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THE BANDWIDTH MINIMIZATION PROBLEM FOR CATERPILLARS WITH HAIR LENGTH 3 IS NP-COMPLETE*

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Abstract. It is shown that the *Bandwidth Minimization problem* remains NP-complete even when restricted to "caterpillars with hairs of length at most three". "Caterpillars" are special trees; they consist of a simple chain (the "body") with various simple chains attached to the vertices of the body (the attached chains are called "hairs"). A previous result in the literature shows that the bandwidth of caterpillars with hairs of length at most 2 can be found in $O(n \log n)$ time (this Journal, 2 (1981), pp. 387–393). We also show that the bandwidth problem is NP-complete when restricted to caterpillars with at most one hair attached to each vertex of the body. The proof is relatively straightforward and thereby also provides an easier proof than found in (SIAM J. Appl. Math., 34 (1978), pp. 477–495) that the bandwidth problem is NP-complete for trees with maximum vertex degree 3.

Key words. computational complexity, NP completeness, graph theory, bandwidth minimization

AMS(MOS) subject classifications. 68C25, 68E10

1. Introduction. An $n \times n$ matrix A is said to have *bandwidth* k if all of its nonzero entries are on one of the $2k+1$ diagonals consisting of the main diagonal and the k diagonals on either side of this main diagonal. The *Bandwidth Minimization problem* is to determine, for a given $n \times n$ matrix A and integer k, whether there exists an $n \times n$ permutation matrix P such that $P \cdot A \cdot P^T$ has bandwidth k. This problem is of great importance in many engineering applications. Typically, the matrices arising in these applications are sparse and matrix operations like inversion and multiplication can be performed with a considerably improved computation time if all the nonzero entries are placed within a small "band". Therefore the problem of reducing the bandwidth of a matrix has been of great interest during the last 20 years. A number of heuristics have been presented in the literature [1], [4], [7], [11]. The Bandwidth Minimization problem itself is NP-complete [13] implying (to our present knowledge) that there exists no efficient algorithm for solving this problem. The Bandwidth Minimization problem is equivalent to the following graph problem: given a graph G and an integer k, determine whether there exists a linear layout of G (i.e. integer labeling of the vertices of G such that each vertex receives a unique integer) such that the maximum difference between adjacent vertices is bounded by k. The problem has been studied also under a graph theoretic viewpoint [3], [4], [5], [6]. It is known to remain NP-complete even for trees with maximum vertex degree 3 [8].

On the positive side, dynamic programming algorithms have been described [10], [14] that can determine whether a graph G with n vertices has bandwidth k in at most $O(n^k)$ steps. It is also known that bandwidth 2 can be determined in linear time [8] and that there is a $O(n \log n)$ algorithm to determine the bandwidth of "caterpillars with hairs of length at most two" [2]. A "caterpillar" is a special kind of tree consisting of a simple chain C (called the "body" or "backbone") with an arbitrary number of simple chains attached by coalescing an endpoint of the added chain with a vertex in C. (The attached chains are called "hairs".) Caterpillars are shown in Fig. 1.1. A caterpillar has hairs of length at most k if all of the simple chains attached to the body have length at most k.

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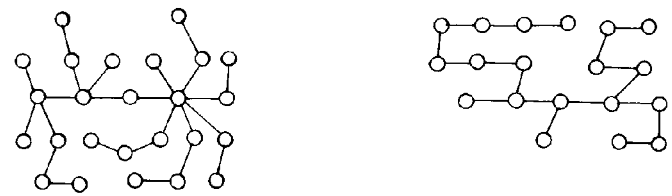


(a) (b)

FIG. 1.1. (a) *A caterpillar with hairs of length at most 3.* (b) *A caterpillar with at most one hair attached to every vertex in the body.*

We show that the Bandwidth Minimization problem is NP-complete even when restricted to caterpillars with hairs of length at most 3 and that it is also NP-complete when restricted to caterpillars with at most one hair attached to every vertex in the body. Caterpillars of this latter type are a special kind of trees with maximum vertex degree 3. Our proof of NP-completeness is relatively straightforward and thereby provides an easier proof for the NP-completeness of the bandwidth problem on trees with maximum degree 3 [8].

