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The Internal Rate of Return of an Investment

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SUMMARY

The concept of internal rate of return can be extended from standard investments or standard loans (with one negative payment at the beginning and only positive payments afterwards) to the general case (where the payments may be positive or negative arbitrarily). This problem from economics is a typical example for the formation of an applied mathematical concept, involving a distinct process of mathematisation. It is easily accessible to students at the upper secondary level, as all that is needed is basic calculus and algebra of polynomials including the analysis of graphs.

1. JUSTIFICATION

In West Germany mathematics student teachers (for the upper secondary level) usually learn about applications of mathematics in their calculus courses, where traditionally the examples are taken from physics. Nowadays however many of these students do not study natural sciences as their second subject so that applications in physics are of no significance and, what is more, are hard to understand and to work with. So they tend to ignore applications in their own studies and, as a consequence, ignore them later in their professional teaching. This tendency has to be countered, for example by an essential extension of the domains of application beyond the natural sciences. In particular, economics should be included much more intensively – in fact, not only as a field for applications, but as a subject whose basic ideas should be studied for their own sake.

One of the basic ideas in economics is the internal rate of return (irr) of a financial transaction (an investment or a financing), which also has the advantage of concerning every citizen in everyday life. Moreover, the fundamental ideas usually laid down at the lower secondary level by dealing with interest and interest rates in mathematics teaching.

The whole unit is a perfect example of applied mathematics including a distinct process of mathematisation, beginning with the planning of some financial transaction (which from now on I will call an investment) and the agreement about the conditions of the loan and the repayment, and ending with the determination of the pure cash flow. This process is led by the so-called *forget-principle* which means: it doesn't matter whether the payments are meant to be rent, redemption, commission, interest or so on, only the amount, the sign and the time of each payment are relevant. This process should be well known from the lower secondary level, and I want to concentrate now on another important aspect of applied mathematics teaching, namely the formation of an applied mathematical concept by extending its domain.

2. THE INTERNAL RATE OF RETURN IN THE STANDARD CASE

In order to estimate the profitability of an investment there is assigned to it a positive real number by a certain (mathematical) function depending only on the cash flow: the *irr*. The higher this number, the better for the lender, and the worse for the borrower. Of course there are also a lot of external factors which influence the profitability.

In the standard case there is a loan a_0 (paid by the lender) at time 0 and several repayments $a_1, a_2, \ldots, a_n$ at times $1, 2, \ldots, n$ (paid by the borrower).

$$\textbf{Cash flow:} \quad (a_0, a_1, \ldots, a_n),\ n \in \mathbb{N},\ a_k \in \mathbb{R},\ a_n \neq 0 \tag{1}$$

$$\textbf{Standard Case:} \quad a_0 < 0; \quad a_1, a_2, \ldots, a_n \geqslant 0 \tag{2}$$

To each payment a_k there belongs its **final value** $a_k(1+x)^{n-k}$ still depending on the rate of interest x. The question is to find out for which x's the *final values* of all payments a_k are balanced, i.e. which x's are solutions of

$$a_0(1+x)^n + a_1(1+x)^{n-1} + \ldots + a_{n-1}(1+x) + a_n = 0 \tag{3}$$

The positive real solution of (3) is called the **internal rate of return** (*irr*) of the investment. In the *standard case* there is uniquely determined one solution $x_0 > -1$. Consider the polynomial

$$p_a(y) = a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-1} y + a_n \quad (y > 0) \tag{4}$$

which has the same zeroes as the continuous, strictly monotonous function

$$f(y) = p_a(y)/y^n = a_0 + a_1/y + \ldots a_{n-1}/y^{n-1} + a_n/y^n$$

possessing positive values near $y = 0$, and tending towards $a_0 < 0$ for large y, with the **net sum**

$$b_n = a_0 + a_1 + \ldots + a_{n-1} + a_n = p_a(1) \tag{5}$$

We have

$$b_n \lesseqgtr 0 \quad \text{iff} \quad x_0 \lesseqgtr 0, \tag{6}$$

which means for instance that the *irr* is positive if and only if the sum of the repayments exceeds the loan.

