TENSOR PRODUCTS OF WEIGHTED SPACES*)

Klaus-D. Bierstedt

In part of our previous paper [6], an exposition of some results on the approximation property (a. p.) for weighted function spaces was given. The methods used to derive the results were then combined with a vector-valued generalized Stone-Weierstrass theorem due to PROLLA to obtain a new theorem on a "localization" of the a. p. for certain subspaces of weighted spaces of type $CV_o(X)$, and the applications of this theorem were demonstrated by a few typical examples. - We shall deal here with a different aspect of the theory, namely with the tensor product representation of weighted spaces of vector-valued functions and of functions of several variables. It is convenient to keep the notation introduced in [6], so we refer to this paper for all terminology.

I. The ε-products $E\varepsilon CV(X)$ resp. $E\varepsilon CV_o(X)$

The key to the method of proof used in [6] was theorem 8. which gave a representation of the ε-product $E\varepsilon CV(X)$ resp. $E\varepsilon CV_o(X)$ as a space of vector-valued functions, in fact as $CV^p(X,E)$ resp. $CV_o(X,E)$, if E was quasi-complete and if $Z \leq V$ or $W \leq V$ and $X = k_{\mathbb{R}}$-space. As this theorem is quite important for the results of this paper, too, we will give its proof here. - So suppose that, from now on, E is a (separated) locally convex space, X a (Hausdorff) completely regular topological space, and $V > 0$ a Nachbin family on X. As a consequence of the assumption $V > 0$, any point evaluation $\delta_x: f \longrightarrow f(x)$, $x \in X$, is continuous on $CV(X)$ resp. $CV_o(X)$, so belongs to $(CV(X))'$ resp. $(CV_o(X))'$.

1. Lemma ([3]). For $Z \leq V$ or $W \leq V$ and $X = k_{\mathbb{R}}$-space, the mapping $\Delta: x \longrightarrow \delta_x$ is continuous from X into $(CV(X))'_c$ resp. $(CV_o(X))'_c$.

Proof. The case $Z \leq V$ is easier to be treated, so we will assume $W \leq V$ and X $k_{\mathbb{R}}$-space. Let C = CV(X) resp. $CV_o(X)$. It is trivial, by definition

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of the weak*-topology, that $\Delta: X \longrightarrow C'[\sigma(C', C)]$ is always continuous.

But for any $K \subset X$ compact : $\Delta(K) \subset \{f \in C; \|f\|_K = \sup_{x \in K} |f(x)| \leq 1\}^\circ$. . And because of $W \leq V$, the set $\{f \in C; \|f\|_K \leq 1\}$ is a neighbourhood of 0 in C, so its polar is equicontinuous in C'. This implies that, for any compact K, $\Delta(K)$ is contained in an equicontinuous subset of C'. The weak*-topology and the topology $\lambda(C', C)$ of C'_c coincide on such sets, as an important result in the duality theory of locally convex spaces proves (cf. Köthe [19]). Hence we can conclude that, for every compact subset K of X, $\Delta|_K: K \longrightarrow C'_c$ is already continuous. C'_c is completely regular, so we obtain the continuity of Δ from our assumption X = $k_{\mathbb{R}}$-space (see the remark following definition 6. in [6]). In the case of $Z \leq V$, the whole of $\Delta(X)$ is contained in an equicontinuous subset of C', so we are done after our first argument already for this assumption. □

Actually a simple exercise on taking polars will show:

2. Lemma ([3]). $\Delta : X \longrightarrow (CV(X))'_c$ resp. $(CV_o(X))'_c$ is continuous if and only if each precompact subset of CV(X) resp. $CV_o(X)$ is equicontinuous.

Now H. BUCHWALTER [12] defines X to be an infra-$k_{\mathbb{R}}$-space, if each precompact subset of (C(X), co) is equicontinuous. So a simple application of 2. proves that $\Delta: X \longrightarrow (CV(X))'_c$ resp. $(CV_o(X))'_c$ is continuous for $W \leq V$ and X only an infra-$k_{\mathbb{R}}$-space. As you can see for instance by taking V=W in 1. and 2., every $k_{\mathbb{R}}$-space is a fortiori an infra-$k_{\mathbb{R}}$-space. The converse, however, is not true, as an example of HAYDON (see [12]) shows. Thus we have a slight refinement of 1. Even more general results of this type are true, but we will not go into the details here.

3. Corollary ([3]). Let u be an element of $E \epsilon CV(X) = \mathcal{L}((CV(X))'_c, E)$ resp. $E \epsilon CV_o(X)$. Then $u \circ \Delta$ belongs to C(X, E), if the conclusion of 1. holds.

We want to prove a little bit more for functions of type $u \circ \Delta$ as in 3.

4. Lemma ([3]). Let u be as in 3. and assume that the conclusion of 1. holds. Then any function $u \circ \Delta$ is even an element of $CV^p(X, E)$ resp.

$CV_o(X, E)$.[1)]

<u>Proof</u>. Start with $u \in E \epsilon CV(X)$ and take $v \in V$ arbitrary. We obtain for $bv(f) = \sup_{x \in X} v(x)|f(x)|$, $f \in CV(X)$, and

$$bv_1 = \{f \in CV(X);\ bv(f) \leq 1\}$$

the inclusion $\{v(x)\delta_x;\ x \in X\} \subset (bv_1)^\circ$. But, by linearity of u,

$$[v(u \circ \Delta)](X) = \{v(x)u(\delta_x);\ x \in X\} = u(\{v(x)\delta_x;\ x \in X\}).$$

Notice that, by the Alaoglu-Bourbaki theorem and by the result from duality theory mentioned in the proof of 1., the equicontinuous and weak*-closed set $(bv_1)^\circ$ is compact in $(CV(X))'_c$. Hence $u \in \mathcal{L}((CV(X))'_c, E)$ takes the subset $\{v(x)\delta_x;\ x \in X\}$ of $(bv_1)^\circ$ onto a relatively compact subset of E. Thus it is obvious that $[v(u \circ \Delta)](X)$ is precompact in E for any $v \in V$, and hence $u \circ \Delta$ belongs to $CV^p(X, E)$.