In the case of caterpillars with at most one hair attached to any vertex of the body, we do not bound the length of the hairs. Caterpillars with maximum degree 3 and with hairs of length at most k have bandwidth at most k and therefore their Bandwidth Minimization problem can be decided in polynomial time [9], [13].

We have said above that the Bandwidth Minimization problem for caterpillars with hairs of length at most 2 is solvable in polynomial time whereas for caterpillars with hairs of length at most 3 it is NP-complete. The proof in § 2 will show that the border line we have determined is even sharper. We will see that the Bandwidth Minimization problem is NP-complete for caterpillars which have at most one node to which hairs of length 3 are attached, while all the other nodes of the body have hairs of length at most 1.

In [12] a weaker form of the NP-completeness result was shown. In the above paper a caterpillar is encoded as a chain with numbers attached to every node of the chain, i.e., the binary encoding is used for the length of the hairs. It is shown that with respect to this encoding the Bandwidth Minimization problem for caterpillars is NP-complete. In this interpretation a caterpillar is not viewed as a graph but as an instance of a special kind of scheduling problem. Note that under this encoding the length of the hairs may grow exponentially in the length of the encoding. In our paper we use the usual graph encoding.

We formulate the two results of this paper as theorems.

THEOREM 1. *The Bandwidth Minimization problem for caterpillars with hairs of length at most 3 is NP-complete.*

THEOREM 2. *The Bandwith Minimization problem for caterpillars with at most one hair attached to every vertex in the body is NP-complete.*

We will prove Theorem 1 in § 2 and Theorem 2 in § 3.

2. Caterpillars with hairs of length at most 3. We prove Theorem 1 by reduction from the Multiprocessor Scheduling problem [8, p. 238]. That is, given a set $T = \{t_1, t_2, \cdots, t_n\}$ of tasks (the ith task in T has execution time t_i), a deadline D, and a number m of processors, we construct a caterpillar C and an integer k such that C has bandwidth k if and only if the tasks in T can be scheduled on the m processors to satisfy the deadline D. The multiprocessor scheduling problem is strong NP-complete and therefore we can assume that all the t_i are polynomially bounded in n.

We first construct two portions of the caterpillar called "barrier" and "turning point". They are shown in Fig. 2.1.

The barrier of height p and the turning point of height p both have bandwidth p (a corresponding layout for the turning point of height 4 is shown in Fig. 2.2). Our construction of the caterpillar C is based on the fact that in every optimal layout of the turning point both nodes a and g either belong to the first half of the layout or to the second half of the layout, i.e., in every optimal layout of the turning point the backbone has to be folded. Because of the importance of this behaviour for our construction, we will give a careful proof below. Let T_p denote the turning point of height p. T_p has exactly $6p+1$ nodes.

LEMMA 1. *Let $T_p = (V, E)$, let $\sigma: V \to \{1, \cdots, 6p+1\}$ be a layout with $|\sigma(i) - \sigma(j)| \leqq$ p for all $\{i, j\} \in E$ and let $p \geqq 4$. Then either $\sigma(a), \sigma(g) < 3p+1$ or $\sigma(a), \sigma(g) > 3p+1$.*

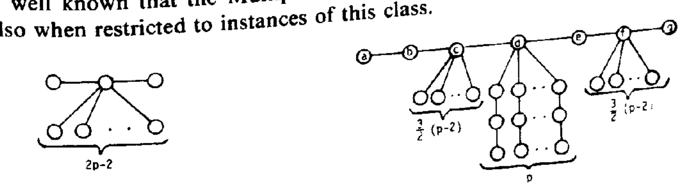
Proof. Let $v_0, \cdots, v_6$ be the nodes with $\sigma(v_i) = i \cdot p + 1$, $1 \leqq i \leqq 6$. We want to show first that $v_0 - v_1 - \cdots - v_6$ form a path in T_p and that $v_3 = d$ holds.

T_p is connected and every path in T_p has length at most 6. Since $\sigma(v_6) - \sigma(v_0) = 6p$ and since $|\sigma(i) - \sigma(j)| \leqq p$ for all $\{i, j\} \in E$, it follows that on the path from v_0 to v_6 any two adjcent nodes have the difference p with regard to σ. Therefore $v_0 - v_1 - \cdots - v_6$ is the only path from v_0 to v_6 of length 6. Every path of length 6 has the node d as its centre. This implies $d = v_3$.

