FUNCTION ALGEBRAS AND A THEOREM OF MERGELYAN FOR VECTOR-VALUED FUNCTIONS

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Abstract

An analogue of Mergelyan's approximation theorem $A(K) = P(K)$ is proved for functions with values in a complete locally convex topological vector space. The result is applied to obtain an approximation theorem for scalar functions on product sets $K = K_1 \times \ldots \times K_n$ of $\mathbb{C}^n$ $(n > 1)$. In the second part, algebras of continuous functions on a compact Hausdorff space with values in a Banach algebra X are considered. There are some conditions, involving the approximation property (which, however, is not easily verified in special cases) that guarantee that such an algebra is isomorphic to the completed (biequicontinuous) tensor product of the algebra X with the corresponding algebra of scalar functions. Applications to vector-valued functions and to scalar functions on product sets are given.

In this paper, a method for dealing with vector-valued functions is developed by using tensor products and L. Schwart's ε-product. The results are stated and there are given indications of how to prove the main theorems. Complete proofs of most of the statements are given in [2].

The theory was stimulated by an effort to prove a vector-valued version of Mergelyan's theorem. Let X be a complete locally convex topological vector space over $\mathbb{C}$ and K a compact Hausdorff space. Then let $C(K)$ be the Banach algebra of all complex-valued continuous functions on K, $C(K,X)$ the complete locally convex space of all functions continuous on K with values in X with the topology of uniform convergence on K. If K is a compact set in the complex plane, let $P(K)$ be the closure of the complex polynomials in $C(K)$ and $A(K)$ the closed subalgebra of $C(K)$ consisting of all functions holomorphic on the interior $\mathring{K}$ of K. Mergelyan's theorem states $A(K) = P(K)$, if the complement of K is connected.

1. Definition. For a compact set K in $\mathbb{C}$, let $A(K,X)$ be the closed subspace of $C(K,X)$ of all functions $f \in C(K,X)$ (weakly) holomorphic on $\mathring{K}$, i.e. for any $x' \in X'$ the scalar function $z \longrightarrow x'(f(z))$ is holomorphic on K. Let $P(K,X)$ be the closure in $C(K,X)$ of all complex polynomials with coefficients in X, that is of all functions $z \longrightarrow \sum_{i=0}^{n} x_i z^i$ $(n \in \mathbb{N}, x_i \in X, i = 0,\ldots,n)$.

What we are going to prove is.

2. Theorem. If the complement of K is connected, then

(1)
$$A(K,X) = P(K,X).$$

(2) If X is a (F) - space, each function continuous on K and holomorphic in $\mathring{K}$ with values in X can be approximated uniformly on K by a sequence of polynomials with coefficients in X.

The following is an immediate consequence of 2 for special X and of [8], 4.4. - G.

3. Corollary. Let X and Y be Banach spaces over $\mathbb{C}$, K a compact set in $\mathbb{C}$ with connected complement. Let for every $z \in K$ a bounded linear operator $A(z)$ from X to Y be given such that

(1) the function $z \longrightarrow A(z)$ is continuous on K with values in the Banach space $\mathcal{L}(X,Y)$ of all continuous linear operators from X to Y,

(2) for any fixed $x \in X$ and any fixed $y' \in Y'$ the scalar function $z \longrightarrow y'(A(z)x)$ is holomorphic on $\mathring{K}$.

Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $A_i \in \mathcal{L}(X,Y)$ $(i = 0,\ldots,n)$ so that

$$\sup_{z \in K} \sup_{\|x\|_X \leqq 1} \|A(z)x - \sum_{i=0}^{n} A_i\, x\, z^i\|_Y \quad .$$

To prove 2, let us first define the ε-product and the tensor product in an abstract way and let us consider some of their properties. Thus for the moment, let X and Y be locally convex spaces over $\mathbb{C}$. The ε-product defined below was first instroduced by L. Schwartz [6] who gave a different definition which is yet proved to be equivalent to 4. for complete X and Y.