In the case of $u \in E \epsilon CV_o(X)$ the proof is a bit more complicated. For $v \in V$ arbitrary we obtain $\{v(x)\delta_x;\ x \in X\} \subset (bv_1)^\circ$ as above, where now $bv_1 = \{f \in CV_o(X);\ bv(f) \leq 1\}$. Then remark (as in the proof of 1.) that on $(bv_1)^\circ$ the topologies $\lambda((CV_o(X))', CV_o(X))$ and $\sigma((CV_o(X))', CV_o(X))$ coincide and that therefore $u|_{(bv_1)^\circ}$ is continuous from the weak*-topology of $(CV_o(X))'$ into E. So, given a continuous seminorm p on E and $\epsilon > 0$, there is a neighbourhood U of 0 in the weak*-topology of $(CV_o(X))'$ such that $u(U \cap (bv_1)^\circ) \subset \{e \in E;\ p(e) < \epsilon\}$. We can assume:

$$U = \{\mu \in CV_o(X))';\ |\mu(f_i)| < \delta\ (i = 1, \dots, n)\}$$

for appropriate $n \in \mathbb{N}$, $\delta > 0$, $f_i \in CV_o(X)$, $i = 1, \dots, n$. As all the f_i belong to $CV_o(X)$, we can find a compact subset K of X with the property that $v(x)|f_i(x)| < \delta$ for $i = 1, \dots, n$ and all $x\ X \setminus K$. This implies that $\{v(x)\delta_x;\ x \in X \setminus K\} \subset U \cap (bv_1)^\circ$ and hence for any $x \in X \setminus K$:

$$p(v(x)(u \circ \Delta)(x)) = p(u(v(x)\delta_x)) < \epsilon.$$

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1) By taking $E = (CV(X))'_c$ resp. $(CV_o(X))'_c$ and $u = id_{(CV(X))'}$ resp. $id_{(CV_o(X))'} \epsilon$ $E \epsilon CV(X)$ resp. $E \epsilon CV_o(X)$ in 4., we obtain: $\Delta = u \circ \Delta \epsilon CV^p(X, (CV(X))'_c)$ <u>resp.</u> $CV_o(X, (CV_o(X))'_c)$, a strengthening of lemma 1.

Thus it is clear that $v(u\circ\Delta)$ vanishes at infinity for any $v\in V$ and hence $u\circ\Delta\in CV_o(X,E)$. □

5. Remark ([3]). If $u\in E\otimes CV(X)\subset E\varepsilon CV(X)$ resp. $E\otimes CV_o(X)\subset E\varepsilon CV_o(X)$, then [without any assumptions (except of course $V>0$)] $u\circ\Delta$ always belongs to $CV^p(X,E)$ resp. $CV_o(X,E)$.

An inspection of the proofs of lemma 1. and 4. shows this, because the elements u of e.g. $E\otimes CV(X)$ are even continuous from $(CV(X))'$ with the weak*-topology into E. The assumptions on X and V were only needed in the proof of 1., i.e. in order to show continuity of $\Delta: X \longrightarrow (CV(X))'_c$. But Δ is always continuous w.r.t. the weak*-topology on $(CV(X))'$.

We will need the information contained in the next lemma, as we go on. (6. is a simple consequence of the bipolar theorem.)

6. Lemma ([3]). We denote by bv_1 the set $\{f\in CV(X)$ resp. $CV_o(X); bv(f)\leq 1\}$ for $v\in V$ arbitrary. Then the absolutely convex hull $\Gamma\{v(x)\delta_x; x\in X\}$ of the set $\{v(x)\delta_x; x\in X\}$ is weak*-dense in bv_1°. (So it is a fortiori dense in the topology of $(CV(X))'_c$ resp. $(CV_o(X))'_c$.)

Proof. In the fixed dual system $\langle CV(X),(CV(X))'\rangle$ $[\langle CV_o(X),(CV_o(X))'\rangle]$ we get: $bv_1 = \{v(x)\delta_x; x\in X\}^\circ$. Hence by the bipolar theorem:

$$(bv_1)^\circ = \{v(x)\delta_x; x\in X\}^{\circ\circ} = \overline{\Gamma\{v(x)\delta_x; x\in X\}},$$

where the closure may be taken in the weak*-topology or in the topology of $(CV(X))'_c$ [resp. $CV_o(X))'_c$]. □

To make use of this lemma, let $\{p_\alpha; \alpha\in A\}$ be a system of seminorms giving the topology of E. Remember that $\{bv_\alpha; v\in V, \alpha\in A\}$ induces the topology of $CV(X,E)$ resp. $CV_o(X,E)$ where

$$bv_\alpha(f) = \sup_{x\in X} p_\alpha(v(x)f(x)) = \sup_{x\in X} v(x)p_\alpha(f(x)), \quad f\in CV(X,E).$$

On the other hand, the topology of $E\varepsilon CV(X)$ resp. $E\varepsilon CV_o(X)$ is given by the system $\{q_{\alpha,v}; \alpha\in A, v\in V\}$:

$$q_{\alpha,v}(u) = \sup_{\mu\in(bv_1)^\circ} p_\alpha(u(\mu)), \quad u\in E\varepsilon CV(X),$$

because the sets of the form $(bv_1)^\circ$, $v\in V$, are a basis for the equicontinuous subsets of $(CV(X))'$ by the properties of Nachbin families.

7. Lemma ([3]). For any $\alpha\in A$ and any $v\in V$, $u\in E\otimes CV(X)$ resp. $E\otimes CV_o(X)$, we obtain $bv_\alpha(u\circ\Delta) = q_{\alpha,v}(u)$, and under the assumptions of 4., this is also true for $u\in E\varepsilon CV(X)$ resp. $E\varepsilon CV_o(X)$.

Proof. By 5., $u\circ\Delta\in CV(X,E)$ - for the second part of 7. apply 4. instead of 5. - and so we can look at $bv_\alpha(u\circ\Delta)$. Applying 6. now, we get:

$$q_{\alpha,v}(u) = \sup_{\mu\in(bv_1)^\circ} p_\alpha(u(\mu)) \overset{(1)}{=} \sup\{p_\alpha(u(\mu));\ \mu\in\Gamma\{v(x)\delta_x;\ x\in X\}\}$$
$$\overset{(2)}{=} \sup\{p_\alpha(u(\mu));\ \mu\in\{v(x)\delta_x;\ x\in X\}\} = \sup_{x\in X} p_\alpha(u(v(x)\delta_x))$$
$$= \sup_{x\in X} p_\alpha(v(x)(u\circ\Delta)(x)) = bv_\alpha(u\circ\Delta),$$

where we used 6. in (1), together with the continuity of $u_{|(bv_1)^\circ}$ from the weak*-topology into E. For (2) we have only to remind that p_α is a seminorm, i.e. that $p_\alpha(\Sigma_{i=1}^n \gamma_i e_i)\le\Sigma_{i=1}^n |\gamma_i| p_\alpha(e_i)\le \max_{i=1,\dots,n} p_\alpha(e_i)$, if only $e_i\in E$, $\gamma_i\in\mathbb{C}$ $(i=1,\dots,n)$, $\Sigma_{i=1}^n |\gamma_i|\le 1$. □

After these preparations we are now ready to combine everything we proved up to now in the following result:

8. Theorem ([3]). By the identification $u\longleftrightarrow u\circ\Delta$, the ε-tensor product $E\otimes_\varepsilon CV(X)$ resp. $E\otimes_\varepsilon CV_o(X)$ is always topologically isomorphic to a subspace of $CV^p(X,E)$ resp. $CV_o(X,E)$. Under the assumptions $Z\le V$ or $W\le V$ and X (infra-) $k_{\mathbb{R}}$-space, we even obtain $E\varepsilon CV(X)\subset CV^p(X,E)$ resp. $E\varepsilon CV_o(X)\subset CV_o(X,E)$ topologically under the same identification.

For the first part of 8., we must only be aware of the definition of the ε-topology on the tensor product as the topology induced by the ε-product. And in both cases we have to realize that the mapping $u\longrightarrow u\circ\Delta$ is, of course, linear and that 7. also proves that this mapping is injective and a topological isomorphism (into).

