}'$ be Hilbert C^*-modules over a C^*-algebra A and T be a densely defined A-linear mapping from $Dom(T) \subset \mathcal{E}$ to $\mathcal{E}'$ such that T is closed, T^* is densely defined and $I + T^*T$ has dense range. Then T is called a regular operator from $\mathcal{E}$ to $\mathcal{E}'$.

Remarks:

1. By Theorem 10.4 of [La:95], we see that affiliated operators on a C^*-algebra A are therefore nothing but regular operators on A, viewed as a Hilbert A-module.

2. By Theorem 10.1 of [La:95], a regular operator T on a Hilbert A-module E is affiliated to the C^*-algebra $\mathcal{K}_A(E)$ with dense domain span$\{\theta_{x,y} | x \in Dom(T), y \in E\}$. See [Pa:99]

Theorem 3.4.1 ([Wo:91] Thm.1.2, [Ku:97] Thm.1.9, Propn. 6.17)**.** *Let $\pi : A \to \mathcal{L}(\mathcal{E})$ be a non-degenerate $*$-homomorphism from a C^*-algebra A to the bounded linear operators on a Hilbert C^*-module $\mathcal{E}$. Then*

(i) π can be extended to a well-defined map $\tilde{\pi} : \eta(A) \to \mathcal{R}(\mathcal{E})$ where $\eta(A)$ is the set of elements affiliated to A and $\mathcal{R}(\mathcal{E})$ is the set of regular operators on $\mathcal{E}$.

(ii) $\pi(Dom(T))\mathcal{E}$ is a core for $\pi(T)$ and $\pi(T)(\pi(b)v) = \pi(T(b))v$ for every $b \in Dom(T)$ and $v \in \mathcal{E}$.

(iii) if T is a normal element affiliated to A then $\pi(T)$ is a normal element in $\mathcal{R}(\mathcal{E})$ and for any bounded continuous function $f : \mathbb{C} \to \mathbb{C}$ we have $f(\pi(T)) = \pi(f(T))$.

We would like to study the functional calculus of a regular operator on interior tensor products of Hilbert C^*-modules. This is done as follows. We follow the treatment in [La:95], pages 104-106, Chapter 9. Let E be a Hilbert A-module, F be a Hilbert B-module and $\alpha : A \to \mathcal{L}(F)$ be a $*$-homomorphism. Let $t : D(t) \to E$ be a regular operator on the Hilbert A-module E with dense domain $D(t)$. Define

$$D_0 := \text{span } \{x \dot{\otimes} y | x \in D(t), y \in F\}$$

where $x\dot{\otimes}y$ denotes the equivalence class of the algebraic tensor product $x \otimes y$ in $E \otimes_\alpha F$. Then D_0 is a dense (algebraic) submodule of $E \otimes_\phi F$. We consider the operator t_1 defined on D_0 as the linear extension of the following operator defined on simple tensors in D_0:

$$t_1(x\dot{\otimes}y) = tx\dot{\otimes}y$$

Similarly, we define $D_0^* :=$ span $\{x\dot{\otimes}y | x \in D(t^*), y \in F\}$ and we define an operator t_2 as

$$t_2(x\dot{\otimes}y) = t^*x\dot{\otimes}y$$

Then we have for $x \in D(t), y \in D(t^*)$, and $u, v \in F$,

$$< t_1(x\dot{\otimes}u), y\dot{\otimes}v >=< x\dot{\otimes}u, t_2(y\dot{\otimes}v) >$$

and therefore by linearity, $< t_1\phi, \psi >=< \phi, t_2\psi >$ for $\phi \in D_0, \psi \in D_0^*$. Therefore $D_0^* \subseteq D(t_1^*)$ and $t_2 \subseteq t_1^*$. So t_1^* is densely-defined and therefore t_1 extends to a closed operator $\alpha_*(t)$.

Proposition 3.4.2 ([La:95], Proposition 9.10)**.** *If t is a regular operator on E then the closed operator $\alpha_*(t)$ is a regular operator on $E \otimes_\alpha F$. Moreover we have*

$$(\alpha_*(t))^* = \alpha_*(t^*)$$

To study the functional calculus of the operator $\alpha_*(t)$ when t is self-adjoint, we consider its bounded transform (or z-transform)

$$f_{\alpha_*(t)} = \alpha_*(t)(1 + \alpha_*(t^*)\alpha_*(t))^{-1/2}$$

Let $f_t = t(1 + t^*t)^{-1/2}$. For $s \in \mathcal{L}(E)$, consider the adjointable operator $s \otimes 1$ in $\mathcal{L}(E \otimes_\alpha F)$ defined in Chapter 4, page 42 of [La:95].

Proposition 3.4.3. *We have $f_t \otimes 1 = f_{\alpha_*(t)}$*

Proof. Let $q = (1 + t^*t)^{-1/2}$. Then $(1 + t^*t)q^2 = 1$. Similarly let $r = (1 + \alpha_*(t^*)\alpha_*(t))^{-1/2}$, then

$$(1 \otimes 1 + \alpha_*(t^*)\alpha_*(t))r^2 = 1 \otimes 1.$$

We wish to prove that $q \otimes 1 = r$. First let us show that if $s \in \mathcal{L}(E)$ and t is a regular operator on E such that $ts \in \mathcal{L}(E)$ then range of $s \otimes 1$ is in $Dom(\alpha_*(t))$ and we have

$$ts \otimes 1 = \alpha_*(t)(s \otimes 1)$$

Since $Im(s) \subseteq Dom(t)$, we have $Im(s \otimes 1) \subseteq D_0 \subseteq Dom(\alpha_*(t))$. So we have on simple tensors

$$\alpha_*(t)(s \otimes 1)(x\dot{\otimes}y) = \alpha_*(t)(sx\dot{\otimes}y) = tsx\dot{\otimes}y = (ts \otimes 1)(x\dot{\otimes}y)$$

Now let $D_1 = \text{span}\{x\dot{\otimes}y | x \in Dom(t^*t), y \in F\}$. Then $D_1 \subseteq D_0$, and we have on D_1,

$$\alpha_*(t^*)\alpha_*(t)(x\dot{\otimes}y) = \alpha_*(t^*)(tx\dot{\otimes}y) = t^*tx\dot{\otimes}y) = \alpha_*(t^*t)(x\dot{\otimes}y)$$

where the second equality is justified by the fact that $x \in Dom(t^*t) \Rightarrow tx\dot{\otimes}y \in D_0^*$. Therefore we have $\alpha_*(t^*)\alpha_*(t) \subseteq \alpha_*(t^*t)$, and since $range(q^2) \subset Dom(t)$, it is easy to see that

$$(1 \otimes 1 + \alpha_*(t^*)\alpha_*(t))(q^2 \otimes 1) = 1 \otimes 1$$

Since the map $s \mapsto s \otimes 1$ is a $*$-homomorphism (cf. [La:95], page 42), we have $r = q \otimes 1$. This implies that

$$f_t \otimes 1 = tq \otimes 1 = \alpha_*(t)(q \otimes 1) = \alpha_*(t)r = f_{\alpha_*(t)}$$

□

Thus by the above proposition we see that the definition of $\alpha_*(t)$ coincides with the image of t under extension of the non-degenerate $*$-homomorphism $\alpha^* : \mathcal{K}_A(E) \to \mathcal{K}_B(E \otimes_\alpha F)$ in (cf. [La:95], page 42) to affiliated operators on $\mathcal{K}(E)$ in Theorem 3.4.1.

3.4.2 Functional calculus for the operator $\mathcal{D}_m$

Since we have an isometric isomorphism (see 3.3.5) $\Psi_{x,reg} : \mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T) \to L^2(\mathcal{G}_x, r^*E)$, there is an induced $*$-isomorphism $\Psi^*_{x,reg} : \mathcal{K}(\mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T)) \to \mathcal{K}(L^2(\mathcal{G}_x, r^*E))$ given by conjugation with $\Psi_{x,reg}$. We have the following

Proposition 3.4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then, we have $f(\tilde{D}_x) \in \mathcal{B}(L^2(\mathcal{G}_x, r^*E))$, and $f(\tilde{D}_x) = \Psi_{x,reg} \circ [f(\mathcal{D}_m) \otimes_{\rho_x^{reg}} Id] \circ \Psi^{-1}_{x,reg}$.*

Proof. By proposition 3.3.7, $\mathcal{D}_m$ is a regular operator on $\mathcal{E}_m$. Therefore $(\rho_x^{reg})_*(\mathcal{D}_m)$ is a regular operator on $\mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T)$ and therefore affiliated to the C^*-algebra $K(\mathcal{E}_m \otimes_{\rho_x^{reg}} l^2(\mathcal{G}_x^T))$. Hence the operator $\Psi^*_{x,reg}((\rho_x^{reg})_*(\mathcal{D}_m))$ is affiliated to $\mathcal{K}(L^2(\mathcal{G}_x, r^*E))$. Now for $\zeta \in C_c^\infty(\mathcal{G}_T, r^*E), \xi \in C_c(\mathcal{G}_x^T), u \in \mathcal{G}_x, \Psi_{x,reg}(\zeta \otimes \xi) \in C_c^\infty(\mathcal{G}_x, r^*E) \subset Dom(\tilde{D}_x)$, and we have by direct computation :

$$\Psi_{x,reg} \circ [\mathcal{D}_m \otimes Id](\zeta \otimes \xi)(u) \quad = \quad [\tilde{D}_x(\Psi_{x,reg}(\zeta \otimes \xi))](u) \tag{3.4.1}$$

Checking domains shows that the closed operators $\mathcal{D}_m \otimes Id := (\rho_x^{reg})_*(\mathcal{D}_m)$ and $\tilde{D}_x$ coincide.

So we get for $f \in C_b(\mathbb{R})$,

$$\begin{aligned} f(\tilde{D}_x) &= f((\Psi^*_{x,reg})(\mathcal{D}_m \otimes Id)) \\ &= (\Psi^*_{x,reg})f((\mathcal{D}_m \otimes Id)) \\ &= (\Psi^*_{x,reg})(f(\mathcal{D}_m) \otimes I) \end{aligned} \tag{3.4.2}$$

where the equalities above are justified by Theorem 3.4.1 and proposition 3.4.3. □

A similar proposition can be stated with similar arguments as before for the leafwise Dirac operator D:

Proposition 3.4.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then, we have $f(D_{L_x}) \in \mathcal{B}(L^2(L_x, E))$, and $f(D_{L_x}) = \Psi_{x,av} \circ [f(\mathcal{D}_m) \otimes_{\rho_x^{av}} Id] \circ \Psi^{-1}_{x,av}$.*

We give now a useful proposition exactly analogous to Theorem 3.19 [BePi:08], which we will use later.

Proposition 3.4.6. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Then $\psi(\mathcal{D}_m) \in \mathcal{K}(\mathcal{E}_m)$ and the following formulas hold:*

(a) $(\pi_x^{reg} \circ \chi_m^{-1})(\psi(\mathcal{D}_m)) = \Psi_{x,reg} \circ (\psi(\mathcal{D}_m) \otimes I) \circ \Psi_{x,reg}^{-1}$ *and* $(\pi_x^{av} \circ \chi_m^{-1})(\psi(\mathcal{D}_m)) = \Psi_{x,av} \circ (\psi(\mathcal{D}_m) \otimes I) \circ \Psi_{x,av}^{-1}$

(b) $\tau_{reg}^{\Lambda}(\chi_m^{-1}(\psi(\mathcal{D}_m))) = \tau^{\Lambda}(\psi(\tilde{D}))$

(c) $\tau_{av}^{\Lambda}(\chi_m^{-1}(\psi(\mathcal{D}_m))) = \tau_{\mathcal{F}}^{\Lambda}(\psi(D))$

in particular the element $\chi_m^{-1}(\psi(\mathcal{D}_m)) \in \mathcal{B}_m^E$ is trace class under both traces τ_{reg}^{Λ} and τ_{av}^{Λ}.

Proof. Since ψ is a Schwartz function we know by the functional calculus for $\mathcal{D}_m$ that the operator $\psi(\mathcal{D}_m)(I + \mathcal{D}_m^2)$ is a bounded adjointable operator on $\mathcal{E}_m$. Also from Proposition 4.2.3 and Proposition 3.4.5 of [Va:01], we have $(I + \mathcal{D}_m^2)^{-1} \in \mathcal{K}(\mathcal{E}_m)$. So one has

$$\psi(\mathcal{D}_m) = \left[\psi(\mathcal{D}_m)(I + \mathcal{D}_m^2)\right] (I + \mathcal{D}_m^2)^{-1} \in \mathcal{K}(\mathcal{E}_m)$$

Now (a) is a consequence of 3.3.13 and the second item is a consequence of 3.3.6. We will now show the formula (c), the proof for (b) is easier.

We have $\tau_{av}^{\Lambda} = \tau_{\mathcal{F}}^{\Lambda} \circ \pi_{av}$. It suffices to prove the result when $\psi(\mathcal{D}_m)$ is positive. Assuming so, we have,

$$\begin{aligned}
(\tau_{av}^{\Lambda} \circ \chi_m^{-1})(\psi(\mathcal{D}_m))) &= (\tau_{\mathcal{F}}^{\Lambda} \circ \pi_{av} \circ \chi_m^{-1})(\psi(\mathcal{D}_m))) \\
&= \tau_{\mathcal{F}}^{\Lambda}((\Psi_{x,av} \circ (\psi(\mathcal{D}_m) \otimes I) \circ \Psi_{x,av}^{-1})_{x \in M}) \\
&= \tau_{\mathcal{F}}^{\Lambda}(\psi(D_x)_{x \in M}) \text{ (from proposition 3.4.5)} \\
&= \tau_{\mathcal{F}}^{\Lambda}(\psi(D)) < \infty
\end{aligned}$$

where in the last line we have used Proposition 7.37 of [MoSc:06]. $\square$

Chapter 4

Stability properties of foliated ρ-invariants

4.1 The foliated η and ρ invariants

Consider the function $\phi_t(x) = x\exp(-tx^2)$ for $t > 0$. Then by the functional calculus, $\phi_t(D)$ and $\phi_t(\tilde{D})$ are well-defined. Since ϕ_t is a Schwartz function, the operators $\phi_t(D)$ and $\phi_t(\tilde{D})$ have smooth Schwartz kernels. Let the Schwartz kernels of e^{-tD^2} and $e^{-t\tilde{D}^2}$ be denoted by k_t and K_t. Then $k_{t,L} \in C^\infty(L \times L, \mathrm{End}(\bigwedge^* T * L))$ and $K_{t,x} \in C^\infty(\mathcal{G}_x \times \mathcal{G}_x, \mathrm{End}(\bigwedge^* T^*\mathcal{G}_x)$. We have the following estimate for these kernels for small t:

Proposition 4.1.1. *(Bismut-Freed estimate) For $0 \leq t < 1$ there exists a $C \geq 0$ such that,*

(1) $|tr_{pt}(k_{t,L}(x, x, n))| = O(\sqrt{t})$

(2) $|tr_{pt}(K_{t,x}(1_x, 1_x)| = O(\sqrt{t})$

for each $x \in L_n$, the leaf through $n \in T$, where tr_{pt} denotes the pointwise trace.

Proof. The proof of (1) is given in [Ra:93]. We give a detailed proof of (2) on the lines of this proof. A parametrix $P(.,.,t;x)$ for the heat kernel satisfies the following equation:

$$K_{t,x}(a, b) = P(a, b, t; x) - \sum_{i \geq 0} (-1)^i P_i(a, b, s; x) \tag{4.1.1}$$

where $P_i = P *_{conv} (R^{*i})$ with $R(a, b, t; x) = (\partial_t + \tilde{\Delta}_b)P(a, b, t; x)$, $*_{conv}$ is the convolution of kernels and $R^{*i} = R *_{conv} \cdots *_{conv} R$ with i factors. The series on the right hand side of 4.1.1 converges uniformly on $\mathcal{G}_x \times \mathcal{G}_x \times [0, T]$ together with all its derivatives. We can construct a parametrix $P(.,.,t;x)$ for the fundamental solution of the heat equation on the possibly noncompact Riemannian manifold $\mathcal{G}_x$ such that P satisfies the following conditions (cf. [Ra:93]):

- $P(a,b,t;x) \in \mathrm{Hom}(\bigwedge^{*} T_a^*\mathcal{G}_x, \bigwedge^{*} T_b^*\mathcal{G}_x)\ \forall t>0$ (4.1.2)
- $(\tilde{\Delta}_b + \dfrac{\partial}{\partial t})P$ is $O(t^m)$ (4.1.3)
- $\tilde{D}_b(\tilde{\Delta}_b + \dfrac{\partial}{\partial t})P$ is $O(t^{m-1})$ (4.1.4)

(4.1.5)

Here m is chosen such that

$$\int_0^t (t-s)^{-n/2}s^{m-1}ds = O(t^{1/2}) \tag{4.1.6}$$

$$||P(a,b,t;x)||_{a,b} \leq Ct^{-n/2} \tag{4.1.7}$$

where $||P(a,b,t;x)||_{a,b}$ denotes the norm on $\mathrm{Hom}(\bigwedge^* T_a^*\mathcal{G}_x, \bigwedge^* T_b^*\mathcal{G}_x)$. The construction of such a parametrix on compact manifolds is classical (see [Pat:71],[MiPle:49]). Choosing the support of P on a precompact ϵ-neighbourhood of the diagonal, where ϵ is less than half the injectivity radius ι[1] of $\mathcal{G}_x$, we can restrict the convolution integrals to metric balls of radius ϵ around one of the variables and use the uniform bound on the volume of these balls due to the bounded geometry property to estimate the integrals instead of the volume of the manifold which is used in estimating for the case of a compact manifold. This estimate is given by

$$|R^{*\lambda}(t,x,y)| \leq \frac{AB^{\lambda-1}vol(B_\epsilon(x))^{\lambda-1}t^{k-(n/2)+\lambda-1}}{(k-\frac{n}{2}+1)(k-\frac{n}{2}+2)...(k-\frac{n}{2}+\lambda-1)}$$

where A, B are constants (cf. [Ros:88]).

The semigroup domination property for the heat kernel gives the following estimate for the norm of $K_{t,x}$ (cf. [Ra:93],[Ros:88]):

$$||K_{t,x}(a,b)||_{a,b} \leq \exp(ct)K_{t,x}^{LB}(a,b) \tag{4.1.8}$$

where K^{LB} is the Schwartz kernel of the operator $e^{-t\Delta_{LB}}$, where Δ_{LB} is the Laplace Beltrami operator on $\mathcal{G}_x$ and $||K_{t,x}(a,b)||_{a,b}$ is the norm on $\mathrm{Hom}(\bigwedge T_a\mathcal{G}_x, \bigwedge T_b\mathcal{G}_x)$.

By Duhamel's principle,

$$(\partial_t + \tilde{\Delta}_b)P *_{conv} R^{*i} = R^{*i} + R^{*(i+1)} \tag{4.1.9}$$

Thus by 4.1.9 and 4.1.1, $K_{t,x}$ is the fundamental solution of the heat equation because the heat operator applied to the sum on the right side of 4.1.1 equals $-R$. So we have from 4.1.1:

$$(\partial_t + \tilde{\Delta}_a)K = 0 \Rightarrow (\partial_t + \tilde{\Delta}_b)P = (\partial_t + \tilde{\Delta}_b)\textstyle\sum_{i\geq 0}(-1)^i P_i(a,b,s;x) \tag{4.1.10}$$

By using Duhamel's principle again we have(cf. [Ro:88, Proposition 7.9, p. 96])

$$\sum_{i\geq 0}(-1)^i P_i(a,b,s;x) = \int_0^t ds \int_{\mathcal{G}_x} K_{t-s,x}(u,w)(\frac{\partial}{\partial s} + \tilde{\Delta}_v)P(w,v,s;x) *_w 1 \tag{4.1.11}$$

where $*_w$ is the hodge operator in the w-variable. So from 4.1.1, we then have,

$$K_{t,x}(u,v) = P(u,v,t;x) + \int_0^t ds \int_{\mathcal{G}_x} K_{t-s,x}(u,w)(\frac{\partial}{\partial s} + \tilde{\Delta}_v)P(w,v,s;x) *_w 1 \tag{4.1.12}$$

[1]We remark that ι is positive as $\mathcal{G}_x$ is the universal cover for the Riemannian manifold $L(x)$ having bounded geometry.

Since the series in 4.1.1 converges uniformly with derivatives of all orders, we can also apply the above computation to $\tilde{D}_b K$ to get

$$\tilde{D}_v K_{t,x}(u,v) = \tilde{D}_v P(u,v,t;x) + \int_0^t ds \int_{\mathcal{G}_x} K_{t-s,x}(u,w)(\frac{\partial}{\partial s} + \tilde{\Delta}_v)\tilde{D}_v P(w,v,s;x) *_w 1 \tag{4.1.13}$$

Applying the pointwise trace functional on both sides of 4.1.13 and using its linearity and normality, we get at $u = v$,

$$tr_{pt}(\tilde{D}K_{t,x}(u,u)) = tr_{pt}(\tilde{D}P(u,u,t;x)) + \int_0^t ds \;\; tr_{pt}\left(\int_{\mathcal{G}_x} K_{t-s,x}(u,w)(\frac{\partial}{\partial s} + \tilde{\Delta})\tilde{D}P(w,u,s;x) *_w 1\right) \tag{4.1.14}$$

where $\tilde{D}K_{t,x}(u,u) = \tilde{D}_v K(u,v)|_{u=v}$, $\tilde{D}P(u,u,t;x) = \tilde{D}_v P(u,v,t;x)|_{u=v}$ and $\tilde{D}P(w,u,s;x) = \tilde{D}_v P(w,u,s;x)|_{u=v}$.

From the estimate of Bismut and Freed(cf. [BiFr:86])

$$|tr_{pt}(\tilde{D}P(u,u,t;x))| = O(\sqrt{t}) \tag{4.1.15}$$

where the inequality constant depends on x. We estimate the integrand for the second term in 4.1.14 for $t \downarrow 0$ following [Ra:93]:

$$\begin{aligned} &\left| tr_{pt}\left(\textstyle\int_{\mathcal{G}_x} K_{t-s,x}(u,w)(\frac{\partial}{\partial s} + \tilde{\Delta})\tilde{D}P(w,u,s;x) *_w 1\right)\right| \\ &\leq C \textstyle\int_{B_x(u,\epsilon)} ||K_{t-s,x}(u,w)||_{u,w} s^{m-1} *_w 1 \\ &\leq C_1 vol(B_x(u,\epsilon))(t-s)^{-n/2} s^{m-1} \end{aligned} \tag{4.1.16}$$

for $C, C_1 > 0$ and where we have used 4.1.4 and the estimate for the kernel of the Laplace Beltrami operator $0 \leq K^{LB}_{t,x}(u,v) \leq Ct^{-n/2}$(cf. [Ch:84], Ch.8§4]). . The bounded geometry of $\mathcal{G}_x$ implies that $\sup_{u \in \mathcal{G}_x} vol(B_x(u,\epsilon)) < \infty$. Thus we get from 4.1.6, 4.1.14, 4.1.15 and 4.1.16,

$$|tr_{pt}(\tilde{D}K_{t,x}(u,u))| < A(x)\sqrt{t} \tag{4.1.17}$$

where the constant $A(x)$ depends on x, the dimension and the local geometry (Christoffel symbols and its derivatives for the Levi-Civita connection) of $\mathcal{G}_x$. However, since M is compact, the constants $A(x)$ are bounded above by a constant A independent of x.

Thus

$$|tr_{pt}(\tilde{D}K_{t,x}(u,u))| = O(\sqrt{t}) \tag{4.1.18}$$

uniformly over $x \in M$. □

Proposition 4.1.2. *The functions* $t \longmapsto \tau^{\Lambda}_{\mathcal{F}}(D\exp(-t^2D^2))$ *and* $t \longmapsto \tau^{\Lambda}(\tilde{D}\exp(-t^2\tilde{D}^2))$ *are Lebesgue integrable on* $(0,\infty)$.

Proof. Since D and $\tilde{D}$ are affiliated to the Von Neumann algebras $W^*(M,\mathcal{F};E)$ and $W^*(\mathcal{G},E)$, so their spectral resolutions E_λ and $\tilde{E}_\lambda$ belong to the respective von Neumann algebras and are $\tau^{\Lambda}_{\mathcal{F}}$-trace class and τ^{Λ}-trace class, respectively. Then corresponding to the positive functionals $F(\lambda) = \tau^{\Lambda}_{\mathcal{F}}(E_\lambda)$ and $\tilde{F}(\lambda) = \tau^{\Lambda}(\tilde{E}_\lambda)$

on $\mathbb{R}$ there exist σ-finite Borel measures on $\mathbb{R}$, denoted α and $\tilde{\alpha}$ respectively, such that for a rapidly decreasing function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\tau^{\Lambda}(f(\tilde{D})) = \int_{\mathbb{R}} f(x) d\tilde{\alpha}(x) \text{ and } \tau^{\Lambda}_{\mathcal{F}}(f(D) = \int_{\mathbb{R}} f(x) d\alpha(x)$$

Now, we have,

$$\begin{aligned}
|\int_1^{\infty} \tau^{\Lambda}_{\mathcal{F}}(D \exp(-t^2 D^2)) dt| &\leq \int_1^{\infty} |\tau^{\Lambda}_{\mathcal{F}}(D \exp(-t^2 D^2))| dt \\
&\leq \int_1^{\infty} \tau^{\Lambda}_{\mathcal{F}}(|D| \exp(-t^2 D^2))| dt \\
&= \int_1^{\infty} \int_0^{\infty} \lambda \exp(-t^2 \lambda^2) d\alpha(\lambda) dt \\
&= \int_0^{\infty} \lambda \exp(-\lambda^2) \int_1^{\infty} \exp(-(t^2-1)\lambda^2) d\alpha \lambda dt \\
&= \int_0^{\infty} \lambda \exp(-\lambda^2) \int_0^{\infty} (u^2+\lambda^2)^{1/2} \lambda^{-1} \exp(-u^2) du d\alpha \lambda \\
&\leq \int_0^{\infty} \exp(-\lambda^2) d\alpha(\lambda) \int_0^{\infty} u \exp(-u^2) du \\
&= \frac{\sqrt{\pi}}{2} \int_0^{\infty} \exp(-\lambda^2) d\alpha(\lambda) \\
&= \frac{\sqrt{\pi}}{2} \tau^{\Lambda}_{\mathcal{F}}(\exp(-D^2)) < \infty \qquad (4.1.19)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\int_0^1 |\tau^{\Lambda}_{\mathcal{F}}(D \exp(-t^2 D^2))| dt &\leq \int_0^1 \sum_{i \in I} \int_{T_i} \int_{L_i} |tr_{pt} k_t(l_i, l_i, t_i)| d\lambda^L(l_i) d\Lambda(t_i) \\
&\leq \int_0^1 \sum_{i \in I} \int_{T_i} \int_{L_i} C d\lambda^L(l_i) d\Lambda(t_i) \qquad \text{by Proposition(4.1.1)} \\
&= C \times vol(M) < \infty \qquad (4.1.20)
\end{aligned}$$

□

Definition We define the foliated η-invariant for the Dirac operator D and its lift $\tilde{D}$ as follows:

$$\eta^{\Lambda}_{\mathcal{F}}(D) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \tau^{\Lambda}_{\mathcal{F}}(D \exp(-t^2 D^2)) dt \text{ and } \eta^{\Lambda}(\tilde{D}) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \tau^{\Lambda}(\tilde{D} \exp(-t^2 \tilde{D}^2)) dt$$

Definition We define the foliated ρ-invariant associated to the longitudinal Dirac operator D on the foliated manifold $(M, \mathcal{F})$ as

$$\rho^{\Lambda}(D; M, \mathcal{F}) = \eta^{\Lambda}(\tilde{D}) - \eta^{\Lambda}_{\mathcal{F}}(D) \qquad (4.1.21)$$

Example 1. Let $\tilde{M} \to M$ be a universal covering of a compact manifold M. Then $(M, \mathcal{F})$ is a foliation with just one leaf, and τ^{Λ} coincides with the L^2-trace Tr_{Γ}, where $\Gamma = \pi_1(M)$ on the von Neumann algebra $W^*(\mathcal{G}) = B(L^2(\tilde{M})^{\Gamma}$, and $\tau^{\Lambda}_{\mathcal{F}}$ coincides with the usual trace on $B(L^2(M))$. Therefore the foliated ρ-invariant coincides with the Cheeger-Gromov ρ-invariant.

2. When the foliation is given by a suspension, the foliated ρ-invariant coincides with the ρ-invariant defined and studied by [BePi:08].

4.2 η-invariant as a determinant

Our main reference for finite projections and finite von Neumann algebras is [Di:57]. Let $\mathcal{M}$ be a von Neumann algebra and $P(\mathcal{M})$ be the set of projections on $\mathcal{M}$. Define a partial order $\leq$ on on $P(\mathcal{M})$ as follows: for two projections p_1 and p_2 of $\mathcal{M}$,

$$p_1 \leq p_2 \Leftrightarrow p_2 p_1 = p_2$$

Also p_1 and p_2 as above are equivalent,denoted $p_1 \sim p_2$ if and only if there exists $u \in \mathcal{M}$ such that $p_1 = u^* u$ and $p_2 = uu^*$.

Definition [Di:57, Chapter I, §6.7, p.97] A von Neumann algebra $\mathcal{M}$ is called finite if given any non zero $T \in \mathcal{M}^+$ there exists a positive, finite, faithful, normal trace ϕ on $\mathcal{M}$ such that $\phi(T) \neq 0$.

Definition [Di:57, Chapter II, §2.1, p. 229] A projection $p \in \mathcal{M}$ is called a finite projection if the the algebra $p\mathcal{M}p$ is finite. The set of finite projections in $\mathcal{M}$ will be denoted by $P^f(\mathcal{M})$.

Definition Let $T \in \mathcal{M}$. Define

$$N_T = sup\{p \in P(\mathcal{M}) | Tp = 0\}$$

and

$$R_T = inf\{p \in P(\mathcal{M}) | pT = T\}$$

N_T is called the null projection of T and R_T is called the range projection of T.

The element T is called finite(or of finite rank) relative to $\mathcal{M}$ if $R_T \in P^f(\mathcal{M})$. Let the set of all finite rank operators relative to $\mathcal{M}$ be denoted by $\mathcal{K}_0$.

Proposition 4.2.1. *$\mathcal{K}_0$ is a two sided $*$-ideal in $\mathcal{M}$.*

Proof. Let $S \in \mathcal{M}$ and $T \in \mathcal{K}_0$. Then $R_{TS} = inf\{p \in P^f(\mathcal{M}) | pTS = TS\}$. We must prove that R_{TS} is finite. Clearly $pT = T \Rightarrow pTS = TS$. Thus $\{p \in P^f(\mathcal{M}) | pT = T\} \subseteq \{p \in P^f(\mathcal{M}) | pTS = TS\}$. Therefore $R_{TS} \leq R_T$. Since R_T is finite, so is R_{TS}.

Now let $T, S \in \mathcal{K}_0$. Then we have to show $T + S \in \mathcal{K}_0$. This follows from the fact that $sup(R_T, R_S)$ is finite [Di:57, Proposition 5, page 231] .

Finally we have to show that if T is finite so is T^*. Consider the polar decomposition of T, $T = W|T|$. Then WW^* is the projection onto the range of T, and W^*W is the projection onto the range of T^*. We show that $R_T = WW^*$. First, we know that $WW^*T = WW^*W|T| = W|T| = T$, so $R_T \leq WW^*$. Also WW^* is the least projection in $\mathcal{M}$ such that $pT = T$. So $WW^* \leq R_T$. Thus $R_T = WW^*$ Similarly we get $R_{T^*} = W^*W$. Therefore $R_T \sim R_{T^*}$ and so R_{T^*} is finite.

$\square$

Definition Let $\mathcal{K}$ denote the norm closure of $\mathcal{K}_0$ in $\mathcal{K}$. Elements of $\mathcal{K}$ are called compact operators relative to $\mathcal{M}$. We will also use the notation $\mathcal{K}(\mathcal{M})$ for $\mathcal{K}$.

Definition ([FaKo:86]) Let τ be a positive, semi-finite, faithful, normal trace on a von Neumann algebra $\mathcal{M}$. Then we define $L^1(\mathcal{M}, \tau) := \{T \in \eta(\mathcal{M}) | \tau(|T|) < \infty\}$, where $\eta(\mathcal{M})$ is the set of densely defined closed operators affiliated to $\mathcal{M}$ and for a positive self-adjoint element $S \in \eta(\mathcal{M})$ we define $\tau(S) = \sup_n \tau(\int_0^n \lambda dE_\lambda) = \int_0^\infty \lambda d\tau(E_\lambda)$.

$L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is then a two-sided $*$-ideal in $\mathcal{M}$.

Definition We define

$$\mathcal{IL}^1 = \{A \in \mathcal{M} | A \text{ invertible and } A - I \in L^1(\mathcal{M}, \tau)\}$$
$$\mathcal{IK} = \{A \in \mathcal{M} | A \text{ invertible and } A - I \in \mathcal{K}(\mathcal{M})\}$$

Proposition 4.2.2. *The spaces $\mathcal{IL}^1$ and $\mathcal{IK}$ are subgroups of the group of invertibles in $\mathcal{M}$, with group operation being composition of operators.*

Proof. Let $A \in \mathcal{IL}^1$. Then, $A = I + B$ for some $B \in L^1(\mathcal{M}, \tau)$. We write

$$\begin{aligned} (I+B)^{-1} &= (I+B)^{-1}(I+B-B) \\ &= I - (I+B)^{-1}B \in \mathcal{IL}^1 \end{aligned}$$

since $B \in L^1(\mathcal{M}, \tau) \cap \mathcal{M} \Rightarrow (I+B)^{-1}B \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$, as $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is an ideal in $\mathcal{M}$.

Now let $A_1, A_2 \in \mathcal{IL}^1$. Then, clearly A_1A_2 is invertible with inverse $A_2^{-1}A_1^{-1}$. We also have $A_i = I + B_i$ for some $B_i \in L^1(\mathcal{M}, \tau), i = 1, 2$

$$\begin{aligned} A_1A_2 &= (I+B_1)(I+B_2) \\ &= I + (B_1 + B_2 + B_1B_2) \end{aligned}$$

But $(B_1 + B_2 + B_1B_2) \in L^1(\mathcal{M}, \tau)$, so $A_1A_2 \in \mathcal{IL}^1$. The proof for the space of operators $\mathcal{IK}$ is similar. $\square$

Now let $\{U_t\}_{t\in[0,1]}$ be a norm continuous path of operators in $\mathcal{IK}$. We have the following lemma:

Lemma 4.2.3. *Let $\epsilon > 0$. There is a piecewise linear path $\{V_t\}_{t\in[0,1]}$ in $\mathcal{IL}^1$ such that $||U_t - V_t|| < \epsilon \forall t \in [0, 1]$.*

Proof. Since U_t is continuous and $[0, 1]$ is compact, U_t is uniformly continuous. Therefore there exists $\delta > 0$ such that

$$|t - t'| < \delta \Rightarrow ||U_t - U_t'|| < \epsilon/4$$

Since each U_t is of the form $I + T_t$, $T_t \in \mathcal{K}(\mathcal{M})$, we get

$$|t - t'| < \delta \Rightarrow ||T_t - T_t'|| < \epsilon/4$$

Now, using the fact that $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is dense in $\mathcal{K}(\mathcal{M})$ in the uniform topology, we can find $S_0, S_{\delta/2}$ such that

$$||S_0 - T_0|| < \epsilon/4, ||S_{\delta/2} - T_{\delta/2}|| < \epsilon/4$$

We define the piecewise linear path for $t \in [0, \delta/2]$ as

$$S_t = tS_{\delta/2} + (1-t)S_0$$

Then we have $\forall t \in [0, \delta/2]$:

$$\begin{aligned} ||S_t - T_t|| &= ||tS_{\delta/2} + (1-t)S_0 - T_t|| \\ &= ||t(S_{\delta/2} - T_{\delta/2}) + tT_{\delta/2} + (1-t)(S_0 - T_0) + (1-t)T_0 - T_t|| \\ &\leq t||S_{\delta/2} - T_{\delta/2}|| + (1-t)||S_0 - T_0|| + t||T_{\delta/2} - T_0|| + ||T_0 - T_t|| \\ &< t\frac{\epsilon}{4} + (1-t)\frac{\epsilon}{4} + t\frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &< \epsilon \qquad\qquad (t \leq 1) \end{aligned} \tag{4.2.1}$$

We now repeat the above construction for the intervals $[(n-1)\delta/2, n\delta/2]$ until $n\delta/2 < 1$, and if $n\delta/2 \geq 1$, then we do the construction for the interval $[(n-1)\delta/2, 1]$. Now put $V_t = I + S_t$ for $t \in [0, 1]$. Thus we get a piecewise linear path of operators $\{V_t\}_{t\in[0,1]} \in \mathcal{IL}^1$ such that $||U_t - V_t|| < \epsilon \ \forall t \in [0, 1]$.

$\square$

Let $\{T_t\}_{t\in[0,1]}$ and $\{S_t\}_{t\in[0,1]}$ be as in the lemma above.

Definition The path $\{S_t\}_{t\in[0,1]}$ is called a piecewise linear ϵ-perturbation of the path of operators $\{T_t\}_{t\in[0,1]}$

Definition Given a continuous piecewise linear path $T = \{T_t\}_{t\in[0,1]}$ in $\mathcal{IK}$, we define its Fuglede-Kadison determinant $w^\tau(T_t)$ as follows:

$$w^\tau(T) := \frac{1}{2\pi i}\int_0^1 \tau(S_t^{-1}\frac{dS_t}{dt})dt$$

for a piecewise linear ϵ-perturbation S of T corresponding to a small enough $\epsilon > 0$.

The following proposition ensures the well-definedness of the above determinant [HiSk:84].

Proposition 4.2.4. *Let $T = \{T_t\}_{t\in[0,1]}$ in $\mathcal{IL}^1$ be a continuous piecewise linear path of operators. Then we have the following:*

1. If $||T_t - I||_1 < 1\ \forall t \in [0,1]$, then for any $t \in [0,1]$ the operator $Log(T_t)$ is well defined in the von Neumann algebra $\mathcal{M}$ and we have

$$w^\tau(T) = \frac{1}{2\pi i}[\tau(Log(T_1)) - \tau(Log(T_0))]$$

2. There exists $\delta > 0$ such that for any continuous piecewise linear path $T' = \{T'_t\}_{t\in[0,1]}$ which satisfies

$$||T'_t - T_t||_1 < \delta,\ \text{and}\ T_0 = T'_0,\ T_1 = T'_1$$

we have $w^\tau(T) = w^\tau(T')$.

3. The determinant for any piecewise linear path continuous in the uniform norm is well-defined and depends only on the homotopy class of the path with fixed endpoints.

We now proceed to interpret the η-invariant as the determinant of a particular path of operators. Our references for this section are [Ke:00], [BePi:08]. Let for $t > 0$,

$$\phi_t(x) := \frac{2}{\sqrt{\pi}}\int_0^{tx} e^{-s^2}ds, \psi_t(x) := -\exp(i\pi\phi_t(x)), f_t(x) := xe^{-t^2x^2}$$

Then the functions $1 - \psi_t, \psi'_t$ and f_t are Schwartz class functions for all $t > 0$. The operators $1 - \psi_t(\mathcal{D}_m), \psi'_t(\mathcal{D}_m)$ and $f_t(\mathcal{D}_m)$ are $C^*(\mathcal{G}_X^X)$-compact operators acting on the Hilbert modules $\mathcal{E}_m$. Furthermore, their images under the representations π^{reg} and π^{av} are τ^Λ and $\tau^\Lambda_{\mathcal{F}}$-trace class operators in the von Neumann algebras $W^*(\mathcal{G}, E)$ and $W^*(M, \mathcal{F}, E)$, respectively. Moreover, $\psi_t(\mathcal{D}_m)$ is an invertible operator with inverse given by $-e^{-i\pi\phi_t(\mathcal{D}_m)}$ for $t \geq 0$, so $(\psi_t(\mathcal{D}_m))_{t>0}$ gives an open path of invertible operators in $\mathcal{IK}_{\mathcal{A}_X^X}(\mathcal{E}_m)$, whose image under $\pi^{reg} \circ \chi_m^{-1}$ is a path in $\mathcal{IL}^1(W^*(\mathcal{G}, E))$. Similarly, its image under $\pi^{av} \circ \chi_m^{-1}$ is a path in $\mathcal{IL}^1(W^*(M, \mathcal{F}, E))$. We define two paths associated to the operator $\mathcal{D}_m$:

$$V^{reg}(\mathcal{D}_m) = (\pi^{reg} \circ \chi_m^{-1}(\psi_t(\mathcal{D}_m)))_{t>0},\ \text{and}$$

$$V^{av}(\mathcal{D}_m) = (\pi^{av} \circ \chi_m^{-1}(\psi_t(\mathcal{D}_m)))_{t>0}$$

Using the traces on the von Neumann algebras $W^*(\mathcal{G}, E)$ and $W^*(M, \mathcal{F}, E)$, we associate to $V^{reg}(\mathcal{D}_m)$ and $V^{av}(\mathcal{D}_m)$ the corresponding determinants. We define

$$w_\epsilon^{reg}(\mathcal{D}_m) := w_\Lambda^{reg}(V_\epsilon^{reg}(\mathcal{D}_m))\ \text{and}\ w_\epsilon^{av}(\mathcal{D}_m) := w_\Lambda^{av}(V_\epsilon^{av}(\mathcal{D}_m))$$

where $V_\epsilon^{reg}(\mathcal{D}_m) = (\pi^{reg} \circ \chi_m^{-1}(\mathcal{D}_m))^{\epsilon\leq t\leq 1/\epsilon}$, and $V_\epsilon^{av}(\mathcal{D}_m) = (\pi^{av} \circ \chi_m^{-1}(\mathcal{D}_m))^{\epsilon\leq t\leq 1/\epsilon}$.

Proposition 4.2.5. *We have the following formulae:*

$$\lim_{\epsilon\to 0} w_\epsilon^{reg}(\mathcal{D}_m) = \frac{1}{2}\eta^\Lambda(\tilde{D}) \text{ and } \lim_{\epsilon\to 0} w_\epsilon^{av}(\mathcal{D}_m) = \frac{1}{2}\eta^\Lambda_{\mathcal{F}}(D)$$

Proof. The proof is exactly as in Theorem 5.13 [BePi:08], we give it here nevertheless for completeness. We have

$$(V_t^{reg}(\mathcal{D}_m))^{-1}\frac{d}{dt}(V_t^{reg}(\mathcal{D}_m)) = (\pi^{reg}\circ\chi_m^{-1})(i\pi\mathcal{D}_m\frac{2}{\sqrt{\pi}}e^{-t^2\mathcal{D}_m^2}) = 2i\sqrt{\pi}(\pi^{reg}\circ\chi_m^{-1})(f_t(\mathcal{D}_m))$$

By Proposition 3.4.4, we have $(\pi_x^{reg}\circ\chi_m^{-1})(f_t(\mathcal{D}_m)) = f_t(\tilde{D}_x)$, so

$$(\pi^{reg}\circ\chi_m^{-1})(f_t(\mathcal{D}_m)) = f_t(\tilde{D})$$

Therefore we have

$$\begin{aligned}
\lim_{\epsilon\to 0} w_\epsilon^{reg}(\mathcal{D}_m) &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_\epsilon^{1/\epsilon}\tau^\Lambda(2i\sqrt{\pi}(\pi^{reg}\circ\chi_m^{-1})(f_t(\mathcal{D}_m)))dt \\
&= \lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi}}\int_\epsilon^{1/\epsilon}\tau^\Lambda(f_t(\tilde{D}))dt \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty \tau^\Lambda(\mathcal{D}_m e^{-t^2\mathcal{D}_m^2})dt \\
&= \frac{1}{2}\eta^\Lambda(\tilde{D})
\end{aligned}$$

The second formula is proved in a similar way using Proposition 3.4.5. □

Corollary 4.2.6. *We have*

$$\rho^\Lambda(D, M, \mathcal{F}) = 2\times\lim_{\epsilon\to 0}(w_\epsilon^{reg}(\mathcal{D}_m) - w_\epsilon^{av}(\mathcal{D}_m)$$

Proof. This is immediate from the definition of $\rho^\Lambda(D, M, \mathcal{F})$. □

4.3 Metric independence of the ρ-invariant

Let g be a leafwise metric on M^{4l-1} which lifts to a $\mathcal{G}$-equivariant family of metrics $(\tilde{g}_x)_{x\in M}$ such that $\tilde{g}_x$ is a metric on $\mathcal{G}_x$. Let $D^{sign} = (D_L^{sign})_{L\in M/\mathcal{F}}$ be the leafwise signature operator on $(M,\mathcal{F})$, and let $\tilde{D}^{sign} = (\tilde{D}_x^{sign})_{x\in M}$ be its lift to the smooth sections on the groupoid as defined in 3.2.1. We can define the foliated ρ-invariant (which coincides up to a sign with the definition 4.1) associated to the signature operators as

$$\rho^\Lambda_{sign}(M,\mathcal{F},g) = \frac{1}{\sqrt{\pi}}\int_0^\infty \tau^\Lambda(\tilde{*}\tilde{d}\exp(-t^2\tilde{\Delta}_{2l-1})) - \tau^\Lambda_{\mathcal{F}}(*d\exp(-t^2\Delta_{2l-1}))dt$$

where $\tilde{\Delta}_{2l-1}$ and Δ_{2l-1} are the Laplacians on $2l-1$ forms associated with the metrics $\tilde{g}$ and g, respectively.