We have seen that $\sigma(d) = 3p+1$ holds. This implies that σ can associate one of the numbers $1, \cdots, p$ or $5p+2, \cdots, 6p+1$ to a node u only if there exists a path of length at least 3 from d to u. This is true only if u is one of the nodes a, g or an endpoint of a hair of length 3 or a point on a hair dangling at the node f. Now let us assume that there exists an optimal layout σ with $\sigma(a) < \sigma(d) < \sigma(g)$. We have to show that this is not possible.

$\sigma(a) < \sigma(d)$ implies that not all the hairs of length 3 can be stretched out to the left. But then $\sigma(a) \leqq p$ must hold and $p-1$ hairs of length 3 together with the nodes a, b, c determine $3p$ nodes which have to be laid out to the left of d. The hairs of length 1 dangling at c and the two remaining neighbours of d have to get values from $[3p+2, 4p+1]$ with respect to σ. This is not possible since $\frac{3}{2}(p-2)+2 > p$ holds for $p \geqq 4$. □

The caterpillar C which we associate to the instance $Y = (\{t_1, \cdots, t_n\}, D, m)$ of the Multiprocessor Scheduling problem is shown in Fig. 2.3. We will see later that the number p has to fulfill some condition. We consider only instances Y with $\sum_{i=1}^{n} t_i =$ $D \cdot m$. It is well known that the Multiprocessor Scheduling problem is strong NP-complete also when restricted to instances of this class.



(a) (b)

FIG. 2.1. (a) *The barrier of height p.* (b) *The turning point of height p, p ≡ 0 mod 2.*

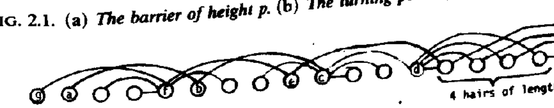


FIG. 2.2. *An optimal layout of the turning point of height 4.*

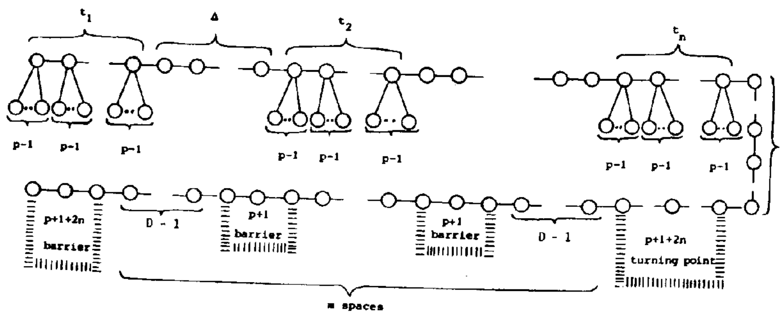


FIG. 2.3. *The caterpillar C, $\Delta = 2 \cdot \{m \cdot (D+2) - 2\}$, p has to be chosen in an appropriate way.*

The caterpillar C consists of an encoding of the sequence of execution times and of some "frame" encoding m "holes" each of size D. These two parts are connected by the turning point of height $p+1+2n$. The behavior of the turning point will force every layout with bandwidth $p+1+2n$ to place these two parts one upon the other. It is a matter of technical details to show that a layout with bandwidth $p+1+2n$ exists if and only if each of the holes can be filled by using all the nodes of the blocks encoding certain execution times (this in turn is equivalent to the instance Y having a solution).

We will give a formal proof in the remainder of this section. In order to make our description less cumbersome, we use some special terminology. The part of the body from the outermost barrier to the turning point we call "ground line" and the part from the turning point to the other end of the body we call "sweeping line". The "ith block", $1 \leqq i \leqq n$, consists of the chain of length t_i of the sweeping line together with the hairs attached to it. The ground line consists of $\lambda = m(D+2)+1$ nodes.

LEMMA 2. *If Y has a solution then C has bandwidth $p+1+2n$.*

Proof. We will define the corresponding layout explicitly. Set $\beta = p+1+2n$.

(i) The ground line is stretched as far as possible, i.e. the points of the ground line get the numbers $i \cdot \beta + 1$, $1 \leqq i \leqq \lambda$. The hairs of the barriers are laid out in such a way that always half of them lie to the left of the center of the barrier and half of them to the right of the center.