It is not hard to show that, by 8., $E\otimes_\varepsilon CV(X)$ resp. $E\otimes_\varepsilon CV_o(X)$ is identified with the subspace of $CV(X,E)$ resp. $CV_o(X,E)$ consisting of all functions f with the additional property that f(X) is contained in a finite dimensional subspace of E. It is remarkable that, in the case of weighted spaces, the space of the vector-valued functions always induces the ϵ-topology on the tensor product (provided $V>0$).

If E is supposed quasi-complete, we will show in a moment that, under the assumptions of the second part of 8., the subspace $E\epsilon CV(X)_{(o)}(X)$ of $CV^p(X,E)$ (resp. $CV_o(X,E)$) actually coincides with the whole space. - Look at any $f\in CV(X,E)$ resp. $CV_o(X,E)$, and take $e'\in E'$ arbitrary. It is trivial that $e'\circ f\in CV(X)$ resp. $CV_o(X)$, so we can define $w_f: e'\longrightarrow e'\circ f$ as a linear mapping from E' into $CV(X)$ resp. $CV_o(X)$.

9. Lemma ([4]). For $f\in CV^p(X,E)$ resp. $CV_o(X,E)$ we always have $w_f\in CV(X)\epsilon E = \mathcal{L}(E'_c, CV(X))$ resp. $CV_o(X)\epsilon E$.

Proof. It is enough to show the assertion for $f\in CV^p(X,E)\supset CV_o(X,E)$. So take a neighbourhood of 0 in CV(X) which, by the properties of Nachbin families, can be assumed of the form bv_1. $P = (vf)(X)$ is a precompact subset of E, so P° is a neighbourhood of 0 in E'_c with the property $w_f(P^\circ)\subset bv_1$, because for $e'\in P^\circ$:

$$bv(w_f(e'))=bv(e'\circ f)=\sup_{x\in X} v(x)|e'(f(x))|=\sup_{x\in X} |e'(v(x)f(x))|\leq 1. \quad \Box$$

10. Lemma ([4]). Let us suppose that E and CV(X) resp. $CV_o(X)$ are quasi-complete. Then the transposed map ${}^t w_f$ of w_f in 9. belongs to $E\epsilon CV(X)$ resp. $E\epsilon CV_o(X)$ and ${}^t w_f(\delta_x) = f(x)$ for all $x\in X$.

Proof. For quasi-complete spaces E and F, $F\epsilon E \cong E\epsilon F$ by transposing. Hence 9. implies the first part of 10. For the last assertion we notice that the definition of ${}^t w_f$ means:

$\langle {}^t w_f(\mu), e'\rangle = \langle \mu, w_f(e')\rangle = \langle \mu, e'\circ f\rangle$ for all $e'\in E'$, $\mu\in(CV(X))'$ resp. $(CV_o(X))'$ and that therefore $e'({}^t w_f(\delta_x)) = e'(f(x))$ for each $e'\in E'$ which proves the equality ${}^t w_f(\delta_x) = f(x)$ for any $x\in X$. $\Box$

Take $f\in CV^p(X,E)$ resp. $CV_o(X,E)$ and suppose E and CV(X) resp. $CV_o(X)$ are quasi-complete. Then $u_f = {}^t w_f\in E\epsilon CV(X)$ resp. $E\epsilon CV_o(X)$ by 10. and $(u_f\circ\Delta)(x) = ({}^t w_f\circ\Delta)(x) = {}^t w_f(\delta_x) = f(x)$ for all $x\in X$. Hence, by the identification $u\longleftrightarrow u\circ\Delta$ in 8., $E\epsilon CV(X)$ resp. $E\epsilon CV_o(X)$ is not only a subspace, but (topologically isomorphic to) the whole space $CV^p(X,E)$ resp. $CV_o(X,E)$, and we have finally proved theorem 8. from [6] which we formulate again for the convenience of the reader:

11. Theorem ([4]). For a quasi-complete locally convex space E and under the assumptions $Z \leq V$ or $W \leq V$ and X $k_{\mathbb{R}}$-space, the following canonical topological isomorphisms hold:

$$CV^p(X,E) = E \epsilon CV(X) = CV(X) \epsilon E,$$
$$CV_o(X,E) = E \epsilon CV_o(X) = CV_o(X) \epsilon E.$$

Denote by E'_{co} the dual E' (of the locally convex space E), equipped with the topology of uniform convergence on all absolutely convex compact subsets of E. The $\tilde{\epsilon}$-product of the two locally convex spaces F and E is then introduced as $F \tilde{\epsilon} E = \mathcal{L}_e(E'_{co}, F)$. Obviously $E'_{co} = E'_c$ and $F \epsilon E = F \tilde{\epsilon} E$, if only E is quasi-complete. With this notation in mind we mention the more general result:

12. Theorem ([4]). Up to topological isomorphisms $E \tilde{\epsilon} CV(X) = CV(X) \tilde{\epsilon} E = \{f \in C(X, E[\sigma(E',E)]); \text{ for each } v \in V \text{ the set } \Gamma(vf(X)) \text{ is relatively compact in } E \text{ (w.r.t. the original topology of } E)\}$,

$E \tilde{\epsilon} CV_o(X) = CV_o(X) \tilde{\epsilon} E = \{f \in C(X, E[\sigma(E,E')]); \text{ for each } v \in V \text{ the set } \Gamma((vf)(X)) \text{ is relatively compact in } E, \text{ and } vf \text{ vanishes at infinity}\}$,

where, on the respective subspaces of $C(X, E[\sigma(E,E')])$, a locally convex topology is introduced by the system $\{bv_\alpha ; \alpha \in A, v \in V\}$ of semi-norms defined exactly as after 6. above (for a system $\{p_\alpha ; \alpha \in A\}$ of semi-norms inducing the topology of E) : $bv_\alpha(f) = \sup\{p_\alpha(v(x)f(x)); x \in X\}$, $f \in C(X, E[\sigma(E,E')])$ with $(vf)(X)$ bounded in E.

Finally we want to remark that weighted spaces of E-valued functions could be defined more generally for arbitrary (Hausdorff) topological vector spaces E. It turns out that, under assumptions as in the second part of 8., (e.g.) $E \epsilon CV_o(X)$ is again a space of continuous E-valued functions and that one can characterize the ϵ-topology as an induced (natural) topology. So a result more general than 8. and somewhat similar to it holds true for general t.v.s. E. In 9. and 10., however, we made use of the dual E′ of E. Consequently 11. cannot be generalized to non locally convex spaces E (with possibly $E' = \{0\}$) in the way indicated here. In fact, a useful characterization of $E \varepsilon CV_o(X)$ as a "nice" space of E-valued functions seems much harder, and only partial results are available, see [3] and [7]. As far as I know, some of the problems mentioned in [3] in this connection are still unsolved.