Theorem 4.3.1. *Assume the data given above. Also let g_0, g_1 be leafwise smooth metrics on $(M, \mathcal{F})$. Then,*

$$\rho^{\Lambda}(M, \mathcal{F}, g_0) = \rho^{\Lambda}(M, \mathcal{F}, g_1)$$

where $\rho^{\Lambda}(M, \mathcal{F}, g)$ is the ρ-invariant associated with the metric g and the corrsponding leafwise signature operator.

Proof. We will extend the method of Cheeger and Gromov [ChGr:85]. Set $g_u = ug_1 + (1-u)g_0$, for $u \in [0,1]$. Let us first compute the variation $\frac{d}{du}\tau_{\mathcal{F}}^{\Lambda}((*d\exp(-t^2\Delta_{2l-1}))$. Denote by $\Delta_{u,k}$ the Laplacian corresponding to the leafwise metric g_u acting on k-forms. Let $\Gamma_u(t) = \exp(-t^2\Delta_{u,2l-1})$. Applying Duhamel's principle, we get,

$$\begin{aligned} *_0\, d\Gamma_u(t-\epsilon)\Gamma_0(\epsilon) - *_0 d\Gamma_u(\epsilon)\Gamma_0(t-\epsilon) &= -\int_{\epsilon}^{t-\epsilon}[*_0 d\Gamma_u(t-s)\Gamma_0(s)]'ds \\ &= -\int_{\epsilon}^{t-\epsilon}[-*_0\, d\Gamma_u'(t-s)\Gamma_0(s) + *_0 d\Gamma_u(t-s)\Gamma_0'(s)]ds \\ &= -\int_{\epsilon}^{t-\epsilon}[*_0 d\Delta_u\Gamma_u(t-s)\Gamma_0(s) - *_0 d\Gamma_u(t-s)\Delta_0\Gamma_0(s)]ds \end{aligned} \tag{4.3.1}$$

Taking the trace $\tau_{\mathcal{F}}^{\Lambda}$ of the term $*_0 d\Gamma_u(t-s)\Delta_0\Gamma_0(s)$ in the last line above, we get,

$$\begin{aligned} \tau_{\mathcal{F}}^{\Lambda}(*_0 d\Gamma_u(t-s)\Delta_0\Gamma_0(s)) &= \tau_{\mathcal{F}}^{\Lambda}([*_0 d\Gamma_u(t-s)\Gamma_0(s/2)][\Gamma_0(s/2)\Delta_0]) \\ &\quad \text{(using the semi-group property of } \Gamma(t) \text{ and commutativity due to functional calculus)} \\ &= \tau_{\mathcal{F}}^{\Lambda}([\Gamma_0(s/2)\Delta_0][*_0 d\Gamma_u(t-s)\Gamma_0(s/2)]) \\ &\quad \text{(using the trace property and boundedness of the two operators in square brackets)} \\ &= \tau_{\mathcal{F}}^{\Lambda}(*_0 d\Delta_0\Gamma_u(t-s)\Gamma_0(s/2)[\Gamma_0(s/2)]) \\ &\quad \text{(using the trace property again and that } \Delta_0 \text{ commutes with } *_0 d) \\ &= \tau_{\mathcal{F}}^{\Lambda}(*_0 d\Delta_0\Gamma_u(t-s)\Gamma_0(s)) \end{aligned} \tag{4.3.2}$$

where we have also used the fact that $Range(\Gamma_u(t-s)) \subseteq Dom(\Delta_0)$, since $\Gamma_u(t-s)$ is a tangentially smoothing operator with uniformly bounded kernel. So from 4.3.1 and 4.3.2 we get,

$$\tau_{\mathcal{F}}^{\Lambda}(*_0 d\Gamma_u(t-\epsilon)\Gamma_0(\epsilon) - *_0 d\Gamma_u(\epsilon)\Gamma_0(t-\epsilon)) = -\int_{\epsilon}^{t-\epsilon}\tau_{\mathcal{F}}^{\Lambda}(*_0 d(\Delta_u - \Delta_0)\Gamma_u(t-s)\Gamma_0(s))ds \tag{4.3.3}$$

Differentiating 4.3.3 with respect to u and setting $u=0$, the right hand side is given by

$$\begin{aligned}
& -\int_{\epsilon}^{t-\epsilon} \tau_{\mathcal{F}}^{\Lambda}(*_0 d\frac{d}{du}[(\Delta_u-\Delta_0)\Gamma_u(t-s)\Gamma_0(s)])ds \\
= & -\int_{\epsilon}^{t-\epsilon} \tau_{\mathcal{F}}^{\Lambda}(*_0 d[\dot{\Delta}_u\Gamma_u(t-s)\Gamma_0(s)+\Delta_u\dot{\Gamma}_u(t-s)\Gamma_0(s)-\Delta_0\dot{\Gamma}_u(t-s)\Gamma_0(s)]|_{u=0})ds \\
= & -\int_{\epsilon}^{t-\epsilon} \tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\Delta}_0\Gamma_0(t-s)\Gamma_0(s))ds \\
= & -\int_{\epsilon}^{t-\epsilon} \tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\Delta}_0\Gamma_0(t))ds \qquad \text{(semi-group property of } \Gamma(t)) \\
= & -(t-2\epsilon)\tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\Delta}_0\Gamma_0(t)) \\
= & -(t-2\epsilon)\tau_{\mathcal{F}}^{\Lambda}(*_0 d(d\dot{\delta}_0+\dot{\delta}_0 d)\Gamma_0(t)) \\
= & -(t-2\epsilon)\tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\delta}_0 d\Gamma_0(t)) \\
= & (t-2\epsilon)\tau_{\mathcal{F}}^{\Lambda}[(*_0 d\dot{*}_0 d *_0 d\Gamma_0(t))+(*_0 d *_0 d\dot{*}_0 d\Gamma_0(t)) \\
= & -2(t-2\epsilon)\tau_{\mathcal{F}}^{\Lambda}(\dot{*}_0 d\Delta_0\Gamma_0(t)) \text{ (by permutation of bounded factors)}
\end{aligned} \tag{4.3.4}$$

Taking the limit as $\epsilon \to 0$ we get the result

$$\lim_{\epsilon\to 0} -\int_{\epsilon}^{t-\epsilon} \tau_{\mathcal{F}}^{\Lambda}(*_0 d\frac{d}{du}[(\Delta_u-\Delta_0)\Gamma_u(t-s)\Gamma_0(s)])ds = 2t\frac{d}{dt}\tau_{\mathcal{F}}^{\Lambda}(\dot{*}_0 d\Gamma_0(t)) \tag{4.3.5}$$

Now to compute the left hand side of 4.3.3, we first note the identity

$$\begin{aligned}
\frac{d}{du}\tau_{\mathcal{F}}^{\Lambda}(*d\Gamma(t)) &= \tau_{\mathcal{F}}^{\Lambda}(\dot{*}d\Gamma(t))+\tau_{\mathcal{F}}^{\Lambda}(*d\dot{\Gamma}(t)) \\
&= \tau_{\mathcal{F}}^{\Lambda}(\dot{*}d\Gamma(t))+\tau_{\mathcal{F}}^{\Lambda}(*d\dot{\Gamma}(t-\epsilon+\epsilon)) \\
&= \tau_{\mathcal{F}}^{\Lambda}(\dot{*}d\Gamma(t))+\tau_{\mathcal{F}}^{\Lambda}(*d\dot{\Gamma}(t-\epsilon)\Gamma(\epsilon))+\tau_{\mathcal{F}}^{\Lambda}(*d\Gamma(t-\epsilon)\dot{\Gamma}(\epsilon)) \\
&= \lim_{\epsilon\to 0}[\tau_{\mathcal{F}}^{\Lambda}(\dot{*}d\Gamma(t))+\tau_{\mathcal{F}}^{\Lambda}(*d\dot{\Gamma}(t-\epsilon)\Gamma(\epsilon))+\tau_{\mathcal{F}}^{\Lambda}(*d\Gamma(t-\epsilon)\dot{\Gamma}(\epsilon))] \\
&= \tau_{\mathcal{F}}^{\Lambda}(\dot{*}d\Gamma(t))+\tau_{\mathcal{F}}^{\Lambda}(*d\dot{\Gamma}(t))+\lim_{\epsilon\to 0}\tau_{\mathcal{F}}^{\Lambda}(*d\Gamma(t)\dot{\Gamma}(\epsilon))
\end{aligned} \tag{4.3.6}$$

which implies that

$$\lim_{\epsilon\to 0}\tau_{\mathcal{F}}^{\Lambda}(*d\Gamma(t)\dot{\Gamma}(\epsilon))=0$$

Using this in the LHS of 4.3.3, we get,

$$\begin{aligned}
& \lim_{\epsilon\to 0}\frac{d}{du}[\tau_{\mathcal{F}}^{\Lambda}(*_0 d\Gamma_u(t-\epsilon)\Gamma_0(\epsilon)-*_0 d\Gamma_u(\epsilon)\Gamma_0(t-\epsilon))]|_{u=0} \\
= & \lim_{\epsilon\to 0}\left(\tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\Gamma}_0(t-\epsilon)\Gamma_0(\epsilon))-\tau_{\mathcal{F}}^{\Lambda}(*_0 d\dot{\Gamma}_0(\epsilon)\Gamma_0(t-\epsilon))\right) \\
= & \frac{d}{du}|_{u=0}\tau_{\mathcal{F}}^{\Lambda}(*_u d\Gamma_u(t))-\tau_{\mathcal{F}}^{\Lambda}(\dot{*}_0 d\Gamma_0(t))
\end{aligned} \tag{4.3.7}$$

So from 4.3.5 and 4.3.7 we get,

$$\frac{d}{du}\tau_{\mathcal{F}}^{\Lambda}(*_u d\Gamma_u(t))|_{u=0} = \tau_{\mathcal{F}}^{\Lambda}(\dot{*}_0 d\Gamma_0(t))+2t\frac{d}{dt}\tau_{\mathcal{F}}^{\Lambda}(\dot{*}_0 d\Gamma_0(t)) \tag{4.3.8}$$

Now, we have,

$$\eta^{\Lambda}_{\mathcal{F}}(M,\mathcal{F},g_u) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\tau^{\Lambda}_{\mathcal{F}}(*_u d\exp(-t^2\Delta_u))dt \tag{4.3.9}$$

$$= \lim_{T\to\infty}\lim_{\epsilon\to 0}\frac{2}{\sqrt{\pi}}\int_{\epsilon}^{T}\tau^{\Lambda}_{\mathcal{F}}(*_u d\exp(-t^2\Delta_u))dt \tag{4.3.10}$$

where both the long time and short time convergence is uniform in u. From 4.3.8 and 4.3.9,

$$\frac{d}{du}\eta^{\Lambda}_{\mathcal{F}}(M,\mathcal{F},g_u)|_{u=0} = \lim_{T\to\infty}\lim_{\epsilon\to 0}\frac{2}{\sqrt{\pi}}\int_{\epsilon}^{T}\frac{d}{du}|_{u=0}\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_u d\exp(-t^2\Delta_u))dt$$

$$= \lim_{T\to\infty}\lim_{\epsilon\to 0}\frac{2}{\sqrt{\pi}}\int_{\epsilon}^{T}[\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(t)) + 2t\frac{d}{dt}\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(t))]dt \tag{4.3.11}$$

whence integration by parts of the second term gives us

$$\frac{d}{du}\eta^{\Lambda}_{\mathcal{F}}(M,\mathcal{F},g_u)|_{u=0} = \frac{2}{\sqrt{\pi}}\left(\lim_{T\to\infty}2T\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(T)) - \lim_{\epsilon\to 0}2\epsilon\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(\epsilon))\right) \tag{4.3.12}$$

We can repeat all the arguments to show that

$$\frac{d}{du}\eta^{\Lambda}(\tilde{D}^{sign}_u)|_{u=0} = \frac{2}{\sqrt{\pi}}\left(\lim_{T\to\infty}2T\tau^{\Lambda}(\tilde{*}_0\tilde{d}\tilde{\Gamma}_0(T)) - \lim_{\epsilon\to 0}2\epsilon\tau^{\Lambda}(\tilde{*}_0\tilde{d}\tilde{\Gamma}_0(\epsilon))\right) \tag{4.3.13}$$

We claim that

$$\lim_{T\to\infty}2T\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(T)) = 0 \text{ and } \lim_{T\to\infty}2T\tau^{\Lambda}(\tilde{*}_0\tilde{d}\tilde{\Gamma}_0(T)) = 0$$

To show this we use the spectral estimate

$$|2T\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(T))| \le C\tau_{\Lambda}((1-E_0)E_{\lambda}(1-E_0)) + 2T\exp(-\lambda(T-1))\tau^{\mathcal{F}}_{\Lambda}(\exp(-\Delta_0))$$

for any $\lambda > 0$ where E_λ is the spectral resolution of Δ_0.

To show this, we note that the operator $\dot{*}*^{-1}$ is a bounded operator acting on the L^2 space of forms. So we have, $\tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 d\Gamma_0(T)) = \tau^{\Lambda}_{\mathcal{F}}(\dot{*}_0 *_0^{-1} *_0 d\Gamma_0(T)) \le ||\dot{*}_0 *_0^{-1}||\tau^{\Lambda}_{\mathcal{F}}(|*_0 d\Gamma_0(T)|) = C_0\tau^{\Lambda}_{\mathcal{F}}(|*_0 d\Gamma_0(T)|)$ Now, as in the proof of 4.1.2, we have

$$\tau^{\Lambda}_{\mathcal{F}}(|*_0 d\Gamma_0(t)|) = \int_0^{\infty}\sqrt{\lambda}\exp(-t^2\lambda)d\alpha(\lambda) \tag{4.3.14}$$

where α corresponds to the positive linear functional $F(\lambda) = \tau^{\Lambda}_{\mathcal{F}}(E_\lambda)$. Since $*_0 d\Gamma_0(t)E_0 = 0$, we have $*_0 d\Gamma_0(t) = *_0 d\Gamma_0(t)(1-E_0)$, so that we can rewrite 4.3.14 as

$$\tau^{\Lambda}_{\mathcal{F}}(*_0 d\Gamma_0(t)) = \int_0^{\infty}\sqrt{\lambda}\exp(-t^2\lambda)(1-\chi_{\{0\}}(\lambda))d\alpha(\lambda) \tag{4.3.15}$$

Now let $g:(0,\infty)\to(0,\infty)$ be defined as $g(x) = \sqrt{x}\exp(-t^2x)$. Then,

$$g'(x) = \frac{1}{2\sqrt{x}}(1-2t^2x)\exp(-t^2x)$$

and

$$g''(x) = \frac{1}{4x^{-3/2}}(1-2t^2x)\exp(-t^2x) - \frac{t^2}{\sqrt{x}}\exp(-t^2x) + \frac{1}{2\sqrt{x}}(1-2t^2x)(-t^2)\exp(-t^2x)$$

Thus solving $g'(x) = 0$ gives $x_* = \frac{2}{t^2}$, and it is a global maximum since

$$g''(x_*) = -\frac{t^3}{\sqrt{2}}\exp(-2) < 0 \text{ for } t > 0$$

We also have $g(x_*) = \frac{\sqrt{2}}{t}\exp(-2)$. So we can estimate the integral in 4.3.14 as follows. Let $\lambda_0 > 0$, so we write

$$\int_0^\infty \sqrt{\lambda}\exp(-t^2\lambda)d\alpha(\lambda) = \int_0^{\lambda_0}\sqrt{\lambda}\exp(-t^2\lambda)(1-\chi_{\{0\}}(\lambda))d\alpha(\lambda) + \int_{\lambda_0}^\infty \sqrt{\lambda}\exp(-t^2\lambda)d\alpha(\lambda) \quad (4.3.16)$$

Let t, λ_0 be such that $0 < \frac{2}{t^2} < \frac{2}{t^2-1}\lambda_0$. Then the first integral on the right hand side of 4.3.16 is bounded above by $\frac{\sqrt{2}}{t}\exp(-2)\int_0^{\lambda_0}(1-\chi_{\{0\}}(\lambda))d\alpha(\lambda) = \frac{\sqrt{2}}{t}\exp(-2)\tau_{\mathcal{F}}^\Lambda(E_{\lambda_0}(1-E_0)) = \frac{\sqrt{2}}{t}\exp(-2)\tau_{\mathcal{F}}^\Lambda((1-E_0)E_{\lambda_0}(1-E_0))$, where we have used the linearity and normality of the trace.

For the second integral, we write

$$\begin{aligned}\int_{\lambda_0}^\infty \sqrt{\lambda}\exp(-t^2\lambda)d\alpha(\lambda) &= \int_{\lambda_0}^\infty \sqrt{\lambda}\exp(-(t^2-1)\lambda)\exp(-\lambda)d\alpha(\lambda)\\ &\leq \int_0^\infty g_{(t^2-1)}(\lambda_0)\exp(-\lambda)d\alpha(\lambda)\\ &\leq \sqrt{\lambda_0}\exp(-(t^2-1)\lambda_0)\tau_{\mathcal{F}}^\Lambda(-e^{\Delta_0}) \quad (4.3.17)\end{aligned}$$

So we get the inequality

$$|2T\tau_{\mathcal{F}}^\Lambda(\dot{*}_0 d\Gamma_0(T))| \leq C_0|2\sqrt{2}\exp(-2)\tau^\Lambda((1-E_0)E_{\lambda_0}(1-E_0))| + |T\sqrt{\lambda_0}\exp(-(t^2-1)\lambda_0)\tau_{\mathcal{F}}^\Lambda(e^{-\Delta_0})| \quad (4.3.18)$$

Letting $T \to \infty$ we have $T\sqrt{\lambda_0}\exp(-\lambda_0(T^2-1)) \to 0$ uniformly with respect to λ_0 for $\lambda_0 \in [0, K]$ for some positive constant K, so that

$$\lim_{T\to\infty}\lim_{\lambda_0\to 0}|T\sqrt{\lambda_0}\exp(-(t^2-1)\lambda_0)\tau_{\mathcal{F}}^\Lambda(-\Delta_0)| = 0$$

Also, since $E_\lambda \to E_0$ strongly as $\lambda \to 0$, using the normality of the trace we get $\lim_{\lambda\to 0}\tau_{\mathcal{F}}^\Lambda(E_0^\perp \cap E_\lambda) = lim_{\lambda\to 0}\tau_{\mathcal{F}}^\Lambda((I-E_0)E_\lambda(I-E_0)) = 0$. So we have the desired result

$$\lim_{T\to\infty} 2T\tau_{\mathcal{F}}^\Lambda(*_0 d\Gamma_0(T)) = 0$$

Similarly, one has

$$\lim_{T\to\infty} 2T\tau^\Lambda(\tilde{*}_0\tilde{d}\tilde{\Gamma}_0(T)) = 0$$

In the second term in 4.3.12 we can replace the operator $\Gamma_0(\epsilon)$ by a suitable c-almost local parametrix for c sufficiently small, so that we get (cf. Proposition3.2.9)

$$\tau_{\mathcal{F}}^\Lambda(*_0 d\Gamma_0(\epsilon)) = \tau^\Lambda(\tilde{*}_0\tilde{d}\tilde{\Gamma}_0(\epsilon))$$

Combining the above two results for the respective small time and large time limits, we get

$$\frac{d}{du}\eta^\Lambda(\tilde{D}_u^{sign})|_{u=0} = \frac{d}{du}\eta_{\mathcal{F}}^\Lambda(M, \mathcal{F}, g_u)|_{u=0} \quad (4.3.19)$$

Thus we get the desired result 4.3.1. $\square$

4.4 Leafwise diffeomorphism invariance of the ρ-invariant

Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be closed foliated smooth manifolds and $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ be a leafwise diffeomorphism. Let g be a leafwise smooth metric on $(M, \mathcal{F})$. Since f is a leafwise diffeomorphism it induces a leafwise smooth metric $f_* g$ on $(M', \mathcal{F}')$. Let Λ be a holonomy-invariant transverse measure on $(M, \mathcal{F})$. Then by Proposition 2.2.1, we have a holonomy-invariant transverse measure $f_*\Lambda$ on $(M', \mathcal{F}')$. The leafwise diffeomorphism f induces a leafwise diffeomorphism between the corresponding monodromy groupoids $\mathcal{G}$ and $\mathcal{G}'$. In particular, there is a diffeomorphism $\tilde{f}_x : \mathcal{G}_x \to \mathcal{G}'_{f(x)}$ for any $x \in M$. Let E (resp. E') be the Grassmanian bundle of tangential forms $\bigwedge^* T^*\mathcal{F}$ (resp. $\bigwedge^* T^*\mathcal{F}'$). Then $\tilde{f}_x$ induces a unitary map $U_x : L^2(\mathcal{G}_x, r^*E) \to L^2(\mathcal{G}'_{f(x)}, r^*E')$ given by

$$U_x \xi(\alpha') = \xi(\tilde{f}_x^{-1}\alpha') \text{ for } \alpha' \in \mathcal{G}'_{f(x)}, \xi \in L^2(\mathcal{G}_x, r^*E).$$

Notice that $f^*E' = E$. The leafwise Hodge operator $\tilde{*}'$ on $\mathcal{G}'$ corresponding to the leafwise metric f_*g then satisfies

$$\tilde{*}'_{f(x)} = U_x \circ \tilde{*}_x \circ U_x^{-1}$$

We also have $U_x \circ \tilde{d}_x = \tilde{d}'_{f(x)} \circ U_x$ and therefore the signature operator $\tilde{D}'_{sign}$ on $\mathcal{G}'$ for the metric f_*g is given by $\tilde{D}'_{sign} = U_x \circ \tilde{D}_{sign} \circ U_x^{-1}$.

Proposition 4.4.1. *Keeping the notations from above, we have*

$$\eta^{\Lambda}(\tilde{D}_{sign}) = \eta^{f_*\Lambda}(\tilde{D}'_{sign}) \text{ and } \eta^{\Lambda}_{\mathcal{F}}(D_{sign}) = \eta^{f_*\Lambda}_{\mathcal{F}'}(D'_{sign})$$

Proof. The functional calculus of $\tilde{D}'_{sign}$ is given by conjugation by U_x, and in particular we have

$$\tilde{D}'_{sign} \exp(-t^2(\tilde{D}'_{sign})^2) = U_x \circ \tilde{D}_{sign} \exp(-t^2(\tilde{D}_{sign})^2) \circ U_x^{-1}$$

Applying the traces on the von Neumann algebras corresponding to E, E', and the holonomy-invariant transverse measures Λ and $f_*\Lambda$, we get for the foliation von Neumann algebra:

$$\tau^{f_*\Lambda}(\tilde{D}'_{sign} \exp(-t^2(\tilde{D}'_{sign})^2)) = \tau^{\Lambda}(U_x \circ \tilde{D}_{sign} \exp(-t^2(\tilde{D}_{sign})^2) \circ U_x^{-1})$$

and similarly for the leafwise signature operators on the foliations corresponding to the metric f_*g we have $D'_{sign} = u_x \circ D_{sign} \circ u_x^{-1}$, where $u_x : L^2(L(x), E_{|_{L_x}}) \to L^2(L'_{f(x)}, E'_{|_{L'_{f(x)}}})$ is the unitary induced by the diffeomorphism $f_x : L_x \to L'_{f(x)}$. Therefore by the functional calculus and applying the trace on the regular von Neumann algebra we have,

$$\tau^{f_*\Lambda}_{\mathcal{F}'}(D'_{sign} \exp(-t^2(D'_{sign})^2)) = \tau^{\Lambda}_{\mathcal{F}}(u_x \circ D_{sign} \exp(-t^2(D_{sign})^2) \circ u_x^{-1})$$

□

Theorem 4.4.2. *Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be foliated manifolds with leafwise smooth metrics g and g' respectively, and let $f : M \to M'$ be a leafwise diffeomorphism. Let Λ be a holonomy-invariant transverse measure on $(M, \mathcal{F})$. Then we have for the leafwise signature operators D_{sign} and D'_{sign} on M and M' corresponding to the metrics g and g', respectively,*

$$\rho^{\Lambda}(M, \mathcal{F}, g) = \rho^{f_*\Lambda}(M', \mathcal{F}', g')$$

Proof. Let $D'_{sign,1}$ be the signature operator on $(M', \mathcal{F}')$ corresponding to the metric $f_* g$. Then by the previous proposition 4.4.1, we have the equality

$$\rho^{\Lambda}(D_{sign}, M, \mathcal{F}) = \rho^{f_*\Lambda}(D'_{sign,1}, M', \mathcal{F}')$$

However from Theorem 4.3.1, we have $\rho^{f_*\Lambda}(D'_{sign,1}, M', \mathcal{F}') = \rho^{f_*\Lambda}(D'_{sign}, M', \mathcal{F}')$, which proves the desired result.

$\square$

Chapter 5

Hilbert-Poincaré complexes on foliations

The results of this chapter are treated in the preprint [BeRo:10].

5.1 Hilbert modules associated with leafwise maps

Let $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a smooth map such that f sends leaves to leaves. Let X (resp. X') be a complete transversal on $(V, \mathcal{F})$ (resp. $(V', \mathcal{F}')$). Denote by $\mathcal{G}$ and $\mathcal{G}'$ the monodromy groupoids of $(V, \mathcal{F})$ and $(V', \mathcal{F}')$, respectively. We use as before the standard notation, setting $\mathcal{G}_X := s^{-1}(X)$, $\mathcal{G}^X := r^{-1}(X)$, $\mathcal{G}_X^X := r^{-1}(X) \cap s^{-1}(X)$ and similarly for $\mathcal{G}'_{X'}, \mathcal{G}'^{X'}$, and $\mathcal{G}'^{X'}_{X'}$. The leafwise map f induces a map by restriction on the transversal X which we also denote by f. Also, f induces a well-defined map $\breve{f} : \mathcal{G} \to \mathcal{G}'$. In the sequel, we will use the same notation for the range and the source maps on the groupoids $\mathcal{G}$ and $\mathcal{G}'$.

Set

$$\mathcal{G}^X_{X'}(f) := \{(x, \gamma') \in X \times \mathcal{G}_{X'} | f(x) = r(\gamma')\}$$

and define the following two maps:

$$r_f : \mathcal{G}^X_{X'}(f) \to X, r_f(x, \gamma') = x$$

and

$$s_f : \mathcal{G}^X_{X'}(f) \to X', s_f(x, \gamma') = s(\gamma')$$

Now, $\mathcal{G}^{X'}_{X'}$ acts freely and properly on $\mathcal{G}^X_{X'}(f)$ as follows: if $\alpha' \in \mathcal{G}^{X'}_{X'}$ is such that $r(\alpha') = s(\gamma')$, then $(x, \gamma')\alpha' = (x, \gamma'\alpha')$. Note that $r(\gamma'\alpha') = f(x)$, so this action is well-defined. This allows us to induce a structure of a right $C_c^\infty(\mathcal{G}'^{X'}_{X'})$-module on $C_c^\infty(\mathcal{G}^X_{X'}(f))$, given by the following rule:

$$(\xi\phi')(x, \gamma') = \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \xi(x, \gamma'\alpha'^{-1})\phi'(\alpha'), \text{ for } \xi \in C_c^\infty(\mathcal{G}^X_{X'}(f)), \phi' \in C_c^\infty(\mathcal{G}'^{X'}_{X'})$$

Lemma 5.1.1. *We have*

$$(\xi\phi')\psi' = \xi(\phi' * \psi'), \text{ for } \xi \in C_c^\infty(\mathcal{G}^X_{X'}(f)), \phi', \psi' \in C_c^\infty(\mathcal{G}'^{X'}_{X'}) \tag{5.1.1}$$

where $*$ is the convolution product in $C_c^\infty(\mathcal{G}'^{X'}_{X'})$.

Proof. Let us compute the left hand side of the above equation first:

$$\begin{aligned}(\xi\phi')\psi'(x,\gamma') &= \sum_{\alpha'_1\in\mathcal{G}'^{X'}_{s(\gamma')}} (\xi\phi')(x,\gamma'\alpha'^{-1}_1)\psi'(\alpha'_1)\\ &= \sum_{\alpha'_1\in\mathcal{G}'^{X'}_{s(\gamma')}} \{\sum_{\alpha'_2\in\mathcal{G}'^{X'}_{r(\alpha'_1)}} \xi(x,\gamma'\alpha'^{-1}_1\alpha'^{-1}_2)\phi'(\alpha'_2)\}\psi'(\alpha'_1)\\ &= \sum_{\alpha'_1\in\mathcal{G}'^{X'}_{s(\gamma')}} \sum_{\alpha'_2\in\mathcal{G}'^{X'}_{r(\alpha'_1)}} \xi(x,\gamma'\alpha'^{-1}_1\alpha'^{-1}_2)\phi'(\alpha'_2)\psi'(\alpha'_1) \qquad (5.1.2)\end{aligned}$$

Now computing the right hand side of 5.1.1, we get

$$\begin{aligned}\xi(\phi'*\psi')(x,\gamma') &= \sum_{\alpha'_3\in\mathcal{G}'^{X'}_{s(\gamma')}} \xi(x,\gamma'\alpha'^{-1}_3)(\phi'*\psi')(\alpha'_3)\\ &= \sum_{\alpha'_3\in\mathcal{G}'^{X'}_{s(\gamma')}} \xi(x,\gamma'\alpha'^{-1}_3) \sum_{\alpha'_4\in\mathcal{G}'^{X'}_{s(\gamma')}} \phi'(\alpha'_3\alpha'^{-1}_4)\psi'(\alpha'_4)\\ &= \sum_{\alpha'_4\in\mathcal{G}'^{X'}_{s(\gamma')}} \sum_{\alpha'_3\in\mathcal{G}'^{X'}_{s(\gamma')}} \xi(x,\gamma'\alpha'^{-1}_3)\phi'(\alpha'_3\alpha'^{-1}_4)\psi'(\alpha'_4)\\ &= \sum_{\alpha'_4\in\mathcal{G}'^{X'}_{s(\gamma')}} \sum_{\alpha'_5\in\mathcal{G}'^{X'}_{r(\alpha'_4)}} \xi(x,\gamma'\alpha'^{-1}_4\alpha'^{-1}_5)\phi'(\alpha'_5)\psi'(\alpha'_4) \qquad (\text{putting } \alpha'_5=\alpha'_3\alpha'^{-1}_4)(5.1.3)\end{aligned}$$

Therefore we get the equality of 5.1.2 and 5.1.3, thus proving 5.1.1.

□

On the other hand, $\mathcal{G}^X_X$ acts on $\mathcal{G}^X_{X'}(f)$ via:

$$\alpha(x,\gamma') = (r(\alpha), \breve{f}(\alpha)\circ\gamma') \text{ for } (x,\gamma')\in\mathcal{G}^X_{X'}(f),\ \alpha\in\mathcal{G}^X_X$$

We note that $r(\breve{f}(\alpha)\circ\gamma') = r(\breve{f}(\alpha)) = f(r(\alpha))$, so this action is well-defined. The left action of $\mathcal{G}^X_X$ on $\mathcal{G}^X_{X'}(f)$ induces in this way a left $C^\infty_c(\mathcal{G}^X_X)$-module structure on $C^\infty_c(\mathcal{G}^X_{X'}(f))$, with the action given by:

$$\pi_f(\phi)\xi(\underbrace{x,\gamma'}_{v}) = \sum_{\alpha\circ u=v}\phi(\alpha)\xi(u) = \sum_{\alpha\in\mathcal{G}^{r_f(v)=x}_X}\phi(\alpha)\xi(s(\alpha),\breve{f}(\alpha^{-1})\circ\gamma') \text{ for } \phi\in C^\infty_c(\mathcal{G}^X_X),\xi\in C^\infty_c(\mathcal{G}^X_{X'}(f))$$

Lemma 5.1.2. *We have the following properties:*

for $\phi,\psi\in C^\infty_c(\mathcal{G}^X_X),\xi\in C^\infty_c(\mathcal{G}^X_{X'}(f)),\phi'\in C^\infty_c(\mathcal{G}^{X'}_{X'})$,

1. $\pi_f(\phi*\psi) = \pi_f(\phi)\pi_f(\psi)$

2. The left and right actions are compatible, i.e. we have

$$\pi_f(\phi)(\xi\phi') = (\pi_f(\phi)\xi)\phi'$$

Proof. 1. Computing the left hand side first, we get:

$$\begin{aligned}
\pi_f(\phi*\psi)(\xi)(\underbrace{x,\gamma'}_{v}) &= \sum_{\alpha\circ u=v}\phi*\psi(\alpha)\xi(u) \\
&= \sum_{\alpha\in\mathcal{G}_X^x}\sum_{\alpha_1\in\mathcal{G}_X^x}\phi(\alpha_1)\psi(\alpha_1^{-1}\alpha)\xi(s(\alpha),\breve{f}(\alpha^{-1})\circ\gamma') \\
&= \sum_{\alpha_1\in\mathcal{G}_X^x}\sum_{\alpha\in\mathcal{G}_X^x}\phi(\alpha_1)\psi(\alpha_1^{-1}\alpha)\xi(s(\alpha),\breve{f}(\alpha^{-1})\circ\gamma') \\
&= \sum_{\alpha_1\in\mathcal{G}_X^x}\sum_{\alpha_2\in\mathcal{G}_X^{s(\alpha_1)}}\phi(\alpha_1)\psi(\alpha_2)\xi(s(\alpha_2),\breve{f}(\alpha_2^{-1}\alpha_1^{-1})\circ\gamma') \qquad (5.1.4)
\end{aligned}$$

Computation of the right hand side gives:

$$\begin{aligned}
\pi_f(\phi)[\pi_f(\psi)\xi](\underbrace{x,\gamma'}_{v}) &= \sum_{\beta\circ w=v}\phi(\beta)[\pi_f(\psi)\xi](w) \\
&= \sum_{\beta\in\mathcal{G}_X^x}\phi(\beta)[\pi_f(\psi)\xi](s(\beta),\breve{f}(\beta^{-1})\circ\gamma') \\
&= \sum_{\beta\in\mathcal{G}_X^x}\phi(\beta)\sum_{\beta_1\in\mathcal{G}_X^{s(\beta)}}\psi(\beta_1)\xi(s(\beta_1),\breve{f}(\beta_1^{-1})\breve{f}(\beta^{-1})\circ\gamma') \\
&= \sum_{\beta\in\mathcal{G}_X^x}\sum_{\beta_1\in\mathcal{G}_X^{s(\beta)}}\phi(\beta)\psi(\beta_1)\xi(s(\beta_1),\breve{f}(\beta_1^{-1}\beta^{-1})\circ\gamma') \qquad (5.1.5)
\end{aligned}$$

comparing 5.1.4 and 5.1.5 gives the result.

2. Starting with the computation of the left hand side first:

$$\begin{aligned}
\pi_f(\phi)(\xi\phi')(x,\gamma') &= \sum_{\alpha\in\mathcal{G}_X^x}\phi(\alpha)(\xi\phi')(s(\alpha),\breve{f}(\alpha^{-1})\gamma') \\
&= \sum_{\alpha\in\mathcal{G}_X^x}\phi(\alpha)\sum_{\alpha'\in\mathcal{G}'^{X'}_{s(\gamma')}}\xi(s(\alpha),\breve{f}(\alpha^{-1})\circ\gamma'\circ\alpha_1'^{-1})\phi'(\alpha_1') \\
&= \sum_{\alpha'\in\mathcal{G}'^{X'}_{s(\gamma')}}\sum_{\alpha\in\mathcal{G}_X^x}\phi(\alpha)\xi(s(\alpha),\breve{f}(\alpha^{-1})\circ\gamma'\circ\alpha_1'^{-1})\phi'(\alpha_1') \\
&= \sum_{\alpha'\in\mathcal{G}'^{X'}_{s(\gamma')}}[\pi_f(\phi)\xi](x,\gamma'\circ\alpha_1'^{-1})\phi'(\alpha_1') \\
&= (\pi_f(\phi)\xi)\phi'(x,\gamma') \qquad (5.1.6)
\end{aligned}$$

□

Assume that f has discrete fibres. We define the following $C_c^\infty(\mathcal{G}'^{X'}_{X'})$- valued inner product on $C_c^\infty(\mathcal{G}^X_{X'})(f)$:

$$<\xi,\eta>(\gamma') = \sum_{\gamma_1'\in\mathcal{G}'^{f(X)}_{r(\gamma')}}\sum_{\{x\in X|f(x)=r(\gamma_1')\}}\overline{\xi(x,\gamma_1')}\eta(x,\gamma_1'\circ\gamma) \text{ for } \xi,\eta\in C_c(\mathcal{G}^X_{X'})(f)$$

Or, equivalently,

$$<\xi,\eta>(\alpha') = \sum_{x_1\in X\cap L_{s(\alpha')}}\sum_{\gamma_1'\in\mathcal{G}'^{f(x_1)}_{r(\alpha')}}\overline{\xi_2(x_1,\gamma_1')}\eta(x_1,\gamma_1'\alpha')$$

Proposition 5.1.3. *With the inner-product defined above, $C_c^\infty(\mathcal{G}_{X'}^X(f))$ is a pre-Hilbert module over $C_c^\infty(\mathcal{G}'^{X'}_{X'})$.*

Proof. We show the following properties for $\xi, \eta \in C_c^\infty(\mathcal{G}_{X'}^X(f))$ and $\phi' \in C_c^\infty(\mathcal{G}'^{X'}_{X'})$

1. $<\xi, \eta\phi'> = <\xi, \eta> *\phi'$
2. $<\xi, \eta>^* = <\eta, \xi>$
3. $<\xi, \xi> \geq 0$ (as a positive element in $C^*(\mathcal{G}'^{X'}_{X'})$).

1. The left hand side computation is as follows:

$$\begin{aligned} <\xi, \eta\phi'>(\gamma') &= \sum_{\gamma_1' \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X | f(x) = r(\gamma_1')\}} \overline{\xi(x, \gamma_1')}(\eta\phi')(x, \gamma_1' \circ \gamma') \\ &= \sum_{\gamma_1' \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X | f(x) = r(\gamma_1')\}} \overline{\xi(x, \gamma_1')} \sum_{\gamma_2' \circ \gamma_3' = \gamma_1' \circ \gamma'} \eta(x, \gamma_2')\phi(\gamma_3') \end{aligned} \tag{5.1.7}$$

Now, computing the RHS:

$$\begin{aligned} <\xi, \eta> *\phi'(\gamma') &= \sum_{\gamma_4' \circ \gamma_5' = \gamma'} <\xi, \eta>(\gamma_4')\phi(\gamma_5') \\ &= \sum_{\gamma_4' \circ \gamma_5' = \gamma'} \sum_{\alpha' \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X | f(x) = r(\alpha')\}} \overline{\xi(x, \alpha')}\eta(x, \alpha'\gamma_4')\phi(\gamma_5') \\ &= \sum_{\alpha' \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X | f(x) = r(\alpha')\}} \sum_{\beta' \circ \gamma_5' = \alpha'\gamma'} \overline{\xi(x, \alpha')}\eta(x, \beta')\phi(\gamma_5') \end{aligned} \tag{5.1.8}$$

5.1.7 and 5.1.8 together give the desired result.

2. We have,

$$\begin{aligned} <\xi, \eta>^*(\gamma') &= \overline{<\xi, \eta>(\gamma'^{-1})} \\ &= \sum_{\gamma_1' \in \mathcal{G}'^{f(X)}_{s(\gamma')}} \sum_{\{x \in X | f(x) = r(\gamma_1')\}} \xi(x, \gamma_1')\overline{(\eta)(x, \gamma_1' \circ \gamma'^{-1})} \\ &= \sum_{\gamma_2' \in \mathcal{G}'^{f(X)}_{r(\gamma')}} \sum_{\{x \in X | f(x) = r(\gamma_2')\}} \overline{(\eta)(x, \gamma_2')}\xi(x, \gamma_2' \circ \gamma') \\ &= <\eta, \xi>(\gamma') \end{aligned} \tag{5.1.9}$$

3. Finally, positivity of the inner-product is classical, it is given in [BeRo:10] and is omitted here.

□

After appropriate completions, we hence get a Hilbert C^*-module over $C^*(\mathcal{G}'^{X'}_{X'})$, the maximal C^*-algebra of the groupoid $\mathcal{G}'^{X'}_{X'}$, and we denote this Hilbert module by $\mathcal{E}_{X'}^X(f)$.

Now let $(V, X, \mathcal{F})$, $(V', X', \mathcal{F}')$ and $(V'', X'', \mathcal{F}'')$ be foliated manifolds with complete trasversals X, X' and X'', respectively. Let $(V, X, \mathcal{F}) \xrightarrow{f} (V', X', \mathcal{F}') \xrightarrow{g} (V'', X'', \mathcal{F}'')$ be smooth leafwise maps. We define

$$\mathcal{G}_{X'}^X(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}_{X''}^{X'}(g) := \{(x, \gamma'); (x', \gamma'') \in \mathcal{G}_{X'}^X(f) \times \mathcal{G}_{X''}^{X'}(g) | x' = s(\gamma')\}/\sim$$

where $((x,\gamma');(x',\gamma''))\sim((x,\gamma')\alpha';\alpha'^{-1}(x',\gamma''))$, for $\alpha'\in\mathcal{G}'^{X'}_{X'}), r(\alpha')=s(\gamma')=x'$.

Proposition 5.1.4. *With the above definition we have a diffeomorphism*

$$\mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g)\cong\mathcal{G}^X_{X''}(g\circ f)$$

Proof. We define a map $\Phi:\mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g)\to\mathcal{G}^X_{X''}(g\circ f)$ in the following way:

$$\Phi([(x,\gamma');(x',\gamma'')])=(x,\breve{g}(\gamma')\circ\gamma'')$$

We note that $g\circ f(x)=g(r(\gamma'))=r(\breve{g}(\gamma'))$, so $(x,g(\gamma')\circ\gamma'')\in\mathcal{G}^X_{X''}(g\circ f)$. Also, this action is well defined, for if $[(x,\gamma');(x',\gamma'')]=[(x_1,\gamma'_1);(x'_1,\gamma''_1)]\Leftrightarrow(x_1,\gamma'_1);(x'_1,\gamma''_1)=((x,\gamma')\alpha';\alpha'^{-1}(x',\gamma''))$ for some $\alpha'\in\mathcal{G}'^{X'}_{X'}$, and we have

$$\begin{aligned}\Phi([(x,\gamma')\alpha';\alpha'^{-1}(x',\gamma'')]) &= \Phi([(x,\gamma'\alpha');(s(\alpha'),\breve{g}(\alpha'^{-1})\gamma'')])\\ &= (x,\breve{g}(\gamma'\alpha')\breve{g}(\alpha'^{-1})\gamma'')\\ &= (x,\breve{g}(\gamma')\gamma'')\\ &= \Phi([(x,\gamma');(x',\gamma'')])\end{aligned}\tag{5.1.10}$$

• Φ is smooth: Let Φ_0 be the map given by

$$\Phi_0((x,\gamma');(x',\gamma''))=(x,\breve{g}(\gamma')\circ\gamma'')$$

Then, Φ_0 can be written as a composition of maps $\Phi_0=(pr_1,m(\breve{g}\circ pr_2,pr_4))$, where $pr_1,pr_2,pr4$ are the projections onto the first, second and fourth coordinates, respectively, and m is the composition map for the groupoid. Since all these maps are smooth, Φ_0 is smooth. Therefore Φ, which is induced by Φ_0, is smooth.

• Φ is injective:

Let $\Phi([(x_1,\gamma'_1);(x'_1,\gamma''_1)])=\Phi([(x_2,\gamma'_2);(x'_2,\gamma''_2)])$. Then we have,

$$x_1=x_2 \text{ and } \breve{g}(\gamma'_1)\gamma''_1=\breve{g}(\gamma'_2)\gamma''_2 \qquad (*)$$

Now let $\alpha'=\gamma'^{-1}_2\circ\gamma'_1\in\mathcal{G}'^{X'}_{X'}$. Then, $s(\alpha')=x'_1, r(\alpha')=x'_2$, and we get,

$$\begin{aligned}((x_2,\gamma'_2)\alpha';\alpha'^{-1}(x'_2,\gamma''_2)) &= ((x_2,\gamma'_2\alpha');(s(\alpha'),\breve{g}(\alpha'^{-1})\gamma''_2))\\ &= ((x_1,\gamma'_1);(x'_1,\breve{g}(\gamma'^{-1}_1)\breve{g}(\gamma'_2)\gamma''_2))\\ &= ((x_1,\gamma'_1);(x'_1,\gamma''_1))\end{aligned}\tag{5.1.11}$$

Hence Φ is injective.