(ii) For the turning point we use an optimal layout which associates with a and g the numbers $\lambda \cdot \beta + 1$ and $\lambda \cdot \beta + 2$. This layout uses the numbers $\lambda \cdot \beta + j$, $1 \leqq j \leqq 6 \cdot \beta + 1$, for the nodes of the turning point and has bandwidth β. All the remaining nodes (i.e. the nodes of the sweeping line and its hairs) get numbers smaller than $\lambda \cdot \beta + 1$.

(iii) Y has a solution and therefore there exist sets I_j, $1 \leqq j \leqq m$, such that $\bigcup_{j=1}^{m} I_j = \{1, \cdots, n\}$ and $\sum_{i \in I_j} t_i = D$ for all $j = 1, \cdots, m$. For every i, if $i \in I_j$ then the nodes of the ith block are placed between the jth and the $(j+1)$st barrier (in the case $j = m$: between the mth barrier and the turning point). Note that by doing so we put exactly $D \cdot p$ nodes between any two barriers. Of course we must bear in mind that adjacent nodes have to get numbers at most β apart. But it is clear that this can be done and we can reach in this way a partial layout which fulfills the bandwidth constraint and which has the property that between any two nodes of the ground line there are laid out so far exactly p nodes.

(iv) Now we have to lay out the chains between the blocks. Every chain has length $\Delta = 2 \cdot (\lambda - 3)$. It can be laid out in such a way that the bandwidth constraint

is fulfilled and that between any two nodes of the ground line there are placed exactly two nodes of the chain (see Fig. 2.4). That is, after we have laid out all the chains exactly $p+2n$ nodes are placed between any two nodes of the ground line, i.e., C has bandwidth $\beta = p+2n+1$. □

LEMMA 3. *If $p > 2n \cdot (d+4)$ and if C has bandwidth $p+1+2n$, then Y has a solution.*

Proof. We will show first that every layout σ of C with bandwidth $\beta = p+1+2n$ numbers the ground line up to symmetry in exactly the same way.

The turning point of height β is a subgraph of C. The turning point has $6\beta+1$ nodes and its longest path is of length 6. Therefore every layout with bandwidth β has to assign to its nodes $6\beta+1$ consecutive numbers. Because of Lemma 1 we know that the ground line and the sweeping line both lie with respect to σ either to the left of the turning point or to the right of the turning point. Therefore the nodes of the turning point have to get the smallest $6\beta+1$ numbers or the largest $6\beta+1$ numbers. We will assume that the turning point is laid out at the right end.

Furthermore the barrier of height β is a subgraph of C. This barrier has $2\beta+1$ nodes and its longest path is of length 2. Therefore σ associates to its nodes $2\beta+1$ consecutive numbers. No edge can cross this barrier and therefore we can conclude from the considerations made above that σ associates with the barrier of height β the numbers $1, \cdots, 2\beta+1$. The current situation is shown in Fig. 2.5.

Note that C has

$$6\beta+1+(D-1) \cdot m + m \cdot (2p+3) + 4n + n \cdot \Delta + p \cdot \sum_{i=1}^{n} t_i = \{6 + m \cdot (D+2)\} \cdot \beta + 1$$

nodes. We already know the numbers associated with the barrier of height β and the turning point. Between these two subgraphs the remaining $\{m \cdot (D+2) - 2\} \cdot \beta - 1$ nodes have to be laid out. The barrier of height β and the turning point are connected by the groundline, i.e. by a path of length $m \cdot (D+2) - 2$. Therefore σ has to associate with the ith node of the ground line $1 \leqq i \leqq \lambda$, the number $(i-1) \cdot \beta + 1$.

We have seen that every layout with bandwidth β numbers the ground line and the turning point in the same way up to symmetry. We have to show now that the sweeping line can be encompassed into the "frame" given by the ground line and its barriers only if the scheduling problem Y has a solution.