II. Tensor products and vector-valued functions

The (completed) ε-tensor product $E \hat{\otimes}_\varepsilon F$ of two locally convex spaces E and F, i.e. the completion of $E \otimes_\varepsilon F$, has proved useful in many applications, see e.g. GROTHENDIECK [16]. The relation between the ε-product and the tensor product is quite simple (involving the a.p., however) as Schwartz's theorem (cf. [6], 3.) shows: In many cases, the ε-product and the ε-tensor product are topologically isomorphic. - We intend to give our results on tensor products $E \hat{\otimes}_\varepsilon Y$ for subspaces Y of CV(X) in this section. The first results we will mention are just a rewording of our former theorems in terms of the ε-tensor product.

13. Theorem ([3]). Let X be any completely regular space, E locally convex and $V>0$ a Nachbin family on X. Then, as a topological isomorphism, $(CV_o(X) \hat{\otimes}_\varepsilon E =)\ E \hat{\otimes}_\varepsilon CV_o(X) = CV_o(X, E)$, iff $CV_o(X, E)$ is only complete.

Proof. By the first part of 8., $E \otimes_\varepsilon CV_o(X)$ is always (identified with) a topological linear subspace of $CV_o(X, E)$. And by [6], 18., this subspace is also dense. So 13. follows immediately. ▭

In the case of CV(X) [instead of $CV_o(X)$], things are not quite so easy. As we remarked in [6] already, a partition of unity argument does not seem to work and hence a general result (corresponding to [6], 18.) about density of $E \otimes CV(X)$ in $CV^p(X, E)$ is not available by now. - Combining 11. and [6], 21. with Schwartz's theorem mentioned above, however, we can derive nevertheless:

14. Theorem ([4]). Let E be complete, assume $Z \leq V$ or $W \leq V$ and X $k_{\mathbb{R}}$-space. Suppose furthermore that the conditions of [6], 21. hold, i.e. (1) for every $v \in V$ the restriction $v|_{\text{supp } v}$ is continuous, and (2) for all $f, f' \in CV(X)$ and

every $v \in V$ there exists $g \in CV(X)$ with $g|_{\text{supp } v} = vff'|_{\text{supp } v}$. Then

$$(CV(X)\check{\otimes}_\varepsilon E =)\ E\check{\otimes}_\varepsilon CV(X) = CV^p(X,E).$$

To formulate an immediate corollary to this, let us introduce the following notation: For any set $X \neq \emptyset$, look at a fixed system V of non-negative real-valued functions on X with the property that for any $\lambda \geq 0$ and all $v_1, v_2 \in V$ you can find $v \in V$ such that $\lambda v_1, \lambda v_2 \leq v$. If then E is some locally convex space, we define $BV^p(X,E) = \{f: X \longrightarrow E;\ (vf)(X)$ precompact in E for every $v \in V\}$. For a system $\{p_\alpha; \alpha \in A\}$ of semi-norms giving the topology of E, the system $\{bv_\alpha; \alpha \in A, v \in V\}$ of semi-norms, as after 6. above, can again be introduced and induces a (canonical) locally convex topology on $BV^p(X,E)$. Put $BV(X) := BV^p(X,\mathbb{C})$.

15. Corollary. If E is complete and if to every $x \in X$ there exists at least one $v \in V$ with $v(x) > 0$, we obtain $BV^p(X,E) = E\check{\otimes}_\varepsilon BV(X)$.

Proof. Take X with the discrete topology. So X is a $k_{\mathbb{R}}$-space. (It is even metrizable and locally compact.) V is a Nachbin family on X satisfying not only $V > 0$, but also $W \leq V$. (Any compact subset of X is finite.) In fact, V consists of continuous functions on X only, and hence (1) and (2) in 14. hold true (cf. the corresponding remark before [6],21.). By 14., $CV^p(X,E) = E\check{\otimes}_\varepsilon CV(X)$, but because of X having the discrete topology, this is nothing else but our assertion. □

As in [6], it is important to observe that a simple argument allows to derive from 13. and 14. theorems on the representation of $Y\check{\otimes}_\varepsilon E$, Y a subspace of $CV(X)$, as a subspace of $CV^p(X,E)$.

To do this, we must first of all remember that the proofs of 9., 10., (and 11.) show that the canonical topological isomorphism of $CV^p(X,E)$ onto $CV(X)\varepsilon E = \mathcal{L}_e(E'_c, CV(X))$ (under the assumptions of 11.) is given by the mapping I with $I(f): e' \longrightarrow e' \circ f$ for any $f \in CV^p(X,E)$. Indeed, each $f \in CV^p(X,E)$ can be represented (cf. 11.) as $f = u \circ \Delta$ for some $u \in E\varepsilon CV(X)$ and then $If = {}^t u \in CV(X)\varepsilon E$ for this u. Hence the function $(If)(e') \in CV(X)$ can be recovered from f by an application of the formula

$$\langle (If)(e'), \delta_x \rangle = \langle {}^t u(e'), \delta_x \rangle = \langle e', u(\delta_x) \rangle = \langle e', (u \circ \Delta)(x) \rangle = \langle e', f(x) \rangle$$

for all $e' \in E'$ and $x \in X$.

If then Y is a topological linear subspace of $CV(X)$, it is easy to see that $Y \varepsilon E$ is a topological linear subspace of $CV(X) \varepsilon E$, too. So under the identification of $CV^p(X,E)$ with $CV(X)\varepsilon E$ as above, $Y\varepsilon E$ is topologically isomorphic to the topological linear subspace $\{f \in CV^p(X,E);\ e' \circ f \in Y \text{ for each } e' \in E'\}$ of $CV^p(X,E)$, as we will now immediately realize. This leads to the following theorem (the first part of which is just [6], 9. and) the last part of which is simply an application of Schwartz's theorem to the first statement of the theorem.

16. Theorem ([4]). Let E be quasi-complete and assume $Z \leq V$ or $W \leq V$ and X $k_{\mathbb{R}}$-space. For a closed topological linear subspace Y of $CV(X)$ resp. $CV_o(X)$, there is a canonical topological isomorphism (as described above) of $E\varepsilon Y = Y \varepsilon E$ with the following topological linear subspace of $CV^p(X,E)$:

$$\{f \in CV^p(X,E) \text{ (resp. } CV_o(X,E));\ e' \circ f \in Y \text{ for each } e' \in E'\}.^{2)}$$

$E \otimes_\varepsilon Y = Y \otimes_\varepsilon E$ is the subspace of all such functions f with the additional property that f takes its values in a finite dimensional subspace of E.

Thus we obtain the topological isomorphism

$E \hat{\otimes}_\varepsilon Y = Y \hat{\otimes}_\varepsilon E = \{f \in CV^p(X,E) \text{ (resp. } CV_o(X,E));\ e' \circ f \in Y \text{ for each } e' \in E'\}$, if Y (or E) has only the approximation property and if E is even complete.