• Φ is surjective:

Let $(x,\gamma'')\in\mathcal{G}^X_{X''}(g\circ f)$. Then as X' is a complete transversal, there exists an element $\gamma'\in\mathcal{G}'^{f(x)}_{X'}$. Set $u=[(x,\gamma');(s(\gamma'),\breve{g}(\gamma'^{-1})\gamma'')]$. Hence $u\in\mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g)$, and we have,

$$\Phi(u)=(x,\breve{g}(\gamma')\breve{g}(\gamma'^{-1})\gamma'')=(x,\gamma'')$$

Thus Φ is surjective.

Hence Φ is a diffeomorphism between $\mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g)$ and $\mathcal{G}^X_{X''}(g\circ f)$. ∎

Proposition 5.1.5. *We have the following isomorphism of Hilbert $C^*(\mathcal{G}_X^X)$-modules :*

$$\mathcal{E}_{X'}^X(f)\otimes_{C^*(\mathcal{G}'^{X'}_{X'})}\mathcal{E}_{X''}^{X'}(g)\cong\mathcal{E}_{X''}^X(g\circ f)$$

Proof. Let $\xi_f\in C_c^\infty(\mathcal{G}_{X'}^X(f)),\eta_g\in C_c^\infty(\mathcal{G}_{X''}^{X'}(g))$. Then we define $\xi_f*\eta_g\in C_c^\infty(\mathcal{G}_{X''}^X(g\circ f))$ as follows. We make the identification $(x,\gamma'')\mapsto[(x,\gamma');(x',\gamma'')]$, and set

$$\xi_f*\eta_g[(x,\gamma');(x',\gamma'')]=\xi_f*\eta_g(x,\gamma''):=\sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}\xi_f(x,\gamma'\alpha')\eta_g(\alpha'^{-1}x',\breve{g}(\alpha'^{-1})\gamma'')\text{ for }(x,\gamma'')\in\mathcal{G}_{X''}^X(g\circ f)$$

We also set

$$s([(x,\gamma');(x',\gamma'')])=s(\gamma'')$$

We check the following properties:

(i) if $[(x_1,\gamma_1');(x_1',\gamma_1'')]=[(x_2,\gamma_2');(x_2',\gamma_2'')]$, then $\xi_f*\eta_g[(x_1,\gamma_1');(x_1',\gamma_1'')]=\xi_f*\eta_g[(x_2,\gamma_2');(x_2',\gamma_2'')]$.

(ii) for $\phi'\in C_c^\infty(\mathcal{G}'^{X'}_{X'})$, we have $\xi_f\phi'*\eta_g=\xi*\pi_g(\phi')\eta_g$.

(iii) $<\xi_f*\eta_g,\xi_f*\eta_g>=<\eta_g,\pi_g(<\xi_f,\xi_f>)\eta_g>$, where the equality is in $C^*(\mathcal{G}_{X''}^{X''})$.

(i) Let $[(x_1,\gamma_1');(x_1',\gamma_1'')]=[(x_2,\gamma_2');(x_2',\gamma_2'')]$. This implies that $(x_2,\gamma_2')=(x_1,\gamma_1')\kappa'$, $(x_2',\gamma_2'')=\kappa'^{-1}(x_1',\gamma_1'')$ for some $\kappa'\in\mathcal{G}'^{X'}_{X'}$ such that $r(\kappa')=s(\gamma_1')=x_1'$.

Then, we have

$$\begin{aligned}
\xi_f*\eta_g[(x_2,\gamma_2');(x_2',\gamma_2'')] &= \sum_{\alpha'\in\mathcal{G}'^{s(\gamma_2')}_{X'}}\xi_f(x_2,\gamma_2'\alpha')\eta_g(\alpha'^{-1}x_2',\breve{g}(\alpha'^{-1})\gamma_2'')\\
&= \sum_{\alpha'\in\mathcal{G}'^{s(\gamma_2')}_{X'}}\xi_f(x_1,\gamma_1'\kappa'\alpha')\eta_g(\alpha'^{-1}\kappa'^{-1}x_1',\breve{g}(\alpha'^{-1})\breve{g}(\kappa'^{-1})\gamma_1'')\\
&= \sum_{\beta'\in\mathcal{G}'^{s(\gamma_1')}_{X'}}\xi_f(x_1,\gamma_1'\beta')\eta_g(\beta'^{-1}x_1',\breve{g}(\beta'^{-1})\gamma_1'')\ \ \text{(putting }\beta'=\kappa'\alpha')\\
&= \xi_f*\eta_g[(x_1,\gamma_1');(x_1',\gamma_1'')]
\end{aligned}\tag{5.1.12}$$

(ii) Let $\phi'\in C_c^\infty(\mathcal{G}'^{X'}_{X'})$. We compute the left hand side:

$$\begin{aligned}
\xi_f\phi'*\eta_g[(x,\gamma');(x',\gamma'')] &= \sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}(\xi_f\phi')(x,\gamma'\alpha')\eta_g(\alpha'^{-1}x',\breve{g}(\alpha'^{-1})\gamma'')\\
&= \sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}\sum_{\alpha_1'\in\mathcal{G}'^{X'}_{s(\alpha')}}(\xi_f)(x,\gamma'\alpha'\alpha_1'^{-1})\phi'(\alpha_1')\eta_g(\alpha'^{-1}x',\breve{g}(\alpha'^{-1})\gamma'')
\end{aligned}\tag{5.1.13}$$

The right hand side is computed as follows:

$$\begin{aligned}
\xi_f * \pi_g(\phi')\eta_g[(x,\gamma');(x',\gamma'')] &= \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')[\pi_f(\phi')\eta_g](\beta'^{-1}x', \breve{g}(\beta'^{-1})\gamma'') \\
&= \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')[\pi_f(\phi')\eta_g](s(\beta'), \breve{g}(\beta'^{-1})\gamma'') \\
&= \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta') \sum_{\beta'_1 \in \mathcal{G}'^{s(\beta')}_{X'}} \eta_g(s(\beta'_1), \breve{g}(\beta_1'^{-1})\breve{g}(\beta'^{-1})\gamma'')\phi'(\beta'_1) \\
&= \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta') \sum_{\beta'_2 \in \mathcal{G}'^{r(\beta')}_{X'}} \eta_g(s(\beta'_2), \breve{g}(\beta_2'^{-1})\gamma'')\phi'(\beta'^{-1}\beta'_2) \quad (\beta'_2 = \beta'\beta'_1) \\
&= \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta') \sum_{\beta'_2 \in \mathcal{G}'^{s(\gamma')}_{X'}} \eta_g(s(\beta'_2), \breve{g}(\beta_2'^{-1})\gamma'')\phi'(\beta'^{-1}\beta'_2) \\
&= \sum_{\beta'_2 \in \mathcal{G}'^{s(\gamma')}_{X'}} \sum_{\beta' \in \mathcal{G}'^{s(\gamma')}_{X'}} \xi_f(x,\gamma'\beta')\eta_g(s(\beta'_2), \breve{g}(\beta_2'^{-1})\gamma'')\phi'(\beta'^{-1}\beta'_2) \\
&= \sum_{\beta'_2 \in \mathcal{G}'^{s(\gamma')}_{X'}} \sum_{\beta'_3 \in \mathcal{G}'^{X'}_{s(\beta'_2)}} \xi_f(x,\gamma'\beta'_2\beta_3'^{-1})\eta_g(s(\beta'_2), \breve{g}(\beta_2'^{-1})\gamma'')\phi'(\beta'_3) \qquad (5.1.14)
\end{aligned}$$

Comparing 5.1.13 and 5.1.14 gives the required equality.

(iii) We compute the left hand side first. Let $\gamma'' \in \mathcal{G}^{X''}_{X''}$. We note that with the identification $\mathcal{G}^X_{X'}(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g) \cong \mathcal{G}^X_{X''}(g \circ f)$ we can write the inner product in $C^\infty_c(\mathcal{G}^X_{X''}(g\circ f))$ as

$$< \xi_f * \eta_g, \xi_f * \eta_g > (\gamma'') = \sum_{u \circ \gamma'' = v} \overline{\xi_f * \eta_g(u)} \xi_f * \eta_g(v) \text{ for } u, v \in \mathcal{G}^X_{X'}(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g)$$

Then, we have,

$$\begin{aligned}
< \xi_f * \eta_g, \xi_f * \eta_g > (\gamma'') &= \sum_{u\circ\gamma''=v} \overline{\xi_f * \eta_g(u)} \xi_f * \eta_g(v) \\
&\quad (u, v \in \mathcal{G}^X_{X'}(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g)) \\
&= \sum_{u\circ\gamma''=v} \left(\sum_{\alpha' \in \mathcal{G}'^{sf(u_1)}_{X'}} \overline{\xi_f(u_1\alpha')\eta_g(\alpha'^{-1}u_2)} \right) (\xi_f * \eta_g(v)) \\
&\quad (u = [u_1, u_2]; u_1 \in \mathcal{G}^X_{X'}(f), u_2 \in \mathcal{G}^{X'}_{X''}(g)) \\
&= \sum_{u\circ\gamma''=v} \left(\sum_{\alpha' \in \mathcal{G}'^{sf(u_1)}_{X'}} \overline{\xi_f(u_1\alpha')\eta_g(\alpha'^{-1}u_2)} \right) \left(\sum_{\alpha'_1 \in \mathcal{G}'^{sf(v_1)}_{X'}} \xi_f(v_1\alpha'_1)\eta_g(\alpha_1'^{-1}v_2) \right) \\
&\quad (v = [v_1, v_2]; v_1 \in \mathcal{G}^X_{X'}(f), v_2 \in \mathcal{G}^{X'}_{X''}(g)) \\
&= \sum_{u; s(u) = r(\gamma'')} \left(\sum_{\alpha' \in \mathcal{G}'^{sf(u_1)}_{X'}} \overline{\xi_f(u_1\alpha')\eta_g(\alpha'^{-1}u_2)} \right) \left(\sum_{\alpha'_1 \in \mathcal{G}'^{sf(u_1)}_{X'}} \xi_f(u_1\alpha'_1)\eta_g(\alpha_1'^{-1}u_2\gamma'') \right) \\
&\quad (u \in \mathcal{G}^X_{X'}(f) \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}^{X'}_{X''}(g), u_1 = v_1, v_2 = u_2\gamma'')
\end{aligned}$$

But we also have

$$\sum_{u\in\mathcal{G}^X_{X'}(f)\times_{\mathcal{G}'^{X'}_{X'}}\mathcal{G}^{X'}_{X''}(g);s(u)=r(\gamma'')}\left(\sum_{\alpha'\in\mathcal{G}'^{s_f(u_1)}_{X'}}\overline{\xi_f(u_1\alpha')\eta_g(\alpha'^{-1}u_2)}\right)A(u_1)B(u_2)=$$
$$\sum_{u_2\in\mathcal{G}^{X'}_{X''}(g);s_g(u_2)=r(\gamma'')}\sum_{u_1\in\mathcal{G}^X_{X'}(f);s_f(u_1)=r_g(u_2)}\overline{\xi_f(u_1)\eta_g(u_2)}A(u_1)B(u_2)$$

for functions $A\in C^\infty_c(\mathcal{G}_X), B\in C^\infty_c(\mathcal{G}^X_{X'}(f))$. Therefore we get

$$< \xi_f*\eta_g,\xi_f*\eta_g > (\gamma'')=$$
$$\sum_{u_2\in\mathcal{G}^{X'}_{X''}(g);s_g(u_2)=r(\gamma'')}\sum_{u_1\in\mathcal{G}^X_{X'}(f);s_f(u_1)=r_g(u_2)}\overline{\xi_f(u_1)\eta_g(u_2)}\sum_{\alpha'_1\in\mathcal{G}'^{s_f(u_1)}_{X'}}\xi_f(u_1\alpha'_1)\eta_g(\alpha'^{-1}_1u_2\gamma'')\quad(5.1.15)$$

Computing now the right hand side,

$$\begin{aligned}
< \eta_g,\pi_g(< \xi_f,\xi_f >)\eta_g > (\gamma'') &= \sum_{a\circ\gamma''=b}\overline{\eta_g(a)}\pi_g(< \xi_f,\xi_f >)\eta_g(b)(a,b\in\mathcal{G}^{X'}_{X''}(g))\\
&= \sum_{a\circ\gamma''=b}\overline{\eta_g(a)}\sum_{\beta'\in\mathcal{G}^{r_g(b)}_{X'}}< \xi_f,\xi_f > (\beta')\eta_g(\beta'^{-1}b)\\
&= \sum_{a;s_g(a)=r(\gamma'')}\overline{\eta_g(a)}\sum_{\beta'\in\mathcal{G}^{r_g(b)}_{X'}}\\
&\quad\sum_{c;s_f(c)=r(\beta')}\left(\overline{\xi_f(c)}\xi_f(c\beta')\right)\eta_g(\beta'^{-1}a\gamma'')(c\in\mathcal{G}^X_{X'}(f))\\
&= \sum_{a;s_g(a)=r(\gamma'')}\overline{\eta_g(a)}\sum_{\beta'\in\mathcal{G}^{r_g(a)}_{X'}}\sum_{c;s_f(c)=r_g(a)}\left(\overline{\xi_f(c)}\xi_f(c\beta')\right)\eta_g(\beta'^{-1}a\gamma'')\\
&= \sum_{a;s_g(a)=r(\gamma'')}\overline{\eta_g(a)}\sum_{\beta'\in\mathcal{G}^{s_f(c)}_{X'}}\sum_{c;s_f(c)=r_g(a)}\left(\overline{\xi_f(c)}\xi_f(c\beta')\right)\eta_g(\beta'^{-1}a\gamma'')\\
&= \sum_{a;s_g(a)=r(\gamma'')}\sum_{c;s_f(c)=r_g(a)}\overline{\eta_g(a)\xi_f(c)}\sum_{\beta'\in\mathcal{G}^{s_f(c)}_{X'}}\xi_f(c\beta')\eta_g(\beta'^{-1}a\gamma'')\quad(5.1.16)
\end{aligned}$$

Comparing 5.1.15 and 5.1.16 we get the result.

Thus from the above properties we see that the map $\xi_f\otimes\eta_g\mapsto\xi_f*\eta_g$ is a well-defined isometric map from $\mathcal{E}^X_{X'}(f)\otimes_{C^*(\mathcal{G}'^{X'}_{X'})}\mathcal{E}^{X'}_{X''}(g)$ to $\mathcal{E}^{X''}_X(g\circ f)$. To see that this map is surjective, we use the fact that $\pi_{g\circ f}:C^*(\mathcal{G}^X_X)\xrightarrow{\cong}\mathcal{K}(\mathcal{E}^{X''}_X(g\circ f))$ and so $\pi_{g\circ f}(C^*(\mathcal{G}^X_X))\mathcal{E}^{X''}_X(g\circ f)$ is dense in $\mathcal{E}^{X''}_X(g\circ f)$. So it suffices to show that an element of the form $\pi_{g\circ f}(h)\xi$ is in the image of this map for all $h\in C^*(\mathcal{G}^X_X)$ and $\xi\in\mathcal{E}^{X''}_X(g\circ f)$. This is done as follows: let $\eta_1,\eta_2\in\mathcal{E}^{X'}_X(f)$ be such that $\theta_{\eta_1,\eta_2}\in\pi_{g\circ f}(C^*(\mathcal{G}^X_X))$. We will prove the following two properties which will suffice to show surjectivity:

1. For $\eta_1,\eta_2,\zeta\in C^\infty_c(\mathcal{G}^X_{X'}(f))$, we have $\theta_{\eta_1,\eta_2}\zeta=\pi_f(\eta_1\star\eta_2)\zeta$, where $\eta_1\star\eta_2\in C^\infty_c(\mathcal{G}^X_X)$ is defined as follows:

$$\eta_1\star\eta_2(\alpha):=\sum_{\alpha'_1\in\mathcal{G}'^{f(r(\alpha))}_{X'}}\eta_1(r(\alpha),\alpha'_1)\overline{\eta_2(s(\alpha),\check{f}(\alpha^{-1})\alpha'_1)}$$

2. We have for $\xi \in C_c^\infty(\mathcal{G}_{X''}^X(g\circ f))$, $\pi_{g\circ f}(\eta_1 \star \eta_2)\xi = \eta_1 * (\eta_2 * \xi)$ where $\eta_2 * \xi \in C_c^\infty(\mathcal{G}_{X''}^{X'}(g))$ is given by

$$\eta_2 * \xi(x', \gamma'') = \sum_{x \in X \cap L_{x'}} \sum_{\gamma' \in \mathcal{G}'^{f(x)}_{x'}} \overline{\eta_2(x, \gamma')}\xi(x, \breve{g}(\gamma')\gamma'')$$

where $L_{x'}$ is the leaf in V which is mapped to the leaf $L'_{x'}$ by f.

We now proceed to verify these properties:

1. Starting our computation from the left hand side, we get,

$$\begin{aligned}
\theta_{\eta_1,\eta_2}\zeta \underbrace{(x,\gamma')}_{f(x)=r(\gamma')} &= (\eta_1. < \eta_2, \zeta >)(x,\gamma') \\
&= \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1}) < \eta_2, \zeta > (\alpha') \\
&= \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1}) \sum_{x_1 \in X \cap L_{s(\alpha')}} \sum_{\gamma'_1 \in \mathcal{G}'^{f(x_1)}_{r(\alpha')}} \overline{\eta_2(x_1, \gamma'_1)}\zeta(x_1, \gamma'_1\alpha') \\
&= \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1}) \sum_{x_1 \in X \cap L_{s(\alpha')}} \sum_{\gamma'_2 \in \mathcal{G}'^{f(x_1)}_{s(\alpha')}} \overline{\eta_2(x_1, \gamma'_2\alpha'^{-1})}\zeta(x_1, \gamma'_2) \\
&= \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1}) \sum_{x_1 \in X \cap L_{s(\gamma')}} \sum_{\gamma'_3 \in \mathcal{G}'^{f(x)}_{f(x_1)}} \overline{\eta_2(x_1, \gamma_3'^{-1}\gamma'\alpha'^{-1})}\zeta(x_1, \gamma_3'^{-1}\gamma') \\
&= \sum_{x_1 \in X \cap L_{s(\gamma')}} \sum_{\gamma'_3 \in \mathcal{G}'^{f(x)}_{f(x_1)}} \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1})\overline{\eta_2(x_1, \gamma_3'^{-1}\gamma'\alpha'^{-1})}\zeta(x_1, \gamma_3'^{-1}\gamma') \\
&= \sum_{x_1 \in X \cap L_{s(\gamma')}} \sum_{\gamma_3 \in \mathcal{G}^x_{x_1}} \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1})\overline{\eta_2(x_1, \breve{f}(\gamma_3^{-1})\gamma'\alpha'^{-1})}\zeta(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma') \\
&\quad \text{(since } \exists \text{ unique } \gamma_3 \in \mathcal{G}^x_{x_1} \text{ such that } \breve{f}(\gamma_3) = \gamma'_3) \\
&= \sum_{\gamma_3 \in \mathcal{G}^x_X} \sum_{\alpha' \in \mathcal{G}'^{X'}_{s(\gamma')}} \eta_1(x, \gamma'\alpha'^{-1})\overline{\eta_2(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma'\alpha'^{-1})}\zeta(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma') \\
&= \sum_{\gamma_3 \in \mathcal{G}^x_X} \sum_{\alpha'_1 \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(x, \alpha'_1)\overline{\eta_2(s(\gamma_3), \breve{f}(\gamma_3^{-1})\alpha_1'^{-1})}\zeta(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma') \\
&= \sum_{\gamma_3 \in \mathcal{G}^x_X} \left(\sum_{\alpha'_1 \in \mathcal{G}'^{f(x)}_{X'}} \eta_1(r(\gamma_3), \alpha'_1)\overline{\eta_2(s(\gamma_3), \breve{f}(\gamma_3^{-1})\alpha_1'^{-1})} \right) \zeta(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma') \\
&= \sum_{\gamma_3 \in \mathcal{G}^x_X} (\eta_1 \star \eta_2)(\gamma_3)\zeta(s(\gamma_3), \breve{f}(\gamma_3^{-1})\gamma') \\
&= \pi_f(\eta_1 \star \eta_2)\zeta(x, \gamma')
\end{aligned}$$

Thus we get the result.

2. We compute the left hand side as follows:

$$\begin{aligned}\pi_{g\circ f}(\eta_1\star\eta_2)\xi\underbrace{(x,\gamma'')}_{g\circ f(x)=r(\gamma'')} &= \sum_{\alpha\in\mathcal{G}_X^x}(\eta_1\star\eta_2)(\alpha)\xi(s(\alpha),\breve{g}\circ\breve{f}(\alpha^{-1})\gamma'')\\ &= \sum_{\alpha\in\mathcal{G}_X^x}\sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\overline{\eta_2(s(\alpha),\breve{f}(\alpha^{-1})\alpha_1')}\xi(s(\alpha),\breve{g}\circ\breve{f}(\alpha^{-1})\gamma'')\end{aligned}\tag{5.1.17}$$

Now computing the right hand side, we have,

$$\begin{aligned}[\eta_1*(\eta_2*\xi)](x,\gamma'') &= [\eta_1*(\eta_2*\xi)][(x,\gamma');(s(\gamma'),\breve{g}(\gamma'^{-1})\gamma'')]\ (\text{for any }\gamma'\in\mathcal{G}'^{f(x)}_{X'})\\ &= \sum_{\alpha'\in\mathcal{G}'^{s(\gamma')}_{X'}}\eta_1(x,\gamma'\alpha')(\eta_2*\xi)(s(\alpha'),\breve{g}(\alpha'^{-1}\gamma'^{-1})\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')(\eta_2*\xi)(s(\alpha'),\breve{g}(\alpha_1'^{-1})\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\sum_{x_1\in X\cap L_{s(\alpha_1')}}\sum_{\gamma_1'\in\mathcal{G}'^{f(x_1)}_{s(\alpha_1')}}\overline{\eta_2(x_1,\gamma_1')}\xi(x_1,\breve{g}(\gamma_1'\alpha_1'^{-1})\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\sum_{x_1\in X\cap L_{s(\alpha_1')}}\sum_{\gamma_2'\in\mathcal{G}'^{f(x_1)}_{f(x)}}\overline{\eta_2(x_1,\gamma_2'\alpha_1')}\xi(x_1,\breve{g}(\gamma_2')\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\sum_{x_1\in X\cap L_x}\sum_{\gamma_2\in\mathcal{G}_x^{x_1}}\overline{\eta_2(x_1,\breve{f}(\gamma_2)\alpha_1')}\xi(x_1,\breve{g}\circ\breve{f}(\gamma_2))\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\sum_{\gamma_2\in\mathcal{G}_x^X}\overline{\eta_2(r(\gamma_2),\breve{f}(\gamma_2)\alpha_1')}\xi(r(\gamma_2),\breve{g}\circ\breve{f}(\gamma_2))\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\eta_1(x,\alpha_1')\sum_{\gamma\in\mathcal{G}_X^x}\overline{\eta_2(s(\gamma),\breve{f}(\gamma^{-1})\alpha_1')}\xi(s(\gamma),\breve{g}\circ\breve{f}(\gamma^{-1}))\gamma'')\\ &= \sum_{\alpha_1'\in\mathcal{G}'^{f(x)}_{X'}}\sum_{\gamma\in\mathcal{G}_X^x}\eta_1(x,\alpha_1')\overline{\eta_2(s(\gamma),\breve{f}(\gamma^{-1})\alpha_1')}\xi(s(\gamma),\breve{g}\circ\breve{f}(\gamma^{-1}))\gamma'')\end{aligned}\tag{5.1.18}$$

Comparing 5.1.17 and 5.1.18 gives the equality. □

5.1.1 Alternative description of $\mathcal{G}^X_{X'}(f)$

Consider the following definition:

$$\mathcal{G}^X\times^f_{\mathcal{G}}\mathcal{G}'_{X'}:=\{(\gamma,\gamma')\in\mathcal{G}^X\times\mathcal{G}_{X'}|f(s(\gamma))=r(\gamma')\}/\sim$$

where $(\gamma_1,\gamma_1')\sim(\gamma\alpha,\breve{f}(\alpha^{-1})\gamma')$ for $\alpha\in\mathcal{G}$ such that $r(\alpha)=s(\gamma)$.

Remark. *For $\alpha\in\mathcal{G}$ such that $r(\alpha)=s(\gamma)$, the action $(\gamma,\gamma')\alpha=(\gamma\alpha,\breve{f}(\alpha^{-1})\gamma')$ is well-defined since $f(s(\gamma\alpha))=r(\breve{f}(\alpha^{-1})\gamma')$.*

We define a map $\Theta:\mathcal{G}^X\times^f_{\mathcal{G}}\mathcal{G}'_{X'}\to\mathcal{G}^X_{X'}(f)$ by $\Theta[\gamma,\gamma']=(r(\gamma),\breve{f}(\gamma)\gamma')$

Proposition 5.1.6. *Θ is a well-defined smooth map and is a diffeomorphism between $\mathcal{G}^X \times^f_{\mathcal{G}} \mathcal{G}'_{X'}$ and $\mathcal{G}^X_{X'}(f)$.*

Proof. • Θ is well-defined and smooth:

We have $\Theta[(\gamma,\gamma')\alpha] = \Theta[\gamma\alpha, \breve{f}(\alpha^{-1})\gamma'] = (r(\gamma\alpha), \breve{f}(\gamma\alpha)\breve{f}(\alpha^{-1})\gamma') = (r(\gamma), \breve{f}(\gamma)\gamma')$

hence Θ is well-defined. Smoothness follows from the smoothness of $r, \breve{f}$ and the composition map in $\mathcal{G}'$.

• Θ is injective:

Let $\Theta[\gamma_1,\gamma_1'] = \Theta[\gamma_2,\gamma_2']$. Then $r(\gamma_1) = r(\gamma_2)$ and $\breve{f}(\gamma_1)\gamma_1' = \breve{f}(\gamma_2)\gamma_2'$. Let $\alpha = \gamma_1^{-1}\gamma_2$. Then $(\gamma_1,\gamma_1')\alpha = (\gamma_2, \breve{f}(\alpha^{-1})\gamma_1') = (\gamma_2, \breve{f}(\gamma_2^{-1})\breve{f}(\gamma_1)\gamma_1') = (\gamma_2,\gamma_2')$.

• Θ is surjective:

Let $(x,\gamma') \in \mathcal{G}^X_{X'}(f)$. Then setting $u = [1_x, \gamma']$, we find $u \in \mathcal{G}^X \times^f_{\mathcal{G}} \mathcal{G}'_{X'}$ as $f(s(1_x)) = f(x) = r(\gamma')$. Then $\Theta([u]) = (x,\gamma')$, and so Θ is surjective.

Hence Θ is a diffeomorphism. ☐

5.1.2 Relation with Connes-Skandalis module

Let f be as before. Let $\mathcal{G}(f)$ denote the right principal $\mathcal{G}'$-bundle (cf. [MkMr:03]) defined by Connes and Skandalis (cf. [CoSk:84],[Co:94],[Co:81]) defined as follows:

$$\mathcal{G}(f) := \{(v,\alpha'); v \in V, \alpha' \in \mathcal{G}' \text{ and } f(x) = r(\alpha')\}$$

$\mathcal{G}(f)$ also has a free action of $\mathcal{G}$ on the left. More precisely, the right action of $\mathcal{G}'$ is given by

$$(v,\alpha')\beta' = (v,\alpha'\beta'), \text{ for } \beta' \in \mathcal{G}' \text{ such that } r(\beta') = s(\alpha')$$

while the left action of $\mathcal{G}$ is given by

$$\lambda(v,\alpha') = (r(\lambda), \breve{f}(\lambda)\alpha') \text{ for } \lambda \in \mathcal{G} \text{ with } s(\lambda) = v\ .$$

It is easy to show that these actions are well-defined.

Let

$$\mathcal{G}(f) \times_{\mathcal{G}'} \mathcal{G}_{X'} := \{((v,\alpha'),\beta'); (v,\alpha') \in \mathcal{G}(f), \beta' \in \mathcal{G}' | s(\alpha') = r(\beta')\}/\sim$$

where $((v,\alpha'),\beta') \sim ((v,\alpha')\lambda', \lambda'^{-1}\beta')$ for any $\lambda' \in \mathcal{G}'$ such that $r(\lambda') = s(\alpha')$.

We define similarly,

$$\mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f) := \{(\alpha,(x,\gamma')); \alpha \in \mathcal{G}, (x,\gamma') \in \mathcal{G}^X_{X'}(f) | s(\alpha) = x\}/\sim$$

where $(\alpha,(x,\gamma')) \sim (\alpha\beta, \beta^{-1}(x,\gamma'))$ for $\beta \in \mathcal{G}^X_X$ such that $r(\beta) = s(\alpha) = x$.

There is a map $\Psi_0 : \mathcal{G}_X \times \mathcal{G}^X_{X'}(f) \to \mathcal{G}(f) \times \mathcal{G}_{X'}$ defined by setting

$$\Psi_0(\alpha,(x,\gamma')) = ((r(\alpha); \breve{f}(\alpha)), \gamma')$$

Proposition 5.1.7. *1. Ψ_0 induces a well-defined map $\Psi : \mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f) \to \mathcal{G}(f) \times_{\mathcal{G}'} \mathcal{G}_{X'}$*

2. The map Ψ is smooth and injective

3. When f is a leafwise homotopy equivalence, Ψ is also surjective.

Proof. 1. Let $\alpha_1 = \alpha\beta \Rightarrow r(\alpha_1) = r(\alpha), (x_1, \gamma_1') = \beta^{-1}(x, \gamma') \Rightarrow x_1 = s(\beta), \gamma_1' = \breve{f}(\beta^{-1})\gamma'$. then we have $(\alpha_1, (x_1, \gamma_1')) \sim (\alpha, (x, \gamma'))$. Now,

$$\Psi_0(\alpha_1, (x_1, \gamma_1')) = ((r(\alpha_1), \breve{f}(\alpha_1)); \gamma_1') = ((r(\alpha), \breve{f}(\alpha_))\breve{f}(\beta); \breve{f}(\beta^{-1})\gamma') = ((r(\alpha), \breve{f}(\alpha_)); \gamma') \circ \breve{f}(\beta)$$

Hence $\Psi_0(\alpha_1, (x_1, \gamma_1')) \sim \Psi_0(\alpha, (x, \gamma'))$, and so Ψ_0 induces a well-defined map $\Psi : \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to \mathcal{G}(f) \times_{\mathcal{G}'} \mathcal{G}_{X'}$ given by

$$\Psi[(\alpha, (x, \gamma'))] = [((r(\alpha); \breve{f}(\alpha)), \gamma')]$$

2. As Ψ_0 can be written in the form $\Psi_0 = ((r \circ pr_1, \breve{f} \circ pr_1); pr_3)$, we see that Ψ_0 is smooth. Hence the induced map Ψ is smooth too. Let $\Psi[(\alpha_1, (x_1, \gamma_1'))] = \Psi[(\alpha_2, (x_2, \gamma_2'))]$. This implies that $r(\alpha_1) = r(\alpha_2)$, and as $\mathcal{G}(f)$ is a principal $\mathcal{G}'$-bundle, this means that there exists a unique $\beta' \in \mathcal{G}'$ with $r(\beta') = s(\breve{f}(\alpha_1))$ such that $\breve{f}(\alpha_1) \circ \beta' = \breve{f}(\alpha_2)$ and $\beta'^{-1}\gamma_1' = \gamma_2'$. Now let $\beta \in \mathcal{G}_X^X$ be the unique homotopy class of a leafwise path connecting $x_2 \in X$ to $x_1 \in X$ such that $\alpha_2 = \alpha_1\beta$. Then we have $\breve{f}(\alpha_1) \circ \breve{f}(\beta) = \breve{f}(\alpha_2)$. Hence $\breve{f}(\beta) = \beta'$, and so $(\alpha_2; (x_2, \gamma_2')) = (\alpha_1\beta; \beta^{-1}(x_1, \gamma_1')) \Rightarrow [\alpha_1; (x_1, \gamma_1')] = [\alpha_2; (x_2, \gamma_2')]$. Thus Ψ is injective.

3. Let $((v, \alpha'); \gamma') \in \mathcal{G}(f) \times \mathcal{G}_{X'}'$ such that $r(\gamma') = s(\alpha')$. Let $g : (V', \mathcal{F}') \to (V, \mathcal{F})$ be the homotopy inverse of f. Denote by H the homotopy between $g \circ f$ and id_V, and H' the homotopy between $f \circ g$ and $id_{V'}$. For any $v \in V$, H gives a leafwise path from v to $g \circ f(v)$, we denote this path by H_v and the homotopy class of H_v as λ_v . Similarly, for any $v' \in V'$, H' gives a leafwise path from v' to $f \circ g(v')$, we denote this path by $H_{v'}'$ and the homotopy class of $H_{v'}'$ as $\lambda_{v'}'$. Using the fact that f is a surjective map (see [BeRo:10]) , we denote the preimage of $\lambda_{v'}'$ as $\lambda_{v'}''$, i.e. $\breve{f}(\lambda_{v'}'') = \lambda_{v'}'$. Now, let $x' = r(\gamma') = s(\alpha')$. Then, as X is a complete transversal, we may choose $\beta \in \mathcal{G}_X^{g(x')}$. Now we set

$$u := (\lambda_{f(v)}''^{-1} \circ \breve{g}(\alpha') \circ \beta; (s(\beta), \breve{f}(\beta)^{-1} \circ \lambda_{x'}' \circ \gamma'))$$

Claim: $\Psi[u] = [(v, \alpha'); \gamma']$.

We have, by definition,

$$\begin{aligned} \Psi_0(u) &= ((v, (\breve{f}(\lambda_{f(v)}''^{-1}) \circ \breve{f}(\breve{g}(\alpha')) \circ \breve{f}(\beta))); \breve{f}(\beta^{-1}) \circ \lambda_{x'}' \circ \gamma') \\ &= ((v, \lambda_{f(v)}'^{-1} \circ \breve{f}(\breve{g}(\alpha')) \circ \breve{f}(\beta))); \breve{f}(\beta^{-1}) \circ \lambda_{x'}' \circ \gamma') \end{aligned} \tag{5.1.19}$$

Now, $((v, (\lambda_{f(v)}'^{-1}) \circ \breve{f}(\breve{g}(\alpha')) \circ \breve{f}(\beta))); \breve{f}(\beta^{-1}) \circ \lambda_{x'}' \circ \gamma') \sim ((v, (\lambda_{f(v)}'^{-1}) \circ \breve{f}(\breve{g}(\alpha')); \lambda_{x'}' \circ \gamma')$, and

$$((v, (\lambda_{f(v)}'^{-1}) \circ \breve{f}(\breve{g}(\alpha')); \lambda_{x'}' \circ \gamma') \sim (v, (\lambda_{f(v)}'^{-1}) \circ \breve{f}(\breve{g}(\alpha')) \circ \lambda_{x'}'; \gamma')$$

So it remains to prove that $(\lambda_{f(v)}'^{-1}) \circ \breve{f}(\breve{g}(\alpha')) \circ \lambda_{x'}' = \alpha'$. To see this, let $\alpha'(s)$, $0 \leq s \leq 1$ be a path representing α' in a leaf of V'. Then we have the following diagram of paths:

$$\begin{tikzcd} f(v) \arrow[r, "H'_{f(v)}"] & f \circ g \circ f(v) \\ x' \arrow[u, "\alpha'"] \arrow[r, "H'_{x'}"] & f \circ g(x') \arrow[u, "f \circ g(\alpha')"'] \end{tikzcd}$$

Now for every $s \in [0,1]$, there is a path given by $H'_{\alpha'(s)}$ connecting $\alpha'(s)$ to $f \circ g((\alpha')(s))$. We call this path $H'(u, \alpha'(s))$, $0 \leq u \leq 1$. For a fixed u_0, the path $\Gamma_{u_0}(s) = H'(u_0, \alpha'(s))$ is a path connecting $H'(u_0, x') = H'_{x'}(u_0)$ to $H'(u_0, f(v)) = H'_{f(v)}(u_0)$. Therefore, the element $\lambda'^{-1}_{f(v)} \circ \breve{f}(\breve{g}(\alpha')) \circ \lambda'_{x'} \in \mathcal{G}'$ is given by the homotopy class of the path $H'(t, f(v))^{-1}_{0 \leq t \leq 1} *_{con} \Gamma_1(\alpha'(s))_{0 \leq s \leq 1} *_{con} H'(t, x')_{0 \leq t \leq 1}$, where $*_{con}$ denotes concatenation of paths. Consider for $0 \leq u \leq 1$,

$$\rho(u) := H'(t, f(v))^{-1}_{0 \leq t \leq u} *_{con} H'(u, \alpha'(s))_{0 \leq s \leq 1} *_{con} H'(t, x')_{0 \leq t \leq u}$$

Then ρ is a path from x' to $f(v)$ and $\rho(0) = H'(0, \alpha'(s))_{0 \leq s \leq 1} = \alpha'(s)_{0 \leq s \leq 1}$, while

$$\rho(1) = H'(t, f(v))^{-1}_{0 \leq t \leq 1} *_{con} H'(u, \alpha'(s))_{0 \leq s \leq 1} *_{con} H'(t, x')_{0 \leq t \leq 1}$$

Hence $[\rho(0)] = \alpha'$, $[\rho(1)] = \lambda'^{-1}_{f(v)} \circ \breve{f}(\breve{g}(\alpha')) \circ \lambda'_{x'}$. Therefore we have proved that

$$\alpha' = \lambda'^{-1}_{f(v)} \circ \breve{f}(\breve{g}(\alpha')) \circ \lambda'_{x'}$$

□

Consider now the following space

$$\mathcal{G}^V_{X'}(f) := \{(v, \gamma') \in V \times \mathcal{G}'_{X'} | r(\gamma') = f(v)\}$$

Then we have

Proposition 5.1.8.

$$\mathcal{G}^V_{X'}(f) \cong \mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f)$$

where $\mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f) := \{(\gamma; (x, \gamma')) \in \mathcal{G}_X \times \mathcal{G}^X_{X'}(f) | x = s(\gamma)\}/\sim$, *and* $((\gamma; (x, \gamma')) \sim (\gamma\alpha; \alpha^{-1}(x, \gamma'))$ *for all* $\alpha \in \mathcal{G}^X_X$ *such that* $r(\alpha) = s(\gamma)$.

Proof. Let $\phi_0 : \mathcal{G}_X \times \mathcal{G}^X_{X'}(f) \to \mathcal{G}^V_{X'}(f)$ be defined as $\phi_0(\gamma; (x, \gamma')) = (r(\gamma), \breve{f}(\gamma)\gamma')$, then ϕ_0 induces a well-defined map $\phi : \mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f) \to \mathcal{G}^V_{X'}(f)$. Indeed, let $\alpha \in \mathcal{G}^X_X$ with $r(\alpha) = s(\gamma)$. Then we have,

$$\begin{aligned} \phi_0(\gamma\alpha; (s(\alpha), \breve{f}(\alpha^{-1})\gamma')) &= (r(\gamma\alpha), \breve{f}(\gamma\alpha)\breve{f}(\alpha^{-1})\gamma') \\ &= (r(\gamma), \breve{f}(\gamma)\gamma') \end{aligned}$$

Thus the induced map ϕ is well-defined and it is clearly smooth.

Assume $\phi[\gamma_1; (x_1, \gamma'_1)] = \phi[\gamma_2; (x_2, \gamma'_2)]$. Then we have

$$r(\gamma_1) = r(\gamma_2) \text{ and } \breve{f}(\gamma_1)\gamma'_1 = \breve{f}(\gamma_2)\gamma'_2$$

Set $\lambda = \gamma_2^{-1}\gamma_1$. Then $s(\lambda) = s(\gamma_1)$ and we have

$$\begin{aligned} (\gamma_2; (s(\gamma_2), \gamma'_2))\lambda &= (\gamma_2\lambda; \lambda^{-1}(s(\gamma_2), \gamma'_2)) \\ &= (\gamma_1, (s(\lambda), \breve{f}(\lambda^{-1})\gamma'_2)) \\ &= (\gamma_1, (s(\gamma_1), \breve{f}(\gamma_1^{-1})\breve{f}(\gamma_2)\gamma'_2)) \\ &= (\gamma_1, (s(\gamma_1), \breve{f}(\gamma_1^{-1})\breve{f}(\gamma_1)\gamma'_1)) \\ &= (\gamma_1, (s(\gamma_1), \gamma'_1)) \end{aligned} \tag{5.1.20}$$

Therefore ϕ is injective.

Let now $(v,\gamma') \in \mathcal{G}^V_{X'}(f)$. Then $r(\gamma') = f(v)$. Since X is a complete transversal, we can find $\lambda \in \mathcal{G}^v_X$. Put $u = (\lambda; (s(\lambda), \breve{f}(\lambda^{-1})\gamma'))$. Then,

$$\phi[u] = (r(\lambda), \breve{f}(\lambda)\breve{f}(\lambda^{-1})\gamma') = (v, \gamma')$$

Hence ϕ is surjective.

Finally, $\phi : \mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f) \to \mathcal{G}^V_{X'}(f)$ is a diffeomorphism.

□

Define projections $\pi_1 : \mathcal{G}^V_{X'}(f) \to V'$ and $\pi_2 : \mathcal{G}^V_{X'}(f) \to \mathcal{G}'_{X'}$ by projecting onto the first and second factor, respectively. Then we state the following corollary which is proved in [BeRo:10].

Corollary 5.1.9. *Let E be a smooth vector bundle over V. Then we have the following isometric isomorphisms between Hilbert modules*

$$\mathcal{E}_{X,E} \otimes_{(\mathcal{A}^X_X)} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}_{X,X';E}(f) \cong \mathcal{E}^V_{X',E}(f)$$

where $\mathcal{E}_{X,X';E}(f)$ is the completion of the pre-Hilbert module $C^\infty_c(\mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f), (r \circ \pi_1)^* E)$ with respect to the maximal norm on $\mathcal{A}^{X'}_{X'}$.

Corollary 5.1.10. *We have an isomorphism of Hilbert $C^*(\mathcal{G}'^{X'}_{X'})$-modules*

$$\mathcal{E}_X \otimes_{C^*(\mathcal{G}^X_X)} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}(f) \otimes_{C^*(\mathcal{G}')} \mathcal{E}_{X'}$$

Proof. Since from the previous corollary we have $\mathcal{E}_X \otimes_{C^*(\mathcal{G}^X_X)} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}_{X,X'}(f)$, it suffices to show that $\mathcal{E}(f) \otimes_{C^*(\mathcal{G}')} \mathcal{E}_{X'} \cong \mathcal{E}_{X,X'}(f)$. Recall that we have a diffeomorphism $\mathcal{G}(f) \times_{\mathcal{G}'} \mathcal{G}'_{X'} \cong \mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f)$. We define a map $\upsilon : C^\infty_c(\mathcal{G}(f)) \otimes_{C^\infty_c(\mathcal{G}^X_X)} C^\infty_c(\mathcal{G}_{X'}) \to C^\infty_c(\mathcal{G}_X \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f))$ given by

$$\upsilon(\xi \otimes \eta)[\gamma; (x, \gamma')] := \int_{\alpha' \in \mathcal{G}'^{r(\gamma')}} \xi(r(\gamma), \breve{f}(\gamma)\alpha')\eta(\alpha'^{-1}\gamma') d\lambda^{r(\gamma')}(\alpha') \text{ for } \xi \in C^\infty_c(\mathcal{G}(f)), \eta \in C^\infty_c(\mathcal{G}_{X'})$$

To check that the above formula is well-defined, we prove the following formulae:

1. If $[\gamma_1; (x_1, \gamma'_1)] = [\gamma_2; (x_2, \gamma'_2)]$, then $\upsilon(\xi \otimes \eta)[\gamma_1; (x_1, \gamma'_1)] = \upsilon(\xi \otimes \eta)[\gamma_2; (x_2, \gamma'_2)]$

2. we have, $\upsilon(\xi\phi' \otimes \eta) = \upsilon(\xi \otimes \chi_m(\phi)\eta)$ for $\phi' \in C^\infty_c(\mathcal{G}')$, where $\phi' \in C^*(\mathcal{G}')$ and $\chi_m : C^*(\mathcal{G}') \to \mathcal{K}(\mathcal{E}'_m)$ is an isomorphism.