Note that the centers of the barriers have got the numbers $Z_j = \beta \cdot (D+2) \cdot j + \beta + 1$, $j = 0, \cdots m-1$. Set $Z_m = \beta(D+2) \cdot m + \beta + 1$. We say that task i, $1 \leqq i \leqq n$, belongs to the jth interval, $1 \leqq j \leqq m$, if and only if $Z_{j-1} < \sigma(u) < Z_j$ holds for some node u of the sweeping line belonging to the ith block (i.e. to the subgraph of C encoding the execution time t_i). We will show first that a task cannot belong to two different intervals.

Let us assume that the task i belongs to two different intervals. Then there exist two adjacent nodes u, v belonging to the sweeping line and to the ith block such that

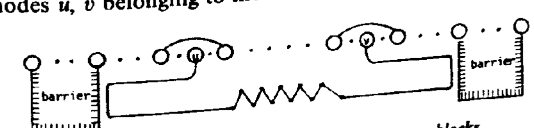


FIG. 2.4. *Layout of the chain connecting two blocks.*

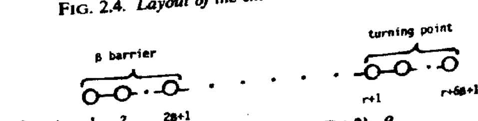


FIG. 2.5. *Layout σ, $r = m \cdot (D+2) \cdot \beta$.*

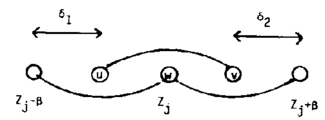


FIG. 2.6. *The situation where a task belongs to two intervals.*

$\sigma(u)<Z_j<\sigma(v)$ holds for some j. Let w be the node with $\sigma(w)=Z_j$. This situation is illustrated in Fig. 2.6. Set $\delta_1=\sigma(u)-(Z_j-\beta)$ and $\delta_2=Z_j+\beta-\sigma(v)$. $\{u,v\}\in E$ implies $\sigma(v)-\sigma(u)\leqq\beta$ and therefore $\delta_1+\delta_2\geqq\beta$ holds. There are $p-1$ hairs dangling at each of the nodes u and v and $2p$ hairs dangling at w. At most $\beta-\delta_1$ hairs dangling at u can get numbers smaller than $Z_j-\beta$ and at most $\beta-\delta_2$ hairs dangling at v can get numbers greater that $Z_j+\beta$. Therefore $5+2p+2(p-1)-(\beta-\delta_1)-(\beta-\delta_2)\geqq 3+4p-\beta=2+3p-2n$ nodes have to get numbers between $Z_j-\beta$ and $Z_j+\beta$. This is not possible since $2\beta+1=3+2p+4n$ and $p\geqq 8n$ hold.

Thus we have shown that every task belongs to exactly one interval. Let I_j, $1\leqq j\leqq m$ be the set of tasks belonging to the jth interval. We have to show that $\sum_{i\in I_j} t_i\leqq D$ holds for all $j=1,\cdots,m$. σ has associated numbers between $Z_{j-1}-\beta$ and $Z_j+\beta$ to all the nodes belonging to a task from I_j (there are $p\cdot\sum_{i\in I_j} t_i$ such nodes) to the hairs of the two barriers ($4p$ nodes) and to the corresponding part of the groundline ($D+5$ nodes). This implies

$$p\cdot\sum_{i\in I_j} t_i+4p+D+5\leqq(D+4)(p+2n+1)+1$$

and therefore

$$\sum_{i\in I_j} t_i\leqq D+\frac{2n(D+4)}{p}.\qquad\square$$

Theorem 1 follows from Lemma 2 and Lemma 3.

3. Caterpillars with at most one hair attached to every vertex in the body. The proof of Theorem 2 does not differ much from the proof of Theorem 1. This time we use a reduction from the 3-Partition problem [8, p. 224] which is a special case of the Multiprocessor Scheduling problem where only instances $(\{t_1,\cdots,t_n\}, D, m)$ with $n=3m$ and $D/4\leqq t_i\leqq D/2$, $1\leqq i\leqq n$, are considered. We construct our caterpillar C again by using barriers and a turning point. The barriers and the turning point have to be defined now in a different way. They are shown in Figs. 3.1 and 3.2.