As a first example of theorem 16., we can apply the result to the closed subspace $CV_o(X)$ of $CV(X)$. We immediately discover in this way a simple remark which, however, is somewhat easier (and in greater generality) to be established in a direct way:

17. Remark. Let X be a Hausdorff space, E locally convex and $f: X \longrightarrow E$ a function with $f(X)$ precompact in E. If $e' \circ f$ vanishes at infinity for every $e' \in E'$, then f itself vanishes at infinity. - Consequently we always have:

$$CV_o(X,E) = \{f \in CV^p(X,E);\ e' \circ f \in CV_o(X) \text{ for each } e' \in E'\}.$$

Proof. The second assertion readily follows from the first. So let f be as in the hypothesis. Then on the precompact set $f(X)$ (its closure in the completion $\hat{E}$ of E being compact) the topology $\sigma(E,E')$ (induced by $\sigma(\hat{E},E') = \sigma(\hat{E},\hat{E}'))$

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2) As Y is closed, our proof also shows :
$Y \varepsilon E = \{f \in CV^p(X,E) \text{ (resp. } CV_o(X,E));\ e' \circ f \in Y \text{ for each } e' \in T'\}$, if T' is some total subset of E'_c.

and the original topology of E (induced by the topology of $\hat{E}$) coincide. To show that f vanishes at infinity, we fix a neighbourhood U of 0 in E and will exhibit a compact set $K\subset X$ with $f(X\setminus K)\subset U$. By our previous argument, there exists a weak neighbourhood of 0, say $V = \{e \in E; |e_i'(e)| < \epsilon,\ i=1,\dots,n\}$ for given $e_i' \in E'$ $(i=1,\dots,n)$, such that $f(X)\cap V \subset f(X)\cap U$. As all $e_i'\circ f$ vanish at infinity, we have already compact sets $K_i \subset X$ $(i=1,\dots,n)$ with $|(e_i'\circ f)(x)| < \epsilon$ for all $x \in X\setminus K_i$. Take $K := \bigcup_{i=1}^{n} K_i$, and we are done. □

Remember that it is not true in general, however, that $f \in CB(X,E)$ with the property $e'\circ f \in C_o(X)$ for each $e' \in E'$ already satisfies $f \in C_o(X,E)$, not even for locally compact X and Banach spaces E. The following simple example illustrates this: For the canonical n-th unit vector e_n in the sequence space c_o, the function $n \longrightarrow e_n$ belongs to $CB(\mathbb{N}, c_o)\setminus C_o(\mathbb{N}, c_o)$, but for each $e' = (f_m)_{m\in\mathbb{N}} \in l^1 = c_o'$ the composed mapping $n \longrightarrow e'(e_n) = f_n$ vanishes at infinity.

Another insight is provided by the next remark:

18. Remark ([4]). Let X be completely regular, E locally convex, $Z \leq V$ or $W \leq V$ and X $k_{\mathbb{R}}$-space. Then a function $f \in BV^p(X,E)$ with $e'\circ f \in C(X)$ for each $e' \in E'$ is even continuous, i.e. belongs to $CV^p(X,E)$.

Direct proof. It is trivial that f is already an element of $C(X, E[\sigma(E,E')])$. Now $Z \leq V$ means that we can find $v \in V$, $v \geq 1$ on X. Therefore f(X), as a subset of $\Gamma((vf)(X))$, is precompact in the topology of E by definition of $BV^p(X,E)$. Therefore, by the argument in the proof of 17., $\sigma(E,E')$ and the initial topology of E coincide on f(X) which proves $f \in C(X,E)$.

In the case of $W \leq V$, we get in the same way that the restriction of f to any compact subset of X is continuous w.r.t. the initial topology of E. The assumption X $k_{\mathbb{R}}$-space is then sufficient to show our assertion in this case. □

Now apply to 16. the simple idea which we already used in the proof of 15. Hence we have as a consequence of 16.:

19. Theorem. Let X be a non-void set and V a system of non-negative real-

valued functions on X with the properties:

(i) for any $\lambda \geq 0$ and all $v_1, v_2 \in V$ there exists $v \in V$ with $\lambda v_1, \lambda v_2 \leq v$,

(ii) for each $x \in X$ we have $v(x) > 0$ for at least one (appropriate) $v \in V$.

Assume furthermore that E is a quasi-complete locally convex space and that Y is a closed topological linear subspace of BV(X). Then

$$Y \epsilon E = \{f \in BV^p(X, E);\ e' \circ f \in Y \text{ for every } e' \in E'\},$$

with the topology induced by $BV^p(X, E)$.

And this, in its turn, leads to the following corollary, where the notation is completely analogous to the terminology in [6], but slightly more general:

20. Corollary. $CV(X) \epsilon E = CV^p(X, E)$ holds under the following conditions:

(1) X is some (Hausdorff) topological space (not necessarily completely regular),

(2) $V > 0$ is a Nachbin family on the space (X, d), d the discrete topology (so the weight functions need not be upper semicontinuous),

(3) E is a quasi-complete locally convex space,

(4) CV(X) is closed in BV(X) (or, equivalently, CV(X) is complete),

(5) a function $f \in BV^p(X, E)$ which is weakly continuous - i.e. has the property $e' \circ f \in C(X)$ for any $e' \in E'$ - belongs to C(X, E) already.

Conditions (1), (2), (4) in 20. are certainly satisfied if X is a (completely regular) $k_{\mathbb{R}}$-space and $W \leq V$ or if $Z \leq V$. And, by 18., in this case condition (5) is also fulfilled. This means that 20. implies (the first part of) 11. So, if we take into account that, by way of 16., 20. was derived from 11., we notice that ϵ-product representations for $BV^p(X, E)$ and $CV^p(X, E)$ are equivalent (in a certain sense).

In fact, 20. is more general than 11. First of all, neither in the proof of completeness for CV(X) nor in the proof of 18. (for X $k_{\mathbb{R}}$-space and $W \leq V$ or for $Z \leq V$) did we actually make use of the upper semi-continuity of the functions in V. Therefore this assumption can be dropped. This could also be done in the following considerations, but to point out another interesting possibility of generalizing 11. with the aid of 20., let us return to the case of X completely regular and $V > 0$ a Nachbin family on X.

21. Definition. X is called a $V_{\mathbb{R}}$-space, if one of the following equivalent conditions holds:

(i) If $f: X \longrightarrow \mathbb{R}$ has the property that $f|_{\{x \in X;\ v(x) \geq 1\}}$ is continuous for every $v \in V$, then $f \in C_{\mathbb{R}}(X)$.

(ii) If E is any completely regular (Hausdorff) space, and if $f: X \longrightarrow E$ satisfies $f|_{\{x \in X;\ v(x) \geq 1\}}$ continuous for every $v \in V$, then f is continuous from X into E.

By our former terminology, a $W_{\mathbb{R}}$-space is nothing else but a $k_{\mathbb{R}}$-space, and each X is a $Z_{\mathbb{R}}$-space. If $V_1 \leq V_2$ for two Nachbin families on X, then of course any $(V_1)_{\mathbb{R}}$-space is again a $(V_2)_{\mathbb{R}}$-space.

22. Proposition. The weighted spaces $CV(X,E)$, $CV^p(X,E)$, and $CV_o(X,E)$ are complete, if E is complete and if X is a $V_{\mathbb{R}}$-space, $V>0$.