1. If $[\gamma_1; (x_1, \gamma'_1)] = [\gamma_2; (x_2, \gamma'_2)]$, then we have, $\gamma_2 = \gamma_1\beta, x_2 = \beta^{-1}x_1; \gamma'_2 = f(\beta^{-1})$ for some $\beta \in \mathcal{G}^{x_2}_{x_1}$. We have,

$$\begin{aligned}
\upsilon(\xi \otimes \eta)[\gamma_1\beta; (\beta^{-1}x_1, \breve{f}(\beta^{-1})\gamma'_1)] &= \int_{\alpha'_1 \in \mathcal{G}'^{s(\breve{f}(\beta))}} \xi(r(\gamma_1), \breve{f}(\gamma_1\beta)\alpha'_1)\eta(\alpha'^{-1}_1 \breve{f}(\beta^{-1})\gamma'_1) d\lambda^{s(\breve{f}(\beta))}(\alpha'_1) \\
&= \int_{\beta' \in \mathcal{G}'^{r(\gamma'_1)}} \xi(r(\gamma_1), \breve{f}(\gamma_1)\beta')\eta(\beta'^{-1}\gamma'_1) d\lambda^{r(\gamma'_1)}(\beta') \\
&= \upsilon(\xi \otimes \eta)[\gamma_1\beta; (\beta^{-1}x_1, \breve{f}(\beta^{-1})\gamma'_1)]
\end{aligned}$$

2. We first compute the left hand side $\upsilon(\xi\phi'\otimes\eta)$. We have,

$$\begin{aligned}\upsilon(\xi\phi'\otimes\eta)[\gamma;(x,\gamma')] &= \int_{\alpha'\in\mathcal{G}'^{r(\gamma')}}(\xi\phi')(r(\gamma),\breve{f}(\gamma)\alpha')\eta(\alpha'^{-1}\gamma')d\lambda^{r(\gamma')}(\alpha')\\ &= \int_{\alpha'\in\mathcal{G}'^{r(\gamma')}}\left(\int_{\beta'\in\mathcal{G}'_{s(\alpha')}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'\beta'^{-1})\phi'(\beta')d\lambda_{s(\alpha')}(\beta')\right)\eta(\alpha'^{-1}\gamma')d\lambda^{r(\gamma')}(\alpha')\\ &= \int_{\alpha'\in\mathcal{G}'^{r(\gamma')}}\int_{\beta'\in\mathcal{G}'_{s(\alpha')}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'\beta'^{-1})\phi'(\beta')\eta(\alpha'^{-1}\gamma')d\lambda_{s(\alpha')}(\beta')d\lambda^{r(\gamma')}(\alpha')\end{aligned}$$

Let us now compute the right hand side $\upsilon(\xi\otimes\chi_m(\phi)\eta)$. We have,

$$\begin{aligned}&\upsilon(\xi\otimes\chi_m(\phi')\eta)[\gamma;(x,\gamma')]\\ =\;& \int_{\alpha'_1\in\mathcal{G}'^{r(\gamma')}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'_1)(\chi_m(\phi')\eta)(\alpha_1'^{-1}\gamma')d\lambda^{r(\gamma')}(\alpha'_1)\\ =\;& \int_{\alpha'_1\in\mathcal{G}'^{r(\gamma')}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'_1)\left(\int_{\beta'_1\in\mathcal{G}'_{s(\alpha'_1)}}\phi'(\beta_1'^{-1})\eta(\beta'_1\alpha_1'^{-1}\gamma')d\lambda_{s(\alpha'_1)}(\beta'_1)\right)d\lambda^{r(\gamma')}(\alpha'_1)\\ =\;& \int_{\alpha'_1\in\mathcal{G}'^{r(\gamma')}}\int_{\beta'_1\in\mathcal{G}'_{s(\alpha'_1)}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'_1)\phi'(\beta_1'^{-1})\eta(\beta'_1\alpha_1'^{-1}\gamma')d\lambda_{s(\alpha'_1)}(\beta'_1)d\lambda^{r(\gamma')}(\alpha'_1)\\ =\;& \int_{\alpha'_1\in\mathcal{G}'^{r(\gamma')}}\int_{\beta'_2\in\mathcal{G}'^{r(\alpha'_1)}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'_1)\phi'(\alpha_1'^{-1}\beta'_2)\eta(\beta_2'^{-1}\gamma')d\lambda^{r(\alpha'_1)}(\beta'_1)d\lambda^{r(\gamma')}(\alpha'_1)\\ =\;& \int_{\beta'_2\in\mathcal{G}'^{r(\alpha'_1)}}\left(\int_{\alpha'_1\in\mathcal{G}'^{r(\gamma')}}\xi(r(\gamma),\breve{f}(\gamma)\alpha'_1)\phi'(\alpha_1'^{-1}\beta'_2)d\lambda^{r(\gamma')}(\alpha'_1)\right)\eta(\beta_2'^{-1}\gamma')d\lambda^{r(\alpha'_1)}(\beta'_1)\\ =\;& \int_{\beta'_2\in\mathcal{G}'^{r(\gamma')}}\left(\int_{\alpha'_2\in\mathcal{G}'_{s(\beta'_2)}}\xi(r(\gamma),\breve{f}(\gamma)\beta'_2\alpha_2'^{-1})\phi'(\alpha'_2)d\lambda_{s(\beta'_2)}(\alpha'_1)\right)\eta(\beta_2'^{-1}\gamma')d\lambda^{r(\alpha'_1)}(\beta'_1)\\ =\;& \int_{\beta'_2\in\mathcal{G}'^{r(\gamma')}}\int_{\alpha'_2\in\mathcal{G}'_{s(\beta'_2)}}\xi(r(\gamma),\breve{f}(\gamma)\beta'_2\alpha_2'^{-1})\phi'(\alpha'_2)\eta(\beta_2'^{-1}\gamma')d\lambda_{s(\beta'_2)}(\alpha'_1)d\lambda^{r(\alpha'_1)}(\beta'_1)\end{aligned}$$

Comparing the last lines of the two computations above gives the result.

We show now that the map υ is an isometry, i.e. $<\upsilon(\xi\otimes\eta),\upsilon(\xi\otimes\eta)>=<\eta,\chi_m(<\xi,\xi>)\eta>$. To show this, we note that with the identification $\mathcal{G}^V_{X'}(f)\cong\mathcal{G}_X\times_{\mathcal{G}^X_X}\mathcal{G}^X_{X'}(f)$, one can rewrite the formula for the map υ as follows:

$$\upsilon(\xi\otimes\eta)(v,\gamma'):=\int_{\alpha'\in\mathcal{G}'^{r(\gamma')}}\xi(v,\alpha')\eta(\alpha'^{-1}\gamma')d\lambda^{r(\gamma')}(\alpha')\text{ for }\xi\in C_c^\infty(\mathcal{G}(f)),\eta\in C_c^\infty(\mathcal{G}_{X'})$$

The $\mathcal{A}^{X'}_{X'}$-valued inner product formula on $C_c^\infty(\mathcal{G}^V_{X'}(f))$ is given by

$$<\xi_1,\xi_1>(\gamma'):=\int_{v\in L_{r(\gamma')}}\sum_{\gamma'_1\in\mathcal{G}'^{f(v)}_{r(\gamma')}}\overline{\xi_1(v,\gamma'_1)}\xi_2(v,\gamma'_1\gamma)d\lambda_{r(\gamma)}(v)$$

where $L_{r(\gamma')}$ is the leaf in V such that $f(L_{r(\gamma')}) = L'_{r(\gamma')}$. Then we have,

$$
\begin{aligned}
& < \upsilon(\xi\otimes\eta), \upsilon(\xi\otimes\eta) > (\gamma') \\
= & \int_{v\in L_{r(\gamma')}} \sum_{\gamma'_1\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \overline{\upsilon(\xi\otimes\eta)(v,\gamma'_1)}\upsilon(\xi\otimes\eta)(v,\gamma'_1\gamma)d\lambda^{L_{\gamma'}}(v) \\
= & \int_{v\in L_{r(\gamma')}} \sum_{\gamma'_1\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \int_{\mathcal{G}'^{r(\gamma'_1)}} \overline{\xi(v,\alpha'_1)\eta(\alpha'^{-1}_1\gamma'_1)}d\lambda^{r(\gamma'_1)}(\alpha'_1)\int_{\mathcal{G}'^{r(\gamma'_1)}}\xi(v,\alpha'_2)\eta(\alpha'^{-1}_2\gamma'_1\gamma')d\lambda^{r(\gamma'_1)}(\alpha'_2)d\lambda^{L_{\gamma'}}(v) \\
& \text{putting } \alpha'_3 = \alpha'^{-1}_1\gamma'_1 \text{ we get} \\
= & \int_{v\in L_{r(\gamma')}} \sum_{\gamma'_1\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \int_{\mathcal{G}'_{s(\gamma'_1)}} \overline{\xi(v,\gamma'_1\alpha'^{-1}_3)\eta(\alpha'_3)}d\lambda_{s(\gamma'_1)}(\alpha'_3)\int_{\mathcal{G}'^{r(\gamma'_1)}}\xi(v,\alpha'_2)\eta(\alpha'^{-1}_2\gamma'_1\gamma')d\lambda^{r(\gamma'_1)}(\alpha'_2)d\lambda^{L_{\gamma'}}(v) \\
& \text{putting } \beta'_1 = \alpha'^{-1}_2\gamma'_1\alpha'^{-1}_3 \text{ we get} \\
= & \int_{v\in L_{r(\gamma')}} \sum_{\gamma'_1\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \int_{\mathcal{G}'_{s(\gamma'_1)}} \overline{\xi(v,\gamma'_1\alpha'^{-1}_3)\eta(\alpha'_3)}\int_{\mathcal{G}'_{r(\alpha'_3)}}\xi(v,\gamma'_1\alpha'^{-1}_3\beta'^{-1}_1)\eta(\beta'_1\alpha'_3\gamma')d\lambda_{r(\alpha'_3)}(\beta'_1)d\lambda_{s(\gamma'_1)}(\alpha'_3)d\lambda^{L_{\gamma'}}(v)
\end{aligned}
\tag{5.1.21}
$$

The $C^*(\mathcal{G}')$-valued inner product on $C_c^\infty(\mathcal{G}(f))$ is given by the following formula:

$$< \xi_1,\xi_2 > (\gamma') = \int_{v\in L_{\gamma'}} \sum_{\gamma'_1\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \overline{\xi_1(v,\gamma'_1)}\xi_2(v,\gamma'_1\gamma')d\lambda^{L_{\gamma'}}(v) \text{ for } \xi_1,\xi_2\in C_c^\infty(\mathcal{G}(f))$$

Computing now the term $< \eta, \chi_m(<\xi,\xi>)\eta >$, given by the inner product in $C_c^\infty(\mathcal{G}'_{X'})$, we have

$$
\begin{aligned}
& < \eta, \chi_m(<\xi,\xi>)\eta > (\gamma') \\
= & \int_{\mathcal{G}'_{r(\gamma')}} \overline{\eta(\alpha')}[\chi_m(<\xi,\xi)\eta](\alpha'\gamma')d\lambda_{r(\gamma')}(\alpha') \\
= & \int_{\mathcal{G}'_{r(\gamma')}} \overline{\eta(\alpha')}\int_{\mathcal{G}'_{r(\alpha')}} <\xi,\xi>(\beta'^{-1})\eta(\beta'\alpha'\gamma')d\lambda_{r(\alpha')}(\beta')d\lambda_{r(\gamma')}(\alpha') \\
= & \int_{\mathcal{G}'_{r(\gamma')}} \overline{\eta(\alpha')}\int_{\mathcal{G}'_{r(\alpha')}}\int_{v\in L_{\beta'}} \sum_{\gamma'_2\in \mathcal{G}'^{f(v)}_{s(\beta')}} \overline{\xi(v,\gamma'_2)}\xi(v,\gamma'_2\beta'^{-1})d\lambda^{L_{\gamma'}}(v)\eta(\beta'\alpha'\gamma')d\lambda_{r(\alpha')}(\beta')d\lambda_{r(\gamma')}(\alpha') \\
& \text{putting } \gamma'_3 = \gamma'_2\alpha' \text{ we get} \\
= & \int_{\mathcal{G}'_{r(\gamma')}} \overline{\eta(\alpha')}\int_{\mathcal{G}'_{r(\alpha')}}\int_{v\in L_{\beta'}} \sum_{\gamma'_3\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \overline{\xi(v,\gamma'_3\alpha'^{-1})}\xi(v,\gamma'_3\alpha'^{-1}\beta'^{-1})d\lambda^{L_{\gamma'}}(v)\eta(\beta'\alpha'\gamma')d\lambda_{r(\alpha')}(\beta')d\lambda_{r(\gamma')}(\alpha') \\
= & \int_{v\in L_{\beta'}} \sum_{\gamma'_3\in \mathcal{G}'^{f(v)}_{r(\gamma')}} \int_{\alpha'\in\mathcal{G}'_{r(\gamma')}}\int_{\mathcal{G}'_{r(\alpha')}} \overline{\eta(\alpha')\xi(v,\gamma'_3\alpha'^{-1})}\xi(v,\gamma'_3\alpha'^{-1}\beta'^{-1})\eta(\beta'\alpha'\gamma')d\lambda_{r(\alpha')}(\beta')d\lambda_{r(\gamma')}(\alpha')d\lambda^{L_{\gamma'}}(v)
\end{aligned}
$$

Comparing the last line above with 5.1.21 we get the result.

Lastly, in order to prove surjectivity of υ, we follow the method of proof in Proposition 5.1.5. Since $\mathcal{E}(f)$ implements the Morita equivalence between $C^*(\mathcal{G})$ and $C^*(\mathcal{G}')$, we have an isomorphism $\pi_f : C^*(\mathcal{G}) \to \mathcal{K}(\mathcal{E}(f))$. Similarly, we have an isomorphism $\pi(f) : C^*(\mathcal{G}) \to \mathcal{K}(\mathcal{E}^V_{X'}(f))$. Let $\xi_1, \xi_2 \in C_c^\infty(\mathcal{G}(f))$. Let $\xi_1 \star \xi_2$ denote the function on $\mathcal{G}$ given by

$$\xi_1\star\xi_2(\gamma) = \int_{\alpha'\in\mathcal{G}'^{f(r(\gamma))}} \xi_1(r(\gamma),\alpha')\overline{\xi_2(s(\gamma),\breve{f}(\gamma^{-1})\alpha')}d\lambda^{f(r(\gamma))}(\alpha')$$

Denote by θ_{ξ_1,ξ_2} the operator in $\mathcal{K}(\mathcal{E}(f))$ given by $\theta_{\xi_1,\xi_2}\zeta := \xi_1 < \xi_2, \zeta >$. Then a straightforward calculation shows that $\theta_{\xi_1,\xi_2} = \pi_f(\xi_1 \star \xi_2)$. Then to prove surjectivity of υ it suffices to show that for any $\kappa \in C_c^\infty(\mathcal{G}_{X'}^V(f))$ we have

$$\pi(f)(\xi_1 \star \xi_2)\kappa = \upsilon(\xi_1 \otimes (\xi_2 \bullet \kappa))$$

where

$$\xi_2 \bullet \kappa(\gamma') = \int_{v \in L_{\gamma'}} \sum_{\gamma_1' \in \mathcal{G}'^{f(v)}_{r(\gamma')}} \overline{\xi_2(v, \gamma_1')}\kappa(v, \gamma_1'\gamma')d\lambda^{L_{\gamma'}}(v) \text{ for } \gamma' \in \mathcal{G}'^{X'}_{X'}.$$

Computing the left hand side, we have

$$\begin{aligned} [\pi(f)(\xi_1 \star \xi_2)\kappa](v, \gamma') &= \int_{\alpha \in \mathcal{G}^v} (\xi_1 \star \xi_2)(\alpha)\kappa(s(\alpha, \check{f}(\alpha^{-1}\gamma'))d\lambda^v(\alpha) \\ &= \int_{\alpha \in \mathcal{G}^v} \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha')\overline{\xi_2(s(\alpha), \check{f}(\alpha^{-1})\alpha')}\kappa(s(\alpha, \check{f}(\alpha^{-1}\gamma')))d\lambda^{f(v)}(\alpha')d\lambda^v(\alpha) \end{aligned} \tag{5.1.22}$$

Now computing the right hand side, we get

$$\begin{aligned} \upsilon(\xi_1 \otimes (\xi_2 \bullet \kappa)))(v, \gamma') &= \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha')(\xi_2 \bullet \kappa)(\alpha'^{-1}\gamma')d\lambda^{f(v)}(\alpha') \\ &= \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha') \int_{v_1 \in L_v} \sum_{\gamma_1' \in \mathcal{G}'^{f(v_1)}_{s(\alpha')}} \overline{\xi_2(v_1, \gamma_1')}\kappa(v_1, \gamma_1'\alpha'^{-1}\gamma')d\lambda^{L_v}(v_1)d\lambda^{f(v)}(\alpha') \\ &= \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha') \int_{v_1 \in L_v} \sum_{\gamma_2' \in \mathcal{G}'^{f(v_1)}_{r(\alpha')}} \overline{\xi_2(v_1, \gamma_2'\alpha')}\kappa(v_1, \gamma_2'\gamma')d\lambda^{L_v}(v_1)d\lambda^{f(v)}(\alpha') \\ &= \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha') \int_{v_1 \in L_v} \sum_{\gamma_2 \in \mathcal{G}_v^{v_1}} \overline{\xi_2(v_1, \check{f}(\gamma_2)\alpha')}\kappa(v_1, \check{f}(\gamma_2)\gamma')d\lambda^{L_v}(v_1)d\lambda^{f(v)}(\alpha') \\ &= \int_{\alpha' \in \mathcal{G}'^{f(v)}} \xi_1(v, \alpha') \int_{\alpha \in \mathcal{G}_v} \overline{\xi_2(r(\alpha), \check{f}(\gamma_2)\alpha')}\kappa(r(\alpha), \check{f}(\gamma_2)\gamma')d\lambda_v(\alpha)d\lambda^{f(v)}(\alpha') \end{aligned}$$

Comparing 5.1.22 with the last line above gives the desired equality.

□

5.2 Hilbert-Poincaré complexes for foliations

We review in the appendix some basic properties of a so-called Hilbert-Poincaré complex and collect some results that are used in the present and next chapter. Our main reference is [HiRoI:05].

Let $(M, \mathcal{F})$ be an odd-dimensional smooth foliation on a closed manifold M. Let X be a complete transversal of the foliation. Denote by $\mathcal{G}$ the monodromy groupoid of the foliation and let $(\lambda_x)_{x \in M}$ be a right-equivariant smooth Haar system on $\mathcal{G}$. We consider the pre-Hilbert module $\mathcal{E}_c^i := C_c^\infty(\mathcal{G}_X, r^* \bigwedge^i T * \mathcal{F}_X)$ with the $\mathcal{A}_c^X := C_c^\infty(\mathcal{G}_X^X)$ valued inner product given by the following formula:

For $\xi_1, \xi_2 \in \mathcal{E}_c^i, u \in \mathcal{G}_X^X$,

$$< \xi_1, \xi_2 > (u) = \int_{v \in \mathcal{G}_{r(u)}} < \xi_1(v), \xi_2(vu) >_{\bigwedge^i T^*_{r(v)}\mathcal{F}} d\lambda_{r(u)}(v) \tag{5.2.1}$$

A right action of $\mathcal{A}_c^X$ on $\mathcal{E}_c^i$ is defined as follows:

For $f \in \mathcal{A}_c^X, \xi \in \mathcal{E}_c^i, \gamma \in \mathcal{G}_X$,

$$(\xi f)(\gamma) = \sum_{\gamma' \in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma)\xi(\gamma'^{-1}) \tag{5.2.2}$$

By taking the completion of $\mathcal{A}_c^X$ with the maximal C^*-norm and then completing the above pre-Hilbert module we obtain a Hilbert $C^*(\mathcal{G}_X^X)$-module $\mathcal{E}_i$ for $0 \le i \le p = dim\mathcal{F}$.

Consider the leafwise de Rham differential $d = (d_x)_{x\in M}$ on $(M, \mathcal{F})$ and for each $x \in M$ denote the $\mathcal{G}_x^x$-equivariant lift of d_x to $\mathcal{G}_x$ by $\tilde{d}_x$. Let $\tilde{d}$ denote the family of operators $(\tilde{d}_x)_{x\in X}$ acting on $\mathcal{E}_c^i$. Then it is easy to see that $\tilde{d}^2 = 0$ and so we can consider the de Rham complex on $\mathcal{G}_X$:

$$\mathcal{E}_c^0 \xrightarrow{\tilde{d}} \mathcal{E}_c^1 ... \xrightarrow{\tilde{d}} \mathcal{E}_c^p$$

The densely defined unbounded operator $\tilde{d}$ then extends to a closed, densely defined unbounded operator on the Hilbert-modules $\mathcal{E}_i$ for $0 \le i \le p$ which we denote by d_X.

We also consider the leafwise Hodge-* operator on $(M, \mathcal{F})$ and denote its lift to $\mathcal{G}_X$ by $T_X : C_c^\infty(\mathcal{G}_X, r^*(\bigwedge^i T * \mathcal{F}) \to C_c^\infty(\mathcal{G}_X, r^*(\bigwedge^{p-i} T * \mathcal{F}))$. T_X is given by the formula:

$$T_X(r^*(dx_1 \wedge dx_2 \wedge ... \wedge dx_k)) = r^*(dx_{k+1} \wedge dx_{k+2} \wedge ... \wedge dx_p)$$

The following computations show that T_X is A_X^X-linear:

we have for $\omega \in \mathcal{E}_c^k, f \in \mathcal{A}_c^X$, one can express ω in local coordinates:

$$\omega = \sum_{i_1 < i_2 < ... < i_k} \omega_I r^*(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})$$

Then we have

$$\begin{aligned}
(\omega f)(\gamma) &= \sum_{\gamma' \in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma)\omega(\gamma'^{-1}) \\
&= \sum_{\gamma' \in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma) \sum_{i_1 < i_2 < ... < i_k} \omega_I(\gamma'^{-1}) r^*(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \\
&= \sum_{i_1 < i_2 < ... < i_k} \sum_{\gamma' \in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma)\omega_I(\gamma'^{-1}) r^*(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \\
&= \sum_{i_1 < i_2 < ... < i_k} (\omega_I f)(\gamma) r^*(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})
\end{aligned}$$

Let $\{j_1, j_2...j_{p-k}\}$ be the complement of the index subset $\{i_1, i_2...i_k\}$ in $\{1, 2, ..., p\}$ sorted in increasing order, and $sign(I, J)$ be the sign of the permutation $\{i_1, i_2, ..., i_k, j_1, j_2, ..., j_{p-k}\}$. Therefore from the above computation we have

$$\begin{aligned}
T_X(\omega f)(\gamma) &= \sum_{i_1<i_2<\ldots<i_k} sign(I,J)(\omega_I f)(\gamma) r^*(dx_{j_1}\wedge dx_{j_2}\wedge \ldots \wedge dx_{j_{p-k}}) \\
&= \sum_{i_1<i_2<\ldots<i_k} sign(I,J) \sum_{\gamma'\in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma)\omega_I(\gamma'^{-1}) r^*(dx_{j_1}\wedge dx_{j_2}\wedge \ldots \wedge dx_{j_{p-k}}) \\
&= \sum_{\gamma'\in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma) \sum_{i_1<i_2<\ldots<i_k} sign(I,J)\omega_I(\gamma'^{-1}) r^*(dx_{j_1}\wedge dx_{j_2}\wedge \ldots \wedge dx_{j_{p-k}}) \\
&= \sum_{\gamma'\in \mathcal{G}^X_{r(\gamma)}} f(\gamma'\gamma) T_X(\omega)(\gamma'^{-1}) \\
&= (T_X(\omega))f(\gamma)
\end{aligned}$$

which proves that T_X is A^X_X-linear.

To check properties (i),(ii), and (iii) in DefinitionA.1 for T_X, we first compute the adjoint of T_X with respect to the inner product given by 5.2.1.

We have for $\omega_1,\omega_2 \in C^\infty_c(\mathcal{G}_X, r^*(\bigwedge^k T * \mathcal{F}))$, we have $< T_X\omega_1, T_X\omega_2 >=< \omega_1,\omega_2 >$ and $T_X(T_X\omega) = (-1)^{k(n-k)}\omega$ for $\omega \in \mathcal{E}^k_c$. Therefore we have,

$$\begin{aligned}
< T_X\omega_1, \omega_2 > &= (-1)^{k(n-k)} < T_X\omega_1, T_X(T_X\omega_2) > \\
&= (-1)^{k(n-k)} < \omega_1, T_X\omega_2 > \\
&= < \omega_1, (-1)^{k(n-k)} T_X\omega_2 >
\end{aligned}$$

Hence we get $T^*_X = (-1)^{k(n-k)}T_X$ on $\mathcal{E}^k_c$. Therefore T_X extends to an adjointable operator on $\mathcal{E}_k$ which satisfies (i) of Definition A.1.

We define the adjoint of $\tilde{d}$ as the operator $\tilde{\delta} : \mathcal{E}^i_c \to \mathcal{E}^{i-1}_c$ by the formula

$$\tilde{\delta} := (-1)^{p(i+1)+1} T_X \tilde{d} T_X$$

Then $\tilde{\delta}$ extends to a closed densely defined unbounded $\mathcal{A}^X_X$-linear operator $\delta_X : \mathcal{E}_\bullet \to \mathcal{E}_{\bullet-1}$.

A straightforward calculation shows that $T_X\tilde{\delta} = (-1)^k\tilde{d}T_X$ on k-forms, and therefore we also have $T_X\delta_X = (-1)^k d_X T_X$ on k-forms. This shows that condition (ii) in Definition A.1 is satisfied.

To see that the third condition is verified, i.e. T_X induces an isomorphism $(T_X)_* : H^*(E,b) \to H^*(E,b^*)$, we first note that due to condition (ii) the map T_X takes $Im(b^*_i)$ to $Im(b_{n-i+1})$ and therefore the induced map $(T_X)_*$ is well-defined.

• $(T_X)_*$ is injective: Let $z \in \mathcal{E}_k$ such that $[T_X z] = 0 \in H^{n-k}(E,b^*)$. Then there exists a sequence $(z_n)_{n\geq 0} \in \mathcal{E}_{n-k}$ such that $T_X z = lim_{n\to\infty} b^* z_n$. Then we have

$$z = \pm T_X(lim_{n\to\infty} b^* z_n) = lim_{n\to\infty} \pm T_X(b^* z_n) = lim_{n\to\infty} b(\pm T_X z_n)$$

Thus $z \in \overline{Im(b)} \Rightarrow [z] = 0 \in H^k(E,b)$. Therefore $(T_X)_*$ is injective. Surjectivity of $(T_X)_*$ follows easily from surjectivity of T_X. Hence $(T_X)_*$ is an isomorphism.

Finally, to check condition (iv) in Definition A.1 we remark that $\tilde{d}+\tilde{\delta}$ is an elliptic operator and therefore extends to a regular Fredholm operator on the Hilbert module , and the extension of $\tilde{d}+\tilde{\delta}$ coincides with

$d_X + \delta_X$(cf. [Va:01],[BePi:08]). Moreover, since $(\tilde{d} + \tilde{\delta} \pm i)^{-1}$ is a pseudo-differential $\mathcal{G}$-operator of negative order, its extension to the Hilbert module is a compact operator. This extension coincides again with $(d_X + \delta_X \pm i)^{-1}$.

Let the Laplacian on the Hilbert-module be defined as $\Delta_X = (d_X + \delta_X)^2 = d_X\delta_X + \delta_X d_X$. Then we have

Proposition 5.2.1. *Let ϕ be a Schwartz function on $\mathbb{R}$ with a compactly supported Fourier transform such that $\phi(0) = 1$. Then on smooth compactly supported forms, we have $Im(\phi(\Delta_X)) \subseteq Dom(d_X)$, and*

$$\phi(\Delta_X)d_X = d_X\phi(\Delta_X)$$

$$\phi(\Delta_X)\delta_X = \delta_X\phi(\Delta_X)$$

Further, $\phi(\Delta_X)$ induces the identity map on cohomology of the Hilbert-Poincaré complex associated to the foliation.

Proof. (i) As ϕ has compactly supported Fourier transform, it takes compactly supported forms to compactly supported forms. Therefore, $Im(\phi(\Delta_X)) \subseteq Dom(d_X)$. Furthermore, since the Fourier transform of a compactly supported smooth function is an entire function whose restriction to $\mathbb{R}$ is Schwartz, and the square of the Fourier transform operator is a constant times identity, we get that ϕ is entire. Then, following the arguments in cf. [HeLa:90], we consider the holomorphic functional calculus for the self-adjoint regular operator Δ_X, which makes sense as the resolvent map $z \mapsto (zI - \Delta_X)^{-1}$ is analytic on the resolvent of Δ_X in $\mathbb{C}$ (cf. Result 5.23 in [Ku:97]). Therefore, choosing a curve γ in $\mathbb{C}$ that does not intersect $\mathbb{R}^+$ and surrounds it, as in cf. [HeLa:90], one can write

$$\phi(\Delta_X) = \frac{1}{2\pi i}\int_\gamma \phi(z)(zI - \Delta_X)^{-1}dz$$

Now, we have $d_X\Delta_X = \Delta_X d_X$ on $\mathcal{E}_c^k$. Therefore for $z \in \mathbb{C}$ in the resolvent of Δ_X, we have $d_X(zI - \Delta_X) = (zI - \Delta_X)d_X$ which in turn implies that $(zI - \Delta_X)^{-1}d_X = d_X(zI - \Delta_X)^{-1}$ and thus $\phi(\Delta_X)d_X = d_X\phi(\Delta_X)$. Similar arguments show that $\phi(\Delta_X)\delta_X = \delta_X\phi(\Delta_X)$

(ii) Now to show that $\phi(\Delta_X)$ induces the identity map on cohomology we proceed as follows. As ϕ is entire with $\phi(0) = 1$, the function ψ given by $\psi(x) = \frac{\phi(x)-1}{x}$ is also entire and in particular smooth on $\mathbb{R}$. Using the facts that Schwartz functions are dense in the space of smooth bounded functions and Schwartz functions with compactly supported Fourier transform are dense in the Schwartz space, one can find a sequence of Schwartz functions with compactly supported Fourier transforms $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_n \xrightarrow{n\to\infty} \psi$ in the $||.||_\infty$ norm. Consequently, $\alpha_n(\Delta_X) \xrightarrow{n\to\infty} \psi(\Delta_X)$ in the strong operator topology and we also have $\alpha_n(\Delta_X)d_X = d_X\alpha_n(\Delta_X)$. So if $v \in \mathcal{E}_c^k$, we get

$$\alpha_n(\Delta_X)v \xrightarrow{n\to\infty} \psi(\Delta_X)v$$

and

$$d_X(\alpha_n(\Delta_X)v) = \alpha_n(\Delta_X)(d_X v) \xrightarrow{n\to\infty} \psi(\Delta_X)(d_X v)$$

Therefore $\psi(\Delta_X)v \in Dom(d_X)$ and $\psi(\Delta_X)d_X = d_X\psi(\Delta_X)$ on $\mathcal{E}_c^k$ for $k = 0, 1, 2, ..., p$.

(iii) Now let $\omega \in Ker(d_X)$. Then there exists a sequence $(\omega_n)_{n\geq 0}$ such that each ω_n is a compactly supported smooth form, $\omega_n \xrightarrow{n\to\infty} \omega$, and $d_X\omega_n \to 0$. We then have,

$$\begin{aligned}
\phi(\Delta_X)\omega_n - \omega_n &= \psi(\Delta_X)\Delta_X(\omega_n) \\
&= \psi(\Delta_X)(d_X\delta_X\omega_n) + \psi(\Delta_X)(\delta_X d_X\omega_n) \text{ (well-defined since } Im(d_X|_{\mathcal{E}_c^k}) \subseteq \mathcal{E}_c^{k+1},\ Im(\delta_X|_{\mathcal{E}_c^k}) \subseteq \mathcal{E}_c^{k-1}) \\
&= d(\psi(\Delta_X)\delta_X\omega_n) + \psi(\Delta_X)\delta_X(d_X\omega_n) \text{ (by (ii) above)} \qquad (5.2.3)
\end{aligned}$$

However, on compactly supported smooth forms we have

$$\begin{aligned}
\psi(\Delta_X)\circ\delta_X &= \psi(\Delta_X)[(I+\Delta_X)(I+\Delta_X)^{-1}]\delta_X \\
&= [\psi(\Delta_X)(I+\Delta_X)][(I+\Delta_X)^{-1}\delta_X] \text{ (since } Im(I+\Delta_X)^{-1} \subseteq Dom(I+\Delta_X)) \\
&= [\psi(\Delta_X)+\phi(\Delta_X)-I][(I+\Delta_X)^{-1}\delta_X]
\end{aligned}$$

But $\psi(\Delta_X)+\phi(\Delta_X)-I$ is clearly bounded as ϕ and ψ are bounded smooth functions and $(I+\Delta_X)^{-1}\delta_X$ is bounded because it is a pseudo-differential operator of negative order. Hence $\psi(\Delta_X)\circ\delta_X$ is a bounded adjointable operator.

Hence, we get

$$\psi(\Delta_X)\delta_X(\omega_n) \xrightarrow{n\to\infty} \psi(\Delta_X)(\delta_X\omega)$$

and

$$\psi(\Delta_X)\delta_X(d\omega_n) \xrightarrow{n\to\infty} 0$$

Hence by 5.2.3, we get

$$\psi(\Delta_X)\delta_X(\omega_n) \xrightarrow{n\to\infty} \psi(\Delta_X)(\delta_X\omega)$$

and

$$d(\psi(\Delta_X)\delta_X\omega_n)) = \phi(\Delta_X)\omega_n - \omega_n - \psi(\Delta_X)\delta_X(d_X\omega_n) \xrightarrow{n\to\infty} \phi(\Delta)\omega - \omega$$

Thus the above two limits together imply $\psi(\Delta_X)\delta_X\omega \in Dom(d_X)$, and $\phi(\Delta_X)\omega - \omega = d_X(\psi(\Delta)\delta\omega) \subseteq Im(d_X)$. So $\phi(\Delta_X)\omega - \omega = 0$ on cohomology and $\phi(\Delta_X)$ is the identity map on cohomology.

□

Proposition 5.2.2. *The closed unbounded operators d_X and δ_X are regular operators.*

Proof. The only thing that one needs to check is that the operators $1+d_X\delta_X$ and $1+\delta_X d_X$ are surjective. We will show that $(1+d_X\delta_X)(1+\delta_X d_X)$ on $Dom(\Delta_X) = Dom(d_X\delta_X)\cap Dom(\delta_X d_X)$ is well-defined and we have $(1+\Delta_X) = (1+d_X\delta_X)(1+\delta_X d_X)$. Then the surjectivity of $(1+d_X\delta_X)$ will follow from the surjectivity of $(1+\Delta_X)$, since Δ_X is a regular operator.

Let $\Delta = \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}$ on $\mathcal{E}_c^k$. Then Δ extends to Δ_X and we have on $\mathcal{E}_c^k$:

$$(1+\tilde{d}\tilde{\delta})(1+\tilde{\delta}\tilde{d}) = (1+\tilde{d}\tilde{\delta}+\tilde{\delta}\tilde{d}) = (1+\Delta)$$

Now let $z \in Dom(\Delta_X)$. Then there exists a sequence $(z_n)_{n\geq 0}$ such that $z_n \in \mathcal{E}_c^k$ and we have

$$z_n \xrightarrow{n\to\infty} z, \text{ and } (1+\Delta)z_n \xrightarrow{n\to\infty} a \in \mathcal{E}$$

But on compactly supported smooth forms we have

$$(1+\Delta)z_n = (1+\tilde{d}\tilde{\delta})(1+\tilde{\delta}\tilde{d})z_n$$

Since $z \in Dom(\delta_X d_X)$ and $\overline{\overline{\tilde{\delta}}}\ \overline{\tilde{d}} = \overline{\tilde{\delta}\tilde{d}}$,the sequence $(1+\tilde{\delta}\tilde{d})z_n$ by definition converges to $(1+\delta_X d_X)z$. This in turn implies that $(1+\delta_X d_X)z$ is in the domain of $(1+d_X\delta_X)$ and $(1+d_X\delta_X)(1+\delta_X d_X)z = a = (1+\Delta_X)z$. Hence $(1+d_X\delta_X)$ is surjective and thus d_X is regular. Since $\delta_X = d_X^*$, By Corollary 9.6 of [La:95] δ_X is also regular.

□

5.3 Homotopy equivalence of HP-complexes on foliations

Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be closed foliated manifolds with complete transversals X and X', respectively, and $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a leafwise smooth homotopy equivalence with leafwise homotopy inverse $g : (V', \mathcal{F}') \to (V, \mathcal{F})$. We now let $E := \bigwedge^* T^*_{\mathbb{C}}\mathcal{F}$ and $E' := \bigwedge^* T^*_{\mathbb{C}}\mathcal{F}'$. We will use the notations from previous sections.

Let $x' \in V'$ and $L'_{x'}$ be the leaf through x'. Let $d'_{x'}$ denote the exterior derivative in $L'_{x'}$ and $\tilde{d}'_{x'}$ its lift to $\mathcal{G}'_{x'}$. Similarly, let d_x be the exterior differential on L_x for $x \in V$ and $\tilde{d}^f_{x'}$ denote its lift to $\mathcal{G}^V_{x'}(f) := \{(v, \gamma') \in V \times \mathcal{G}_{x'} | r(\gamma') = f(v)\}$. We note that for $h \in C^\infty_c(\mathcal{G}'_{x'})$, $\tilde{d}'_{x'}h(\gamma') \in T^*_{r(\gamma')}\mathcal{F}'^*$ is given for $X' \in T_{r(\gamma')}L'_{x'}$ as follows:

$$(\tilde{d}_{x'}h)_{\gamma'}(X'_{r(\gamma')}) := (\tilde{X}'h)(\gamma')$$

where $\tilde{X}'$ is the unique horizontal lift of X' on $\mathcal{G}'_{x'}$, i.e. $\tilde{X}'$ is such that $r_{*,\gamma'}\tilde{X}'_{\gamma'} = X'_{r(\gamma')}$. Similarly for $X \in T_v(f^{-1}(L'_{x'}))$ we define $\tilde{X} \in T_{(v,\gamma')}\mathcal{G}^V_{X'}(f)$ as the unique lift of X (via π_1) such that

$$(\pi_1)_{*,(v,\gamma')}\tilde{X}_{(v,\gamma')} = X_{\pi_1(v,\gamma')} = X_v$$

and for $u \in C^\infty_c(\mathcal{G}^V_{x'}(f), \pi_1^*E)$, $\tilde{d}^f_{x'}u(v,\gamma') \in T^*_v\mathcal{F}$ is given by

$$(\tilde{d}^f_{x'}u)_{(v,\gamma')}(X_v) := (\tilde{X}u)(v,\gamma')$$

Definition We define a map $\Psi_f : C^\infty_c(\mathcal{G}'_{X'}, r^*E') \to C^\infty_c(\mathcal{G}^V_{X'}(f), \pi_1^*E)$ by the following formula:

$$\Psi_f(\omega')(v,\gamma') = ({}^t f_{*v})[(\pi_2^!\omega')(v,\gamma')] \text{ for } \omega' \in C^\infty_c(\mathcal{G}'_{X'}, r^*E'), (v,\gamma') \in \mathcal{G}^V_{X'}(f)$$

where

• ${}^t f_{*v} : \bigwedge^* T^*_{f(v)}\mathcal{F}' \to \bigwedge^* T^*_v\mathcal{F}$ is the transpose of the pushout map $f_{*,v} : T_v\mathcal{F} \to T_{f(v)}\mathcal{F}'$ and is given by the formula:

$${}^t f_{*v}(\alpha'_{f(v)}) = (f^*\alpha')_v$$

with f^* being the pullback of differential forms via f.

• $\pi_2^!$ is the pullback via π_2 of elements of $C^\infty_c(\mathcal{G}'_{X'}, r^*E')$ given by $(\pi_2^!\omega')(v,\gamma') = \omega'(\gamma') \in E'_{r(\gamma')=f(v)}$. So ${}^t f_{*v}[(\pi_2^!\omega')(v,\gamma')] \in E_v = (\pi_1^*E)_{(v,\gamma')}$ and the map Ψ_f is well-defined.

Proposition 5.3.1. *We have the following properties:*

1. for $\alpha' \in C^\infty_c(L'_{x'}, E')$, $\pi_1^!(f^*\alpha') = \Psi_f(r^!\alpha')$

2. for $h \in C^\infty_c(\mathcal{G}'_{x'}), \tilde{d}^f_{x'}(\pi_2^!h) = \Psi_f(\tilde{d}'_{x'}h)$

3. $\tilde{d}^f_{x'} \circ \Psi_f = \Psi_f \circ \tilde{d}'_{x'}$ on $C^\infty_c(\mathcal{G}'_{X'}, r^*E')$.

where we have denoted pullbacks via π_1 and r by $\pi_1^!$ and $r^!$ respectively.

Proof. For $(v,\gamma')\in\mathcal{G}^V_{X'}(f)$,

$$f\circ\pi_1(v,\gamma')=f(v)=r(\gamma')=r\circ\pi_2(v,\gamma') \tag{5.3.1}$$

1. We compute as follows:

$$\begin{array}{rcl}
\pi_1^!(f^*\alpha')(v,\gamma') & = & (f^*\alpha')(v)\\
 & = & {}^t f_{*_v}(\alpha'_{f(v)})\\
 & = & {}^t f_{*_v}[\alpha'_{(f\circ\pi_1)(v,\gamma')}]\\
 & = & {}^t f_{*_v}[(f\circ\pi_1)^!(\alpha'_{(v,\gamma')})]\\
 & = & {}^t f_{*_v}[(r\circ\pi_2)^!(\alpha'_{(v,\gamma')})] \quad \text{(by 5.3.1)}\\
 & = & {}^t f_{*_v}[\pi_2^!\circ r^!(\alpha'_{(v,\gamma')})]\\
 & = & \Psi_f(r^!\alpha')(v,\gamma')
\end{array}$$

2. We have $r_{*,\gamma'}(\widetilde{f_{*,v}X})=(f_{*,v}X)_{r(\gamma')}$ and

$$\begin{aligned}(r_{*,\gamma'}[(\pi_2)_{*,(v,\gamma')}]\tilde{X})_{r(\gamma')})&=((r\circ\pi_2)_{*,(v,\gamma')}\tilde{X})_{r(\gamma')}\\
&=((f\circ\pi_1)_{*,(v,\gamma')}\tilde{X})_{r(\gamma')}=(f_{*,v}(\pi_1)_{*,(v,\gamma')}\tilde{X})_{r(\gamma')}=(f_{*,v}X)_{r(\gamma')}\end{aligned}$$

But as r is an immersion we get $(\widetilde{f_{*,v}X})_{\gamma'}=(\pi_2)_{*,(v,\gamma')}\tilde{X}$

$$\begin{array}{rcl}
<\Psi_f(\tilde{d}_{x'}h)(v,\gamma'),X> & = & <{}^t f_{*_v}[(\tilde{d}_{x'}h)(\gamma')],X>\\
 & = & <(\tilde{d}_{x'}h)(\gamma'),f_{*,v}X>\\
 & = & (\widetilde{f_{*,v}X})_{\gamma'}h(\gamma')\\
 & = & ((\pi_2)_{*,(v,\gamma')}\tilde{X})_{\gamma'}h(\gamma')=\tilde{X}\pi_2^!h(v,\gamma')\\
 & = & <\tilde{d}^f_{x'}\pi_2^!h(v,\gamma'),X>
\end{array} \tag{5.3.2}$$

Hence $\tilde{d}^f_{x'}(\pi_2^!h)=\Psi_f(\tilde{d}'_{x'}h)$

3. First, we note that it is easy to verify the following two properties:

$$\Psi_f(\omega\wedge\alpha)=\Psi_f(\omega)\wedge\Psi_f(\alpha)\text{ and }\Psi_f(h)=\pi_2^!h$$

for $\omega,\alpha\in C_c^\infty(\mathcal{G}'_{X'},r^*E')$ and $h\in C_c^\infty(\mathcal{G}'_{X'})$. Now, let $\alpha'\in C^\infty(L'_{x'},E')$. Then $r^!\alpha'\in C^\infty(\mathcal{G}'_{x'},r^*E')$ and we have,

$$\begin{array}{rcl}
(\tilde{d}^f_{x'}\circ\Psi_f)(r^!\alpha') & = & \tilde{d}^f_{x'}[\pi_1^!(f^*\alpha')]\\
 & = & \pi_1^!(d(f^*\alpha'))\\
 & = & \pi_1^!(f^*(d\alpha'))\\
 & = & \Psi_f(r^!(d\alpha'))\\
 & = & \Psi_f(\tilde{d}_{x'}r^!\alpha'))\\
 & = & (\Psi_f\circ\tilde{d}_{x'})(r^!\alpha')
\end{array}$$

Now for any section $s\in C_c^\infty(\mathcal{G}_{x'},r^*E')$, we can write s in the following form:

$$s=\sum_i h_i\ r^!\alpha'_i$$

where $h_i \in C_c^\infty(\mathcal{G}'_{x'}), \alpha'_i \in C_c^\infty(L'_{x'}, E')$. Then we have

$$\begin{aligned}(\tilde{d}^f_{x'} \circ \Psi_f)(\sum_i h_i r^! \alpha'_i) &= \tilde{d}^f_{x'}[\sum_i \pi_2^! h_i (\Psi_f r^! \alpha')] \\ &= \sum_i \tilde{d}^f_{x'} \pi_2^! h_i \wedge \Psi_f r^! \alpha'_i + \pi_2^! h_i \tilde{d}^f_{x'} \Psi_f r^! \alpha'_i \\ &= \sum_i \tilde{d}^f_{x'} \pi_2^! h_i \wedge \Psi_f r^! \alpha'_i + \pi_2^! h_i \Psi_f (\tilde{d}_{x'} r^! \alpha'_i) \end{aligned} \quad (5.3.3)$$

We also have,

$$\begin{aligned}\Psi_f \tilde{d}_{x'}(\sum_i h_i r^! \alpha'_i) &= \Psi_f[\sum_i \tilde{d}_{x'} h_i \wedge r^! \alpha'_i + h_i \tilde{d}_{x'} r^! \alpha'_i] \\ &= \sum_i \Psi_f(\tilde{d}_{x'} h_i \wedge r^! \alpha'_i) + \Psi_f(h_i \tilde{d}_{x'} r^! \alpha'_i) \\ &= \sum_i \tilde{d}^f_{x'} \pi_2^! h_i \wedge \Psi_f r^! \alpha'_i + \pi_2^! h_i \Psi_f(\tilde{d}_{x'} r^! \alpha'_i)\end{aligned} \quad (5.3.4)$$

Hence from 5.3.3 and 5.3.4 we get the desired result. □

We now define a map $\Phi_f : C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^* E) \to C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f), (r \circ pr_1)^* E)$ as follows:

$$\Phi_f(\xi)[\gamma; (s(\gamma), \gamma')] = \xi(r(\gamma), \breve{f}(\gamma)\gamma') \text{ for } \xi \in C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^* E) \quad (5.3.5)$$

Proposition 5.3.2. *Φ_f is a chain map.*

Proof. The proof is technical but straightforward. We refer the reader to [BeRo:10] for the details.