4. Definition. By Y'_c is denoted the (topological) dual space Y' of Y with the topology of uniform convergence on all absolutely convex precompact subsets of Y. Let then $X \varepsilon Y = \mathcal{L}_e(Y'_c, X)$ be the locally convex space consisting of all continuous linear mappings from Y'_c into X with the topology of uniform convergence on all equicontinuous subsets of Y'.

If X and Y are complete, so is $X \varepsilon Y$. If X and Y are Banach spaces, $X \varepsilon Y$ is the Banach subspace of $\mathcal{L}(Y', X)$ consisting of all linear mappings from Y' into X whose restrictions to the unit ball Y'_1 of Y' are continuous from the $\sigma(Y', Y)$ - topology on Y'_1 into X. For complete X and Y the space $X \varepsilon Y$ is canonically isomorphic (by transposition) to $Y \varepsilon X$.

5. Lemma. If the topologies of X resp. Y are given by the directed systems $\{q_\alpha, \alpha \in A\}$ resp. $\{p_\beta, \beta \in B\}$ of seminorms, then the topology of $X \varepsilon Y$ can be given by the system $\{q_{\alpha\beta}, \alpha \in A, \beta \in B\}$ of seminorms: $q_{\alpha\beta}(u) = \sup_{y' \in \zeta_\beta} q_\alpha(u(y'))$ for every $u \in X \varepsilon Y$, where ζ_β is the set of extreme points of the (absolute) polar E^o_β of $E_\beta = \{y \in Y, p_\beta(y) \leq 1\}$.

The proof consists in an application of several theorems on locally convex spaces including e.g. those of Alaoglu-Bourbaki and Krein-Milman.

By using the Hahn-Banach theorem and the universal property of the tensor product one can see that

$$\sum_{i=1}^{n} x_i \otimes y_i \longrightarrow (y' \longrightarrow \sum_{i=1}^{n} x_i\, y'(y_i)) \quad (n \in \mathbb{N}, x_i \in X, y_i \in Y, \; i = 1, \ldots, n, \; y' \in Y')$$

defines a one-to-one linear map of the algebraic tensor product $X \otimes Y$ into $X \varepsilon Y$. We identify $X \otimes Y$ with the corresponding subspace of $X \varepsilon Y$. The completion of $X \otimes Y$ in the induced topology is denoted by $X \check{\otimes} Y$ and called the injective tensor product of X and Y. If X and Y are complete, $X \check{\otimes} Y$ can be considered as the closure of $X \otimes Y$ in $X \varepsilon Y$.

6. Proposition (L. Schwartz [6]). If X is complete and $X \otimes Y$ is dense in $X \varepsilon Y$ for each locally convex space Y, then X has the approximation property (i.e. the identity of X can be approximated uniformly on precompact subsets of X by operators of finite rank). Conversely, if X has the approximation property and Y is any complete locally convex space, then $X \otimes Y$ is dense in $X \varepsilon Y$. If X has the approximation property and X and Y are complete, then of course $X \check{\otimes} Y = X \varepsilon Y$.

L. Waelbroeck [9] proved that for a Banach space X the fact that $X \otimes Y$ is dense in $X \varepsilon Y$ for all closed subspaces Y of (c_o), the space of sequences converging to zero (with the sup-norm), implies that X possesses the approximation property. Schwartz's proof of 6. indicates that for complete X from $X \otimes X'$ dense in $X \varepsilon X'_c$ already follows that X has the approximation property, but in general one cannot handle the space X'_c easily.

We now look at function spaces and denote by Y a closed linear subspace of $C(K)$, K some fixed compact Hausdorff space.

7. Theorem. By $u \longrightarrow (t \longrightarrow u(\delta_t))$ $(u \in X \varepsilon Y,\ t \in K, \delta_t$ the corresponding delta-functional) an injective map of $X \varepsilon Y$ into $C(K,X)$ is defined.

The topology on $X \varepsilon Y$ is given by the system $\{r_\alpha ; \alpha \in A\}$ of seminorms: $r_\alpha(u) = \sup_{t \in K} q_\alpha(u(\delta_t))$ for every $u \in X \varepsilon Y$, i.e. the topology of $X \varepsilon Y$ is induced by $C(K,X)$.