Proof. It is enough to prove our assertion for $CV(X,E)$, as $CV^p(X,E)$ and $CV_o(X,E)$ are closed subspaces. A generalized Cauchy-sequence $\{f_\alpha\}_{\alpha \in A}$ in $CV(X,E)$, however, converges pointwise (because of $V > 0$) to a function $f: X \longrightarrow E$ (as E is complete). And $vf_\alpha \longrightarrow vf$ uniformly for every $v \in V$, so $(vf)(X)$ is always bounded. Finally, f_α tends to f uniformly on every set $\{x \in X;\ v(x) \geq 1\}$, $v \in V$, by definition of the topology of $CV(X,E)$, and f_α and f are bounded on such sets. We immediately conclude: $f|_{\{x \in X;\ v(x) \geq 1\}}$ continuous for each $v \in V$, and our assumption $X = V_{\mathbb{R}}$-space is enough to obtain $f \in C(X,E)$, because E is completely regular (as a Hausdorff topological vector space). This finishes the proof. □

21. and 22. were suggested by the article [14] of A. GOULLET DE RUGY. In this article, Goullet de Rugy obtained a partial converse to 22., namely that under some (mild) restrictions the completeness of $CV_o(X)$ already implies $X = V_{\mathbb{R}}$-space. A theorem of BLANCHARD and JOURLIN [10] on completeness of $CV(X)$ in the case V = the Nachbin family of all positive multiples of characteristic functions of the so-called "bounded" (=relatively pseudocompact, «borné») subsets of X, however, suggests that for completeness of $CV(X)$, $X = V_{\mathbb{R}}$-space might not be necessary in general.

23. Proposition. If E is locally convex and X a $V_{\mathbb{R}}$-space, any function $f \in BV^p(X,E)$ with $e' \circ f \in C(X)$ for each $e' \in E'$ belongs to $C(X,E)$.
The proof of 23. is the same as the last part of the proof of 18. (We have to replace the compact subsets of X by the sets $\{x \in X;\ v(x) \geq 1\}$, $v \in V$, of course.)

It should also be pointed out that lemma 1. remains true for a Nachbin family V 0 on any (completely regular) $V_{\mathbb{R}}$-space, so by 2. each precompact subset of $CV(X)$ is equicontinuous in this case, too. - But instead of going through the whole proof of 11. once more, we immediately obtain from 20., 22., and 23. the following generalization:

24. Theorem. Let X be completely regular, $V>0$ a Nachbin family on X and E a quasi-complete locally convex space. If X is even a $V_{\mathbb{R}}$-space, $CV(X)\varepsilon E = CV^p(X,E)$ and $CV_o(X)\varepsilon E = CV_o(X,E)$.

(The last part of 24. is clear from the first conclusion, from e.g. 19. and 17.) As an easy consequence of 24., by the way, we get:

25. Corollary. Assume that the conditions of 24. hold and that E is even complete.

(1) Then $CV_o(X) \check{\otimes}_\varepsilon E = CV_o(X,E)$, and $CV_o(X)$ has the a.p.

(2) If, furthermore, (1) and (2) in 14. are satisfied, $CV(X) \check{\otimes}_\varepsilon E = CV^p(X,E)$ is true, too.

Proof. The first part of (1) follows from 22. and 13., and the second part of (1) is then obvious after [6], 3. (2) is implied by 24. and [6], 21. □

It is still possible to give a more general result than 24. along the same lines of proof. We will only sketch the ideas therefore.

26. Definition. Let (e.g.) X be some completely regular space and E a locally convex space. We introduce the following notation for a Nachbin family $V>0$ on X: $\mathcal{F}_V$ is the system of all (closed) subsets $F_v := \{x \in X;\ v(x) \geq 1\}$ $(v \in V)$ of X. And $RV^p(X,E) := \{f \in BV^p(X,E);\ f_{|S}$ continuous for each $S \in \mathcal{F}_V\}$, $RV_o(X,E) := \{f \in RV^p(X,E);\ vf$ vanishes at infinity for each $v \in V\}$, where both linear spaces are equipped with the locally convex topology induced from $BV^p(X,E)$. Again put $RV(X) := RV^p(X,\mathbb{C})$, $RV_o(X) := RV_o(X,\mathbb{C})$.

We immediately realize that $RV^p(X,E) = CV^p(X,E)$ and $RV_o(X,E) = CV_o(X,E)$, if X is a $V_{\mathbb{R}}$-space. (This holds also true, if only each function $f \in BV^p(X,E)$ with $f_{|F_v}$ continuous for every $v \in V$ is already continuous on X.)

It is readily shown that $RV^p(X,E)$ and $RV_o(X,E)$ are always closed subspaces of $BV^p(X,E)$, so they are a fortiori complete if E is only complete (but X an arbitrary completely regular space). Furthermore, as in 18. and 17., we can derive that a function $f \in BV^p(X,E)$ with the property $e' \circ f \in RV(X)$ resp. $RV_o(X)$ for all $e' \in E'$ must necessarily be an element of $RV^p(X,E)$ resp. $RV_o(X,E)$. Thus 19. implies:

27. Theorem. For any Nachbin family $V > 0$ on an arbitrary completely regular space X and for quasi-complete E, $RV(X) \varepsilon E = RV^p(X,E)$ and $RV_o(X) \varepsilon E = RV_o(X,E)$.

We note in passing that for instance the completion $\widehat{CV(X)}$ of $CV(X)$ always equals $\overline{CV(X)} \subset RV(X)$. So, if $CV(X)$ is also dense in $RV(X)$, we have, of course, $\widehat{CV(X)} = RV(X)$, and then $CV(X)$ is complete iff any function $f \in BV(X)$ with $f_{|S}$ continuous for each $S \in \mathcal{F}_V$ is already continuous on X.

Let e.g. V consist of the positive multiples of characteristic functions for a given system $\mathcal{S}$ of closed subsets of X (with the property that $S_1, S_2 \in \mathcal{S}$ implies the existence of $S \in \mathcal{S}$ such that $S_1 \cup S_2 \subset S$). Then $CV(X)$ is certainly dense in $RV(X)$, if for each $S \in \mathcal{F}_V$ and each $f \in RV(X)$ the (bounded) function $f_{|S}$ can be extended to some $g \in CV(X)$. But the last hypothesis is true, if the system $\mathcal{S}$ consists of compact sets only or if X = normal. [Compare Blanchard and Jourlin [10], and see Goullet de Rugy [14] for $CV_o(X)$ instead of $CV(X)$.]

We turn to some other typical applications of 16. without trying to give the most general formulations. So we will assume from now on (for the rest of this section) that E is a quasi-complete locally convex space, X a fixed completely regular space, V a Nachbin family on X with $Z \leq V$, or with $W \leq V$ and then X even a $k_{\mathbb{R}}$-space.

28. Definition. Let P be a property that a function $f \in CV^p(X,E)$ resp. $CV_o(X,E)$ may have (or not). Define $PV^p(X,E)$ resp. $PV_o(X,E) := \{f \in CV^p(X,E)$ resp. $CV_o(X,E);$ f has property $P\}$, with the topology induced by $CV^p(X,E)$. We put for convenience $PV(X) := PV^p(X,\mathbb{C})$, $PV_o(X) := PV_o(X,\mathbb{C})$. (So we omit E, if it equals $\mathbb{C}$.) We will only admit linear properties P, i.e. such properties P to make a linear space out of $PV^p(X,E)$ resp. $PV_o(X,E)$.