3. THE INTERNAL RATE OF RETURN IN THE GENERAL CASE

Boulding (1936) extended this notion of *irr* to the general case of an investment, where the payments a_k (in (1)) may be arbitrarily negative or positive: it is defined to be 'the' solution of (3), thus still taking into account the amount, the sign and the time of each payment in an appropriate manner. In the general case it can happen that some of the payments out are paid later than (some of) the payments in. Economically speaking, this means just a rise of the profit, but mathematically speaking there is a fundamental change, as (3) now can have no solution or more than one solution $x_0 > -1$.

It is well known that (3) has n (complex) solutions (even in the *standard case*), but non-real solutions and real solutions $x_0 \leqslant -1$ make no economical sense. So only real solutions $x_0 > -1$ come into question, and that's why in the *standard case* x_0 is really uniquely determined. In the following there is considered the variant where only *non-negative rates of return* are admissible.

What then is to be done with a *cash flow* with a negative *net sum*? Whereas in the *standard case* the roles of the lender and the borrower are clearly distinguished by the sequence of their payments, one needs a different criterion in the general case: that party is the borrower, whose sum of payments is higher, because he repays the loan *and* some interest, and the other one is the lender. So, if the *cash flow* yields a negative *net sum*, the roles of the lender and the borrower have to be changed, as well as the sign of every payment, and the new *cash flow* has got a positive *net sum*.

For example, in the *standard case*, a negative *net sum* may result from the 'borrower's' premature stopping of his payments, and this means that in fact the 'lender' pays some interest. After changing the procedure the situation is described reasonably anew. Instead of, for example a *cash flow* of (−100, 40, 40) one has to consider (100, −40, −40) with $b_n = 20 > 0$, which means that the new borrower 're'pays the loan with interest in advance. For the new lender the deal proves to be a perfect investment (no matter how it is mathematised) with a profit higher than any finite one. So the *irr* can only be put as ∞ (which corresponds to the fact that equation (3) has no solution in $\mathbb{R}^+$). In the general case it can also happen that (3) has two or more positive solutions, for instance (400, −1300, 1000) yields $x_1 = 0.25$ and $x_2 = 1$.

This ambiguity of the *irr* has been known from the beginning and couldn't be removed in a satisfactory way; in Bender (1988) there is proposed a modified definition where it is avoided. In the general case one has to take real intervals instead of real numbers, but the class of *cash flows* with a uniquely determined real number as reasonably behaving *irr* is much larger than the *standard case* with positive *net sum*.

What does 'reasonable' mean in this context? The *irr*, taken as a function on the set of all *cash flows* into $\mathbb{R}^*: = \mathbb{R}^+ \cup \{0, \infty\}$, must have the following properties.

(a) It must be uniquely defined (**uniqueness, existence**).
(b) In the *standard case* it must coincide with the ordinary rate of return (**permanence of the concept**).
(c) If one investment is obviously more profitable than the other (the borrower's payments are higher or earlier, the lender's payments are lower or later), then its *irr* must be higher, unless it is 0 or ∞ (**positive ordinality**).
(d) If there are only small changes in the entries of the *cash flow*, there are also only small changes in the *irr*, unless it is 0 or ∞ (**reduced continuity**).

3.1 The transformed cash flow

For the general discussion one needs information about the polynomial (4) $p_a(y)$ in the domain $y \geqslant 1$ or, even better, of the polynomial $p_c(x) = p_a(1+x) = p_a(y)$ in the domain $x \geqslant 0$. By performing all multiplications we get

$$p_c(x) = c_0 x^n + c_1 x^{n-1} + \ldots + c_{n-1} x + c_n \tag{7}$$

$$\left(\text{with} \quad c_k = \sum_{j=0}^{k} \binom{n-j}{k-j} a_j, \quad \text{in particular } c_n = b_n\right),$$

and equation (3) is equivalent to $p_c(x) = 0$. The **original cash flow (ocf)** (1) is now replaced by the **transformed cash flow (tcf)** $(c_0, c_1, \ldots, c_n)$ which has some nice properties. Let $(c_0^{(1)}, c_1^{(1)}, \ldots, c_n^{(1)})$ and $(c_0^{(2)}, c_1^{(2)}, \ldots, c_n^{(2)})$ be the *tcfs* of two investments $I^{(1)}$ and $I^{(2)}$; then $I^{(1)}$ is more profitable than $I^{(2)}$ (in the sense of (*c*)), if $c_k^{(1)} \geqslant c_k^{(2)}$ for all k; and any *tcf* has at most as many changes of sign as its *ocf*.