The proof uses 5. and the Arens-Kelley characterisation of the extreme points of the unit ball Y'_1 of Y' as delta-functionals. The main ideas in the proof of 5. and 7. are due to B. Gramsch and D. Vogt [4]. Their tensor product arguments could be carried over to the ε-product with some minor changes.

For a complete X we get $X \otimes Y \subset X \check{\otimes} Y \subset X \varepsilon Y \subset C(K,X)$ by identifying some topologically isomorphic spaces. If then $Y = C(K)$, it is easy to see (by using the partition of unity argument) that $X \check{\otimes} C(K) = X \varepsilon C(K) = C(K,X)$.

Now return to the actual proof of 2. and take the notations as defined there. By a simple argument using 7. we have

$$\begin{array}{ccc} X \check{\otimes} A(K) & \subset & A(K,X) \\ \| & & \cup \\ X \check{\otimes} P(K) & \subset & P(K,X). \end{array}$$

All we now have to show is $X \check{\otimes} A(K) = A(K,X)$. By 6. and a theorem of L.Eifler [3] proving that $A(K)$ possesses the approximation property (if K does not separate the plane), all that is left to prove is $A(K,X) \subset X \varepsilon A(K)$. Looking at the identifications above, the equality $C(K) \varepsilon X = C(K,X)$ means that by the mapping

$$f \longrightarrow (x' \longrightarrow (z \longrightarrow x'(f(z)))) \quad (f \in C(K,X),\ x' \in X',\ z \in K)$$

$f \in C(K,X)$ is mapped into $C(K) \varepsilon X$. But by definition of $A(K,X)$, each $f \in A(K,X)$ is mapped into $A(K) \varepsilon X$, that is, in our notation one gets $A(K,X) \subset X \varepsilon A(K)$, which completes the proof.

To give an application of 2. to scalar function algebras, let us take for some compact $K' \subset \mathbb{C}$ with connected complement $X = A(K')$. Then, by the proof of 2., $A(K) \check{\otimes} A(K') = A(K', A(K))$ and by identifying $A(K', A(K))$ with an appropriate space (defined below) of functions of two variables on the product set $K \times K'$ and proceeding by induction one obtains an approximation theorem of Mergelyan type for product sets of $\mathbb{C}^n$ $(n > 1)$.

<u>8. Definition</u>. For a compact K in $\mathbb{C}^n$ $(n \geq 1)$ let $P(K)$ and $A(K)$ be defined as usual. If $K = K_1 \times \ldots \times K_n$ with K_i compact in $\mathbb{C}$, define

$A_{\pi}(K) = \{ f \in C(K);$ the function of one variable $z_i \longrightarrow f(z_1, \ldots, z_i, \ldots, z_n)$ belongs to $A(K_i)$ for arbitrary fixed $z_j \in K_j$ $(j \neq i)$, $i,j = 1, \ldots, n$.

$A_{\pi}(K)$ is a closed subspace of $C(K)$ and it is obvious that in general, for $n > 1$, $A_{\pi}(K) \subsetneqq A(K)$, but $A_{\pi}(K) = A(K)$, if K is a fat set (i.e. K is the closure of its interior).

The theorem mentioned above looks like follows:

<u>9. Corollary</u>. For $K = K_1 \times \ldots \times K_n$ as in the definition of $A_{\pi}(K)$, if $\mathbb{C} \setminus K_i$ is connected $(i = 1, \ldots n)$, one obtains

(1) $$A_{\pi}(K) = A(K_1) \check{\otimes} \ldots \check{\otimes} A(K_n) = P(K_1) \check{\otimes} \ldots \check{\otimes} P(X_n) = P(K)$$

(2) Each function in $A_{\pi}(K)$, for fat K of course each function in $A(K)$, can be approximated uniformly on K by polynomials in n variables.

Another proof of 9. can be provided by theorems in [3]. 9. generalises a result in [7], it seems to be quite interesting because counterexamples by E. Kallin [5] show that the equality $A(K) = P(K)$ is false in general even for fat compact polynomially convex sets K in $\mathbb{C}^n$ $(n > 1)$.