□

Notice that $\Phi_f(\xi)[\gamma; (s(\gamma), \gamma')] \in E_{r(\gamma)}$. Now the map $\Psi_f : C_c^\infty(\mathcal{G}'_{X'}, r^* E') \to C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^* E)$ defined for f in can also be written for the map g, giving a map

$$\Psi_g : C_c^\infty(\mathcal{G}_X, r^* E) \to C_c^\infty(\mathcal{G}^{V'}_X(g), \pi_1^* E')$$

So,

$$\Psi_g(\omega)(v', \gamma) = ({}^t g_{*_{v'}})[(\pi_2^! \omega)(v', \gamma)] \text{ for } \omega \in C_c^\infty(\mathcal{G}_X, r^* E), (v', \gamma) \in \mathcal{G}^{V'}_X(g)$$

We consider the map

$$\rho : \mathcal{G}'_{X'} \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}^{X'}_X(g) \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f) \to \mathcal{G}^{V'}_X(g) \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f)$$

given by

$$[[\gamma'_1; (s(\gamma'_1), \gamma_1)]; (s(\gamma_1), \gamma'_2)] \mapsto [(r(\gamma'_1), \breve{g}(\gamma'_1)\gamma_1); (s(\gamma_1), \gamma'_2)]$$

Then, by Proposition 5.1.7 ρ is a diffeomorphism. We define the tensor product $\Psi_g \otimes I$:

$$\Psi_g \otimes I : C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f), (r \circ \pi_1)^* E) \to C_c^\infty(\mathcal{G}^{V'}_X(g) \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f))$$

as

$$(\Psi_g \otimes I)(\alpha)[(v', \gamma); (s(\gamma), \gamma')] = {}^t g_{*,v'}(\alpha[\gamma; (s(\gamma), \gamma')]) \text{ for } \alpha \in C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}^X_{X'}(f), (r \circ pr_1)^* E)$$

With the identification ρ given above, this can also be written as:

$$(\Psi_g \otimes I)(\alpha)(\rho)[[\gamma'_1; (s(\gamma'_1), \gamma_1)]; (s(\gamma_1), \gamma'_2)] = {}^t g_{*, r(\gamma'_1)}(\alpha[\breve{g}(\gamma'_1)\gamma_1; (s(\gamma_1), \gamma'_2)])$$

Then the following formula holds for the composite map $(\Psi_g \otimes I) \circ \Phi_f$:

$$(\Psi_g \otimes I)(\Phi_f \xi)[(v', \gamma); (s(\gamma), \gamma')] = {}^t g_{*,v'}(\xi(r(\gamma), \breve{f}(\gamma)\gamma')) \in E'_{v'} \text{ for } \xi \in C_c^\infty(\mathcal{G}_{X'}^{V}(f), \pi_1^* E)$$

Now consider the diffeomorphism (the proof of this is analogous to the one given in 5.1.4), $\lambda : \mathcal{G}_X^{V'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to \mathcal{G}_X^{V'}(f \circ g)$ given by

$$\lambda[(v', \gamma); (s(\gamma), \gamma')] = (v', \breve{f}(\gamma)\gamma')$$

Then λ induces a map $\Lambda : C_c^\infty(\mathcal{G}_{X'}^{V'}(f \circ g), \pi_1'^* E') \to C_c^\infty(\mathcal{G}_X^{V'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (\pi_1' \circ pr_1)^* E')$ given by:

$$(\Lambda\alpha')[(v', \gamma); (s(\gamma), \gamma')] := \alpha'(v', \breve{f}(\gamma)\gamma') \in E'_{v'} \text{ for } \alpha' \in C_c^\infty(\mathcal{G}_{X'}^{V'}(f \circ g), \pi_1'^* E')$$

Then Λ induces an isometry at the level of Hilbert modules.

Let $\Psi_{f\circ g} : C_c^\infty(\mathcal{G}'_{X'}, r^* E') \to C_c^\infty(\mathcal{G}_{X'}^{V'}(f \circ g), \pi_1'^* E')$ be defined analogously as Ψ_f, replacing f by $f \circ g$. We then have the following

Proposition 5.3.3. *The following diagram is commutative:*

$$\begin{tikzcd}
C_c^\infty(\mathcal{G}'_{X'}, r^* E') \arrow[r, "\Psi_{f\circ g}"] \arrow[d, "\Psi_f"] & C_c^\infty(\mathcal{G}_{X'}^{V'}(f \circ g), \pi_1'^* E') \arrow[d, "\Lambda"] \\
C_c^\infty(\mathcal{G}_{X'}^{V}(f), \pi_1'^* E) \arrow[r, "\Lambda'"] & C_c^\infty(\mathcal{G}_X^{V'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (\pi_1' \circ pr_1)^* E')
\end{tikzcd}$$

i.e., $\Lambda' \circ \Psi_f = \Lambda \circ \Psi_{f\circ g}$, *where* $\Lambda' := (\Psi_g \otimes I) \circ \Phi_f$. .

Proof. Let $\beta' \in C_c^\infty(\mathcal{G}_{x'}, r^* E')$. Then we have

$$\begin{aligned}
(\Lambda' \circ \Psi_f)(\beta')[(v', \gamma); (s(\gamma), \gamma')] &= ({}^t g_{*,v'})[\Psi_f(\beta'(r(\gamma), \breve{f}(\gamma) \circ \gamma')] \\
&= ({}^t g_{*,v'})[{}^t f_{*,r(\gamma)}(\beta'(\breve{f}(\gamma)\gamma'))] \\
&= ({}^t f_{*,g(v')} \circ g_{*,v'})(\beta'(\breve{f}(\gamma)\gamma'))] \\
&= ({}^t f_{*,g(v')} \circ g_{*,v'})(\beta'(\breve{f}(\gamma)\gamma'))] \\
&= ({}^t (f \circ g)_{*,v'})(\beta'(\breve{f}(\gamma)\gamma'))]
\end{aligned} \tag{5.3.6}$$

We also have,

$$\begin{aligned}
(\Lambda \circ \Psi_{f\circ g})(\beta')[(v, \gamma); (s(\gamma), \gamma')] &= \Psi_{f\circ g}(v', \breve{f}(\gamma)\gamma') \\
&= ({}^t (f \circ g)_{*,v'})(\beta'(\breve{f}(\gamma)\gamma'))]
\end{aligned} \tag{5.3.7}$$

So we get the statement of the proposition from 5.3.6 and 5.3.7. □

Now, let $\delta : \mathcal{G}_X^{X'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \to \mathcal{G}_{X'}^{X'}(f \circ g)$ be the diffeomorphism as in Proposition 5.1.4. It induces an isomorphism

$$\Omega : C_c^\infty(\mathcal{G}_{X'}^{X'}(f \circ g)) \to C_c^\infty(\mathcal{G}_X^{X'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)) \text{ given by } \Omega(\xi) = \delta^! \xi = \xi \circ \delta$$

We make the identification

$$\zeta : \mathcal{G}'_{X'} \times_{\mathcal{G}'^{X'}_{X'}} [\mathcal{G}_X^{X'}(g) \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f)] \xrightarrow{\cong} \mathcal{G}'_{X'} \times_{\mathcal{G}_{X'}^{X'}} \mathcal{G}_{X'}^{X'}(f \circ g)$$

given by

$$\zeta[\gamma_1'; [(s(\gamma'), \gamma); (s(\gamma), \gamma')]] = [\gamma_1'; (s(\gamma_1'), \breve{f}(\gamma)\gamma')]$$

The map ζ induces the operator

$$I \otimes \Omega : C_c^\infty(\mathcal{G}'_{X'}, \pi_1'^* E') \otimes_{C\infty_c(\mathcal{G}'^{X'}_{X'})} C_c^\infty(\mathcal{G}^{X'}_{X'}(f \circ g)) \to C_c^\infty(\mathcal{G}'_{X'}, \pi_1'^* E') \otimes_{C_c^\infty(\mathcal{G}'^{X'}_{X'})} C_c^\infty(\mathcal{G}^{X'}_X(g) \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f))$$

.

Proposition 5.3.4. *The operator $I \otimes \Omega$ is well-defined, i.e. the following property holds:*

$$\Omega(\pi(\phi)\xi) = \pi(\phi)\Omega(\xi) \text{ for } \phi \in C_c^\infty(\mathcal{G}'^{X'}_{X'}), \xi \in C_c^\infty(\mathcal{G}^{X'}_{X'}(f \circ g))$$

Proof. Computing the left hand side first:

$$\begin{aligned}
\Omega(\pi(\phi)\xi)[(x', \gamma); (s(\gamma), \gamma')] &= [\pi(\phi)\xi](x', \breve{f}(\gamma)\gamma') \\
&= \sum_{\alpha' \in \mathcal{G}'^{x'}_{X'}} \phi(\alpha')\xi(s(\alpha'), \breve{f}(\breve{g}(\alpha'^{-1}))\gamma)\gamma')
\end{aligned}$$

Now computing the right hand side:

$$\begin{aligned}
\pi(\phi)(\Omega\xi)[(x', \gamma); (s(\gamma), \gamma')] &= \sum_{\alpha' \in \mathcal{G}'^{x'}_{X'}} \phi(\alpha')(\Omega\xi)[(s(\alpha'), \breve{g}(\alpha'^{-1})\gamma)\gamma'); (s(\gamma), \gamma')] \\
&= \sum_{\alpha' \in \mathcal{G}'^{x'}_{X'}} \phi(\alpha')\xi(s(\alpha'), \breve{f}(\breve{g}(\alpha'^{-1})\gamma)\gamma')
\end{aligned}$$

Thus from 5.3.8 and 5.3.8 we see that the operator $I \otimes \Omega$ is indeed well-defined. ☐

Using now the definition 5.3.5 of the map Φ_f for g, we get a map

$$\Phi_g : C_c^\infty(\mathcal{G}^{V'}_X(g), \pi_1'^* E') \to C_c^\infty(\mathcal{G}'_{X'} \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}'^{X'}_X(g), (r \circ pr_1)^* E')$$

as follows:

$$\Phi_g(\xi')[\gamma'; (s(\gamma'), \gamma)] = \xi'(r(\gamma'), \breve{g}(\gamma')\gamma) \text{ for } \xi' \in C_c^\infty(\mathcal{G}^{V'}_X(g), \pi_1'^* E')$$

Similarly we define the map $\Phi_{f\circ g}$. Then we have the following

Lemma 5.3.5. *The following diagram is commutative:*

$$\begin{tikzcd}
C_c^\infty(\mathcal{G}^{V'}_{X'}(f \circ g), \pi_1'^* E') \arrow[r, "\Lambda"] \arrow[d, "\Phi_{f\circ g}"] & C_c^\infty(\mathcal{G}^{V'}_X(g) \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f), (\pi_1' \circ pr_1)^* E') \arrow[d, "\Phi_g \otimes I"] \\
C_c^\infty(\mathcal{G}'_{X'} \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}'^{X'}_{X'}(f \circ g), (r \circ pr_1)^* E') \arrow[r, "I\otimes\Omega"] & C_c^\infty(\mathcal{G}'_{X'} \times_{\mathcal{G}'^{X'}_{X'}} \mathcal{G}'^{X'}_X(g) \times_{\mathcal{G}^X_X} \mathcal{G}^X_{X'}(f), (\pi_1' \circ pr_1)^* E')
\end{tikzcd}$$

i.e. we have $(\Phi_g \otimes I) \circ \Lambda = (I \otimes \Omega) \circ \Phi_{f\circ g}$.

Proof. We compute the left hand side as follows: Let $\alpha \in C_c^\infty(\mathcal{G}_{X'}^{V'}, \pi_1'^* E')$

$$\begin{aligned}((\Phi_g \otimes I) \circ \Lambda)(\alpha)[\gamma_1'; [(s(\gamma_1'), \gamma); (s(\gamma), \gamma')]] &= (\Lambda\alpha)[(r(\gamma_1'), \breve{g}(\gamma_1')\gamma)); (s(\gamma), \gamma')] \\ &= \alpha(r(\gamma_1'), \breve{f}(\breve{g}(\gamma_1')\gamma)\gamma')) \\ &= \alpha(r(\gamma_1'), (\breve{f} \circ \breve{g})(\gamma_1')\breve{f}(\gamma)\gamma'))\end{aligned}$$

Computing the right hand side, we get,

$$\begin{aligned}((I \otimes \Omega) \circ \Phi_{f\circ g})(\alpha)[\gamma_1'; [(s(\gamma_1'), \gamma); (s(\gamma), \gamma')]] &= (\Phi_{f\circ g}\alpha)[\gamma_1'; (s(\gamma_1'), \breve{f}(\gamma)\gamma')] \\ &= \alpha(r(\gamma_1'), (\breve{f} \circ \breve{g})(\gamma_1')\breve{f}(\gamma)\gamma')\end{aligned}$$

Thus from 5.3.8 and 5.3.8 we get the desired equality.

□

Now let $\Theta_f := \Phi_f \circ \Psi_f : C_c^\infty(\mathcal{G}_{X'}, r^*E') \to C_c^\infty(\mathcal{G}_X, r^*E) \otimes_{C_c^\infty(\mathcal{G}_X^X)} C_c^\infty(\mathcal{G}_{X'}^X(f))$. Similarly we define the maps Θ_g and $\Theta_{f\circ g}$.

Then from Proposition 5.3.3 and Lemma 5.3.5 we get the following

Theorem 5.3.6. *The following diagram is commutative:*

$$\begin{tikzcd} \mathcal{E}^{c,\infty}_{X',E'} \arrow[r, "\Theta_f"] \arrow[d, "\Theta_{f\circ g}"] & \mathcal{E}^{c,\infty}_{X,E} \otimes \mathcal{E}^{X,c,\infty}_{X'}(f) \arrow[d, "\Theta_g \otimes I"] \\ \mathcal{E}^{c,\infty}_{X',E'} \otimes \mathcal{E}^{X',c,\infty}_{X'}(f\circ g) \arrow[r, "I\otimes\Omega"] & \mathcal{E}^{c,\infty}_{X',E'} \otimes \mathcal{E}^{X',c,\infty}_{X}(g) \otimes \mathcal{E}^{X,c,\infty}_{X'}(f) \end{tikzcd}$$

i.e. we have

$(I_{\mathcal{E}^{c,\infty}_{X',E'}} \otimes \Omega) \circ \Theta_{f\circ g} = (\Theta_g \otimes I_{\mathcal{E}^{X,c,\infty}_{X'}(f)}) \circ \Theta_f$

where $\mathcal{E}^{c,\infty}_{X',E'} := C_c^\infty(\mathcal{G}_{X'}, r^*E')$, $\mathcal{E}^{X,c,\infty}_{X'}(f) := C_c^\infty(\mathcal{G}^X_{X'}(f))$ *and so on.*

Proof. We have from Lemma 5.3.5,

$$(I \otimes \Omega) \circ \Phi_{f\circ g} = (\Phi_g \otimes I) \circ \Lambda.$$

So we have, $(I \otimes \Omega) \circ \Phi_{f\circ g} \circ \Psi_{f\circ g} = (\Phi_g \otimes I) \circ \Lambda \circ \Psi_{f\circ g}$. Now using Proposition 5.3.3 and the definition for $\Theta_{f\circ g}$ we have

$$(I \otimes \Omega) \circ \Theta_{f\circ g} = (\Phi_g \otimes I) \circ (\Psi_g \otimes I) \circ \Phi_f \circ \Psi_f$$

Since $(\Phi_g \otimes I) \circ (\Psi_g \otimes I) = \Theta_g \otimes I$ and $\Theta_f = \Phi_f \circ \Psi_f$ we get the result.

□

5.3.1 Regularization and extension to adjointable operators on Hilbert-modules

In the previous section all computations were done in dense subspaces of the Hilbert modules. In this section we will extend each of the maps defined to the maximal completion of the pre-Hilbert modules.

We use the following notations:

- $\mathcal{A}_X^X := \overline{C_c^\infty(\mathcal{G}_X^X)}^{max}$
- $\mathcal{A}_{X'}^{X'} := \overline{C_c^\infty(\mathcal{G}'^{X'}_{X'})}^{max}$
- $\mathcal{E}_{X,E} := \overline{C_c^\infty(\mathcal{G}_X, r^*E)}^{<.,.>}$
- $\mathcal{E}_{X',E'} := \overline{C_c^\infty(\mathcal{G}'_{X'}, r^*E')}^{<.,.>}$
- $\mathcal{E}_X^{X'}(f) := \overline{C_c^\infty(\mathcal{G}_X^{X'}(f))}^{<.,.>}$
- $\mathcal{E}_{X'}^X(g) := \overline{C_c^\infty(\mathcal{G}_{X'}^X(g))}^{<.,.>}$
- $\mathcal{E}_{X',E}^V(f) := \overline{C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)}^{<.,.>}$
- $\mathcal{E}_{X,E'}^{V'}(g) := \overline{C_c^\infty(\mathcal{G}_X^{V'}(g), \pi_1'^*E')}^{<.,.>}$

where $\overline{\bullet}^{<.,.>}$ denotes the completion of a pre-hilbert module with respect to a suitable C^*-valued inner product and $\overline{\bullet}^{max}$ denotes the maximal completion of the C^*-algebra.

We will show the following results:

Proposition 5.3.7. *The maps Φ_f, Ω, Λ are isometric isomorphisms and therefore extend to adjointable operators of Hilbert modules.*

Proof. We have defined Φ_f as a map $\Phi_f : C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E) \to C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r \circ pr_1)^*E)$ as follows: for $\xi \in C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)$,

$$\Phi_f(\xi)[\gamma; (s(\gamma), \gamma')] = \xi(r(\gamma), \breve{f}(\gamma)\gamma')$$

Using the diffeomorphism $\phi_f : \mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f) \xrightarrow{\cong} \mathcal{G}_{X'}^V(f)$ we can see that $\Phi_f(\xi) = \xi \circ \phi_f$. Then the inner product on $C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r \circ pr_1)^*E)$ is defined as

$$< \xi_1, \xi_2 > := < \xi_1 \circ \phi_f, \xi_2 \circ \phi_f >_{C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)}$$

where we recall that the inner product on $C_c^\infty(\mathcal{G}_{X'}^V(f), \pi_1^*E)$ is given by the following formula:

$$< \xi, \eta > (\gamma') := \int_{v \in L_{r(\gamma')}} \sum_{\gamma_1' \in \mathcal{G}'^{f(v)}_{r(\gamma')}} < \xi(v, \gamma_1'), \eta(v, \gamma_1'\gamma) >_{E_v} d\lambda_{r(\gamma)}(v)$$

where $L_{r(\gamma')}$ is the leaf in V such that $f(L_{r(\gamma')}) = L'_{r(\gamma')}$.

Therefore Φ_f is an isometry. Clearly, Φ_f is also surjective with the inverse image given by $\xi := \eta \circ \phi_f^{-1}$. Therefore Φ_f is an isometric isomorphism of Hilbert modules.

Also, as in the proof of lemma 5.1.4, one defines a map $\omega : C_c^\infty(\mathcal{G}_X, r^*E) \otimes_{C_c^\infty(\mathcal{G}_X^X)} C_c^\infty(\mathcal{G}_{X'}^X(f)) \to C_c^\infty(\mathcal{G}_X \times_{\mathcal{G}_X^X} \mathcal{G}_{X'}^X(f), (r \circ pr_1)^*E)$

$$\omega(\xi \otimes \eta)[\gamma; (x, \gamma')] := \sum_{\alpha \in \mathcal{G}_X^{s(\gamma)}} \xi(\gamma\alpha)\eta(\alpha^{-1}x', \breve{f}(\alpha^{-1})\gamma')$$

We can follow the proof of 5.1.4 to prove that ω extends to an isometric isomorphism of Hilbert modules, and we have an isometric isomorphism $\Phi_f^{-1} \circ \omega : \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} \mathcal{E}_{X'}^X(f) \to \mathcal{E}_{X',E}^V(f)$.

In a similar way it can be shown that Λ and Ω are isometric isomorphisms of Hilbert modules and further there are isometric isomorphisms (by abusing notation) $\Omega : \mathcal{E}_{X'}^X(f) \otimes_{\mathcal{A}_{X'}^{X'}} \mathcal{E}_X^{X'}(g) \to \mathcal{E}_X^X(g \circ f)$ and $\Lambda : \mathcal{E}_{X',E'}^{V'}(f \circ g) \to \mathcal{E}_{X,E}^{V'}(g) \otimes_{\mathcal{A}_X^X} \mathcal{E}_{X'}^X(f)$.

Finally, we remark that any isometric isomorphism of Hilbert modules is adjointable with the adjoint map simply being its inverse. ☐

Proposition 5.3.8. *Let $\phi \in C_c^\infty(\mathbb{R})$ be such that $\phi(0) = 1$ and $\hat{\phi} \in C_c^\infty(\mathbb{R})$. We define*

$$\Psi_f^\phi := \Psi_f \circ \phi(\Delta')$$

where Δ' is the Laplacian. Ψ_f^ϕ extends to an adjointable map on the Hilbert module.

Proof. To see that $\Psi_f^\phi := \Psi_f \circ \phi(\Delta')$ extends to an adjointable operator on Hilbert modules, we compute its Schwartz kernel on the dense subspaces. Let k_ϕ denote the Schwartz kernel of $\phi(\Delta')$. Since ϕ is chosen such that it is rapidly decreasing and its Fourier transform is compactly supported, k_ϕ is smooth and is uniformly supported, i.e. it has compact support in $\mathcal{G}'$.

Let k_f denote the Schwartz kernel of Ψ_f on $C_c^\infty(\mathcal{G}'_{X'}, r^*E')$. So we have for $\omega' \in C_c^\infty(\mathcal{G}'_{X'}, r^*E')$,

$$\Psi_f(\omega')(v,\gamma') = \int_{\gamma'_1 \in \mathcal{G}'^{s(\gamma')}_{X'}} k_f((v,\gamma'),\gamma'_1)(\omega'(\gamma'_1))d\lambda(\gamma'_1) \tag{5.3.8}$$

Here $k_f((v,\gamma'),\gamma'_1) : E'_{r(\gamma'_1)} \to E_v$, i.e. $k_f((v,\gamma'),\gamma'_1) \in Hom(E'_{r(\gamma'_1)}, E_v)$.

Claim: $k_f((v,\gamma'),\gamma'_1) = {}^t f_{*,v} \circ \delta(\gamma',\gamma'_1)$, where $\delta(\gamma',\gamma'_1)$ is the distribution on $\mathcal{G}'_{X'} \times \mathcal{G}'_{X'}$ satisfying the formula:

$$\int \delta(\gamma',\gamma'_1)\omega(\gamma'_1) = \omega(\gamma')$$

The claim is clear from the following computation:

$$\begin{aligned}
\int_{\gamma'_1 \in \mathcal{G}'^{s(\gamma')}_{X'}} {}^t f_{*,v} \circ \delta(\gamma',\gamma'_1)(\omega'(\gamma'_1))d\lambda(\gamma'_1) &= {}^t f_{*,v}\left(\int_{\gamma'_1 \in \mathcal{G}'^{s(\gamma')}_{X'}} \delta(\gamma',\gamma'_1)(\omega'(\gamma'_1))d\lambda(\gamma'_1)\right) \\
&= {}^t f_{*,v}(\omega'(\gamma')) =: \Psi_f(\omega')(v,\gamma')
\end{aligned}$$

Therefore the Schwartz kernel of the operator Ψ_f^ϕ is the convolution of the kernels of Ψ_f and $\phi(\Delta')$. Let this Schwartz kernel be denoted by K_F. Therefore we have,

$$\begin{aligned}
K_F((v,\gamma'),\gamma'_1) &= \int_{\beta'} {}^t f_{*,v} \circ \delta(\gamma',\beta') \circ k_\phi(\beta',\gamma'_1)d\lambda(\beta') \\
&= {}^t f_{*,v} \circ k_\phi(\gamma',\gamma'_1) \in Hom(E'_{r(\gamma'_1)}, E_v)
\end{aligned} \tag{5.3.9}$$

Since k_ϕ is smooth with compact support in $\mathcal{G}'$, K_F also is smooth with compact support in $\mathcal{G}^V_{X'}(f)$. Hence Ψ_f^ϕ is a bounded operator and therefore adjointable as a map between the dense subspaces by classical arguments using the Riesz representation theorem. This is seen as follows: $K_F((v,\gamma'),\gamma'_1)$ acts as a bounded linear transformation between the inner product spaces $E'_{r(\gamma'_1)}$ and E_v. Now for a fixed $w \in E_v$ we define a bounded linear functional ψ_w on $E'_{r(\gamma'_1)}$ by $\psi_w(u) =< K_F((v,\gamma'),\gamma'_1)u, w >_{E_v}$. Then by the Riesz representation theorem there exists a unique $h \in E'_{r(\gamma'_1)}$ such that $\phi_w(u) =< u, h >_{E'_{r(\gamma'_1)}}$. Then we define the adjoint of the homomorphism $K_F((v,\gamma'),\gamma_1')$ as $K_F^*((v,\gamma'),\gamma_1')w = h$ which clearly satisfies $< K_F((v,\gamma'),\gamma_1')u, w >_{E_v} =< u, K_F^*((v,\gamma'),\gamma_1')w >_{E'_{r(\gamma_1'}}$.

Now, since an adjointable operator between pre-Hilbert modules extends by continuity to an adjointable operator on the Hilbert module, Ψ_f^ϕ extends to an adjointable operator $\Psi_f^\phi : \mathcal{E}_{X',E'} \to \mathcal{E}_{X',E}^V(f)$.

□

Now let us consider the leafwise homotopy $H : [0,1] \times V \to V$ and $H' : [0,1] \times V' \to V'$, and denote as before for $x \in V$ the homotopy class of the path $t \to H(t,x), 0 \le t \le s$ by γ_x^s. For $0 \le s \le 1$, let $H_s := H \circ i_s$, where $i_s : V \hookrightarrow [0,1] \times V$ is the map $i_s(v) = (s,v)$. Then we have the following

Proposition 5.3.9. *For all* $0 \le s \le 1$, $\mathcal{E}_X^X(H_s)$ *is isomorphic as a Hilbert module to* $\mathcal{A}_X^X$.

Proof. We define a map $\theta_H^s : C_c^\infty(\mathcal{G}_X^X) \to C_c^\infty(\mathcal{G}_X^X(H_s))$ by the following formula:

$$\theta_H^s(\xi)(x,\gamma) = \xi((\gamma_x^s)^{-1}\gamma) \text{ for } \xi \in C_c^\infty(\mathcal{G}_X^X)$$

• θ_H^s is $C_c^\infty(\mathcal{G}_X^X)$-linear:

Let $\phi \in C_c^\infty(\mathcal{G}_X^X)$. Then we have,

$$\begin{aligned}
\theta_H^s(\xi * \phi)(x,\gamma) &= (\xi * \phi)((\gamma_x^s)^{-1}\gamma) \\
&= \sum_{\gamma_1 \in \mathcal{G}_{r(\gamma)}^X} \xi(\gamma_1^{-1})\phi(\gamma_1(\gamma_x^s)^{-1}\gamma) \\
&= \sum_{\gamma_2 \in \mathcal{G}_{r(\gamma)}^X} \xi((\gamma_x^s)^{-1}\gamma_2^{-1})\phi(\gamma_2\gamma) \text{ (putting } \gamma_2 = \gamma_1(\gamma_x^s)^{-1} \text{)} \\
&= \sum_{\gamma_2 \in \mathcal{G}_{r(\gamma)}^X} (\theta_H\xi)(x,\gamma_2^{-1})\phi(\gamma_2\gamma) \\
&= \sum_{\gamma_3 \in \mathcal{G}_{s(\gamma)}^X} (\theta_H\xi)(x,\gamma\gamma_3^{-1})\phi(\gamma_3) \text{ (putting } \gamma_3 = \gamma_2\gamma \text{)} \\
&= [(\theta_H\xi)\phi](x,\gamma)
\end{aligned}$$

Thus θ_H^s is $C_c^\infty(\mathcal{G}_X^X)$-linear.

• θ_H^s is an isometry:

Let $\xi \in C_c^\infty(\mathcal{G}_X^X)$. Then we have,

$$\begin{aligned}
< \theta_H^s \xi, \theta_H^s \xi > (\gamma) &= \sum_{\gamma_1 \in \mathcal{G}_{r(\gamma)}^{H_s(X)}} \sum_{H_s(x)=r(\gamma_1)} \overline{(\theta_H^s \xi)(x,\gamma_1)} (\theta_H^s \xi)(x, \gamma_1 \gamma) \\
&= \sum_{\gamma_1 \in \mathcal{G}_{r(\gamma)}^{H_s(X)}} \sum_{H_s(x)=r(\gamma_1)} \overline{\xi((\gamma_x^s)^{-1}\gamma_1)} \xi((\gamma_x^s)^{-1}\gamma_1\gamma) \\
&= \sum_{\gamma_2 \in \mathcal{G}_{r(\gamma)}^{X}} \overline{\xi(\gamma_2)} \xi(\gamma_2 \gamma) \text{ (putting } \gamma_2 = (\gamma_x^s)^{-1}\gamma_1) \\
&= \sum_{\gamma_2 \in \mathcal{G}_{r(\gamma)}^{X}} \overline{\xi(\gamma_2)} \xi(\gamma_2 \gamma) \\
&= \sum_{\gamma_2 \in \mathcal{G}_{r(\gamma)}^{X}} \xi^*(\gamma_2^{-1}) \xi(\gamma_2 \gamma) \\
&= (\xi^* * \xi)(\gamma)
\end{aligned}$$

• θ_H^s has dense image:

Let $\eta \in C_c^\infty(\mathcal{G}_X^X(g \circ f))$. Then define for $\gamma \in \mathcal{G}_X^X$,

$$(\theta_H^s)^* \eta(\gamma) = \eta(r(\gamma), \gamma_{r(\gamma)}^s \gamma)$$

Thus $(\theta_H^s)^* \eta \in C_c^\infty(\mathcal{G}_X^X)$, and we have,

$$\theta_H^s((\theta_H^s)^*\eta)(x,\gamma) = ((\theta_H^s)^*\eta)((\gamma_x^s)^{-1}\gamma) = \eta(x, \gamma_x^s(\gamma_x^s)^{-1}\gamma) = \eta(x,\gamma)$$

Thus θ_H^s has dense image.

Therefore, θ_H^s is an isomorphism of Hilbert modules. □

Corollary 5.3.10. *$\mathcal{E}_{X'}^{X'}(g \circ f)$ is isomorphic to $\mathcal{A}_X^X$ as a Hilbert module.*

Proof. The isomorphism is given by $\theta_H := \theta_H^1$ as $H_1 = g \circ f$.

□

Corollary 5.3.11. *$\mathcal{E}_{X'}^{X'}(f \circ g)$ is isomorphic to $\mathcal{A}_{X'}^{X'}$ as a Hilbert module.*

Proof. We use the homotopy H' to define the isomorphism in a similar way as in the previous proposition. □

Remark. *We note that $\mathcal{G}_X^X(id_V) \xrightarrow{\pi_2} \mathcal{G}_X^X$ is a diffeomorphism, where π_2 is the projection onto the second factor. We also note that there is a canonical isomorphism of Hilbert modules $\delta : \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} \mathcal{A}_X^X \cong \mathcal{E}_{X,E}$*

Consider the Hilbert-Poincaré complex associated to the odd-dimensional foliation $(V, X, \mathcal{F}^p)$ given as follows:

Set $\mathcal{E}_c^k := C_c^\infty(\mathcal{G}_X, r^* \bigwedge^k T^*\mathcal{F})$. We denote the lift of the leafwise de Rham derivative on V to $\mathcal{G}_X$ by d_X and the lift of the leafwise Hodge operator by T_X. As proved in the previous section, the complex $(\mathcal{E}_X, d_X, T_X)$ given by:

$$\mathcal{E}_0 \xrightarrow{d} \mathcal{E}_1 \xrightarrow{d} \mathcal{E}_2 ... \xrightarrow{d} \mathcal{E}_p$$

is a Hilbert Poincaré complex as per the definition in subsection A.1.

Proposition 5.3.12. *The map*

$$I \otimes \theta_H^{-1} : \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} \mathcal{E}_X^X(g \circ f) \to \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} \mathcal{A}_X^X$$

is a well-defined map of Hilbert-modules. Further the composition map

$$\rho_H := \delta \circ (I \otimes \theta_H^{-1}) : \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} \mathcal{E}_X^X(g \circ f) \to \mathcal{E}_{X,E}$$

commutes with the differential on $\mathcal{E}_{X,E}$, i.e. $\rho_H(d_X \otimes I) = d_X \circ \rho_H$.

Proof. Let us prove the left linearity of θ_H, i.e. that $\theta_H(\phi * \xi) = \pi(\phi)\theta_H(\xi)$. Let $\phi, \xi \in C_c^\infty(\mathcal{G}_X^X)$. Then we have,

$$\begin{aligned}\theta_H(\phi * \xi)(x, \gamma) &= (\phi * \xi)((\gamma_x^s)^{-1}\gamma) \\ &= \sum_{\gamma_1 \in \mathcal{G}_X^x} \phi(\gamma_1)\xi(\gamma_1^{-1}(\gamma_x^s)^{-1}\gamma)\end{aligned}$$

Computing the right hand side, we get:

$$\pi(\phi)\theta_H(\xi)(x, \gamma) = \sum_{\alpha \in \mathcal{G}_X^x} \phi(\alpha)\xi(\gamma_{s(\alpha)}^{-1} \circ (\check{f} \circ \check{g})(\alpha^{-1}) \circ \gamma) \tag{5.3.10}$$

Put $\gamma_1 = \gamma_x^{-1} \circ (\check{f} \circ \check{g})(\alpha) \circ \gamma_{s(\alpha)}$. Then from the proof of Proposition 5.1.7 we see that $\gamma_1 = \alpha$. Therefore we have,

$$\pi(\phi)\theta_H(\xi)(x, \gamma) = \sum_{\gamma_1 \in \mathcal{G}_X^x} \phi(\gamma_1)\xi(\gamma_1^{-1} \circ (\gamma_x^s)^{-1}\gamma) \tag{5.3.11}$$

which is equal to the left hand side.

To see the ρ_H is a chain map, we compute as follows:

$$\begin{aligned}\rho_H \circ (d_X \otimes I) &= \delta \circ (I \otimes \theta_H^{-1}) \circ (d_X \otimes I) \\ &= \delta \circ (d_X \otimes I) \circ (I \otimes \theta_H^{-1}) \text{ (since } I \otimes \theta_H^{-1} \text{ and } d_X \otimes I \text{ commute)}\end{aligned} \tag{5.3.12}$$

But then we have for $\xi \in C_c^\infty(\mathcal{G}_X, r^*E), \phi \in C_c^\infty(\mathcal{G}_X^X)$,

$$\begin{aligned}\delta \circ (d_X \otimes I)(\xi \otimes \phi) &= \delta(d_X\xi \otimes \phi) \\ &= d_X(\xi)\phi \text{ (since } \delta \text{ is given by right multiplication)} \\ &= d_X(\xi\phi) \text{ (since } d_X \text{ is } \mathcal{A}_X^X\text{-linear)} \\ &= d_X[\delta(\xi \otimes \phi)] = d_X \circ \delta(\xi \otimes \phi)\end{aligned}$$

Therefore from the two computations above we get the desired result.

□

Definition We set $f_\phi^* := \Phi_f \circ \Psi_f^\phi$.

Notice that f_ϕ^* is an adjointable operator since both Φ_f and Ψ_f^ϕ are adjointable. As $\hat{\phi}$ is compactly supported, $\phi(\Delta)$ preserves the dense space $C_c^\infty(\mathcal{G}_X, r^*E)$, and by Proposition 5.2.1, $\phi(\Delta)$ induces the identity map on cohomology of the complex $(\mathcal{E}_X, d_X, T_X)$.

Theorem 5.3.13. *On cohomology, we have the following functoriality formula:*

$$(I \otimes \Omega)^{-1}(g^*_\phi \otimes I)f^*_\phi = (f \circ g)^*_\phi$$

Furthermore, f^*_ϕ induces an isomorphism on cohomology as a chain map between the complexes $(\mathcal{E}'_{X'}, d'_{X'}, T'_{X'})$ and $(\mathcal{E}_X \otimes \mathcal{E}^X_{X'}(f), d_X \otimes I, T_X \otimes I)$.

Proof. Since Θ_f, Θ_g are chain maps, we get from Theorem 5.3.6,

$$\begin{aligned} (I \otimes \Omega)^{-1}(g^*_\phi \otimes I)f^*_\phi &= (I \otimes \Omega)^{-1}(\Theta_g \otimes I)(\phi(\Delta) \otimes I)\Theta_f \phi(\Delta) \\ &= (I \otimes \Omega)^{-1}(\Theta_g \otimes I)\Theta_f \phi(\Delta)^2 \\ &= \Theta_{f\circ g}\phi(\Delta)^2 \\ &= (f \circ g)^*_\phi \circ \phi(\Delta) \end{aligned} \tag{5.3.13}$$

But as $\phi(\Delta)$ is identity on cohomology, we deduce the relation.

Let now ρ^*_H denote the map induces by ρ_H on cohomology. Then the Poincaré lemma for Hilbert modules states that (see next section)

$$\rho^*_H \circ (f \circ g)^*_\phi = Id_{H^*(\mathcal{E}, d_X, T_X)}$$

So we get

$$\rho^*_H (I \otimes \Omega)(g^*_\phi \otimes I)f^*_\phi = Id_{H^*(\mathcal{E}, d_X, T_X)}$$

Therefore f^*_ϕ is injective. Also, since $\rho^*_H(I \otimes \Omega)$ is an isomorphism, $g^*_\phi \otimes I$ is surjective. Applying the same argument reversing the roles of g and f, we get the existence of operators A_1, A_2, such that on cohomology:

$$(f^*_\phi \otimes I) \circ A_1 = Id, \qquad A_2 \circ f^*_\phi = Id$$

The two equalities above together imply that

$$(f^*_\phi \otimes I) : \mathcal{E}_{X',E'} \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^{X'}_X(g) \to \mathcal{E}_{X,E} \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^X_{X'}(f) \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^{X'}_X(g)$$

is an isomorphism on cohomology with inverse $A_2 \otimes I$. However, using the fact that $\mathcal{E}^{X'}_X(g) \otimes_{\mathcal{A}^X_X} \mathcal{E}^X_{X'}(f) \cong \mathcal{E}^{X'}_{X'}(f \circ g) \cong^{\rho_{h'}} \mathcal{A}^{X'}_{X'}$, we get that the map

$$(f^*_\phi \otimes I \otimes I) : \mathcal{E}_{X',E'} \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^{X'}_X(g) \otimes_{\mathcal{A}^X_X} \mathcal{E}^X_{X'}(f) \to \mathcal{E}_{X,E} \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^X_{X'}(f) \otimes_{\mathcal{A}^{X'}_{X'}} \mathcal{E}^{X'}_X(g) \otimes_{\mathcal{A}^X_X} \mathcal{E}^X_{X'}(f)$$

is conjugated via isomorphisms to the map f^*_ϕ. Therefore the map f^*_ϕ induces an isomorphism on cohomology.

□

5.3.2 Poincaré lemma for Hilbert modules

Let $X_0 := \{0\} \times X \subset [0,1] \times V$ be the complete transversal of the product foliation on $[0,1] \times V$, for which each leaf is the product $[0,1] \times L$ where L is a leaf in V. We denote this foliation by $[0,1] \times \mathcal{F}$. Let $\hat{\mathcal{G}}$ be the monodromy groupoid of the foliation $([0,1] \times V, [0,1] \times \mathcal{F})$. Then we have

$$\hat{\mathcal{G}} = \mathcal{G} \times [0,1]^2$$

We denote by $\hat{\mathcal{A}}^{X_0}_{X_0}$ the maximal C^*-algebra of $\hat{\mathcal{G}}^{X_0}_{X_0}$.

Define maps Φ_H and Ψ_H as in the definition 5.3.5. We note that using a similar proof as for Lemma 5.3.9, one has an isomorphism $\rho_H : \mathcal{E}_{X_0,\hat{E}} \otimes_{\hat{\mathcal{A}}^{X_0}_{X_0}} \mathcal{E}^{X_0}_X(h) \to \mathcal{E}_{X_0,\hat{E}}$. Then we set $H^* := \rho_H \circ \Phi_H \circ \Psi_H = \rho_H \circ \Theta_H$, so we have a map

$$\mathcal{E}_{X,E} \xrightarrow{\Psi_H} \mathcal{E}^{[0,1]\times V}_X(H) \xrightarrow{\Phi_H} \mathcal{E}_{X_0,\hat{E}} \otimes_{\hat{\mathcal{A}}^{X_0}_{X_0}} \mathcal{E}^{X_0}_X(H) \xrightarrow{\rho_H} \mathcal{E}_{X_0,\hat{E}}$$

where $\hat{E} := r^* \bigwedge^* T^*([0,1] \times \mathcal{F})$.