The number s of changes of sign in the *tcf*, again, is an upper bound for the number z of positive zeroes of the polynomial $p_c(x)$ (which are also called **'positive zeroes'**, or **'solutions** of the investment', 'of $p_a(1+x)$', 'of the *ocf*', 'of the *tcf*'), and, what is more, $z-s$ is an even number. (This is the famous Descartes' rule of signs, which usually is not included in the syllabus, but should be, as it is a powerful means for analysing the graphs of real polynomials.)

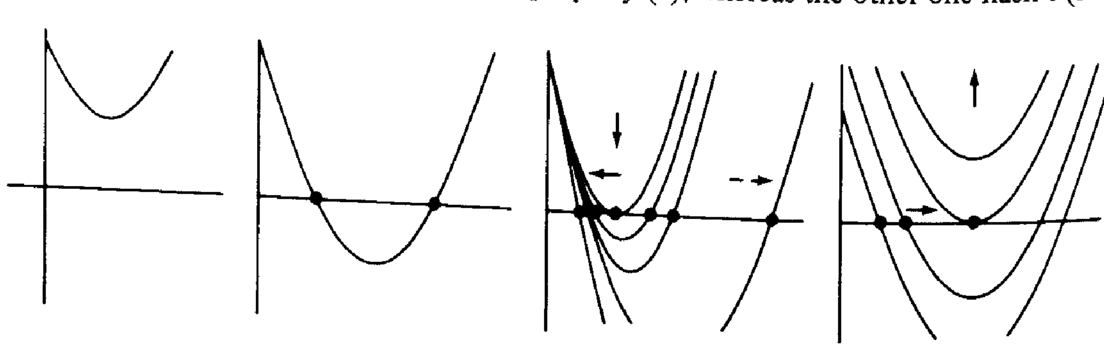
In the standard case one has $c_0 < 0$ and either $s = 1 = z$ (if $b_n > 0$) or $s = 0 = z$ (if $b_n \leqslant 0$), as one knows already from (6). The use of the *tcf* can be seen from examples like this. The *ocf* $(-1, 4, -7, 6, 1, -1)$ has four changes of sign, its *tcf* is $(-1, -1, -1, -1, 3, 2)$ with $s = 1$, and one knows immediately that there is one and only one *positive solution.*

3.2 Cash flows with two or more positive solutions

Throughout this section $b_n (= c_n)$ is taken to be positive. Let $(c_0, c_1, \ldots, c_n)$ be a *tcf* with $s = 2$, that is $z = 0$ or $z = 2$.

$z = 0$ means that at any rate (no matter how large it is) the sum of the *final values* of the negative payments is smaller than the respective sum of the positive payments (because $p_c(x) > 0$ for all $x > 0$, see Fig. 29.1(a)), and the *irr* has to be put ∞.

If there are two (possibly identical) *positive zeroes*, one has to choose, of course, the smaller one as the *irr*, because this one has property (*c*), whereas the other one hasn't (see



(*a*) (*b*) (*c*) (*d*)

Fig. 29.1

Fig. 29.1(b)), and: if the *cash flow* is changed continuously into the *standard case*, the smaller zero changes continuously into the conventional *irr*, whereas the larger zero tends towards ∞ (see Fig. 29.1(c)).

It is clear by intuition (but cannot be proved at the secondary level) that the zeroes depend continuously on the coefficients of the polynomial – unless they disappear (literally), which happens when these coefficients grow (see Fig. 29.1(d)). In this case the *irr* becomes ∞. So, in fact, there is no complete continuity but, at least in the case $s \leqslant 2$, there are no finite, hence inconspicuous, jumps. The points of discontinuity are well marked, as they always have the value ∞ in their neighbourhood. This fact is essential for the usefulness of the concept.