Let $\tilde{P}$ be a linear property that the functions in CV(X) resp. $CV_o(X)$ may have and denote by $\tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$, as defined above, the corresponding subspaces of CV(X). We say that the property P is weakly determined by $\tilde{P}$ if a function f in $CV^p(X,E)$ resp. $CV_o(X,E)$ has property P iff for each $e' \in E'$ the function $e' \circ f$ (belonging to CV(X) resp. $CV_o(X)$) has property $\tilde{P}$.[3)]

So if P is weakly determined by $\tilde{P}$, we get in our terminology:
$PV^p(X,E)$ resp. $PV_o(X,E) = \{f \in CV^p(X,E)$ resp. $CV_o(X,E);\ e' \circ f \in \tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$ for each $e' \in E'\}$.

Remark that $\tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$ closed in CV(X) resp. $CV_o(X)$ and P weakly determined by $\tilde{P}$ implies that $PV^p(X,E)$ resp. $PV_o(X,E)$ is a closed subspace of $CV^p(X,E)$ resp $CV_o(X,E)$. To prove this, assume the generalized

3) By note 1) after lemma 4., $\Delta: x \longrightarrow \delta_x$ is an element of $CV^p(X,(CV(X))'_c)$ resp. $CV_o(X,(CV_o(X))'_c)$. Composing with the transposed mapping ${}^tI$ of the canonical injection I: $\tilde{P}V(X) \longrightarrow CV(X)$ resp. $\tilde{P}V_o(X) \longrightarrow CV_o(X)$, and noting that ${}^tI$ is continuous w.r.t. the $'_c$-topologies on the duals, we obtain that Δ, as an application into $(\tilde{P}V(X))'$ resp. $(\tilde{P}V_o(X))'$, belongs to $CV^p(X,(\tilde{P}V(X))'_c)$ resp. $CV_o(X,(\tilde{P}V_o(X))'_c)$. Now assume that for $E = (\tilde{P}V(X))'_c$ resp. $(\tilde{P}V_o(X))'_c$, property P is weakly determined by $\tilde{P}$ (as defined here) and remark $E' = \tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$, if the last space is only closed in CV(X) resp. $CV_o(X)$ (which we will also assume). It is then clear that Δ even belongs to $PV^p(X,(\tilde{P}V(X))'_c)$ resp. $PV_o(X,(\tilde{P}V_o(X))'_c)$.

sequence $\{f_\alpha\}_{\alpha \in A}$ converges to f in $CV^p(X,E)$ resp. $CV_o(X,E)$ and that all f_α have property P. Then for each $e' \in E'$, $e' \circ f_\alpha \longrightarrow e' \circ f$ in $CV(X)$ and $e' \circ f_\alpha \in PV(X)$ resp. $PV_o(X)$ for all $\alpha \in A$. Our assumption implies $e' \circ f \in PV(X)$ resp. $PV_o(X)$, and as this is true for each $e' \in E'$ and as P is weakly determined by $\tilde{P}$, $f \in PV^p(X,E)$ resp. $PV_o(X,E)$.

As a reformulation of 16. in the terminology we have just introduced, we find now:

29. Theorem. Suppose property P as in 28. is weakly determined by $\tilde{P}$. Suppose moreover that $\tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$ is a closed subspace of $CV(X)$. Then $PV^p(X,E) = E \varepsilon \tilde{P}V(X)$ resp. $PV_o(X,E) = E \varepsilon \tilde{P}V_o(X)$.

We can even replace the ε-product by the ε-tensor product in these equalities, if E is complete and if $\tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$ (or E) has the a. p.

(From the ε-product representation of $PV^p(X,E)$ resp. $PV_o(X,E)$ it would have been clear anyway that, for closed $\tilde{P}V(X)$ resp. $\tilde{P}V_o(X)$ in $CV(X)$, $PV^p(X,E)$ resp. $PV_o(X,E)$ must be closed in $CV^p(X,E)$, as one can easily see.)

A variation of the same idea which gave rise to 29. was presented in [4], 3.5 and 3.6. There, for a given topological linear subspace Y_E of $CV^p(X,E)$ resp. $CV_o(X,E)$, a corresponding space Y_o of scalar functions was defined by:

$Y_o := \{g \in CV(X)$ resp. $CV_o(X);\ g = e' \circ f$ for some $f \in Y_E$ and some $e' \in E'\}$.

The following condition was imposed on Y_E:
For any $e \in E$ and any $g \in Y_o$, the function $f: X \longrightarrow E$, defined by $f(x) = eg(x)$ for all $x \in X$, is an element of Y_E.

It was then proved that Y_o is a linear subspace of $CV(X)$ resp. $CV_o(X)$ which is closed, if only Y_E is closed in $CV^p(X,E)$. And for complete E, closed Y_E, we obtained $Y_E = Y_o \tilde{\otimes}_\varepsilon E = Y_o \varepsilon E$, if Y_o had the a. p.

The formulation we gave in 29. might be more suitable for some concrete applications to which we proceed now. To start with, let me exhibit a more or less trivial example: For a (Hausdorff) topological vector $k_{\mathbb{R}}$-space X (e.g. X metrizable), let V denote a Nachbin family on X consisting of the positive multiples of the characteristic functions for a given system $\mathcal{S}$ of subsets of X with the following properties:

(i) each $S \in \mathcal{S}$ is closed and bounded,

(ii) for every compact subset K of X, we can find $S \in \mathcal{S}$ with $K \subset S$,

(iii) for $S_1, S_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ such that $S_1 \cup S_2 \subset S$.

Then, for $CV^p(X,E)$ and $CV(X)$, the properties P and $\tilde{P}$ of linearity of functions in $CV^p(X,E)$ and $CV(X)$, resp., are well-defined, and P is, of course, weakly determined by $\tilde{P}$. In the usual notation, $\tilde{P}V(X) = X'_{\mathcal{S}}$, and $PV^p(X,E) = \mathcal{C}_{\mathcal{S}}(X,E) := \{u \in \mathcal{L}(X,E); u(S) \text{ precompact in } E \text{ for each } S \in \mathcal{S}\}$, both with the topology of uniform convergence on all sets $S \in \mathcal{S}$. Now 29. gives, for instance, for complete E: $\mathcal{C}_{\mathcal{S}}(X,E) = X'_{\mathcal{S}} \check{\otimes}_{\varepsilon} E$, if E or (the locally convex space) $X'_{\mathcal{S}}$ has the a.p. The assumptions here are a little bit too strong, and somewhat weaker conditions are sufficient according to our former considerations. (The result we obtained is related to Grothendieck [16], "Prop." 37. (d).) Continuous affine (instead of linear) functions could be treated in a similar way.

What we originally had in mind when writing [4] were holomorphic functions. So assume that X is an open subset of $\mathbb{C}^N$ $(N \geq 1)$, $W \leq V$ for a Nachbin family V on X, and E a quasi-complete locally convex space. It is well-known that for functions in $CV^p(X,E)$ [or $CV_0(X,E)$] and $CV(X)$ [or $CV_0(X)$] the properties P = H and $\tilde{P}$ = H of being holomorphic are related in the way which we call "P is weakly determined by $\tilde{P}$". As an immediate consequence of 29., we can therefore state:

30. Corollary ([4]). $HV^p(X,E) = E \varepsilon HV(X)$ and $HV_0(X,E) = E \varepsilon HV_0(X)$, with $\check{\otimes}_{\varepsilon}$ replacing ε for complete E, if E or HV(X) resp. $HV_0(X)$ has the a.p.