Since we also have

$$\hat{\mathcal{G}}_{X_0} = [0,1] \times \mathcal{G}_X,\ \hat{\mathcal{G}}^{X_0}_{X_0} = \mathcal{G}^X_X$$

there is a canonical isomorphism of Hilbert modules between the completions of $C^\infty_c(\hat{\mathcal{G}}_{X_0} \times_{\hat{\mathcal{G}}^{X_0}_{X_0}} \mathcal{G}^{X_0}_X(H), (r \circ \pi_1)^*\hat{E})$ and the completion of $C^\infty_c(\hat{\mathcal{G}}_{X_0}, r^*\hat{E}) \otimes_{C^\infty_c(\mathcal{G}^X_X)} C^\infty_c(\mathcal{G}^X_X)$. Therefore we can finally construct an adjointable map of Hilbert modules:

$$H^\sharp := H^* \circ \phi(\Delta) : \mathcal{E}_{X,E} \to \mathcal{E}_{X_0,\hat{E}}$$

Remark. *Using the same arguments as in the previous section, we can show that $H^\sharp$ is adjointable as $\Psi^\phi_H = \Psi_H \circ \phi(\Delta)$ extends to an adjointable map and Φ_h extends to an isometric isomorphism and therefore is also adjointable (with adjoint Φ^{-1}_H).*

Remark. *Notice that for any $x \in X$, if $H^*_x := (\rho_H \circ \Theta_h)_x$, a map*

$$H^*_x : C^\infty_c(\mathcal{G}_x, r^*E) \to C^\infty_c(\hat{\mathcal{G}}_{(0,x)}, r^*\hat{E}) = C^\infty_c([0,1] \times \mathcal{G}_x, r^*\hat{E})$$

then H^*_x is simply given by the formula:

$$H^*_x(\xi)(t,\gamma) = ({}^tH_*)_{(t,r(\gamma)}(\xi(H(t,\gamma) \circ \gamma^t_{s(\gamma)}))$$

where $({}^tH_*)_{t,v} : T^*_{H(t,v)}V \to T^*_{t,v}([0,1]\times V)$ is the transpose of the differential of H and $H(t,\gamma) := \gamma^t_{r(\gamma)}\gamma(\gamma^t_s(\gamma))^{-1}$. Indeed, and as before, on dense spaces, we take the following map for H^*_x:

$$C^\infty_c(\mathcal{G}_X, r^*E) \xrightarrow{\Psi_H} C^\infty_c(\mathcal{G}^{[0,1]\times V}_X(H), (r \circ pr_1)^*\hat{E}) \xrightarrow{\Phi_H} C^\infty_c(\hat{\mathcal{G}}_{X_0} \times_{\hat{\mathcal{G}}^{X_0}_{X_0}} \mathcal{G}^{X_0}_X(H), (r \circ pr_1)^*\hat{E}) \xrightarrow{\rho_H} C^\infty_c(\hat{\mathcal{G}}_{X_0}, \hat{E})$$

Thus we have,

$$\begin{aligned}(I \otimes \theta^{-1}_H)(\Phi_H \circ \Psi_H)(\xi)[(t,\gamma),(0,\alpha)] &= (\Phi_H \circ \Psi_H)(\xi)[(t,\gamma),(s(t,\gamma),(0,\alpha))] \qquad (5.3.14)\\ &\quad \textit{(as } \theta_H : \mathcal{G}^{X_0}_X(h) \to \mathcal{G}^{X_0}_{X_0} \textit{ is given by } ((0,x),\gamma) \mapsto (0,0,\gamma)\textit{)}\\ &= ({}^tH_*)_{(t,r(\gamma)}(\xi(\breve{H}(t,\gamma)\alpha)) \qquad (5.3.15)\end{aligned}$$

And,

$$\begin{aligned}\delta \circ (I \otimes \theta^{-1}_H)(\Phi_H \circ \Psi_H)(\xi)(t,\gamma) &= (I \otimes \theta^{-1}_H)(\Phi_H \circ \Psi_H)(\xi)[(t,\gamma), 1_{(0,\gamma)}]\\ &= ({}^tH_*)_{(t,r(\gamma)}(\xi(\breve{H}(t,\gamma)1_{0,\gamma}))\\ &= ({}^tH_*)_{(t,r(\gamma)}(\xi(H(t,\gamma)\gamma^t_{s(\gamma)}))\end{aligned}$$

where in the last line we have used the following: since $\breve{H}(t,s,\gamma) = \gamma^t_{r(\gamma)}\gamma(\gamma^s_{s(\gamma)})^{-1}$, so for $(t,\gamma) \in \hat{\mathcal{G}}^{X_0}_{X_0}$, $\breve{H}(t,\gamma) := \breve{H}(t,0,\gamma) = \gamma^t_{r(\gamma)}\gamma$. But we have $\gamma^t_{r(\gamma)}\gamma = H(u,\gamma) \circ \gamma^t_{s(\gamma)}$.

Now if $\xi \in C_c^\infty([0,1] \times \mathcal{G}_x, r^*\hat{E})$ is a k-form, it can be expressed using a local chart $(U, x_1, x_2, ..., x_p)$ in the leaf L_x where p is the dimension of $\mathcal{F}$, as follows:

$$\xi = \sum_{i_1<i_2<...<i_k} (\xi_1)_I r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) + \sum_{j_1<j_2<...<j_{k-1}} (\xi_2)_J dt \wedge r^!(dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_{k-1}})$$

Then there is a well-defined map

$$\int_0^1 : C_c^\infty([0,1] \times \mathcal{G}_x, r^*\hat{E}) \to C_c^\infty(\mathcal{G}_x, r^*E)$$

given by

$$\int_0^1 \xi := \sum_{j_1<j_2<...<j_{k-1}} (\int_1^0 \xi_2(x,t)dt) r^!(dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_{k-1}})$$

Let for $0 \leq s \leq 1$, $i_s : \mathcal{G}_x \hookrightarrow [0,1] \times \mathcal{G}_x$ be the map $i_s(\gamma) = (s, \gamma)$

Lemma 5.3.14. *We have,*

$$d_{\mathcal{G}_x}(\int_0^1 \xi) + \int_0^1 (d_{\hat{\mathcal{G}}_x} \xi) = \xi_1 \circ i_1 - \xi_1 \circ i_0$$

where $\xi_1 = \sum_{i_1<i_2<...<i_k} (\xi_1)_I r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})$.

Proof. Let ξ be a k-form in $C_c^\infty([0,1] \times \mathcal{G}_x, r^*(\bigwedge^k T^*([0,1] \times \mathcal{F})))$. It is sufficient to check the formula in local coordinates and where ξ is in either of the following forms:

1. $\xi(t,x) = \xi_1(t,x) r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})$, or
2. $\xi(t,x) = \xi_2(t,x) dt \wedge r^!(dx_{j_1} \wedge dx_{j_2} \wedge ... \wedge dx_{j_{k-1}})$

For the first case, we have:

$$d_{\hat{\mathcal{G}}_x} \xi = \sum_{m=1}^p \frac{\partial \xi_1}{\partial x_m} r^!(dx_m \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) + \frac{\partial \xi_1}{\partial t} dt \wedge r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \tag{5.3.16}$$

so we get

$$\begin{aligned} (\int_0^1 d_{\hat{\mathcal{G}}_x} \xi)(x) &= (\int_0^1 \frac{\partial \xi_1}{\partial t} dt) r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \\ &= (\xi_1(1,x) - \xi_1(0,x)) r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \\ &= \xi_1 \circ i_1(x) - \xi_1 \circ i_0(x) \end{aligned}$$

Since in this case $\int_0^1 \xi = 0$, we get the desired result. Now for the second case, we have:

$$\begin{aligned} \int_0^1 d_{\hat{\mathcal{G}}_x} \xi &= \int_0^1 (\sum_{m=1}^p -\frac{\partial \xi_2}{\partial x_m} dt \wedge r^!(dx_m \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})) \\ &= \sum_{m=1}^p (\int_0^1 -\frac{\partial \xi_2}{\partial x_m} dt) r^!(dx_m \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})) \end{aligned}$$

And on the other hand,

$$\begin{aligned} d_{\mathcal{G}_x}(\int_0^1 \xi) &= d_{\mathcal{G}_x}((\int_0^1 \xi_2(t,x)dt)r^!(dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k})) \\ &= \sum_{m=1}^{p}(\int_0^1 \frac{\partial \xi_2}{\partial x_m}dt)r^!(dx_m \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}) \end{aligned}$$

So in this case $\int_0^1 d_{\hat{\mathcal{G}}_x}\xi + d_{\mathcal{G}_x}(\int_0^1 \xi) = 0$ which finishes the proof as $\xi_1 = 0$ in this case. □

If $\xi = H_x^*(\eta)$ for $\eta \in C_c^\infty(\mathcal{G}_x, r^*E)$, we have

$$\int_0^1 d_{\hat{\mathcal{G}}_x} H_x^*(\eta) + d_{\hat{\mathcal{G}}_x}(\int_0^1 H_x^*(\eta)) = H^*(\eta)_1|_{\{1\}\times\mathcal{G}_x} - H^*(\eta)_1|_{\{0\}\times\mathcal{G}_x}$$

But we know from above remarks that,

$H^*(\eta)(1,\gamma) = ({}^tH_*)_{(1,r(\gamma))}[\eta(H(1,\gamma)\circ\gamma^1_{s(\gamma)})]$, and $H_1^*(\eta)(\gamma) = ({}^tH_{1,*})_{r(\gamma)}[\eta(H_1(\gamma)\circ\gamma_{s(\gamma)})]$

where $H_1^* = (g\circ f)^* = (\rho_h \circ \Theta_{g\circ f})$.

By lemma 5.3.16 below, we have:

$$H_1^*(\eta) - H_0^*(\eta) = \int_0^1 d_{\hat{\mathcal{G}}_x} H_x^*(\eta) + d_{\hat{\mathcal{G}}_x}(\int_0^1 H_x^*(\eta)) = d_{\hat{\mathcal{G}}_x}\int_0^1 H_x^*(\eta) + d_{\hat{\mathcal{G}}_x}(\int_0^1 H_x^*(\eta))$$

Hence setting $K^*\eta := \int_0^1 H^*\eta$, we get

$$H^*_{(1,x)}(\eta) - H^*_{(0,x)}(\eta) = K_x^* \circ d_{\mathcal{G}_x} + d_{\mathcal{G}_x} \circ K_x^* \tag{5.3.17}$$

Let $K^\sharp = K^* \circ \phi(\Delta)$. Then we have the following

Proposition 5.3.15. *$K^\sharp$ is an adjointable operator on $\mathcal{E}_{X,E}$ and we have*

$$(g\circ f)^\sharp - \phi(\Delta) = K^\sharp \circ d + d \circ K^\sharp$$

Proof. We have already shown, modulo Lemma 5.3.16, in 5.3.17 that

$$(g\circ f)^* - Id = K^* \circ d + d \circ K^*$$

We now check the following:

• $\int_0^1 : C_c^\infty([0,1]\times\mathcal{G}_x, r^*\hat{E}) \to C_c^\infty(\mathcal{G}_x, r^*E)$ is well-defined map of Hilbert modules and extends to an adjointable operator on $\mathcal{E}_{X_0,\hat{E}}$.

We check the following properties of the map:

1. $\int_0^1$ is a $C_c^\infty(\mathcal{G}_X^X)$-linear map with the identification $\hat{\mathcal{G}}_X^X \cong \mathcal{G}_X^X$ so that

$$\int_0^1 (\xi\phi) = (\int_0^1 \xi)\phi$$

2. The adjoint map is given by $D\eta = \eta \wedge dt$. i.e. we have

$$< \int_0^1 \xi, \eta > = < \xi, \eta \wedge dt >$$

1. We compute as follows: Let $\xi = \xi_1 + \xi_2 \wedge dt$ so that $\xi\phi = \xi_1\phi + \xi_2\phi \wedge dt$. Then,

$$\begin{aligned}
[\int_0^1 (\xi\phi)] &= \int_0^1 \frac{\partial \xi_2 \phi}{\partial t} \wedge dt \\
&= \int_0^1 \frac{\partial \xi_2}{\partial t}\phi \wedge dt \\
&= (\int_0^1 \frac{\partial \xi_2}{\partial t} \wedge dt)\phi \\
&= (\int_0^1 \xi)\phi
\end{aligned}$$

So we get the desired result.

2. Let $\xi(t,\gamma) = \eta_1(t,\gamma) + \eta_2(t,\gamma) \wedge dt, \eta(t,\gamma) = \eta_1(t,\gamma) + \eta_2(t,\gamma) \wedge dt$ be differential forms. Let the hodge $*$-operators on $\hat{\mathcal{G}}_{X_0}$ be denoted by $*_{\bigwedge^\bullet \hat{\mathcal{F}}}$ and on $\mathcal{G}_X$ by $*_{\bigwedge^\bullet \mathcal{F}}$, respectively. Then we have,

$$*_{\bigwedge^\bullet \hat{\mathcal{F}}}(\eta_1(t,\gamma)) = (*_{\bigwedge^\bullet \mathcal{F}}\eta_1(t,\gamma)) \wedge dt$$

and

$$*_{\bigwedge^\bullet \hat{\mathcal{F}}}(\eta_2(t,\gamma) \wedge dt) = (-1)^{p-\partial\eta_2}(*_{\bigwedge^\bullet \mathcal{F}}\eta_2(t,\gamma))$$

Then, we have,

$$\begin{aligned}
\int_0^1 \xi(t,\gamma) \wedge *_{\bigwedge^\bullet \hat{\mathcal{F}}}\eta(t,\gamma) &= \int_0^1 (\xi_1(t,\gamma) \wedge *_{\bigwedge^\bullet \mathcal{F}}\eta_1(t,\gamma)) \wedge dt + (-1)^{p-\partial\eta_2}\int_0^1 \xi_2(t,\gamma) \wedge dt \wedge (*_{\bigwedge^\bullet \mathcal{F}}\eta_2(t,\gamma)) \\
&= \int_0^1 (\xi_1(t,\gamma) \wedge *_{\bigwedge^\bullet \mathcal{F}}\eta_1(t,\gamma)) \wedge dt + \int_0^1 \xi_2(t,\gamma) \wedge (*_{\bigwedge^\bullet \mathcal{F}}\eta_2(t,\gamma)) \wedge dt
\end{aligned} \tag{5.3.18}$$

Now we have,

$$\begin{aligned}
< \int_0^1 \xi, \alpha > (\gamma) &= \int_{\mathcal{G}_{r(\gamma)}} < \int_0^1 \xi(\gamma_1), \alpha(\gamma_1\gamma) >_{E_{r(\gamma_1)}} d\lambda_{r(\gamma)}(\gamma_1) \\
&= \int_{\mathcal{G}_{r(\gamma)}} < \int_0^1 \xi_1(t,\gamma_1) \wedge dt, \alpha(\gamma_1\gamma) >_{E_{r(\gamma_1)}} d\lambda_{r(\gamma)}(\gamma_1) \\
&= \int_{\mathcal{G}_{r(\gamma)}} \int_0^1 < \xi_1(t,\gamma_1)\wedge, \alpha(\gamma_1\gamma) >_{E_{r(\gamma_1)}} dt d\lambda_{r(\gamma)}(\gamma_1)
\end{aligned} \tag{5.3.19}$$

Letting $\eta(t,\gamma) = \alpha(\gamma) \wedge dt$ we get:

$$\int_0^1 < \xi(t,\gamma_1), \eta(t,\gamma_1\gamma) > dt = \pm \int_0^1 \xi_1(t,\gamma_1) \wedge *_{\bigwedge^\bullet \mathcal{F}}\alpha(\gamma_1\gamma) > \wedge dt = \int_0^1 < \xi_1(t,\gamma_1), \alpha(\gamma_1\gamma) > dt$$

Therefore we get,

$$
\begin{aligned}
< \int_0^1 \xi, \alpha > (\gamma) &= \int_{\mathcal{G}_{r(\gamma)}} \int_0^1 < \xi(t,\gamma_1), \eta(t,\gamma_1\gamma) >_{E_{r(\gamma_1)}} dt d\lambda_{r(\gamma)}(\gamma_1) \\
&= \int_{\mathcal{G}_{r(\gamma)}\times[0,1]} < \xi(t,\gamma_1), \eta(t,\gamma_1\gamma) >_{E_{r(\gamma_1)}} dt d\lambda_{r(\gamma)}(\gamma_1) \\
&= < \xi, \eta > (\gamma) =< \xi, \alpha \wedge dt >
\end{aligned}
$$

Hence the desired equality. Thus $\int_0^1$ extends to an adjointable operator on Hilbert modules, and as a result $K^\sharp$ as well.

□

We have used above the following lemma.

Lemma 5.3.16. *We have,* $H^*(\eta)_1|_{(1,\gamma)} = H_1^*(\eta)(\gamma)$

Proof. We first show that for $\xi = \xi_1 + \xi_2 \wedge dt$, we have $\xi_1 = ({}^t i_{1,*})\xi$.

Let $X \in T_x L$, then, $({}^t i_{1,*}\xi_x)(X) = \xi_{(1,x)}(i_{1,*})(X) = \xi_{(1,x)}(X_{(1,x)}) = (\xi_1)_{(1,x)}(X_{(1,x)})$, as X does not have $\frac{\partial}{\partial t}$ terms.

Now, we note that as $H_1 = H \circ i_1$, we have $({}^t i_{1,*})_x \circ ({}^t H_*)_{(1,x)} = ({}^t H_{1,*})_x$, and so

$$
\begin{aligned}
H^*(\eta)_1|_{(1,\gamma)} &= ({}^t i_{1,*})_{r(\gamma)}(H^*(\eta)(1,\gamma)) \\
&= ({}^t i_{1,*})_{r(\gamma)} \circ ({}^t H_*)_{(1,r(\gamma))}(\eta(H(1,\gamma)\gamma_{s(\gamma)}) \\
&= ({}^t H_{1,*})_{r(\gamma)}(\eta(H(1,\gamma)\gamma_{s(\gamma)}) \\
&= H_1^*(\eta)(\gamma)
\end{aligned}
$$

Thus proving the claim. Therefore we have

$$(g \circ f)^* - Id = K^* \circ d + d \circ K^*$$

Multiplying by $\phi(\Delta)$ on the right in the above equation, we get the desired result:

$$(g \circ f)^\sharp - \phi(\Delta) = K^\sharp \circ d + d \circ K^\sharp$$

since $\phi(\Delta)$ commutes with d.

□

Corollary 5.3.17. $(g \circ f)^\sharp$ *induces the identity on cohomology of the complex* $(\mathcal{E}_X, d_X)$.

Proof. This is immediate from the previous proposition and the fact that $\phi(\Delta)$ induces the identity on cohomology, while $K^\sharp \circ d + d \circ K^\sharp$ is zero on cohomology. □

Chapter 6

Applications: Extending Keswani's proof for foliations

6.1 von Neumann algebras associated with a leafwise homotopy equivalence

Recall that $f:(V,\mathcal{F}) \to (V',\mathcal{F}')$ is a leafwise homotopy equivalence. Recall also that for $v' \in V'$, $\mathcal{G}_{v'}^{V}(f)$ is the inverse image of the Connes-Skandalis principal bundle $\mathcal{G}(f) := \{(v,\alpha') \in V \times \mathcal{G}' | f(v) = r(\alpha')\}$ under the map s_f given by $s_f(v,\alpha') = s(\alpha')$ (cf. [CoSk:84]). Let $L^2(\mathcal{G}_{v'}^{V}(f), \pi_1^* E)$ be the Hilbert space defined as the completion of $C_c^\infty(\mathcal{G}_{v'}^{V}(f), \pi_1^* E)$ with the inner product given by:

$$< \xi_{v'}, \eta_{v'} >:= \int_{v \in L_{v'}} \sum_{\gamma' \in \mathcal{G}'^{f(v)}_{v'}} < \xi(v,\gamma'), \eta(v,\gamma') >_{E_v} d\lambda_{v'}(\gamma') \text{ for } \xi_{v'}, \eta_{v'} \in C_c^\infty(\mathcal{G}_{v'}^{V}(f), \pi_1^* E),$$

where $L_{v'}$ is the (unique) leaf in $(V',\mathcal{F}')$ whose image under f is in $L'_{v'}$ in $(V',\mathcal{F}')$. Then the family of Hilbert spaces $\mathcal{H}(f) := (L^2(\mathcal{G}_{v'}^{V}(f), \pi_1^* E))_{v' \in V'}$ is a measurable field of Hilbert spaces (cf. [Di:57]). For every $\gamma' \in \mathcal{G}'^{v'_1}_{v'_2}$ there is an isometric isomorphism $U_{\gamma'} : L^2(\mathcal{G}_{v'_2}^{V}(f), \pi_1^* E) \to L^2(\mathcal{G}_{v'_1}^{V}(f), \pi_1^* E)$ given by

$$U_{\gamma'} \xi_{v'_2}(v',\alpha') := \xi_{v'_2}(v', \alpha' \gamma')$$

Then the measurable field of Hilbert spaces $\mathcal{H}(f)$ has a square-integrable representation of $\mathcal{G}'$ (for definitions see [Co:79], definition 5.11, page 37). So by Theorem 6.2, page 40 of [Co:79],

$$\text{End}_{\Lambda'}(\mathcal{H}(f)) := \{[T] | T = (T_{v'})_{v' \in V'} \text{measurable family of } \Lambda'\text{-essentially bounded operators s.t. } T_{v'_1} U_{\gamma'} = U_{\gamma'} T_{v'_2}\}$$

is a von Neumann algebra, where $T_{v'} \in B(L^2(\mathcal{G}_{v'}^{V}(f), \pi_1^* E))$ for each $v' \in V'$ and $\Lambda' = f_* \Lambda$ is the holonomy invariant transverse measure on $(V',\mathcal{F}')$ associated with the holonomy invariant transverse measure Λ on $(V,\mathcal{F})$ and the equivalence classes $[.]$ are given by equality of operators Λ'-*a.e.*.

Now, we consider the leafwise graph of f for $v' \in V'$,

$$\Gamma(f,v') := \{(v, f(v)) | v \in V, f(v) \in L'_{v'}\}$$

Then $H(f) = (L^2(\Gamma(f, v'), \pi_1^* E))_{v' \in V'}$ is a measurable field of Hilbert spaces, with inner-product given by

$$< \xi_1, \xi_2 >= \int_{v \in L_{v'}} < \xi_1(v, f(v)), \xi_2(v, f(v)) >_{E_v} d\lambda^L(v)$$

for $\xi_1, \xi_2 \in L^2(\Gamma(f, v'), \pi_1^* E)$, where $L_{v'}$ is as before. Then we define as above $\mathrm{End}_{\Lambda'}(H(f))$ as the von Neumann algebra associated with this field: it is the set of measurable families of Λ'-essentially bounded operators $T = (T_{v'})_{v' \in V'}$ s.t. $T_{v'_1} = T_{v'_2}$ for v'_1, v'_2 in the same leaf, where $T_{v'} \in B(L^2(\Gamma(f, v'), \pi_1^* E))$ for each $v' \in V'$.

6.2 Traces

Let $T_f = (T_{f,v'})_{v' \in V'} \in \mathrm{End}_{\Lambda'}(\mathcal{H}(f))$ be a positive operator such that each $T_{f,v'}$ is positive and given by a kernel as follows:

$$T_{f,v'} \xi(v, \gamma') = \int_{v_1 \in L_v} \sum_{\gamma'_1 \in \mathcal{G}'^{f(v)}_{v'}} k_{T_f, v'}((v, \gamma'), (v_1, \gamma'_1)) \xi(v_1, \gamma'_1) d\lambda_{L_v}(v_1)$$

where $k_{T_f, v'} \in C_c^\infty(\mathcal{G}^V_{v'}(f) \times \mathcal{G}^V_{v'}(f), Hom(\pi_1^* E, \pi_3^* E))$.

Then the trace of T_f is defined as follows. Let $(U'_\alpha)_{\alpha \in A}$ be a distinguished open cover on $(V', \mathcal{F}')$. Let X'_α denote the local transversal of U'_α. Without loss of generality one can assume that $\overline{X'}_\alpha \cap \overline{X'}_\beta = \emptyset$ for $\alpha \neq \beta$(cf. [HiSk:83]). Then we can choose an distinguished open cover $(U_i)_{i \in I}$ of $(V, \mathcal{F})$ such that for $i \in I$ there exists $\alpha(i) \in A$ such that $f(U_i) \subseteq U'_{\alpha(i)}$. Let $U_i \cong W_i \times X_i$, where X_i is transversal to the plaques W_i. One can also assume without loss of generality that the induced map on the transversal $\hat{f} : X_i \to \hat{f}(X_i)$ is a diffeomorphism onto its image (cf [CoSk:84], [BePi:08]). Let $\pi_{\alpha(i)} : \hat{f}(X_i) \to X'_{\alpha(i)}$ be the map which projects to the local transversal. Denote $X'_i := \pi_{\alpha(i)}(\hat{f}(X_i))$. Then it can be easily seen that $X' := \bigcup_{i \in I} X'_i$ is a complete transversal for $(V', \mathcal{F}')$.

Let $(\phi_i^2)_{i \in I}$ be a partition of unity in V subordinate to U_i, $\sum_{i \in I} \phi_i^2 = 1$. Let α_i be the restriction of the leafwise measures λ on V to the plaques of W_i.

Definition The trace $\tau_{\Lambda', f}(T_f)$ of T_f is defined by

$$\tau_{\Lambda', f}(T_f) := \sum_{i \in I} \int_{v' \in X'_i} \int_{v \in W_{i,v'}} tr(K^i_{T_f}(v, 1_{f(v)}, v, 1_{f(v)})) d\alpha_i(v) d\Lambda'(v')$$

where $W_{i,v'}$ is the unique plaque in U_i corresponding to the plaque $W'_{\alpha(i),v'}$ in $U'_{\alpha(i)}$ by the image of f, $1_{f(v)}$ is the homotopy class of the constant path at $f(v)$ and $K^i_{T_f}(v_1, \gamma'_1, v_2, \gamma'_2) = \phi_i(v_1)\phi_i(v_2) K_{T_f}(v_1, \gamma'_1, v_2, \gamma'_2)$.

The following propositions give some properties of the trace $\tau_{\Lambda', f}$:

Proposition 6.2.1. *The above formula for $\tau_{\Lambda', f}$ does not depend on the choices of $(U'_\alpha)_{\alpha \in A}, (X'_\alpha)_{\alpha \in A}$, $(U_i)_{i \in I}, (X_i)_{i \in I}$ and $(\phi_i^2)_{i \in I}$.*

Proof. Let us choose another distinguished open cover $\{\tilde{U}'_\beta\}_{\beta \in B}$ of V' with local transversal X'_β and a corresponding distinguished open cover of V $\{\tilde{U}_j\}_{j \in J}$ with $\tilde{U}_j \cong \tilde{W}_j \times \tilde{X}_j$ such that f induces a diffeomorphism $\hat{f} : \tilde{X}_j \to \hat{f}(\tilde{X}_j)$. We consider the complete transversal $\tilde{X}' = \cup_{j \in J} \tilde{X}'_j$ where $\tilde{X}'_j = \pi_{\beta(j)}(\hat{f}(\tilde{X}_j))$ as before.

Let

$$\tilde{\tau}_{\Lambda',f}(T_f) := \sum_{j\in J}\int_{v'\in\tilde{X}'_j}\int_{v\in\tilde{W}_{j,v'}} tr(K^j_{T_f}(v,1_{f(v)},v,1_{f(v)}))d\alpha_j(v)d\Lambda'(v')$$

By considering locally finite refinements of the cover $\{\tilde{U}'_\beta\}_{\beta\in B}$, we can assume without loss of generality that $U'_\alpha = \cup_{\gamma\in B(\alpha)}\tilde{U}_\gamma$ with $\tilde{U}_\beta\cap\tilde{U}_{\beta'} = \emptyset$ if and only if $\beta\neq\beta'$ for $\beta,\beta'\in B(\alpha)$. Corresponding to this refinement one can choose a refinement $\{U_j\}_{j\in J}$ of the cover $\{U_i\}_{i\in I}$ such that for all $j\in J, \exists\beta\in B$ for which $f(U_j)\subseteq\tilde{U}_\beta$ and $U_j\cap U_{j'}=\emptyset$ if and only if $j\neq j'$. Let $j(i)=\{j\in J|U_i\cap U_j\neq\emptyset\}$. We claim that

$$\begin{aligned}\int_{v'\in\tilde{X}'_i}\int_{v\in\tilde{W}_{i,v'}} & tr(K^i_{T_f}(v,1_{f(v)},v,1_{f(v)}))d\alpha_i(v)d\Lambda'(v')\\ = & \sum_{j(i)\in J(i)}\int_{v'\in\tilde{X}'_{j(i)}}\int_{v\in\tilde{W}_{j(i),v'}} tr(K^{j(i)}_{T_f}(v,1_{f(v)},v,1_{f(v)}))d\alpha_{j(i)}(v)d\Lambda'(v')\end{aligned}$$

Evidently it is enough to consider the case when $V'\cong W'\times X'$ and $V\cong W\times X$ such that $\hat{f}(X)$ is diffeomorphic to X. We choose a finite open foliated good cover of V' denoting it $\mathcal{U}'=\{U'_\beta\}_{\beta\in B}$ and a corresponding finite open foliated good cover of V, $\mathcal{U}=\{U_j\}_{j\in J}$ such that conditions on the previous paragraph are satisfied. In this case partitions of unity functions are identically 1 on open charts.

Now let $J_X=\{j\in J|X\cap U_j\neq\emptyset\}$. We also let for $j\in J_X, X(j)=X\cap U_j$. For $j\in J_X$ consider the set $S(j)=\{k\in J|\exists\beta\in B$ and $v'\in X'_j$ such that $f(U_j)\subseteq U'_\beta, W_{k,v'}\subseteq W_{v'}\}$ where $W_{k,v'}$ is the plaque through v in the foliated chart U_k such that $f(v)\in W'_{v'}$ and $W_{v'}$ is the corresponding 'leaf' in V. It is not difficult to see that we can assume w.l.o.g. that J_X has only one element. Now for $v'\in X$, let $J_{W,v'}:=\{k\in J|W'_v\cap U_k\neq\emptyset\}$. Then we can divide X into equivalence classes of subsets with the relation given by $v'_1\sim v'_2\Leftrightarrow J_{W,v'_1}=J_{W,v'_2}$. Denote these subsets by $X'_1,X'_2,...,X'_m$. Each X'_i is a connected open subset of X'. Then we clearly have

$$\begin{aligned}\int_{v'\in X'}\int_{v\in W_{v'}} & tr(K_{T_f}(v,1_{f(v)},v,1_{f(v)}))d\alpha(v)d\Lambda'(v')\\ = & \sum_{i=1}^m\int_{v'\in X'_i}\int_{v\in W_{v'}} tr(K_{T_f}(v,1_{f(v)},v,1_{f(v)}))d\alpha(v)d\Lambda'(v')\end{aligned}$$

Let $J=\{j_0,j_1,...,j_N\}$. Set $h(v,v')=tr(K_{T_f}(v,1_{f(v)},v,1_{f(v)}))$ for $v\in W_{v'}$. We want to prove

$$\int_{v'\in X'} h(v,v')d\alpha(v)d\Lambda'(v') = \sum_{l=1}^N\int_{v'\in X'_{j_l}}\int_{v\in W_{j_l,v'}} h(v,v')d\alpha_{j_l}(v)d\Lambda'(v') \qquad (6.2.1)$$

Then we have,

$$\begin{aligned}
\int_{v'\in X'} h(v,v')d\alpha(v)d\Lambda'(v') &= \sum_{i=1}^{m}\int_{v'\in X'_i}\int_{v\in W_{v'}} h(v,v')d\alpha(v)d\Lambda'(v') \\
&= \sum_{i=1}^{m}\int_{v'\in X'_i}\sum_{k\in J_{W,v'}}\int_{v\in W_{k,v'}} h(v,v')d\alpha_k(v)d\Lambda'(v') \\
&= \int_{v'\in X'_1}\sum_{k\in J_{W,v'}}\int_{v\in W_{k,v'}} h(v,v')d\alpha_k(v)d\Lambda'(v') + I_2 \\
&= \sum_{k\in J_{W,v'}}\int_{\pi_k(X'_1)}\int_{v\in W_{k,v'}} h(v,v')d\alpha_k(v)d\Lambda'(v') + I_2
\end{aligned}$$

where

$$I_2 = \sum_{i=2}^{m}\int_{v'\in X'_i}\sum_{k\in J_{W,v'}}\int_{v\in W_{k,v'}} h(v,v')d\alpha_k(v)d\Lambda'(v')$$

and π_k is the projecction onto X'_k. We have used the fact that $X'_1 \cong \pi_k(X'_1)$ in the last line. Now choosing all i such that $j_0 \in J_{W,v'}, v' \in X'_i$ in the above sum, we get

$$\sum_i \int_{X'_{j_0}\cap\pi_{j_0}(X'_i)}\int_{v\in W_{j_0,v'}} h(v,v')d\alpha_k d\Lambda' + \text{ other terms}$$

However, we have $X'_{j_0} = \cup_{i\in I|j_0\in J_{W,v'},v'\in X'_i}(X'_{j_0}\cap\pi_{j_0}(X'_i))$. So the first term in the above line equals

$$\int_{X'_{j_0}}\int_{v\in W'_{j_0,v'}} h(v,v')d\alpha_{j_0}d\Lambda'(v')$$

Therefore we can 'extract' individual terms in the sum appearing in 6.2.1 from the original integral. Since X' is a complete transversal such individual terms corresponding to every index in J can be extracted and no residual terms are left.

$\square$

Proposition 6.2.2. *$\tau_{\Lambda',f}$ satisfies the following property: for $T \in \mathrm{End}_{\Lambda'}(\mathcal{H}(f))^+$ such that $\tau_{\Lambda',f}(T) < \infty$ and $S \in \mathrm{End}_{\Lambda'}(\mathcal{H}(f))^+$ such that TS and ST are operators with smooth, compactly supported Schwartz kernels, we have: $\tau_{\Lambda',f}(ST) = \tau_{\Lambda',f}(TS)$.*

Proof. We note that the Schwartz kernel of TS is given by convolution of the Schwartz kernels of T and S as follows:

$$K_{(TS)_{v'}}((v,\gamma'),(v_1,\gamma'_1)) = \int_{v_2\in L_v}\sum_{\gamma'_2\in\mathcal{G}'^{f(v_2)}_{v'}} K_{T,v'}((v,\gamma'),(v_2,\gamma'_2))K_{S,v'}((v_2,\gamma'_2),(v_1,\gamma'_1))d\lambda^L(v_2) \quad (6.2.2)$$

The Schwartz kernel of ST is given by a similar convolution formula. Therefore we have,

$$\begin{aligned}
\tau_{\Lambda',f}(TS) = \sum_{i\in I}\int_{v'\in X'_i}\int_{v\in W_{i,v'}} \\
\int_{v_2\in L_v}\sum_{\gamma'_2\in\mathcal{G}'^{f(v_2)}_{v'}} tr(K^i_{T,v'}((v,1_{f(v)}),(v_2,\gamma'_2))K_{S,v'}((v_2,\gamma'_2),(v,1_{f(v)})))d\lambda^L(v_2)d\alpha_i(v)d\Lambda'(v') \quad (6.2.3)
\end{aligned}$$

Since support of $K^i_{T,v'} \subseteq U_i$, the right hand side above can be written as

$$\begin{aligned}
\tau_{\Lambda',f}(TS) &= \sum_{i\in I}\int_{v'\in X'_i}\int_{v\in W_{i,v'}}\int_{v_2\in W_{i,v'}} \\
&\quad \sum_{\gamma'_2\in \mathcal{G}'^{f(v_2)}_{v'}} tr(K^i_{T,v'}((v,1_{f(v)}),(v_2,\gamma'_2))K_{S,v'}((v_2,\gamma'_2),(v,1_{f(v)})))d\alpha_i(v_2)d\alpha_i(v)d\Lambda'(v') \\
&= \sum_{i\in I}\int_{v'\in X'_i}\int_{v_2\in W_{i,v'}} \\
&\quad \sum_{\gamma'_2\in \mathcal{G}'^{f(v_2)}_{v'}} \left[\int_{v\in W_{i,v'}} tr(K_{S,v'}((v_2,\gamma'_2),(v,1_{f(v)}))K^i_{T,v'}((v,1_{f(v)}),(v_2,\gamma'_2)))d\alpha_i(v)\right] d\alpha_i(v_2)d\Lambda'(v') \\
&= \sum_{i\in I}\int_{v'\in X'_i}\int_{v_2\in W_{i,v'}} \\
&\quad \left[\int_{v\in W_{i,v'}} \sum_{\gamma'_2\in \mathcal{G}'^{f(v_2)}_{v'}} tr(K_{S,v'}((v_2,1_f(v_2)),(v,\gamma'_2))K^i_{T,v'}((v,\gamma'_2),(v_2,1_{f(v_2)})))d\alpha_i(v)\right] d\alpha_i(v_2)d\Lambda'(v') \\
&= \sum_{i\in I}\int_{v'\in X'_i}\int_{v_2\in W_{i,v'}} tr(K^i_{ST}(v_2,1_{f(v_2)},v_2,1_{f(v_2)}))d\alpha_i(v_2)d\Lambda'(v') \\
&= \tau_{\Lambda',f}(ST)
\end{aligned}$$

where we have used the holonomy invariance of Λ' and the $\mathcal{G}'$-equivariance property for the Schwartz kernels.

□

We define in a similar way a trace functional $\tau^{\mathcal{F}}_{\Lambda',f}$ on $\mathrm{End}_{\Lambda'}(H(f))$ as follows. Let $t_f = (t_{f,v'})_{v'\in V'} \in \mathrm{End}_{\Lambda'}(H(f))$ be such that almost each $t_{f,v'}$ is positive (w.r.t the measure on V') and given by a kernel as follows:

$$t_{f,v'}\xi(v,f(v)) = \int_{v_1\in L_v} k_{t_f,v'}((v,f(v)),(v_1,f(v_1)))\xi(v_1,f(v_1))d\lambda_{L_v}(v_1)$$

where $k_{t_f,v'} \in C^\infty_c(\Gamma(f,v')\times\Gamma(f,v'),Hom(\pi_1^*E,\pi_3^*E))$.

Definition The trace $\tau^{\mathcal{F}}_{\Lambda',f}(t_f)$ of t_f is defined by

$$\tau^{\mathcal{F}}_{\Lambda',f}(t_f) := \sum_{i\in I}\int_{v'\in X'_i}\int_{v\in W_{i,v'}} tr(K^i_{t_f}((v,f(v)),(v,f(v)))d\alpha_i(v)d\Lambda'(v')$$

where $W_{i,v'}$ is the unique plaque in U_i corresponding to the plaque $W'_{\alpha(i),v'}$ in $U'_{\alpha(i)}$ by the image of f and $K^i_{t_f}((v_1,f(v_1)),(v_2,f(v_2))) = \phi_i(v_1)\phi_i(v_2)K_{t_f}((v_1,f(v_1)),(v_2,f(v_2)))$.

A proposition analogous to the two propositions above can be stated for $\tau^{\mathcal{F}}_{\Lambda',f}$ as follows.

Proposition 6.2.3. *(i) The above formula for $\tau^{\mathcal{F}}_{\Lambda',f}$ does not depend on the choices of $(U'_\alpha)_{\alpha\in A}$, $(X'_\alpha)_{\alpha\in A}$, $(U_i)_{i\in I}$, $(X_i)_{i\in I}$ and $(\phi_i^2)_{i\in I}$.*

(ii) For $t \in \mathrm{End}_{\Lambda'}(H(f))^+$ *such that* $\tau^{\mathcal{F}}_{\Lambda',f}(t) < \infty$ *and* $s \in \mathrm{End}_{\Lambda'}(H(f))^+$ *such that* ts *and* st *are positive operators with smooth, compactly supported Schwartz kernels, we have:* $\tau^{\mathcal{F}}_{\Lambda',f}(st) = \tau^{\mathcal{F}}_{\Lambda',f}(ts)$.

Proof. The proofs are similar to the proofs in Proposition 6.2.1 and Proposition 6.2.2. □

Recall the von Neumann algebra defined on the measurable field of Hilbert spaces $\mathcal{H} := (L^2(\mathcal{G}_v, r^*E))_{v\in V}$ where the inner product on $L^2(\mathcal{G}_v, r^*E)$ is given by

$$< \xi_v, \eta_v > = \int_{\alpha\in\mathcal{G}_v} < \xi_v(\alpha), \eta_v(\alpha) >_{E_{r(\alpha)}} d\lambda_v(\alpha) \text{ for } \xi_v, \eta_v \in C_c^\infty(\mathcal{G}_v, r^*E)$$

Then the von Neumann algebra $\mathrm{End}_\Lambda(\mathcal{H})$ is denoted by $W^*(\mathcal{G}, E)$ (see section 2.3.4). This von Neumann algebra has a positive, semifinite, faithful, normal trace τ^Λ.

Proposition 6.2.4. *Let* T *be a positive trace-class operator in* $W^*(\mathcal{G}, E)$ *with compactly smoothing kernel* k_T. *Let* T_f *be the operator in* $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$ *whose Schwartz kernel is given by* $K_{T_f}(v_1, v_2, \gamma') = k_T(\gamma)$, *where* $\gamma \in \mathcal{G}_{v_2}^{v_1}$ *is unique such that* $f(\gamma) = \gamma'$. *Then we have*

$$\tau_{\Lambda',f}(T_f) = \tau^\Lambda(T)$$

Proof. One has $K_{T_f}(v_1, \gamma_1', v_2, \gamma_2') = k_T(\gamma_{12})$, where $\gamma_{12} \in \mathcal{G}_{v_2}^{v_1}$ is unique such that $f(\gamma_{12}) = \gamma_1'\gamma_2'^{-1}$. As T is a Λ-essentially bounded operator, its kernel is also a compactly supported measurable function on $\mathcal{G}$, and therefore K_{T_f} is also a measurable function with compact support viewed as a section on $\mathcal{G}(f)$. Hence T_f is Λ'-essentially bounded.

K_{T_f} is $\mathcal{G}'$-equivariant since we have $K_{T_f}(v_1, \gamma_1'\alpha', v_2, \gamma_2'\alpha') = k_T(\gamma_{12}) = K_{T_f}(v_1, \gamma_1', v_2, \gamma_2')$ since $\gamma_1'\gamma_2'^{-1} = \gamma_1'\alpha'\alpha'^{-1}\gamma_2'^{-1}$, for $\alpha' \in \mathcal{G}'$ such that $r(\alpha') = s(\gamma_1') = s(\gamma_2')$. Therefore T_f interwines the representation of $\mathcal{G}'$. Hence T_f is a positive operator in $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$.

We compute as follows:

$$\begin{aligned}
\tau_{\Lambda',f}(T_f) &= \sum_{i\in I}\int_{v'\in T_i'}\int_{v\in W_{i,v'}} tr(K^i_{T_f}(v, 1_{f(v)}, v, 1_{f(v)}))d\alpha_i(v)d\Lambda'(v') \\
&= \sum_{i\in I}\int_{v'\in \hat{f}(T_i)}\int_{v\in W_{i,v'}} tr(K^i_{T_f}(v, 1_{f(v)}, v, 1_{f(v)}))d\alpha_i(v)d\Lambda'(v') \text{ (since } T_i' = \pi_{\alpha(i)}(\hat{f}(T_i)) \text{)} \\
&= \sum_{i\in I}\int_{\theta\in T_i}\int_{v\in W_{i,\theta}} tr(K^i_{T_f}(v, 1_{f(v)}, v, 1_{f(v)}))d\alpha_i(v)d\Lambda(\theta) \text{ (since } \Lambda' = f_*\Lambda \text{)} \\
&= \sum_{i\in I}\int_{\theta\in T_i}\int_{v\in W_{i,\theta}} \phi_i^2(v)tr(K_T(1_v))d\alpha_i(v)d\Lambda(\theta) \\
&= \tau^\Lambda(T)
\end{aligned}$$

□

A similar proposition relating operators between foliation von Neumann algebra $W^*(V, \mathcal{F}; E)$ and $\mathrm{End}_{\Lambda'}(H(f)$ is given as follows. Recall that $W^*(M, \mathcal{F}; E)$ has a positive, semifinite, faithful, normal trace $\tau^\Lambda_{\mathcal{F}}$.

Proposition 6.2.5. *Let* $t = (t_L)_{L\in V/\mathcal{F}}$ *be a positive trace-class operator in* $W^*(V, \mathcal{F}; E)$ *such that for each* $L \in V/\mathcal{F}$, *the Schwartz kernels* $k_{t,L} \in C_c^\infty(L\times L, E_{|L})$. *Let* t_f *be the operator in* $\mathrm{End}_{\Lambda'}(H(f))$ *whose Schwartz* *kernel is given by* $K_{t_f}((v_1, f(v_1)), (v_2, f(v_2))) = k_{t,L_{v_1}}(v_1, v_2)$*, where* $v_1, v_2 \in V$ *are in the same leaf. Then we have*

$$\tau^{\mathcal{F}}_{\Lambda',f}(t_f) = \tau^{\Lambda}_{\mathcal{F}}(t)$$

Proof. The proof is similar to the proof in 6.2.4.

□

6.3 Operators on Hilbert modules

Recall that for every $v \in V$ we have an isometric isomorphism of Hilbert spaces $\Psi_{v,reg} : \mathcal{E}_{X,E} \otimes_{\mathcal{A}_X^X} l^2(\mathcal{G}_v^X) \to L^2(\mathcal{G}_v, r^*E)$ given by the following formula:

$$[\Psi_{v,reg}(\zeta \otimes \xi)](\gamma) = \sum_{\alpha \in \mathcal{G}_v^X} \xi(\alpha)\zeta(\gamma\alpha^{-1}) \tag{6.3.1}$$

where $\zeta \in \mathcal{E}_X^c, \xi \in l^2(\mathcal{G}_v^X), \gamma \in \mathcal{G}_v$.