Obviously it is not the number s, but the number z, which determines how to fix the *irr*. For any investment with positive *net sum* and at most two *positive zeroes* the *irr* is

$$\begin{aligned}&\infty, \text{ if there is no } \textit{positive zero},\\ &x_0, \text{ if there is on and only one } \textit{positive zero } x_0,\\ &x_1, \text{ if there are two } \textit{positive zeroes } x_1 \leqslant x_2.\end{aligned} \quad (8)$$

This concept of *irr* is provided with properties (*a*)–(*d*), but it cannot be extended to the case with three or more *positive zeroes*. Let, for example, $(-1000, 400, -50, c_3)$ be the *tcf* of some investment. Obviously, the smallest and the largest zero (x_s and x_1) would have properties (*a*)–(*c*), but not (*d*), as $c_3 = 2$ and $c_3 = \frac{50}{27}$ are points of discontinuity, where x_s resp. x_1 would have finite jumps.

If one wants to define the *irr* for the general case, one must take complete intervals, thus loosening property (*a*); in the example above the interval would be $[x_s, x_1]$ (Fig. 29.2).

In practice, every fairly normal and reasonable investment has positive *net sum* and belongs to the case $z = 1$, or at least to the case $z = 2$, even if the *ocf* has more than two changes of sign. In West Germany there is one well-known exception: The effective conditions of present *building loan contracts* are rather bad for the borrowers: they get

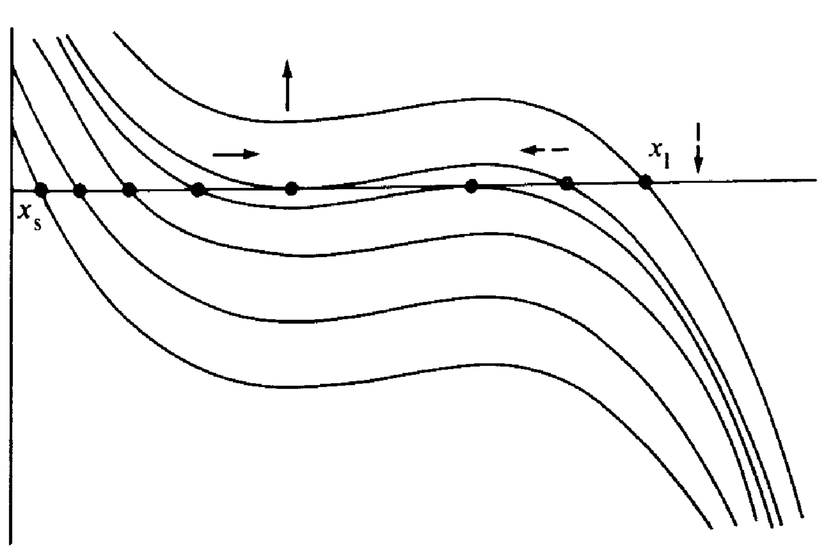


Fig. 29.2 –

the loan only after a long starting phase, where they have to save a high portion of the loan in advance. A typical *ocf* with a nominal loan of 120 000 is (2750, 7 times 6000, −112 790, 9 times 8640, 5700, 2820) with $s = 2$ and $z = 0$, hence $irr = \infty$.

3.3 Cash flows with net sum zero

Finally, in order to complete the system, one has to consider the case $b_n = 0$, which usually does not occur in practice. As the *irr* cannot be extended to this case continously, it is constantly put equal to 0, thus delivering no additional information, and again, the points of discontinuity are well marked.

4. FINAL REMARKS WITH REGARD TO MATHEMATICS TEACHING

This exposition of the general concept of *irr* can be taken as the framework of a teaching unit which, of course, has to deal with certain aspects in much more detail and can omit others.

When trying to expand the domain of the *irr* beyond the *standard case* one meets a surprising phenomenon, namely the loss of uniqueness, and one has to modify the concept of *irr*, which presupposes a careful analysis of the purpose and the properties of the ordinary concept as well as developing criteria which the new concept shall fulfil. Following the *deep-end-principle* one should start with the most general case and accept the weakening of several properties: the values 0 and ∞ can occur; there is left only a reduced concept of continuity; and in the end even the condition of uniqueness has to be moderated.

Besides its significance for mathematics and applied mathematics teaching, this teaching unit can also be relevant for students' everyday lives, as it clarifies the ordinary concept of *irr* and shows how to judge extreme cases. Its strong point is, furthermore, the close interlocking of mathematical and economical arguments and the perpetual subordination of mathematics to economics.

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