From 2. one can see that in our proof of the important relation $A(K,X) = X \check{\otimes} A(K)$ we used the approximation property for $A(K)$, 6. and 7., which apply to any closed subspace Y of $C(K)$, and the definition of weak holomorphy. An analogue of this property can be found in many spaces of vector-valued functions, too. Therefore, without changing the proof really, we can proceed to generalise the theorem.

To fix the ideas, let K be a compact Hausdorff space and X be a Banach algebra with identity e $(\|e\| = 1)$. By the map $\lambda \longrightarrow \lambda e$ consider $\mathbb{C}$ as a subalgebra of X. Let $C(K,X)$ be the Banach algebra of continuous X-valued functions on K and identify X with the subalgebra of $C(K,X)$ of all X-valued constant functions on K. If S is any subalgebra of $C(K,X)$, denote by

$$S_o = \{ f \in C(K); \text{ the function } t \longrightarrow f(t)\,e \text{ belongs to } S \}$$

the corresponding algebra of scalar functions in S. We shall not make any difference between the scalar function $t \longrightarrow f(t)$ and the vector-valued one $t \longrightarrow f(t)\,e$. Now consider the following properties that a subalgebra S of $C(K,X)$ can possess:

(1) S is closed in $C(K,X)$,

(2) S contains the constants (in our notation $X \subset S$),

(3) S_o separates points,

(4) for an arbitrary $f \in S$ and an arbitrary $x' \in X'$ the function $t \longrightarrow x'(f(t))$ belongs to S_o,

(5) X or S_o has the approximation property.

If S satisfies (1), (2) and (3), S_o is a scalar function algebra. Conversely, if S_o is a function algebra on K, then $X \check{\otimes} S_o$ is an algebra of continuous X-valued functions with properties (1)-(4). Other examples of subalgebras of $C(K,X)$ satisfying

(1)-(4) are for compact $K \subset \mathbb{C}$ the algebras $A(K,X)$, $P(K,X)$.

Condition (4) is very restrictive. It allows to prove the following theorem by just the same method as 2.

10. Theorem. Let S be a subalgebra of $C(K,X)$ with properties (1), (2), (4) and (5). Then

$$S = X \check{\otimes} S_o .$$

This is an approximation theorem for vector-valued functions which reduces the vector-valued case to the scalar one.

To state a few applications of our method for vector-valued functions, the following corollaries are immediate from 10. and well-known theorems on function algebras, tensor products and vector-valued functions.

11. Corollary. Let S be a subalgebra of $C(K,X)$ with properties (1)-(3). Then each of the following conditions ensures $S = C(K,X)$:

(i) S_o is self-adjoint (i.e. with $f \in S_o$ the complex conjugate $\bar{f}$ belongs to S_o, too),

(ii) $\mathrm{Re}\, S_o = \{ \mathrm{Re}\, f ; f \in S_o \}$ is closed in $C(K)$ or (real) subalgebra of $C(K)$,

(iii) K is finite or countable,

(iv) K is totally disconnected, and the maximal ideal space $\mathfrak{M}(S_o)$ of S_o is equal to K,

(v) $K \subset \mathbb{C}$, $S = P(K,X)$, $\mathring{K} = \emptyset$, and $\mathbb{C} \setminus K$ is connected.

12. Corollary. Let S be a subalgebra of $C(K,X)$ with properties (1), (2), (4) and (5) and let X be commutative. Then

(i) If X is semi-simple, so is S.

(ii) $\mathfrak{M}(S) = \mathfrak{M}(X) \times \mathfrak{M}(S_o)$ (up to a homeomorphism).

(iii) When identifying the maximal ideal spaces by the homeomorphism in (ii), the Silov-boundaries coincide: $\Gamma_S = \Gamma_X \times \Gamma_{S_o}$.