Here the representation of $\mathcal{A}_X^X$ on $l^2(\mathcal{G}_v^X)$, $\rho_v^{reg} : \mathcal{A}_{X,c}^X \to B(l^2(\mathcal{G}_v^X))$ is defined as

$$[\rho_v^{reg}(f)](\xi)(\gamma) = \sum_{\gamma_1 \in \mathcal{G}_v^X} \xi(\gamma_1) f(\gamma\gamma_1^{-1}), \tag{6.3.2}$$

Let as before $\mathcal{B}_m^E := C^*(\mathcal{G}, E)$. We have an isomorphism $\chi_m : \mathcal{B}_m^E \to \mathcal{K}_{\mathcal{A}_X^X}(\mathcal{E}_{X.E})$. Then we have (see Proposition 3.3.5):

Proposition 6.3.1. *Let* $v \in V$*. Then we have for* $S \in \mathcal{B}_m^E$*,*

$$\pi_v^{reg}(S) = \Psi_{v,reg} \circ [\chi_m(S) \otimes Id_{B(l^2(\mathcal{G}_v^X))}] \circ \Psi_{v,reg}^{-1}$$

In a similar way, for every $v' \in V'$ there is a representation $\rho_{v',reg}$ of $\mathcal{A}_{X'}^{X'}$ on $l^2(\mathcal{G}'^{X'}_{v'})$. Then we have the interior tensor product $\mathcal{E}^V_{X',E}(f) \otimes_{\rho_{v',reg}} l^2(\mathcal{G}'^{X'}_{v'})$ which is a Hilbert space. Consider the map $\Psi^f_{v',reg} : \mathcal{E}^V_{X',E}(f) \otimes_{\rho_{v',reg}} l^2(\mathcal{G}'^{X'}_{v'}) \to L^2(\mathcal{G}^V_{v'}(f), \pi_1^* E)$ given by

$$[\Psi^f_{v',reg}(\zeta \otimes \xi)](v, \gamma') = \sum_{\alpha' \in \mathcal{G}'^{X'}_{v'}} \xi(\alpha')\zeta(v, \gamma'\alpha'^{-1}) \tag{6.3.3}$$

where $\zeta \in \mathcal{E}^{V,c}_{X',E'}, \xi \in l^2(\mathcal{G}'^{X'}_{v}), \gamma \in \mathcal{G}'_{v'}$ such that $f(v) = r(\gamma')$.

Proposition 6.3.2. $\Psi^f_{v',reg}$ *is a well-defined map and an isometric isomorphism.*

Proof. • $\Psi^f_{v',reg}$ is well-defined:

Let $\delta_{\gamma'}$ denote the delta function at $\gamma' \in \mathcal{G}'^{X'}_{v'}$. Letting $\gamma'\xi(v, \gamma'') = \xi(v, \gamma''\gamma'^{-1})$, we have for $\phi' \in C_c^\infty(\mathcal{G}'^{X'}_{X'})$,

$$\begin{aligned}
\Psi^f_{v',reg}(\xi\phi' \otimes \delta_{\gamma'})(v,\gamma'') &= (\xi\phi')(v,\gamma''\gamma'^{-1}) \\
&= \sum_{\alpha' \in \mathcal{G}'^{X'}_{r(\gamma')}} \xi(v,\gamma''\gamma'^{-1}\alpha'^{-1})\phi'(\alpha') \\
&= \sum_{\beta' \in \mathcal{G}'^{X'}_{x'}} \xi(v,\gamma''\beta'^{-1})\phi'(\beta'\gamma'^{-1}) \\
&= [\xi(\phi' * \delta_{\gamma'})](v,\gamma'') \\
&= [\Psi^f_{v',reg}(\xi \otimes (\phi' * \zeta))](v,\gamma')
\end{aligned}$$

Therefore $\Psi^f_{v',reg}$ is well-defined.

• $\Psi^f_{v',reg}$ is an isometry:

We have for $\xi_1, \xi_2 \in \mathcal{E}^{V,c}_{X',E}(f)$,

$$\begin{aligned}
< \xi_1 \otimes \delta_{\gamma'_1}, \xi_2 \otimes \delta_{\gamma'_2} > &= < \delta_{\gamma'_1}, < \xi_1, \xi_2 > * \delta_{\gamma'_2} > \\
&= < \delta_{\gamma'_1}, \sum_{\beta'} < \xi_1, \xi_2 > (\beta'\gamma_2'^{-1})\delta_{\beta'} \\
&= < \xi_1, \xi_2 > (\gamma'_1\gamma_2'^{-1}) \\
&= \int_{v \in L_{r(\gamma'_1)}} \sum_{\gamma' \in \mathcal{G}'^{f(v)}_{r(\gamma'_1)}} < \xi_1(v,\gamma'), \xi_2(v,\gamma'\gamma'_1\gamma_2'^{-1}) >_{E_v} d\lambda^L(v)
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
< \gamma'_1\xi_1, \gamma'_2\xi_2 > &= \int_{v \in L_{r(\gamma'_1)}} \sum_{\gamma' \in \mathcal{G}'^{f(v)}_{r(\gamma'_1)}} < (\gamma'_1\xi_1)(v,\gamma'), (\gamma'_2\xi_2)(v,\gamma') >_{E_v} d\lambda^L(v) \\
&= \int_{v \in L_{r(\gamma'_1)}} \sum_{\gamma' \in \mathcal{G}'^{f(v)}_{r(\gamma'_1)}} < \xi_1(v,\gamma'\gamma_1'^{-1}), \xi_2(v,\gamma'\gamma_2'^{-1}) >_{E_v} d\lambda^L(v) \\
&= \int_{v \in L_{r(\gamma'_1)}} \sum_{\gamma'' \in \mathcal{G}'^{f(v)}_{r(\gamma'_1)}} < \xi_1(v,\gamma''), \xi_2(v,\gamma''\gamma'_1\gamma_2'^{-1}) >_{E_v} d\lambda^L(v)
\end{aligned}$$

which proves that $\Psi^f_{v',reg}$ is an isometry.

• $\Psi^f_{v',reg}$ is surjective: It suffices to prove this for $v' \in X'$ since there is an isomorphism of Hilbert spaces $L^2(\mathcal{G}^V_{v'_1}(f), \pi_1^*E) \cong L^2(\mathcal{G}^V_{v'_2}(f), \pi_1^*E)$ for v'_1, v'_2 in the same leaf. Consider $\eta \in C_c^\infty(\mathcal{G}^V_{v'}(f), \pi_1^*E)$. Then η can be extended to $\tilde\eta \in C_c^\infty(\mathcal{G}^V_{X'}(f), \pi_1^*E)$. Let $\delta_{v'}$ be the delta function at v', which can be seen as an l^2 function on $\mathcal{G}'^{X'}_{v'}$. Then η is the image of $\tilde\eta \otimes \delta_{v'}$ under $\Psi^f_{v',reg}$. Hence $\Psi^f_{v',reg}$ is surjective. □

Recall that we have an isomorphism $\chi^f_m : \mathcal{B}^E_m \xrightarrow{\cong} \mathcal{K}_{\mathcal{A}^{X'}_{X'}}(\mathcal{E}^V_{X',E}(f))$. We define a representation for $v' \in V'$, $\pi^{f,reg}_{v'} : \mathcal{B}^E_m \to \mathrm{End}_{\Lambda'}(\mathcal{H}(f))$ by the following formula:

$$[\pi^{f,reg}_{v'}(h)\xi](v,\gamma') = \int_{v_1\in L_v} \sum_{\gamma'_1\in \mathcal{G}'^{f(v_1)}_{v'}} h(\gamma_1)\xi(v_1,\gamma'_1)d\lambda^L_v(v_1)$$

where given $v_1 \in L_v, \gamma'_1 \in \mathcal{G}'^{f(v_1)}_{v'}$, γ_1 is the unique element in $\mathcal{G}^v_{v_1}$ such that $f(\gamma_1) = \gamma'\gamma_1'^{-1}$.

Then we have the following proposition:

Proposition 6.3.3. *Let $v' \in V'$ and $S \in \mathcal{B}^E_m$. Then,*

$$\pi^{f,reg}_{v'}(S) = \Psi^f_{v',reg} \circ [\chi^f_m(S) \otimes Id_{B(l^2(\mathcal{G}'^{X'}_{v'}))}] \circ (\Psi^f_{v',reg})^{-1}$$

Proof. We first note that the isomorphism χ^f_m is given by the following formula, for $\phi \in \mathcal{B}^E_{m,c}$, $\zeta \in \mathcal{E}^{V,c}_{X',E}(f)$,

$$\chi^f_m(\phi)(\zeta)(v,\gamma') = \int_{v_1\in L_v} \sum_{\gamma'_1\in \mathcal{G}'^{f(v_1)}_{s(\gamma')}} \phi(\gamma_1)\zeta(v,\gamma'_1)d\lambda^L_v(v_1)$$

where $\gamma_1 \in \mathcal{G}^v_{v_1}$ is unique such that $f(\gamma_1) = \gamma'\gamma_1'^{-1}$. Then computing the left hand side, we get,

$$\begin{aligned}
[\pi^{f,reg}_{v'}(\phi)](\Psi_{v'}(\zeta\otimes\xi)(v,\gamma') &= \int_{v_1\in L_v} \sum_{\gamma'_1\in \mathcal{G}'^{f(v_1)}_{v'}} \phi(\gamma_1)(\Psi_{v'}(\zeta\otimes\xi)(v_1,\gamma'_1)d\lambda^L_v(v_1) \\
&= \int_{v_1\in L_v} \sum_{\gamma'_1\in \mathcal{G}'^{f(v_1)}_{v'}} \phi(\gamma_1) \sum_{\alpha'\in \mathcal{G}'^{X'}_{v'}} \xi(\alpha')\zeta(v_1,\gamma'_1\alpha'^{-1})d\lambda^L_v(v_1)
\end{aligned}$$

Computing the right hand side, we get,

$$\begin{aligned}
(\Psi_{v'}(\chi^f_m(\phi)\otimes I)(\zeta\otimes\xi))(v,\gamma') &= (\Psi_{v'}(\chi_m(\phi)\zeta\otimes\xi))(v,\gamma') \\
&= \sum_{\beta'\in \mathcal{G}'^{X'}_{v'}} \xi(\beta')[\chi^f_m(\phi)\zeta](v,\gamma'\beta'^{-1}) \\
&= \sum_{\beta'\in \mathcal{G}'^{X'}_{v'}} \xi(\beta') \int_{v_1\in L_v} \sum \gamma'_1 \in \mathcal{G}'^{f(v_1)}_{r(\beta')} \phi(\gamma_1)\zeta(v_1,\gamma'_1) \\
&\quad (\ \gamma_1 \in \mathcal{G}^v_{v_1}!\ \text{s.t. } f(\gamma_1) = \gamma'\gamma_1'^{-1}\ ) \\
&= \sum_{\beta'\in \mathcal{G}'^{X'}_{v'}} \xi(\beta') \int_{v_1\in L_v} \sum \gamma'_2 \in \mathcal{G}'^{f(v_1)}_{s(\beta')} \phi(\gamma_2)\zeta(v_1,\gamma'_2\beta'^{-1}) \\
&\quad (\ \gamma'_2 = \gamma'_1\beta', \gamma_2 = \gamma_1 \in \mathcal{G}^v_{v_1}!s.t.f(\gamma_2) =\ \gamma'\gamma_2'^{-1})
\end{aligned}$$

Comparing the last lines of the above computations gives the result. ∎

Now consider the representation $\rho_{v',av}$ of $\mathcal{A}^{X'}_{X'}$ on $l^2(\mathcal{G}'^{X'}_{v'})$ given in Section 3.3.2. Consider the interior tensor product $\mathcal{E}^V_{X',E}(f) \otimes_{\rho_{v',av}} l^2(\mathcal{G}'^{X'}_{v'}/\mathcal{G}'^{v'}_{v'})$ which is a Hilbert space. Consider the map $\Psi^f_{v',av} : \mathcal{E}^{V,c}_{X',E}(f) \otimes_{\rho_{v',av}} l^2(\mathcal{G}'^{X'}_{v'}/\mathcal{G}'^{v'}_{v'}) \to L^2(\Gamma(f,v'),\pi_1^*E)$ given on simple tensors by

$$[\Psi^f_{v',av}(\zeta\otimes\xi)](v,f(v)) = \sum_{\alpha'\in \mathcal{G}'^{X'}_{v'}} \xi([\alpha'])\zeta(v,\gamma'\alpha'^{-1}) \tag{6.3.4}$$

where $[\alpha']$ is the class of α in $\mathcal{G}'^{X'}_{v'}/\mathcal{G}'^{v'}_{v'}$, $\gamma' \in \mathcal{G}'^{f(v)}_{v'}$ and the formula does not depend on the choice of γ'. Then we have

Proposition 6.3.4. *$\Psi^f_{v',av}$ is a well-defined map and an isometric isomorphism.*

Proof. The proof is similar to the proof in 6.3.2. $\square$

We also define a representation $\pi^{f,av}_{v'} : \mathcal{B}^E_m \to \mathrm{End}_{\Lambda'}(H(f))$ by the following formula:

$$[\pi^{f,av}_{v'}(h)\xi](v,f(v)) = \int_{v_1\in L_v} \sum_{\gamma_1\in\mathcal{G}^{v_1}_{v_1}} h(\gamma_1)\xi(v_1,f(v_1))d\lambda^L_v(v_1)$$

Proposition 6.3.5. *Let $v'\in V'$ and $S\in\mathcal{B}^E_m$. Then,*

$$\pi^{f,av}_{v'}(S) = \Psi^f_{v',av}\circ[\chi^f_m(S)\otimes Id_{B(l^2(\mathcal{G}'^{X'}_{v'}/\mathcal{G}'^{v'}_{v'}))}]\circ(\Psi^f_{v',av})^{-1}$$

Proof. The proof is similar to the proof in 6.3.3. $\square$

Let D be the leafwise signature operator on $(V,\mathcal{F})$, $\tilde{D}=(\tilde{D}_v)_{v\in V}$ be its lift on the monodromy groupoid and $\mathcal{D}_m$ be the associated self-adjoint regular operator on $\mathcal{E}_{X,E}$. Then we have the following result relating the functional calculi of $\mathcal{D}_m$ and $\tilde{D}$ (cf. 3.4.4, [BePi:08])

Proposition 6.3.6. *Let $\psi:\mathbb{R}\to\mathbb{R}$ be a bounded continuous function. Then, for each $v\in V$, we have $\psi(\tilde{D}_v)\in\mathcal{B}(L^2(\mathcal{G}_v,r^*E))$, and*

$$\psi(\tilde{D}_v) = \Psi_{v,reg}\circ[\psi(\mathcal{D}_m)\otimes_{\rho^{reg}_v} Id]\circ\Psi^{-1}_{v,reg}$$

Now consider the leafwise graph of f, for $v'\in V'$,

$$\Gamma(f,v') := \{(v,f(v))|v\in V, f(v)\in L'_{v'}\}$$

Then, we define the densely defined closed unbounded operator $(D_f)_{v'} : L^2(\mathcal{G}^V_{v'}(f),\pi_1^*E)\to L^2(\mathcal{G}^V_{v'}(f),\pi_1^*E)$ as the $\mathcal{G}'^{v'}_{v'}$-invariant lift of the operator $\underline{D_f}_{v'} : L^2(\Gamma(f,v'),\pi_1^*E)\to L^2(\Gamma(f,v'),\pi_1^*E)$ which is a densely defined unbounded operator given by:

$$\underline{D_f}\phi(v,f(v)) = D\tilde{\phi}(v) \text{ for } \phi\in C^\infty_c(\Gamma(f,v'),\pi_1^*E)\to C^\infty_c(\Gamma(f,v'),\pi_1^*E) \tag{6.3.5}$$

where $D_v : L^2(L_v,E)\to L^2(L_v,E)$ is the leafwise signature operator on V on the leaf L_v and $\tilde{\phi}(v) = \phi(v,f(v))$. Then one can check that the Schwartz kernel of D_f is given by $K_{D_f}(v_1,v_2,\gamma') = K_{\tilde{D}}(\gamma)$ where $\gamma\in\mathcal{G}^{v_1}_{v_2}$ such that $f(\gamma)=\gamma'$. This follows from the fact that the Schwartz kernel of $\underline{D_f}$ is given by

$$K_{\underline{D_f}}(v_1,f(v_1),v_2,f(v_2)) = \sum_{\gamma'\in\mathcal{G}'^{f(v_1)}_{f(v_2)}} K_{D_f}(v_1,v_2,\gamma')$$

and so we have

$$\begin{aligned} K_{\underline{D_f}}(v_1,f(v_1),v_2,f(v_2)) &= \sum_{\gamma'\in\mathcal{G}'^{f(v_1)}_{f(v_2)}} K_{D_f}(v_1,v_2,\gamma') \\ &= \sum_{\gamma\in\mathcal{G}^{v_1}_{v_2}} K_{\tilde{D}}(\gamma) \\ &= K_D(v_1,v_2) \end{aligned}$$

Therefore by the uniqueness of the Schwartz kernel and that of the lift of the operator $\underline{D_f}$ we see that $K_{D_f}(v_1, v_2, \gamma') = K_{\tilde{D}}(\gamma)$. The ellipticity of the operator $\tilde{D}$ in turn implies that the operator D_f is an elliptic operator (i.e. each $(D_f)_{v'}$ is elliptic for $v' \in V'$) and that for any bounded measurable function g, the Schwartz kernel of $g(D_f)$ is a bounded measurable section and belongs to the von Neumann algebra $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$.

Proposition 6.3.7. *The operator $D_f = ((D_f)_{v'})_{v' \in V'}$ is family of self-adjoint elliptic operators with each $(D_f)_{v'}$ acting on sections over $\mathcal{G}^V_{v'}(f)$. Moreover, for any bounded measurable function g, $g(D_f)$ belongs to the von Neumann algebra $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$.*

Proof. On a local chart, one can express the Schwartz kernel of D_f as the Fourier transform of its symbol. Then from the above remarks it is clear that the local family of symbols for D_f coincides with the family of symbols of D as an endomorphism on E. Since D is elliptic, its family of symbols is invertible, hence the same is true for D_f. Therefore D_f is elliptic.

Since the family of operators $((D_f)_{v'})v' \in V'$ is a measurable family of self-adjoint operators, the measurable family spectral theorem (cf. Theorem XIII.85 in [ReSiIV:78]) then implies that for a bounded measurable function g, we have $g(D_f) = (g(D_f)_{v'})_{v' \in V'}$ is a measurable field of uniformly bounded intertwining operators. Therefore $g(D_f) \in \mathrm{End}_{\Lambda'}(\mathcal{H}(f))$. □

The elliptic operator D_f therefore defines a closable operator $\mathcal{D}_f$ on $\mathcal{E}^V_{X',E}(f)$ which extends to an unbounded regular operator due to the ellipticity of D_f as in Proposition 3.3.7.

Proposition 6.3.8. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then, for each $v \in V$, we have $\psi((D_f)_v) \in \mathcal{B}(L^2(\mathcal{G}^V_{v'}(f), \pi_1^* E))$, and*

$$\psi((D_f)_{v'}) = \Psi^f_{v',reg} \circ [\psi(\mathcal{D}_f) \otimes_{\rho^{reg}_{v'}} Id] \circ (\Psi^f_{v',reg})^{-1} = \pi^f_{v',reg} \circ (\chi^f_m)^{-1}(\psi(\mathcal{D}_f))$$

Proof. The first equality is proved as in the proof of Proposition 3.4.4 and the second equality is a corollary of Proposition 6.3.3. □

Parallelly, we have similar results for the average representations and the operator $\underline{D}_f$.

Proposition 6.3.9. *We have,*

(i) The operator $\underline{D}_f = ((\underline{D}_f)_{v'})_{v' \in V'}$ is a family of self-adjoint elliptic operators with each $(\underline{D}_f)_{v'}$ acting on sections of $\Gamma(f, v')$. Moreover, for any bounded measurable function g, $g(\underline{D}_f)$ belongs to the von Neumann algebra $\mathrm{End}_{\Lambda'}(H(f))$.

(ii) Let $\psi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then, for each $v \in V$, we have $\psi((\underline{D}_f)_v) \in \mathcal{B}(L^2(\Gamma(f, v'), \pi_1^* E))$, and

$$\psi((\underline{D}_f)_{v'}) = \Psi^f_{v',av} \circ [\psi(\mathcal{D}_f) \otimes_{\rho^{av}_{v'}} Id] \circ (\Psi^f_{v',av})^{-1} = \pi^f_{v',av} \circ (\chi^f_m)^{-1}(\psi(\mathcal{D}_f))$$

6.4 Determinants and the Large time path

6.4.1 Determinants of paths

Using the representation $\pi^{f,reg} : \mathcal{B}^E_m \to \mathrm{End}_{\Lambda'}(\mathcal{H}(f))$ and the isomorphism $\chi^f_m : \mathcal{B}^E_m \to \mathcal{K}_{\mathcal{A}^{X'}_{X'}}(\mathcal{E}^V_{X',E})$ we can define a map $\sigma^{f,reg} : \mathcal{IK}_{\mathcal{A}^{X'}_{X'}}(\mathcal{E}^V_{X',E}) \to \mathcal{IK}(\mathrm{End}_{\Lambda'}(\mathcal{H}(f)))$ by $\sigma^{f,reg} := \pi^{f,reg} \circ (\chi^f_m)^{-1}$. In addition, we note that for a Schwartz function ψ, $\sigma^{f,reg}(\psi(\mathcal{D}_f)) = \psi(D_f)$ is a $\tau_{\Lambda',f}$-trace class operator in $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$. We denote the determinant on $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$ by $w^f_{\Lambda'}$.

Let $\mathcal{B}_t, a \leq t \leq b$ be a norm continuous path of operators in $\mathcal{IK}_{\mathcal{A}^{X'}_{X'}}(\mathcal{E}^V_{X',E})$ such that the end points $\mathcal{B}_a$ and $\mathcal{B}_b$ map to $\tau_{\Lambda',f}$-trace class operators in $\mathrm{End}_{\Lambda'}(\mathcal{H}(f))$. Then we define the determinant w^f of this path as follows:

Definition $w^f((\mathcal{B}_t)_{a\leq t\leq b}) := w^f_{\Lambda'}(\sigma^{f,reg}((\mathcal{B}_t)_{a\leq t\leq b}))$

Similarly, one can define a map $\sigma^{f,av} : \mathcal{IK}_{\mathcal{A}^{X'}_{X'}}(\mathcal{E}^V_{X',E}) \to \mathcal{IK}(\mathrm{End}_{\Lambda'}(H(f)))$ by $\sigma^{f,av} := \pi^{f,av} \circ (\chi^f_m)^{-1}$ and another determinant $w^f_{\mathcal{F}}$ of $\mathcal{B}_t$ is defined as

Definition $w^f_{\mathcal{F}}((\mathcal{B}_t)_{a\leq t\leq b}) := w^f_{\Lambda',\mathcal{F}}(\sigma^{f,av}((\mathcal{B}_t)_{a\leq t\leq b}))$

where $w^f_{\Lambda',\mathcal{F}}$ is the determinant on $\mathrm{End}_{\Lambda'}(H(f))$.

6.4.2 The Large time path

Our goal in this section is to furnish a path of operators $\psi_\epsilon(\mathcal{D}'_m) \oplus \psi_\epsilon(\mathcal{D}_f)$ on the Hilbert module $\mathcal{J} := \mathcal{E}_{X',E'} \oplus \mathcal{E}^V_{X',E}(f)$, and to compute its determinant as defined in the Section 4.2. This path of operators will connect $-\exp(i\pi\phi_\epsilon(\mathcal{D}_f)) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}'))$ to the identity on $\mathcal{J}$, where

$$\phi_\epsilon(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-t^2} dt$$

Recall that we have a homotopy equivalence f^*_ϕ between Hilbert-Poincaré complexes $(\mathcal{E}_{X',E'}, d'_{X'}, T'_{X'})$ and $(\mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f), d_X \otimes I, T_X \otimes I)$, where E, E' are the longitudinal exterior bundles on V and V', respectively, and T_X (resp. $T'_{X'}$) is the lift of the Hodge $*$-operator along the leaves on $\mathcal{G}_X$ (resp. $\mathcal{G}_{X'}$). Since there is an isometric isomorphism of Hilbert modules $\Phi : \mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f) \cong \mathcal{E}^V_{X',E}(f)$ which is also a chain map, we have a homotopy equivalence A_f between the chain complexes $(\mathcal{E}_{X',E'}, d'_{X'}, T'_{X'})$ and $(\mathcal{E}^V_{X',E}(f), d_f, T_f)$, where d_f and T_f correspond through conjugation by the isomorphism Φ to $d_X \otimes I$ and $T_X \otimes I$, respectively.

We denote by S the grading operator which is defined on k-forms of $\mathcal{E}_{X,E_k}$ as,

$$S = i^{k(k-1)+l} T_X$$

Denote the operator on $\mathcal{E}^V_{X',E}(f)$ corresponding to $S \otimes I$ on $\mathcal{E}_{X,E} \otimes \mathcal{E}^X_{X'}(f)$ by S_f. Similarly, define the grading operator S' on $\mathcal{E}_{X',E'_p}$

Now we define, as in [KeI:00], [HiRoI:05], a path of grading operators on $\mathcal{J}_k := \mathcal{E}_{X',E'_k} \oplus \mathcal{E}^V_{X',E}(f)$:

$$\Sigma_1(t) = \begin{pmatrix} tA^*_f S_f A_f + (1-t)S' & 0 \\ 0 & -S_f \end{pmatrix} (0 \leq t \leq 1)$$

$$\Sigma_2(t) = \begin{pmatrix} -\cos(\pi t)A^*_f S_f A_f & \sin(\pi t)A^*_f S_f \\ \sin(\pi t) S_f A_f & \cos(\pi t) S_f \end{pmatrix} (1 \leq t \leq \frac{3}{2})$$

$$\Sigma_3(t) = \begin{pmatrix} 0 & e^{2\pi i t} A^*_f S_f \\ e^{2\pi i t} S_f A_f & 0 \end{pmatrix} (\frac{3}{2} \leq t \leq 2)$$

We denote by $\Sigma(t)$ the concatenation of the paths $\Sigma_1(t), \Sigma_2(t), \Sigma_3(t)$. Let $\mathbf{B} = \begin{pmatrix} B' & 0 \\ 0 & B_f \end{pmatrix}$ where $B' = d'_{X'} + (d'_{X'})^*$, and B_f corresponds to the operator $d_X \otimes I + (d_X)^* \otimes I$.

Lemma 6.4.1. *The operators $\mathbf{B} \pm \Sigma(t)$ are invertible for all $t \in [0, 2]$.*

Proof. We have for $t \in [0,1], \mathbf{B} + \Sigma(t) = \begin{pmatrix} B' + tA_f^* S_f A + (1-t)S' & 0 \\ 0 & B_f - S_f \end{pmatrix}$

Consider the mapping cone complex of the chain map $K = tA_f^* S_f A + (1-t)S' : (\mathcal{E}'_k, d'_{X'}) \to (\mathcal{E}'_{p-k}, -(d'_{X'})^*)$. Its differential is

$$d_K = \begin{pmatrix} d'_{X'} & 0 \\ K & (d'_{X'})^* \end{pmatrix}$$

Since K is an isomorphism on cohomology, its mapping cone complex is acyclic, i.e. all the cohomology groups are zero. Therefore the operator $B_K = d_K + d_K^*$ is invertible on $\mathcal{J}$. Now, $B_K = \begin{pmatrix} B' & K \\ K & B' \end{pmatrix}$

As $B' + K$ identifies with B_K on the $+1$ eigenspace of the involution which interchanges the copies of $\mathcal{J}$. Thus $B + K$ is an invertible operator. We can use similar arguments to show the invertibility of the operators $\mathbf{B} + \Sigma(t)$ for $t \in [1, 2]$. ∎

Then define the path of operators

$$\mathcal{W}(t) = \left\{ \begin{array}{l} (\mathbf{B} + \Sigma(t))(\mathbf{B} - \Sigma(t))^{-1} \text{ for } 0 \le t \le \frac{3}{2}, \text{and} \\ \\ (\mathbf{B} + \mathbf{e}(t)\Sigma(t))(\mathbf{B} - \Sigma(t))^{-1} \text{ for } \frac{3}{2} \le t \le 2 \end{array} \right\} \tag{6.4.1}$$

where

$$\mathbf{e}(t) = -\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix}$$

Since $(\mathcal{D} + iI)(\mathcal{D} - iI)^{-1} = (iBS + iI)(iBS - iI)^{-1} = (B + S)SS^{-1}(B - S)^{-1} = (B + S)(B - S)^{-1}$,we have $\mathcal{W}(0) = \mathcal{U}' \oplus \mathcal{U}_f$, where $\mathcal{U}'$ (resp. $\mathcal{U}_f$) is the Cayley transform of $\mathcal{D}'$ (resp. $\mathcal{D}_f$).

We also have, $\mathcal{W}(2) = Id_{\mathcal{J}}$. So the path $\mathcal{W}(t)$ connects $\mathcal{U}' \oplus \mathcal{U}_f$ to the identity. Recall that our goal is to connect $-\exp(i\pi\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}))$ to the identity, where

$$\phi_\epsilon(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-t^2} dt$$

To this end, we will connect $\mathcal{U}'_\epsilon \oplus \mathcal{U}_{f,\epsilon}$ to $-\exp(i\pi\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_f))$ using chopping functions, where $\mathcal{U}'_\epsilon$ (resp.$\mathcal{U}_{f,\epsilon}$) is the Cayley transform of $\frac{1}{\epsilon}\mathcal{D}'$ (resp. $\frac{1}{\epsilon}\mathcal{D}_f$).

Recall that a chopping function is a continuous odd function on $\mathbb{R}$ which tends to ± 1 at $\pm\infty$ and absolute value bounded by 1. We note that $\mathcal{U}$ can be written as $-\exp(i\pi\chi(\mathcal{D}))$ where $\chi(x) = \frac{2}{\pi}\arctan(x)$ which is a chopping function. Since $\phi_\epsilon(x)$ is also a chopping function, there is a linear homotopy between the two. Let

$$\gamma_\epsilon(t, s) = (1 - s)\phi_\epsilon(t) + s\chi_\epsilon(t) \qquad (\chi_\epsilon(t) = \chi(\frac{t}{\epsilon}))$$

be this linear homotopy. Then, $\Gamma_\epsilon(s) = -\exp(i\pi\gamma_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\gamma_\epsilon(\mathcal{D}_f))$ is a path of operators that connects $\mathcal{U}' \oplus \mathcal{U}_f$ to $-\exp(i\pi\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_f))$. Let $\mathcal{W}_\epsilon(t)$ be the path $\mathcal{W}(t)$ with $\mathbf{B}$ replaced by $\frac{1}{\epsilon}\mathbf{B}$.

Definition The concatenation of the paths $\mathcal{W}_\epsilon(t)$ and $\Gamma_\epsilon(s)$ gives a continuous path of operators that connects $-\exp(i\pi\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_f))$ to the identity on $\mathcal{J}$. We call this the large time path and denote it by LT_ϵ.

6.4.3 The determinant of the large time path

In this section our goal is to show that the large time path as defined in the previous section has a well-defined Fuglede-Kadison determinant and thereby calculate this determinant.

Consider the continuous field of Hilbert spaces $J = (J_{v'} := L^2(\mathcal{G}'_{v'}, r^*E'))_{v'\in V'} \oplus L^2(\mathcal{G}^V_{v'}(f), \pi_1^*E)$. From the discussion in the previous sections, there is an isometric isomorphism of Hilbert spaces

$$\Xi_{v',reg} : \mathcal{J} \otimes l^2(\mathcal{G}'^{X'}_{v'}) \to J_{v'}$$

As before, we have a von Neumann algebra $End_{\Lambda'}(J)$, which we denote by $W^*(f)$. The trace of an element $T \in W^*(f)^+$ which is of the form $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ such that T_{11} is $\tau_{\Lambda',f}$-trace class and T_{22} is $\tau^{\Lambda'}$-trace class is given by

$$\tilde{\tau}_{\Lambda',f}(T) = \tau_{\Lambda',f}(T_{11}) + \tau^{\Lambda'}(T_{22})$$

We define a map $\varpi_{reg} : \mathcal{K}(\mathcal{J}) \to W^*(f)$ as follows: for $\mathcal{T} \in \mathcal{K}(\mathcal{J})$, $\varpi(\mathcal{T})$ is a family of operators $(T_{v'})_{v'\in V'}$ such that $T_{v'} = \varpi_{v',reg}(\mathcal{T})$, where

$$T'_v(\mathcal{T}) := \Xi_{v',reg} \circ (\mathcal{T} \otimes I) \circ (\Xi_{v',reg})^{-1}$$

Lemma 6.4.2. *(i) The map $\varpi_{reg} : \mathcal{K}(\mathcal{J}) \to W^*(f)$ is well-defined, i.e. $\varpi_{reg}(\mathcal{K}(\mathcal{J})) \subseteq W^*(f)$.*

(ii) Further, $\varpi_{reg}(\mathcal{K}(\mathcal{J})) \subseteq \mathcal{K}W^*(f)$, where $\mathcal{K}W^*(f)$ is the set of compact operators in the von Neumann algebra $W^*(f)$.

Proof. i) We prove the following two properties:

a) for $v'_1, v'_2 \in V', \gamma' \in \mathcal{G}'^{v'_2}_{v'_1}$, we have $\varpi_{v'_1,reg}(\mathcal{T}) \circ U_{\gamma'} = U_{\gamma'} \circ \varpi_{v'_2,reg}(\mathcal{T})$

b) Ess-sup$_{\Lambda'}||\varpi_{v',reg}(\mathcal{T})|| < \infty$

a) We note that for $v'_1, v'_2 \in V', \gamma' \in \mathcal{G}'^{v'_2}_{v'_1}$, we have $U_{\gamma'} \circ \Xi_{v'_2,reg} = \Xi_{v'_1,reg}$. Then we have,

$$\begin{aligned}
\varpi_{v'_1,reg}(\mathcal{T}) \circ U_{\gamma'} &= (\Xi_{v'_1,reg} \circ (\mathcal{T} \otimes I) \circ (\Xi_{v'_1,reg})^{-1}) \circ U_{\gamma'} \\
&= (U_{\gamma'} \circ \Xi_{v'_2,reg}) \circ (\mathcal{T} \otimes I) \circ (\Xi_{v'_2,reg})^{-1} \\
&= U_{\gamma'} \circ \varpi_{v'_2,reg}(\mathcal{T})
\end{aligned}$$

b) Denote by $I_{v'}$ the identity on $B(l^2(\mathcal{G}'^{X'}_{v'}))$. Let $\xi \in L^2(J)$ be a measurable section and $\xi_{v'}$ be its restriction to $L^2(J_{v'})$. Then we have,

$$\begin{aligned}
| < \varpi_{v',reg}(\mathcal{T})\xi_{v'}, \xi_{v'} > | &= | < \Xi_{v',reg} \circ (\mathcal{T} \otimes I_{v'}) \circ (\Xi_{v',reg})^{-1})\xi_{v'}, \xi_{v'} > | \\
&= | < \Xi_{v',reg} \circ (\mathcal{T} \otimes I_{v'}) \circ (\Xi_{v',reg})^{-1})\xi_{v'}, \Xi_{v',reg}\Xi^{-1}_{v',reg}\xi_{v'} > | \\
&= | < (\mathcal{T} \otimes I_{v'}) \circ (\Xi_{v',reg})^{-1})\xi_{v'}, \Xi^{-1}_{v',reg}\xi_{v'} > | \text{ (since } \Xi_{v'} \text{ is an isometry)} \\
&\leq ||\mathcal{T} \otimes I_{v'}||||(\Xi_{v',reg})^{-1})\xi_{v'}||^2 \\
&\leq ||\mathcal{T}||||(\Xi_{v',reg})^{-1})\xi_{v'}||^2
\end{aligned}$$

Therefore, we get $||\varpi_{v',reg}(\mathcal{T})|| \leq ||\mathcal{T}||$ for all $v' \in V'$. The result is then immediate.

(ii) Let $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in L^2(J)$ such that $\eta_1, \xi_1 \in C_c^\infty(\mathcal{G}_{v'}^V(f), \pi_1^* E)$, and $\eta_2, \xi_2 \in C_c^\infty(\mathcal{G}'_{v'}, r^* E')$. Define the operator $\Theta_{\eta,\xi} \in \mathcal{K}(\mathcal{J})$ by

$$\Theta_{\eta,\xi} = \begin{pmatrix} \theta_{\eta_1,\xi_1} & \theta_{\eta_1,\xi_2} \\ \theta_{\eta_2,\xi_1} & \theta_{\eta_2,\xi_2} \end{pmatrix}$$

where $\theta_{\eta_i,\xi_j}(\zeta_j) = \eta_i. < \xi_j, \zeta_j >$, for $i, j \in \{1, 2\}$. Then $\varpi_{v',reg}(\Theta_{\eta,\xi})$ is of the form

$$\varpi_{v',reg}(\Theta_{\eta,\xi}) = \begin{pmatrix} (\pi_{v'}^{f,reg} \circ (\chi_m^f)^{-1})(\theta_{\eta_1,\xi_1}) & \bullet \\ \bullet & (\pi_{v',reg} \circ \chi_m^{-1})(\theta_{\eta_2,\xi_2}) \end{pmatrix}$$

However, since η_1, ξ_1 are smooth sections with compact support, $(\pi_{v'}^{f,reg} \circ (\chi_m^f)^{-1})(\theta_{\eta_1,\xi_1})$ is $\tau_{\Lambda',f}$-trace class, and similary $(\pi_{v',reg} \circ \chi_m^{-1})(\theta_{\eta_2,\xi_2})$ is $\tau^{\Lambda'}$-trace class. Hence $\varpi_{v',reg}(\Theta_{\eta,\xi}) \in \mathcal{K}W^*(f)$. Since operators of the form $\Theta_{\eta,\xi}$ generate $\mathcal{K}(\mathcal{J})$, we get the result.

□

The map ϖ_{reg} induces a map $\mathcal{IK}(\mathcal{J}) \to \mathcal{IK}W^*(f)$. Unless there is ambiguity we will use the same notation ϖ for the induced map. Now consider the large time path LT_ϵ defined in the previous section, which is a path of operators in $\mathcal{L}(\mathcal{J})$ which connects $-\exp(i\pi\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_f))$ to the identity on $\mathcal{J}$. In fact, since $(\mathbf{B} \pm i)^{-1} \in \mathcal{K}(\mathcal{J})$, we see that the operators $(\mathbf{B} - \Sigma(t))^{-1} \in \mathcal{K}(\mathcal{J})$. Hence $(\mathbf{B} + \Sigma(t))(\mathbf{B} - \Sigma(t))^{-1} = Id_{\mathcal{J}} - 2\Sigma(t)(\mathbf{B} - \Sigma(t))^{-1} \in \mathcal{IK}(\mathcal{J})$ for $t \in [0, 2]$. We also note that the derivatives of $\phi_\epsilon(x)$ and $\chi(x)$ are Schwartz functions. Therefore $1 - \exp(i\pi\gamma_\epsilon(x, s))$ is a Schwartz class function, and hence the path of operators $(\Gamma_\epsilon(s))_{s\in[0,1]}$ lies entirely in $\mathcal{IK}(\mathcal{J})$.

So the large time path LT_ϵ consists entirely of operators in $\mathcal{IK}(\mathcal{J})$. Its image under ϖ_{reg} lies in $\mathcal{IK}W^*(f)$ and has end-points which are trace-class perturbations of the identity. Therefore there is a well-defined determinant of this path which is given as

$$\omega_{reg}(LT_\epsilon) = \tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_\epsilon))$$

where $\tilde{w}^{\Lambda',f}$ is the determinant for a path in $\mathcal{IK}W^*(f)$ associated with the trace $\tilde{\tau}_{\Lambda',f}$.

Proposition 6.4.3. *Let $V_\epsilon(\mathcal{D}_f)$ (resp. $V_\epsilon(\mathcal{D}'_m)$) be the path $(\psi_t(\mathcal{D}_f)_{\epsilon\leq t\leq 1/\epsilon}$ (resp. $(\psi_t(\mathcal{D}'_m)_{\epsilon\leq t\leq 1/\epsilon})$. We have*

$$(\tilde{w}^{\Lambda',f} \circ \varpi_{reg}) \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix} = (w^{\Lambda'} \circ \pi^{reg} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}'_m)) - (w^\Lambda \circ \pi^{reg} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}_m))$$

Proof. We have from the definition of ϖ_{reg},

$$\varpi_{reg} \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix} = \begin{pmatrix} (\pi^{reg} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}'_m)) & 0 \\ 0 & (\pi^{f,reg} \circ (\chi_m^f)^{-1})V_\epsilon(-\mathcal{D}_f) \end{pmatrix}$$

Since the trace $\tilde{\tau}_{\Lambda',f}$ is given by $\tilde{\tau}_{\Lambda',f} = \tau^{\Lambda'} \oplus \tau_{\Lambda',f}$, we easily get

$$(\tilde{w}^{\Lambda',f} \circ \varpi_{reg}) \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix} = (w^{\Lambda'} \circ \pi^{reg} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}'_m)) - (w^{\Lambda',f} \circ \pi^{f,reg} \circ (\chi_m^f)^{-1})(V_\epsilon(\mathcal{D}_f))$$

However, from Proposition 6.2.4 and the definition of the determinants, we get $(w^{\Lambda',f} \circ \pi^{f,reg} \circ (\chi_m^f)^{-1})(V_\epsilon(\mathcal{D}_f)) = (w^\Lambda \circ \pi^{reg} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}_m))$. This finishes the proof.

□

Similarly we define a von Neumann algebra associated to the field of Hilbert spaces $\underline{J} = (L^2(\mathcal{G}'_{v'}, r^*E'))_{v'\in V'} \oplus L^2(\Gamma(f, v';, \pi_1^*E))$ which we denote by $W^*_{\mathcal{F}}(f)$. We also define, analogous to ϖ_{reg}, a map $\varpi_{av} : \mathcal{K}(\mathcal{J}) \to \mathcal{K}W^*_{\mathcal{F}}(f)$ given by

$$\varpi_{v',av}(\mathcal{T}) := \Xi_{v',av} \circ (\mathcal{T} \otimes I) \circ (\Xi_{v',av})^{-1} \text{ for } \mathcal{T} \in \mathcal{K}(\mathcal{J})$$

where $\Xi_{v',av}$ is an isometric isomorphism of Hilbert spaces

$$\Xi_{v',av} : \mathcal{J} \otimes l^2(\mathcal{G}'^{X'}_{v'}/\mathcal{G}'^{v'}_{v'}) \to \underline{J}_{v'}$$

Using the trace $\tilde{\tau}^{\mathcal{F}}_{\Lambda',f}$ on $W^*_{\mathcal{F}}(f)$ one can define a determinant $\tilde{w}^{\Lambda',f}_{\mathcal{F}}$, and we define another determinant of the Large Time Path

$$\omega_{av}(LT_\epsilon) = \tilde{w}^{\Lambda',f}_{\mathcal{F}}(\varpi_{av}(LT_\epsilon))$$

We have the following proposition similar to Proposition 6.4.3:

Proposition 6.4.4. *Let $V_\epsilon(\mathcal{D}_f)$ and $V_\epsilon(\mathcal{D}'_m))$ be as before. We have*

$$(\tilde{w}^{\Lambda',f}_{\mathcal{F}} \circ \varpi_{av}) \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix} = (w^{\Lambda'}_{\mathcal{F}} \circ \pi^{av} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}'_m)) - (w^{\Lambda}_{\mathcal{F}} \circ \pi^{av} \circ \chi_m^{-1})(V_\epsilon(\mathcal{D}_m))$$

Then as a consequence of Proposition 6.4.3, Proposition 6.4.4 and Corollary 4.2.6 we have

Corollary 6.4.5. *The following relation holds*

$$\rho_\Lambda(D) - \rho_{\Lambda'}(D') = 2 \times \lim_{\epsilon\to 0}(\tilde{w}^{\Lambda',f} \circ \varpi_{reg} - \tilde{w}^{\Lambda',f}_{\mathcal{F}} \circ \varpi_{av}) \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix}$$

Proof. We get the result easily by first subtracting the left hand sides of the equations in the statements of Propositions 6.4.3 and 6.4.4 and then applying Corollary 4.2.6. □

We now estimate the determinant of the Large Time Path with the following proposition.