The barrier of height p has $(2p-1)^2$ nodes and the length of its body is equal to $4p-4$. It has bandwidth p (note that $p\cdot(4p-4)+1=(2p-1)^2$) and it is easy to see

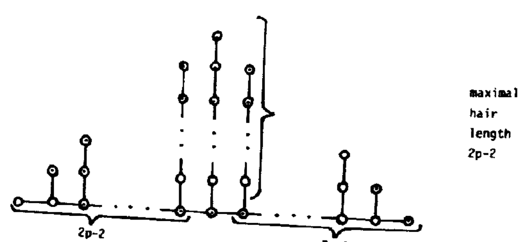


FIG. 3.1. *The barrier of height p.*

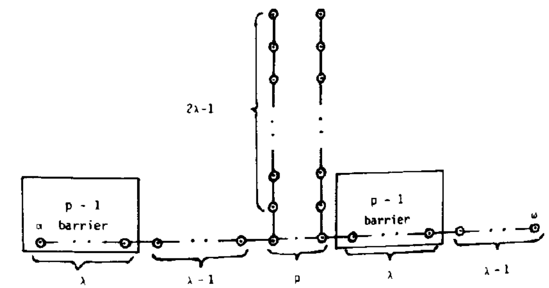


FIG. 3.2. *The turning point of height p, $\lambda=4p-5$.*

that the strategy which lays out all hairs of the left part as far as possible to the left and all hairs of the right part as far as possible to the right leads to an optimal layout (in doing so we have to go from the outer hairs to the inner hairs and to pay attention to the bandwidth restriction). The turning point of the height p also has bandwidth p. This time we get an optimal layout by stretching all hairs of length $2\lambda-1$ to the left and organizing the layout of the two barriers and the two chains such that both times one barrier and one chain overlap. Furthermore it is not difficult to see that for $p\geqq 5$ an analogue of Lemma 1 holds, i.e. for every layout with bandwidth p both nodes α and ω either belong to the left half of the layout or to the right half of the layout.

The caterpillar C which we associate this time to the instance $Y=(\{t_1,\cdots,t_n\}, D, m)$ of the 3-Partition problem is shown in Fig. 3.3. We can also apply the proof of Lemma 2 with only technical changes in this case showing that if Y has a solution then C has bandwidth $p+1+2n$.

In order to prove the other direction we follow the proof of Lemma 3. Again the ground line together with its barriers and its turning point defines a frame into which the sweeping line together with its hairs has to be embedded. As in the proof of Lemma 3 we define the notion of a task belonging to some interval.

A simple calculation shows that a task belongs to exactly one interval if $p\geqq 6n$ holds. Let us assume that a task i belongs to two intervals. Since the body of a barrier has length $4p+1$ and since nodes of the sweeping line belonging to task i are laid out as well to the left as to the right of the center of the barrier, there are more than p^2 nodes belonging to task i laid out within the region of the barrier. This is not possible if $p\geqq 6n$ holds.

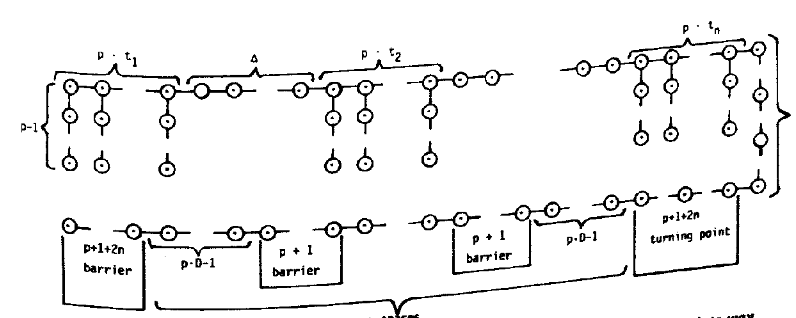


FIG. 3.3. *The caterpillar C, $\Delta=2p(m\cdot(D+4)-4)$, p has to be chosen in an approprriate way.*

A similar straightforward computation shows that if I_j, $1 \leqq j \leqq m$, denotes the set of tasks belonging to the jth interval, then $\sum_{i \in I_j} t_i \leqq D$ holds for all $j = 1, \cdots, m$ provided p fulfills $p \geqq 2n \cdot (D+4)$. Thus we have shown that if C has bandwidth $p+1+2n$ and if $p \geqq 2n \cdot (D+4)$ holds then Υ has a solution. This completes the proof of Theorem 2. □

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