The case of non-commutative X seems to be of special interest because of its applications to the space $X = \mathcal{L}(Y)$ of bounded linear operators on a Banach space Y. The following is obtained by applying [1], 3.14:

13. Corollary. Let S be a subalgebra of $C(K,X)$ satisfying (1)-(5), and let $\mathfrak{M}(S_o)$ be equal to K. Then S has the fixed ideal property (see [1]).

Let f_k $(k = 1,\ldots,m,\ m \in \mathbb{N})$ be functions in S such that for arbitrary $t \in K$ there exist $x_k \in X$ $(k = 1,\ldots,m)$ with

$$\sum_{k=1}^{m} x_k\, f_k(t) = e.$$

Then there exist $g_k \in S$ $(k = 1,\ldots,m)$ with

$$\sum_{k=1}^{m} g_k(t)\, f_k(t) = e$$

for all $t \in K$. Especially, a function $f \in S$ is left (right) invertible in S if and only if for every $t \in K$ the value $f(t)$ is left (right) invertible in X.

Notice that, in a Banach algebra, one-sided inverses are not defined uniquely (in general).

Corollaries 12. and 13. apply to the algebra $A(K,X)$, if $\mathbb{C} \setminus K$ is connected.

Added July 16th, 1969:

At the "Summer Gathering on Function Algebras" in Aarhus, the author was informed that quite the same result as theorem 2. was independently shown by E. Briem, K.B. Laursen and N.W. Pedersen in a preprint: "Mergelyan's theorem for vector-valued functions with an application to slice algebras", Aarhus, Dec. 1968. Their methods are very different from ours and do not apply to more general spaces of vector-valued functions: they show that Rudin's proof of Mergelyan's theorem can be imitated in the vector-valued case.

Results similar to 2. and 9. were, as I was told in Aarhus, announced by T.W. Gamelin and J. Garnett in Sec. 6 of their article: "Constructive techniques in rational approximation". Their method consists in imitating the scalar case, too; their result corresponding to 9. seems to need some minor correction.

The following theorem shows the connection between the ε-product and the slice-product:

<u>14. Definition (L. Eifler [3])</u>. Let K_1 and K_2 be compact Hausdorff spaces and A resp. B closed linear subspaces of $C(K_1)$ resp. $C(K_2)$.

Then define the slice product of A and B by

$$A \# B = \{ f \in C(K_1 \times K_2);$$

the function $t_1 \longrightarrow f(t_1,t_2)$ belongs to A for every fixed $t_2 \in K_2$. and the function $t_2 \longrightarrow f(t_1,t_2)$ belongs to B for every fixed $t_1 \in K_1 \}$.

<u>15. Proposition</u>. Under the conditions of 14., if $A \varepsilon B$ is identified with a subspace of

$$C(K_1 \times K_2), \quad \text{then} \quad A \varepsilon B = A \# B.$$

If one looks at the embedding proved in 7. and identifies $C(K_2,C(K_1))$ with $C(K_1 \times K_2)$ in the obvious way, then

$$A \varepsilon B = \{ f \in C(K_1 \times K_2);$$

the function $t_1 \longrightarrow b'(f(t_1,t_2))$ (where b' is applied with respect to the second variable t_2) belongs to A for arbitrary $b' \in C'(K_2)$, and the function $t_2 \longrightarrow a'(f(t_1,t_2))$ (a' applied with respect to t_1) belongs to B for arbitrary $a' \in C'(K_1) \}$.

Now the proposition is an easy consequence of e.g. the standard Krein-Milman and Arens-Kelley argument above.

<u>The old slice algebra problem</u>

Given two function algebras A and B, must then $A \check{\otimes} B = A \# B$? is now by 6. seen to be (perhaps not equivalent, but) very close to the problem, whether every function algebra possesses the approximation property. 6. and 15. imply a result on the slice product first announced by L. Eifler [3] and then also shown in a different way in the paper by Briem, Laursen and Pedersen. 6. and 15. also provide a partial converse of this theorem.

Some, but not many function algebras (e.g., in which concerns published results, C(K), P(K), and for "good" K, A(K)) are known to possess the approximation property and it seems to be an extremely difficult problem to find out in each single case whether a given function algebra possesses the approximation property.

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