Proposition 6.4.6. *We have, $\tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_\epsilon)) \to 0$ and $\tilde{w}^{\Lambda',f}_{\mathcal{F}}(\varpi_{av}(LT_\epsilon)) \to 0$ as $\epsilon \downarrow 0$.*

Proof. We shall prove the result for $\tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_\epsilon))$, the proof for $\tilde{w}^{\Lambda',f}_{\mathcal{F}}(\varpi_{av}(LT_\epsilon))$ is similar. Recall that LT_ϵ is the concatenation of the paths $\mathcal{W}_\epsilon(t) = \mathcal{W}(\frac{t}{\epsilon})$ and $\Gamma_\epsilon(s)$, where $\mathcal{W}_\epsilon(t)$ connects $-\exp(i\pi\chi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\chi_\epsilon(\mathcal{D}_f)) = \mathcal{U}'_\epsilon \oplus \mathcal{U}^{-1}_{f,\epsilon}$ to the identity, and $\Gamma_\epsilon(s)$ connects $-\exp(i\pi\chi_\epsilon(\mathcal{D}')) \oplus -\exp(-i\pi\chi_\epsilon(\mathcal{D}_f))$ to $-\exp(2\pi i\phi_\epsilon(\mathcal{D}')) \oplus -\exp(-2\pi i\phi_\epsilon(\mathcal{D}_f))$.

Let $\tilde{\mathbf{\Pi}} \in \mathcal{K}(\mathcal{J})$ denote the projection onto the kernel of $\mathbf{B}$, whose image under ϖ is a τ^J-trace class operator in $\mathcal{K}(W^*(f))$. Then, we claim that $\varpi(LT_\epsilon)$ converges strongly to the path

$$\mathcal{W}_\infty(t) = \begin{cases} \varpi(-\tilde{\mathbf{\Pi}} + \tilde{\mathbf{\Pi}}^\perp), & -1 \le t \le \frac{3}{2} \\ \varpi(-\mathbf{e}(t)\tilde{\mathbf{\Pi}} + \tilde{\mathbf{\Pi}}^\perp), & \frac{3}{2} \le t \le 2 \end{cases} \tag{6.4.2}$$

where

$$\mathbf{e}(t) = -\begin{pmatrix} e^{2\pi it} & 0 \\ 0 & e^{-2\pi it} \end{pmatrix}$$

To see this, we first claim that the path $\Gamma_\epsilon(s), s \in [0,1]$ converges strongly to $\mathcal{W}_\infty(t), t \in [-1, \frac{3}{2}]$ as $\epsilon \to 0$. We have the following lemma:

Lemma 6.4.7. *As $\epsilon \to 0$ the image under ϖ_{reg} of $\mathcal{U}'_\epsilon \oplus \mathcal{U}_{f,\epsilon}^{-1}$ and $\exp(i\pi\phi_\epsilon(\mathcal{D})) \oplus -\exp(-2\pi i\phi_\epsilon(\mathcal{D}_f))$ converge strongly to $\varpi_{reg}(-\tilde{\mathbf{\Pi}} + \tilde{\mathbf{\Pi}}^\perp)$.*

Proof. This is a consequence of the spectral theorem and the fact that the functions $(x \pm i\epsilon)(x \mp i\epsilon)^{-1}$ and $\exp(i\pi\phi_\epsilon(x))$ converge to $1 - 2\chi_0(x)$ as $\epsilon \to 0$. ☐

Therefore the above lemma implies that the path $\varpi_{reg}(\Gamma_\epsilon(s))$ converges strongly to $\varpi_{reg}(-\tilde{\mathbf{\Pi}} + \tilde{\mathbf{\Pi}}^\perp)$. Now, extend the path $\Sigma(t)$ by setting $\Sigma(t) = \Sigma(0)$ for $-1 \leq t \leq 0$. We define an operator

$$\Sigma'(t) = \begin{cases} \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}} - t\tilde{\mathbf{\Pi}}^\perp\Sigma(t)\tilde{\mathbf{\Pi}}^\perp, & -1 \leq t \leq 0 \\ \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}, & 0 \leq t \leq \frac{3}{2} \end{cases}$$

We consider the following path of operators

$$\mathcal{W}'(t) = \begin{cases} (\mathbf{B} + \Sigma'(t))(\mathbf{B} - \Sigma'(t))^{-1} \text{ for } -1 \leq t \leq \frac{3}{2}, \text{and} \\ \\ (\mathbf{B} + \mathbf{e}(t)\Sigma'(\frac{3}{2}))(\mathbf{B} - \Sigma'(\frac{3}{2}))^{-1} \text{ for } \frac{3}{2} \leq t \leq 2 \end{cases}$$

We claim that the image under ϖ_{reg} of the path $\mathcal{W}'_\epsilon(t)$, which is the path $\mathcal{W}'(t)$ in which $\mathbf{B}$ is replaced by $\frac{1}{\epsilon}\mathbf{B}$, converges strongly to the path $\mathcal{W}_\infty(t)$. We note here for future use that $\mathcal{W}_\infty(t)$ connects $\varpi_{reg}(\tilde{\mathbf{\Pi}} + \tilde{\mathbf{\Pi}}^\perp)$ to the identity, and that the operators $\tilde{\Sigma}(s,t) = (1-s)\Sigma(t) + s\Sigma'(t)$ give a fixed-point homotopy between the paths $\mathcal{W}'(t)$ and $\mathcal{W}(t)$, so that $\mathcal{W}'(t)$ and $\mathcal{W}(t)$ have the same determinant.

From now onwards, unless stated otherwise, we use the same notation for an operator and its image under ϖ_{reg}. To prove the claim, we compress $\mathcal{W}'(t)$ to the range of $\tilde{\mathbf{\Pi}}$. We have , for $-1 \leq t \leq \frac{3}{2}, \tilde{\mathbf{\Pi}}\Sigma'(t)\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}$. Then, we get for $-1 \leq t \leq \frac{3}{2}$

$$\tilde{\mathbf{\Pi}}(\mathbf{B} + \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}\tilde{\mathbf{\Pi}} \tag{6.4.3}$$

But as $(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})\tilde{\mathbf{\Pi}} = -\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}$, we have

$$-\tilde{\mathbf{\Pi}} = (\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}$$

where we have used the fact that $\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}$ and $(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}$ are self-adjoint. So $\tilde{\mathbf{\Pi}}(\mathbf{B} + \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})(\mathbf{B} - \tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}})^{-1}\tilde{\mathbf{\Pi}} = -\tilde{\mathbf{\Pi}}\tilde{\mathbf{\Pi}} = -\tilde{\mathbf{\Pi}}$, for $-1 \leq t \leq \frac{3}{2}$.

For $\frac{3}{2} \leq t \leq 2$, since $\tilde{\mathbf{\Pi}}\Sigma'(\frac{3}{2})\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\Sigma(\frac{3}{2})\tilde{\mathbf{\Pi}}$, the above arguments can be applied again to get

$$\tilde{\mathbf{\Pi}}\mathcal{W}'(t)\tilde{\mathbf{\Pi}} = -\mathbf{e}(t)\tilde{\mathbf{\Pi}}$$

Thus we have $\tilde{\mathbf{\Pi}}\mathcal{W}'(t)\tilde{\mathbf{\Pi}} = \tilde{\mathbf{\Pi}}\mathcal{W}_\infty(t)\tilde{\mathbf{\Pi}}$, for $-1 \leq t \leq 2$.

Now we compress $\mathcal{W}'(t)$ to the range of $\tilde{\mathbf{\Pi}}^\perp$. For $-1 \leq t \leq 0$, we have

$$\tilde{\mathbf{\Pi}}^\perp\Sigma'(t)\tilde{\mathbf{\Pi}}^\perp = -t\tilde{\mathbf{\Pi}}^\perp\Sigma(t)\tilde{\mathbf{\Pi}}^\perp$$

Therefore,

$$\begin{aligned}\tilde{\mathbf{\Pi}}^{\perp}\mathcal{W}'(t)\tilde{\mathbf{\Pi}}^{\perp} &= \tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B}+\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}-t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp})(\mathbf{B}-\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}+t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp})^{-1}\tilde{\mathbf{\Pi}}^{\perp} \\ &= \tilde{\mathbf{\Pi}}^{\perp}-2t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B}+\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}+t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp})^{-1}\end{aligned}$$

Now, $\tilde{\mathbf{\Pi}}^{\perp}\mathcal{W}'_{\epsilon}(t)\tilde{\mathbf{\Pi}}^{\perp} = \tilde{\mathbf{\Pi}}^{\perp}-2t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp}(\epsilon^{-1}\mathbf{B}+\tilde{\mathbf{\Pi}}\Sigma(t)\tilde{\mathbf{\Pi}}+t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp})^{-1}$

However, as B anticommutes with S, $(\epsilon^{-1}B+S)^2 = \epsilon^{-2}B^2+I$ is bounded below by a multiple of ϵ^{-1}, so the norm of $(\epsilon^{-1}\mathbf{B}+t\tilde{\mathbf{\Pi}}^{\perp}\Sigma(t)\tilde{\mathbf{\Pi}}^{\perp})^{-1}$ is bounded above by a multiple of ϵ on the range of $\tilde{\mathbf{\Pi}}^{\perp}$. Hence we get

$$s\lim_{\epsilon\to 0}\tilde{\mathbf{\Pi}}^{\perp}\mathcal{W}'_{\epsilon}(t)\tilde{\mathbf{\Pi}}^{\perp} = \tilde{\mathbf{\Pi}}^{\perp}$$

For the interval $0\le t\le \frac{3}{2}$, we have $\tilde{\mathbf{\Pi}}^{\perp}\Sigma'(t)\tilde{\mathbf{\Pi}}^{\perp}=0$, and $(\mathbf{B}+\Sigma(t))\tilde{\mathbf{\Pi}}^{\perp} = \mathbf{B}\tilde{\mathbf{\Pi}}^{\perp}$ so $\tilde{\mathbf{\Pi}}^{\perp} = \mathbf{B}\tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B}+\Sigma(t))^{-1}$

$$\begin{aligned}\tilde{\mathbf{\Pi}}^{\perp}\mathcal{W}'(t)\tilde{\mathbf{\Pi}}^{\perp} &= \tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B})(\mathbf{B}+\Sigma'(t))^{-1}\tilde{\mathbf{\Pi}}^{\perp} \\ &= \tilde{\mathbf{\Pi}}^{\perp}\end{aligned}$$

Lastly, for the interval $\frac{3}{2}\le t\le 2$, we have $\tilde{\mathbf{\Pi}}^{\perp}\Sigma'(\frac{3}{2})\tilde{\mathbf{\Pi}}^{\perp}=0$, and $(\mathbf{B}-\Sigma'(\frac{3}{2}))\tilde{\mathbf{\Pi}}^{\perp} = \mathbf{B}\tilde{\mathbf{\Pi}}^{\perp}$ so $\tilde{\mathbf{\Pi}}^{\perp} = \mathbf{B}\tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B}-\Sigma(\frac{3}{2}))^{-1}$

Therefore we get

$$\begin{aligned}\tilde{\mathbf{\Pi}}^{\perp}\mathcal{W}'(t)\tilde{\mathbf{\Pi}}^{\perp} &= \mathbf{B}\tilde{\mathbf{\Pi}}^{\perp}(\mathbf{B}-\Sigma(\frac{3}{2}))^{-1}\tilde{\mathbf{\Pi}}^{\perp} \\ &= \tilde{\mathbf{\Pi}}^{\perp}\end{aligned} \tag{6.4.4}$$

So, we have as $\epsilon\to 0$, $\varpi_{reg}(\mathcal{W}'_{\epsilon}(t))\to\mathcal{W}_{\infty}(t)$ strongly.

Let $\rho(x)$ be the signum function of x, i.e.

$$\rho(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0\end{cases}$$

Then there is a straight line homotopy h_{ϵ} connecting the functions $\phi_{\epsilon}(x)$ and $\rho(x)$,i.e. for $0\le t\le 1$,

$$h_{\epsilon}(t) = (1-t)\phi_{\epsilon}+t\rho$$

Consider the path $X_{\epsilon}(t) := \exp(i\pi h_{\epsilon}(t)(\tilde{D}'))\oplus\exp(i\pi h_{\epsilon}(t)(\tilde{D}_f))$. Then $X_{\epsilon}(t)$ connects $\varpi_{reg}(-\exp(i\pi\phi_{\epsilon}(\mathcal{D}'))\oplus -\exp(-i\phi_{\epsilon}(\mathcal{D}_f)))$ to $\varpi_{reg}(-\tilde{\mathbf{\Pi}}+\tilde{\mathbf{\Pi}}^{\perp})$, as $\exp(i\pi\rho(x)) = 1-2\chi_0(x)$. Therefore, we get a loop of operators $\mathcal{O}(t)$ by concatenating the paths $X_{\epsilon}(t)$, $\mathcal{W}_{\infty}(t)$ and the reverse of $\varpi_{reg}(LT_{\epsilon})$. Now as $\phi_{\epsilon}\to\rho$ as $\epsilon\to 0$, and the reverse of $\varpi_{reg}(LT_{\epsilon})$ converges strongly to the reverse of $\mathcal{W}_{\infty}(t)$, the loop $\mathcal{O}(t)$ is strongly null-homotopic. Therefore the determinant of $\mathcal{O}(t)$ is zero. So from the additivity of the determinant we get

$$\tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_{\epsilon})) = \tilde{w}^{\Lambda',f}(X_{\epsilon})+\tilde{w}^{\Lambda',f}(\mathcal{W}_{\infty})$$

However, as $\epsilon\to 0$, X_{ϵ} converges strongly to the constant path $\varpi_{reg}(-\tilde{\mathbf{\Pi}}+\tilde{\mathbf{\Pi}}^{\perp})$, we have $\tilde{w}^{\Lambda',f}(X_{\epsilon})\to 0$. Also, as

$$\mathcal{W}_{\infty}(t)^{-1}\frac{d\mathcal{W}_{\infty}(t)}{dt} = \begin{cases} 0 \quad -1\le t\le\frac{3}{2} \\ -\begin{pmatrix} Id & 0 \\ 0 & -Id\end{pmatrix}\varpi_{reg}(\tilde{\mathbf{\Pi}})\end{cases} \tag{6.4.5}$$

So we have $\tilde{\tau}_{\Lambda',f}(\mathcal{W}_\infty(t)^{-1}\frac{d\mathcal{W}_\infty(t)}{dt}) = \tau^{\Lambda'}(\tilde{\Pi}') - \tau_{\Lambda',f}(\tilde{\Pi}_f)$, where $\tilde{\Pi}_f$ (resp. $\tilde{\Pi}'$) is the projection onto the kernel of $\tilde{D}_f$ (resp. $\tilde{D}'$). However, $\tau_{\Lambda,f}(\tilde{\Pi}_f) = \tau^{\Lambda}(\tilde{\Pi})$, where $\tilde{\Pi}$ is the projection onto the kernel of $\tilde{D}$. Also, from the homotopy invariance of foliated Betti numbers [HeLa:91], $\tau^{\Lambda}(\tilde{\Pi}) = \tau^{\Lambda'}(\tilde{\Pi}')$. So $\tilde{w}^{\Lambda',f}(\mathcal{W}_\infty(t)) = 0$. Thus $\tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_\epsilon)) \to 0$ as $\epsilon \to 0$ and we conclude the proof. $\square$

6.5 Remarks on the Small Time Path and homotopy invariance

From the previous section we have a path $V_\epsilon = \begin{pmatrix} V_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & V_\epsilon(-\mathcal{D}_f) \end{pmatrix}$ which connects $\begin{pmatrix} \psi_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & \psi_\epsilon(-\mathcal{D}_f) \end{pmatrix}$ to $\begin{pmatrix} \psi_{1/\epsilon}(\mathcal{D}'_m) & 0 \\ 0 & \psi_{1/\epsilon}(-\mathcal{D}_f) \end{pmatrix}$. Moreover, we have the Large Time Path that connects $\begin{pmatrix} \psi_{1/\epsilon}(\mathcal{D}'_m) & 0 \\ 0 & \psi_{1/\epsilon}(-\mathcal{D}_f) \end{pmatrix}$ to the identity. As in the proof of Keswani [Ke:99], the next step is to construct a Small Time Path ST_ϵ, which connects $\begin{pmatrix} \psi_\epsilon(\mathcal{D}'_m) & 0 \\ 0 & \psi_\epsilon(-\mathcal{D}_f) \end{pmatrix}$ to the identity. Then we would have a loop l at the identity whose determinant would converge to $\frac{1}{2}(\rho_\Lambda(D) - \rho_{\Lambda'}(D'))$ provided we have the following estimate for the determinant of the Small Time Path ST_ϵ:

Proposition 6.5.1. *Assume that $\mathcal{G}$ is torsion-free (i.e. all isotropy groups $\mathcal{G}_x^x$ are torsion free for $x \in V$) and the maximal Baum-Connes map $\mu_{max} : K_*(B\mathcal{G}) \to K_*(C^*(\mathcal{G}))$ is bijective. Then we have $(\tilde{w}^{\Lambda',f}(\varpi_{reg}(ST_\epsilon)) - \tilde{w}_{\mathcal{F}}^{\Lambda',f}(\varpi_{av}(ST_\epsilon))) \to 0$ as $\epsilon \downarrow 0$.*

Using the surjectivity of μ_{max} and Proposition 3.3.12, one can prove, as in [BePi:08], the following equality:

$$\tilde{w}^{\Lambda',f}(\varpi_{reg}(l)) - \tilde{w}_{\mathcal{F}}^{\Lambda',f}(\varpi_{av}(l)) = 0$$

The above equation implies

$$\tilde{w}^{\Lambda',f}(\varpi_{reg}(V_\epsilon)) - \tilde{w}_{\mathcal{F}}^{\Lambda',f}(\varpi_{av}(V_\epsilon)) + \tilde{w}^{\Lambda',f}(\varpi_{reg}(LT_\epsilon)) - \tilde{w}_{\mathcal{F}}^{\Lambda',f}(\varpi_{av}(LT_\epsilon)) + \tilde{w}^{\Lambda',f}(\varpi_{reg}(ST_\epsilon)) - \tilde{w}_{\mathcal{F}}^{\Lambda',f}(\varpi_{av}(ST_\epsilon)) = 0$$

Then, from Corollary 6.4.5, Proposition 6.4.6 and Proposition 6.5.1, we have

$$\rho_\Lambda(D) = \rho_{\Lambda'}(D')$$

We summarize the result in the following

Theorem 6.5.2. *Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be smooth foliations on closed manifolds V and V', respectively, and $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a leafwise homotopy equivalence. Let Λ be a holonomy invariant transverse measure on $(V, \mathcal{F})$ and $\Lambda' = f_*\Lambda$ be the associated holonomy invariant transverse measure on $(V', \mathcal{F}')$. Assume that the maximal Baum-Connes map for the monodromy groupoid $\mathcal{G}$ of $(V, \mathcal{F})$ is surjective. Then for the leafwise signature operators D and D' on $(V, \mathcal{F})$ and $(V', \mathcal{F}')$, respectively, we have*

$$\rho_\Lambda(D) = \rho_{\Lambda'}(D')$$

Appendix A

Signatures and homotopy equivalences of Hilbert-Poincaré complexes

A.1 Unbounded HP-complexes

Definition An n-dimensional Hilbert-Poincaré complex (abbreviated HP-complex) over a C^*- algebra A is a complex (E, b) of countably generated Hilbert C-modules

$$E_0 \xrightarrow{b_0} E_1 \xrightarrow{b_1} E_2 ... \xrightarrow{b_n} E_n$$

where each b_i is a densely defined closed unbounded regular operator with a densely defined regular adjoint $b^*_\bullet : E_{\bullet+1} \to E_\bullet$ such that successive operators in the complex are composable (i.e. the image of one is contained in the domain of the other) and $b_{i+1} \circ b_i = 0$, together with adjointable operators $T : E_\bullet \to E_{n-\bullet}$ satisfying the following properties:

1. For $v \in E_p$,
$$T^* v = (-1)^{(n-p)p} T v$$

2. T maps $Dom(b^*)$ to $Dom(b)$, and we have for $v \in Dom(b^*) \subset E_p$,
$$T b^*_{p-1} v + (-1)^{p+1} b_{n-p} T v = 0$$

3. T induces an isomorphism between the cohomology of the complex (E, b) and that of the dual complex (E, b^*):
$$E_n \xrightarrow{b^*_{n-1}} E_{n-1} \xrightarrow{b^*_{n-2}} E_{n-2} ... \xrightarrow{b^*_0} E_0$$
i.e. the induced map $T_* : H^k(E, b) \to H^k(E, b^*)$ is an isomorphism.

4. The operator $B := b + b^* : E \to E$ is a regular Fredholm operator (i.e. it has an inverse modulo compacts) and $(B \pm i)^{-1} \in \mathcal{K}_A(\mathcal{E})$.

Recall that the cohomology of the complex (E, b) is defined here to be the unreduced one given by

$$H^k(E, b) := \frac{Ker\ b_k}{Im\ (b_{k-1})}.$$

Recall also that a regular Fredholm operator is a regular operator t which has a pseudo-left inverse and a pseudo-right inverse. A pseudo-left inverse for t is an operator $G \in \mathcal{L}(\mathcal{E})$ such that Gt is closable, $\overline{Gt} \in \mathcal{L}(\mathcal{E})$, and $\overline{Gt} = 1 \bmod \mathcal{K}(\mathcal{E})$. Similarly a pseudo-right inverse for t is an operator $G' \in \mathcal{L}(\mathcal{E})$ such that tG' is closable, $\overline{tG'} \in \mathcal{L}(\mathcal{E})$, and $\overline{tG'} = 1 \bmod \mathcal{K}(\mathcal{E})$.

Remark. *The complex (E, b) given in the definition is viewed as an two-sided infinite complex with finitely many non-zero entries.*

We consider E as the direct sum $\oplus_{0\leq i\leq n} E_i$ and $b = \oplus_{0\leq i\leq n} b_i$ and similarly for b^*.

Definition Let $dimE = n = 2l + 1$ be odd. Define on E_p,

$$S = i^{p(p-1)+l}T \text{ and } D = iBS.$$

Then D is the signature operator of the HP-complex (E, b, T).

Proposition A.1.1 ([HiRoI:05], Lemma 3.4)**.** *With the notations as in A.1 we have $S^* = S$ and $bS+Sb^* = 0$.*

Proof. We have $S^* : E_{n-p} \to E_p$,

$$S^* = (-i)^{(n-p)(n-p+1)+l}T^* = (-i)^{(n-p)(n-p+1)+l}(-1)^{p(n-p)}T$$

Now

$$\begin{aligned} (n-p)(n-p+1)+l &= (2l+1-p)(2l+1-p+1) \\ &= (2l+1-p)(2l+2-p) \\ &\equiv p(p-1)+l \bmod 2 \end{aligned}$$

So that

$$\begin{aligned} (-i)^{(n-p)(n-p+1)+l}(-1)^{p(n-p)} &= (-1)^{\frac{3}{2}[p(p-1)+l]+1+(n-p)p} \\ &= (-1)^{\frac{[p(p-1)+l]}{2}+1+(n-p)p} \\ &= (-1)^{\frac{[p(p-1)+l]}{2}} = i^{p(p-1)+l} \end{aligned}$$

Therefore $S^* = (i)^{[p(p-1)+l]}T = S$. Again we have, for $v \in Dom(b^*) \subset E_p$, since $b^*v \in E_{p+1}$,

$$\begin{aligned} (Sb^* + bS)v &= (i^{p(p+1)+l}T)b^*v + b(i^{p(p-1)+l}Tv) \\ &= i^{(p(p+1)+l}(Tb^*v + i^{-2p}bTv) \\ &= i^{(p(p+1)+l}(Tb^*v + (-1)^p bTv) = 0 \end{aligned}$$

where we have used property (ii) of T. ☐

Recall that a regular operator t is adjointably invertible if there exists an adjointable operator s such that $st \subseteq ts = 1$. Notice that for a self-adjoint t, this is equivalent to the surjectivity of t [Ku:97].

Proposition A.1.2 ([HiRoI:05], Proposition 2.1)**.** *A Hilbert-Poincaré complex is acyclic, i.e. its cohomology groups are all zero, if and only if the operator B is adjointably invertible. Moreover, $B^{-1} \in \mathcal{K}_{\mathcal{A}}(\mathcal{E})$.*

Proof. Let the HP-complex be acyclic. To prove that B is adjointably invertible it suffices to prove that B is surjective. Since all the cohomologies are trivial, $Im(b) = Ker(b)$, so the range of b is closed. Since the differentials $b_k, k = 0, 1..., n$ are regular operators, $Q(b) = b(1 + b^*b)^{-1/2}$ is a bounded adjointable operator and we have $Im(b) = Im(Q(b))$, $Ker(b) = Ker(Q(b))$. Then by Theorem 3.2 in [La:95] (which is an application of the Open Mapping theorem), $Q(b)Q(b^*)$ is bounded below on $Im(Q(b))$ and therefore $Im(Q(b)) \subseteq Im(Q(b)Q(b^*))$. Similarly, $Im(Q(b^*)) \subseteq Im(Q(b^*)Q(b))$.

Now, as $Im(Q(b))$ is closed, $Ker(Q(b))$ is an orthocomplemented submodule with $Ker(Q(b))^\perp = Im(Q(b)^*) = Im(Q(b^*))$. Hence we have $E = Im(Q(b)) \oplus Im(Q(b^*))$. So for any $v \in E$, we have $v = Q(b)v_1 + Q(b^*)v_2$ for some $v_1, v_2 \in E$. However $Q(b)v_1 = Q(b)Q(b^*)w_1$, and $Q(b^*)v_2 = Q(b^*)Q(b)w_2$ for some $w_1, w_2 \in E$. Hence we have for any $v \in E$, $v = Q(b)Q(b^*)w_1 + Q(b^*)Q(b)w_2$. We will now prove the following lemma

Lemma A.1.3. *We have*

$$Q(b)^2 = Q(b^*)^2 = 0 \text{ and } Q(b + b^*) = Q(b) + Q(b^*).$$

Proof. 1. Let $f = Q(b)$. Then we have $b = f(1 - f^*f)^{-1/2}$, $(1 + b^*b)^{-1/2} = (1 - f^*f)^{1/2}$, and since $fp(f^*f) = p(ff^*)f$ for any polynomial p, by continuity it also holds for any $p \in C([0, 1])$. So in particular we have

$$f(1 - f^*f)^{1/2} = (1 - ff^*)^{1/2}f \tag{A.1.1}$$

We compute

$$\begin{aligned} f^2 &= [b(1 + b^*b)^{-1/2}][b(1 + b^*b)^{-1/2}] \\ &= [b(1 - f^*f)^{1/2}]f \\ &= b[(1 - f^*f)^{1/2}f] \text{ (since } Im((1 - f^*f)^{1/2}) = Im(1 + b^*b)^{-1/2} \subseteq Dom(b) \text{)} \\ &= b[f(1 - ff^*)^{1/2}] \\ &= 0 \end{aligned}$$

since $bf = b(b(1 + b^*b)^{-1/2}) = (b^2)(1 + b^*b)^{-1/2} = 0$, the computation justified by the facts that $Im(b) \subseteq Dom(b)$ and $Im(1 + b^*b)^{-1/2} \subseteq Dom(b)$. Similarly one can show that $(f^*)^2 = 0$.

2. We will show that $f = Q(b) = b(1 + b^*b + bb^*)^{-1/2}$ and $f^* = b^*(1 + b^*b + bb^*)^{-1/2}$ so that we will have

$$f + f^* = Q(b) + Q(b^*) = (b + b^*)(1 + b^*b + bb^*)^{-1/2} = (b + b^*)(1 + (b + b^*)^2)^{-1/2} = Q(b + b^*)$$

We proceed as follows. We have $b = f(1 - f^*f)^{-1/2}$ and $b^* = f^*(1 - ff^*)^{-1/2}$. We note that for any polynomial p with $p(0) = 1$, we have $fp(ff^*) = f$, since $f^2 = 0$. So the equality also holds by continuity for any $p \in C([0, 1])$ for which $p(0) = 1$. In particular we note for later use that

$$f(1 - ff^*)^{1/2} = f \tag{A.1.2}$$

Let $\Delta_b = bb^* + b^*b$. Then $(1 + \Delta_b)^{-1}$ and $(1 + \Delta_b)^{-1/2}$ are bounded operators since $\Delta_b = B^2$ is regular. Let $G = (1 + \Delta_b)^{-1/2}$. We show first that $(1 - f^*f)^{-1/2}G$ is bounded:

$$\begin{aligned} < (1 - f^*f)^{-1/2}Gx, (1 - f^*f)^{-1/2}Gx > &= < Gx, (1 - f^*f)^{-1}Gx > \\ &= < Gx, (1 + b^*b)Gx > \\ &= < Gx, Gx > + < Gx, b^*bGx > \\ &= < Gx, Gx > + < bGx, bGx > \end{aligned}$$

So that $(1-f^*f)^{-1/2}G$ would be bounded if bG has a bounded extension. We compute:

$$\begin{aligned}
< bG(Gx), bG(Gx) > &= < G^2x, b^*bG^2x > \\
&\leq < G^2x, (1+b^*b)G^2x > \\
&\leq < G^2x, (1+b^*b)G^2x > + < b^*G^2x, b^*G^2x > \\
&= < G^2x, (1+b^*b+bb^*)G^2x > \\
&= < G^2x, x > \leq ||G||^2||x||^2
\end{aligned}$$

So bG is bounded on the range of G. However, $Im(G) = Dom(\Delta_b)$ is dense, so bG extends to a bounded operator. Thus $(1-f^*f)^{-1/2}G$ has a bounded extension as well. Now we compute as follows on $Dom(\Delta_b)$

$$\begin{aligned}
& (1-f^*f)(1+\Delta_b) \\
= \ & (1-f^*f)(1+f(1-f^*f)^{-1/2}f^*(1-ff^*)^{-1/2}+f^*(1-ff^*)^{-1/2}f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+f(1-f^*f)^{-1/2}f^*(1-ff^*)^{-1/2}+(1-f^*f)f^*(1-ff^*)^{-1/2}f(1-f^*f)^{-1/2} \text{ (since } f^2=0) \\
= \ & (1-f^*f)+f(1-f^*f)^{-1/2}f^*(1-ff^*)^{-1/2}+(1-f^*f)^{1/2}((1-f^*f)^{1/2}f^*(1-ff^*)^{-1/2})(f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+bb^*+(1-f^*f)^{1/2}((1-f^*f)^{1/2}f^*)((1-ff^*)^{-1/2}f(1-f^*f)^{-1/2})
\end{aligned}$$

Now, as $Dom(1+bb^*)^{1/2} = Dom(b^*)$ (cf. Theorem 10.7 in [La:95]), $(1+bb^*)^{1/2}b$ is well-defined on $Dom(\Delta_b) = Dom(b^*b) \cap Dom(bb^*)$. As $(1-ff^*)^{-1/2} = (1+bb^*)^{1/2}$ as regular operators with domain $Dom(b^*)$, we also have $((1-ff^*)^{1/2}(1-ff^*)^{-1/2}) = (1+bb^*)^{-1/2}(1+bb^*)^{1/2} = 1$ on $Dom(\Delta_b)$. So on $Dom(\Delta_b)$,

$$((1-f^*f)^{1/2}f^*(1-ff^*)^{-1/2})(f(1-f^*f)^{-1/2}) = ((1-f^*f)^{1/2}f^*)((1-ff^*)^{-1/2})(f(1-f^*f)^{-1/2})$$

Therefore, we have on $Dom(\Delta_b)$,

$$\begin{aligned}
& (1-f^*f)(1+f(1-f^*f)^{-1/2}f^*(1-ff^*)^{-1/2}+f^*(1-ff^*)^{-1/2}f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+bb^*+(1-f^*f)^{1/2}((1-f^*f)^{1/2}f^*)((1-ff^*)^{-1/2}f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+bb^*+(1-f^*f)^{1/2}(f^*(1-ff^*)^{1/2}((1-ff^*)^{-1/2}f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+bb^*+(1-f^*f)^{1/2}f^*((1-ff^*)^{1/2}(1-ff^*)^{-1/2})f(1-f^*f)^{-1/2}) \\
= \ & (1-f^*f)+bb^*+(1-f^*f)^{1/2}f^*f(1-f^*f)^{-1/2})
\end{aligned}$$

However, by the functional calculus for the self-adjoint operator f^*f, we have

$$(1-f^*f)^{1/2}f^*f(1-f^*f)^{-1/2} = f^*f$$

Finally we get on $Dom(\Delta_b)$

$$(1-f^*f)(1+\Delta_b) = 1-f^*f+bb^*+f^*f = 1+bb^*$$

However, as $(1-f^*f)^{-1/2}(1+\Delta_b)^{-1/2}$ is bounded, we get

$$(1-f^*f)^{-1/2}(1+\Delta_b)^{-1/2} = (1+bb^*)^{-1/2} \Rightarrow (1+\Delta_b)^{-1/2} = (1-f^*f)^{1/2}(1-ff^*)^{1/2}$$

. So we have

$$\begin{aligned}
b(1+\Delta_b)^{-1/2} &= b(1-f^*f)^{1/2}(1-ff^*)^{1/2} \\
&= f(1-f^*f)^{-1/2}(1-f^*f)^{1/2}(1-ff^*)^{1/2} \\
&= f(1-ff^*)^{1/2} \\
&= f \text{ (from equation A.1.2)}
\end{aligned}$$

Therefore we have proved $Q(b+b^*) = Q(b)+Q(b^*)$. $\square$

Finally, to prove that B is invertible we proceed as follows. We have already established that for any $v \in E$, there exist $w_1, w_2 \in E$ such that $v = Q(b)Q(b^*)w_1 + Q(b^*)Q(b)w_2$. However, since $Q(b)^2 = 0$, and $Q(b^*)^2 = 0$ we have

$$v = (Q(b) + Q(b^*))(Q(b^*)w_1 + Q(b)w_2)$$

which shows that $Q(b) + Q(b^*)$ is surjective and hence so is $Q(b + b^*)$. However, $Im(Q(b + b^*)) = Im(B)$ and hence B is surjective and thus invertible.

Conversely, let B be invertible. Then for $v \in Ker(b)$, there exists $w \in Dom(B)$ such that $v = Bw$. Then

$$||b^*w||^2 = || < b^*w, b^*w > || = || < w, bb^*w > || = || < w, bBw > || = 0$$

hence $v = Bw = bw \in Im(b)$. Therefore $Ker(b) = Im(b)$ and thus the complex is acyclic. □

A.2 Signatures of HP-complexes

Definition A chain map between HP-complexes (E, b) and (E', b') is a family of adjointable maps $A = (A_i)_{i \geq 0}$ such that each A_i is an adjointable operator in $\mathcal{L}_A(E_i, E'_i)$ and we have $b'_i A_i = A_i b_i$. A chain map is denoted $A : (E, b) \to (E', b')$.

The mapping cone complex of a chain map $A : (E, b) \to (E', b')$ is the complex

$$E''_0 \xrightarrow{b^0_A} E''_1 \xrightarrow{b^1_A} E''_2 ... \xrightarrow{b^{n-1}_A} E''_n$$

where $E''_i = E_{i+1} \oplus E'_i$ and $b^i_A := \begin{pmatrix} -b_{i+1} & 0 \\ A_{i+1} & b'_i \end{pmatrix}$.

Proposition A.2.1 ([HiRoI:05], Lemma 3.5)**.** *The self-adjoint operators $B \pm S : E \to E$ are invertible.*

Proof. Consider the mapping cone complex of the chain map $S : (E, b) \to (E, b^*)$. Its differential is

$$d_S = \begin{pmatrix} -b & 0 \\ S & b^* \end{pmatrix}$$

Since S is an isomorphism on cohomology, its mapping cone complex is acyclic, i.e. all the cohomology groups are zero. Therefore the operator $B_S = d_S + d^*_S$ is invertible on $E \oplus E$. Now, $B_S = \begin{pmatrix} B & S \\ S & B \end{pmatrix}$

As B_S identifies with $B + S$ on the $+1$ eigenspace of the involution which interchanges the copies of E and with $B - S$ on the -1 eigenspace. Thus $B \pm S$ is an invertible operator. □

Proposition A.2.2. *A chain map between HP complexes induces an isomorphism on cohomology if and only if its mapping cone complex is acyclic.*

Proof. This follows from the general theory of (Co)homological algebra that a chain map induces a long exact sequence of cohomology groups: if $f : C \to C'$ is a cochain map of cochain complexes and $C(f)$ is its mapping cone complex, then we have a long exact sequence

$$... \to H^n(C(f)) \xrightarrow{i_*} H^n(C') \xrightarrow{f_*} H^n(C) \xrightarrow{\delta_*} H^{n+1}(C(f)) \to ...$$

where $i : C' \to C(f)$ is the map given by $y \mapsto (0, y)$ and $\delta : C(f) \to C$ is given by $(b, c) \mapsto -b$. □

Definition Let (E, b) be an odd-dimensional Hilbert-Poincaré complex. Then the signature of (E, b) is defined as the class of the self-adjoint invertible operator $(B + S)(B - S)^{-1} \in K_1(\mathcal{K}_A(E)$. We denote this class by $\sigma(E, b)$.

Lemma A.2.3 ([HiRoI:05], Proposition 3.8)**.** *If an HP-complex (E, b) over a C^*-algebra A is acyclic, its signature is zero.*

Proof. Since all the cohomology groups are zero, tT is an admissible duality operator for $t \in [-1, 1]$ (i.e. it satisfies the assumptions (i),(ii),(iii) in the definition of an HP-complex). Therefore the operators $B - tS$ is adjointably invertible for $t \in [-1, 1]$. Therefore the path of operators $(B + S)(B - tS)^{-1}, -1 \leq t \leq 1$, is a norm continuous path of invertible operators connection $(B + S)(B - S)^{-1}$ to the identity. Therefor the class of $(B + S)(B - S)^{-1}$ is trivial in $K_1(\mathcal{K}_A(E))$. □

A.3 Homotopy invariance of the signature

Definition Let (E, b) be a complex of Hilbert-modules. An *operator homotopy* between Hilbert-Poincaré complexes (E, b, T_1) and (E, b, T_2) is a norm-continuous family of adjointable operators $T_s, s \in [0, 1]$ such that each (E, b, T_s) is a Hilbert-Poincaré complex.

Lemma A.3.1 ([HiRoI:05], Lemma 4.5)**.** *Operator homotopic HP-complexes have the same signature.*

Proof. Let (E, b) be a complex of Hilbert-modules and $T_s, s \in [0, 1]$ be a norm-continuous family of duality operators acting on (E, b) and S_s be the self-adjoint operators defined from T_s as in definition A.1. First we note from Result 5.22 in [Ku:97] that for a regular operator t the map $\mathbb{C} \supseteq \rho(t) \ni \lambda \mapsto (t - \lambda)^{-1}$ is continuous. Since $(B \pm S)$ is an invertible self-adjoint regular operator, we have

$$(B + S)^{-1} = \lim_{\mu \to 0} (B + S + i\mu)^{-1}$$

Now for a fixed $\mu \in \mathbb{R}$ and any $s_1, s_2 \in \mathbb{R}$, the resolvent identity holds:

$$(B + S_{s_1} + i\mu)^{-1} - (B + S_{s_2} + i\mu)^{-1} = (B + S_{s_1} + i\mu)^{-1}(S_{s_2} - S_{s_1})(B + S_{s_2} + i\mu)^{-1}$$

One can use techniques in Theorem $VI.5$ of [ReSiI:80] to show that the above identity implies that $(B + S_s + i\mu)^{-1}$ is norm-continuous in $s \in [0, 1]$. Then $M_\mu := \sup_{s \in [0,1]} (B + S_s + i\mu)^{-1} < \infty$ for alll $\mu \in \mathbb{R}$, and we have

$$\begin{aligned}
||(B + S_s)^{-1} - (B + S_{s_0})^{-1}|| &= ||\lim_{\mu \to 0}((B + S_s + i\mu)^{-1} - (B + S_{s_0} + i\mu)^{-1})|| \\
&= \lim_{\mu \to 0} ||((B + S_s + i\mu)^{-1} - (B + S_{s_0} + i\mu)^{-1})|| \\
&\leq \lim_{\mu \to 0} M_\mu^2 ||S_s - S_{s_0}|| \\
&\leq M||S_s - S_{s_0}||
\end{aligned}$$

Hence using the norm continuity of the family S_s we get the norm continuity of $(B + S_s)^{-1}$. Similary we can prove that the family $(B - S_s)^{-1}$ is continuous in norm. Therefore the families $B(B + S_s)^{-1}$ and $S_s(B + S_s)^{-1}$ are also norm continuous, and hence $(B + S_s)(B - S_s)^{-1}$ is a norm continuous family of bounded adjointable operators which gives an operator homotopy between $(B + S_0)(B - S_0)^{-1}$ and $(B + S_1)(B - S_1)^{-1}$ and thus they lie in the same class in K_1 by the homotopy invariance property of K_1. □

Lemma A.3.2 ([HiRoI:05], Lemma 4.6)**.** *If a duality operator T is operator homotopic to $-T$ then the signature of the HP-complex is zero.*

Proof. Let $T_s, s \in [0,1]$ be the operator homotopy between T and $-T$. Then from the proof of the previous lemma one can show that $(B+S)(B-S_s)^{-1}$ is a norm-continuous path of adjointable operators. This path implements the operator homotopy in K-theory connecting $(B+S)(B-S)^{-1}$ to the identity. So the class $\sigma(E,b)$ in K-theory is zero. □

Definition A homotopy equivalence between two HP-complexes (E,b,T) and (E',b',T') is a chain map $A:(E,b)\to(E',b')$ which induces an isomorphism on cohomology and for which the maps ATA^* and T' between the complex (E',b') and its dual (E',b'^*) induce the same map on cohomology.

Theorem A.3.3 ([HiRoI:05], Theorem 4.3)**.** *If two odd-dimensional HP-complexes (E,b,T) and (E',b',T') are homotopy equivalent then their signatures are equal in $K_1(\mathcal{K}_A(E))$.*

Proof. The proof of the above theorem as given in [HiRoI:05] works word by word in this case. □

A.4 Morita equivalence of HP-complexes

Let now A, B be σ-unital C^*-algebras which are Morita-equivalent, with Morita bimodule ${}_AE_B$. So we have $A \cong \mathcal{K}_B({}_AE_B)$, and so there is a $*$-homomorphism $\phi: A \to \mathcal{L}({}_AE_B)$. Let now (E,b) be a Hilbert-Poincaré complex of countably generated A-modules. We also assume that there exists a duality T on (E,b) such that the associated operator S satisfies $S^2=1$. We then form a Hilbert-Poincaré complex $(E\otimes{}_AE_B, b\otimes I)$:

$$E_0\otimes{}_AE_B \xrightarrow{b\otimes I} E_1\otimes{}_AE_B \xrightarrow{b\otimes I} E_2\otimes{}_AE_B ... \xrightarrow{b\otimes I} E_n\otimes{}_AE_B$$

Let $\mathcal{M}: K_1(A)\to K_1(B)$ be the isomorphism induced by the Morita equivalence between A and B. Then we have

Proposition A.4.1. *$\mathcal{M}[\sigma(E,b)] = \sigma(E\otimes{}_AE_B, b\otimes I)$.*

Proof. 1. We note that

$$(D+iI)(D-iI)^{-1} = (iBS+iI)(iBS-iI)^{-1} = (B+S)SS^{-1}(B-S)^{-1} = (B+S)(B-S)^{-1}$$

Let $U(D) = (D+iI)(D-iI)^{-1}$ and $E_A := \oplus_p E_p$.

2. The class of $U(D)$ in $K_1(A)$ can be identified with the class of the KK-cycle in $KK(\mathbb{C},A)$ given by $(E_A, \lambda, U(D))$, where λ is the scalar multiplication by complex numbers on the left. Then, $\mathcal{M}[\sigma(E,b)]$ can be identified with an element of $KK(\mathbb{C},B)$ which will be given by the Kasparov product of $(E_A,\lambda,U(D))$ with the Morita KK-cycle $({}_AE_B,\phi,0)$.

3. We claim that this Kasparov-product is given by the KK-cycle $(E_A\otimes{}_AE_B, \lambda\otimes I, U(D)\otimes I)$. This can be seen by the characterization of the Kasparov product with connections as $U(D)\otimes I$ is a 0-connection[1] on $E_A\otimes{}_AE_B$.

4. Since D is self-adjoint regular operator, by the uniqueness of the functional calculus we have $U(D)\otimes I = U(D\otimes I)$. But then we can identify the class of $U(D\otimes I)$ in $K_1(B)$ with the cycle $(E_A\otimes{}_AE_B, \lambda\otimes I, U(D)\otimes I)$ in $KK(\mathbb{C},B)$ (we refer the reader to [Bl:98, Section 17.5, page 154] for details on this identification). This finishes the proof. □

[1]for definitions and properties see [Bl:98, Chapter VIII, Section 18.3]

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