As it turns out, much more is true. We could take in 30. (resp. 29.) X = an open subset of a topological vector space Y (over $\mathbb{C}$) with the property that X is a $k_{\mathbb{R}}$-space under the topology induced from Y (so e.g. Y metrizable is enough[4]), and then P and $\tilde{P}$ might denote certain kinds of analyticity with the only restriction that, for E and V as above, a function $f \in CV^p(X,E)$ resp. $CV_o(X,E)$ is analytic in the sense of property P if and only if $e' \circ f \in CV(X)$ resp. $CV_o(X)$ is analytic in the sense of property $\tilde{P}$ for each $e' \in E'$.

Only one possible type of analyticity shall be investigated here in more detail: Let X, Y, V, E satisfy the assumptions made up to now and take Gâteaux - analyticity as P and $\tilde{P}$ (see e.g. HERVE [17], III. 1.3.). By the very definition, it is already clear that P is always weakly determined by $\tilde{P}$ ([17], p. 66). We will call holomorphic (Fréchet-analytic in [17], T-analytic in PIZANELLI [22]) all continuous Gâteaux-analytic mappings and therefore denote by $HV^p(X,E)$, $HV_o(X,E)$, $HV(X)$, $HV_o(X)$ the spaces

$PV^p(X,E) = \{f \in CV^p(X,E);\ f \text{ is Gâteaux-analytic}\}$,

$PV_o(X,E)$, $\tilde{P}V(X)$, and $\tilde{P}V_o(X)$, respectively. Using [17], III. 2.2., Theorem 2, we have obtained:

31. Corollary. $HV^p(X,E) = E\varepsilon HV(X)$ *and* $HV_o(X,E) = E\varepsilon HV_o(X)$ *in this case, too.*

Note that the conditions on X (or Y) can be relaxed, if the Nachbin family V on X is considerably "stronger" than W.

In the case V = W, the ε-product representation in 31. is due to M. SCHOTTENLOHER [23] who also gave many interesting examples and related results. As in general the space HW(X) is no longer nuclear, the question of the a.p. for HW(X) arose. This question (among other problems) was investigated by R. ARON (his paper is [1]) and SCHOTTENLOHER. They found out, for instance, that a Banach space X has the a.p. if and only if HW(X) has the a.p. In their paper [2], they prove generalizations and many other beautiful results, among them equivalences for the a.p. of the space of holomorphic functions (on a Banach space), equipped with stronger topologies than co (e.g. with Nachbin's ported topology etc.) which are not

4) By the Banach-Dieudonné theorem (cf. Köthe [19], §21, 10.(1)), every $'_c$-dual of a Fréchet space is a k-space, so a fortiori a $k_{\mathbb{R}}$-space. Hence, for instance, (separated) (LS)-spaces Y are always $k_{\mathbb{R}}$-spaces, too, and so a fortiori, are open subsets of such spaces.

weighted topologies in our sense.

Instead of looking at holomorphic functions only, as we did in 30., we could also have drawn attention to harmonic functions or to solutions of certain linear partial differential equations. Results of this type are related to our research in [9].

We want to mention some simple spaces of "mixed" type next. These are the spaces of [6], 16. and 17., first introduced by B. GRAMSCH [15]. Let X be a locally compact space, G a non-void open subset of $\mathbb{C}^N$ $(N\geq 1)$, and $\Lambda\neq\emptyset$ open in $G\times X$. As in [6], we use the notation $\Lambda_x := \{z\in G;\ (z,x)\in\Lambda\}$ and define for a quasi-complete locally convex space E:

$CH(\Lambda, E) := \{f\in C(\Lambda, E);\ f(.,x)$ holomorphic on Λ_x for each $x\in X\}$,

equipped with the topology co. Similarly, if $\Lambda\neq\emptyset$ is a closed subspace of $\mathbb{C}\times X$ with Λ_x compact for each $x\in X$, we can define the space

$CA(\Lambda, E) := (\{f\in C(\Lambda, E);\ f(.,x)$ analytic on $\mathring{\Lambda}_x$ for each $x\in X\}$, co).

Condition (*) in [6] requires that for each $x\in X$, Λ_x does not separate the plane.

32. Corollary. $CH(\Lambda, E) = E\,\varepsilon\,CH(\Lambda)$ $(=E\check{\otimes}_\varepsilon CH(\Lambda)$ for complete $E)$; and $CA(\Lambda, E) = E\,\varepsilon\,CA(\Lambda)$, which equals $E\check{\otimes}_\varepsilon CA(\Lambda)$ if E is complete and if E has the a.p. or condition (*) holds.

Proof. For a continuous function $f\colon \Lambda \longrightarrow E$, the property that $f(.,x)$ is holomorphic on Λ_x resp. $\mathring{\Lambda}_x$ for every $x\in X$ is weakly determined. Hence, by 29., the equations for the ε-products are true. For the rest just remember [6], 16. and 17. □

Gramsch [15], 1.11 assumed for open $\Lambda\subset G\times X\subset\mathbb{C}^N\times X$ the following regularity condition:

(R) Λ_x is $H(G)$-convex for each $x\in X$ (with $\Lambda_x\neq\emptyset$), i.e. we always have density of $H(G)|_{\Lambda_x}$ in $(H(\Lambda_x), co)$.

Under this assumption, he directly established (among other things) the density of $(H(G)\otimes C(X))|_{\Lambda}\otimes E$ in $CH(\Lambda, E)$. We shall give another proof of this fact by applying the approximation results of PROLLA and SUMMERS. (For instance a scalar version of [6], 12. is enough for our purpose.) Take $V = C_c^+(\Lambda)$, $A = (\{\text{constants on } G\} \otimes C(X))_{|\Lambda}$ and set $W := (H(G) \otimes C(X))_{|\Lambda}$. Then by [6], 12., $f \in C(\Lambda)$ belongs to $\overline{W}^{(C(\Lambda), co)}$ iff for each $x \in X$, $f(.,x) \in \overline{H(G)_{|\Lambda_x}}^{(C(\Lambda_x), co)}$. So condition (R) assures density of $(H(G) \otimes C(X))_{|\Lambda}$ in $CH(\Lambda)$. Hence $(H(G) \otimes C(X))_{|\Lambda} \otimes E$ dense in $CH(\Lambda, E)$ can be deduced from 32.

We conclude this section with two remarks on applications of the ϵ-product representations we have proved here.

First, L. SCHWARTZ [24], Cor. 2, p. 48 shows that the ϵ-product $E \epsilon F$ (or the ϵ-tensor product $E \check{\otimes}_\epsilon F$) of two complete locally convex spaces E and F with a.p. also possesses the a.p. Thus a corollary to the results in this section and in [6], we have the a.p. for many spaces of functions with values in a complete locally convex space with a.p.

Finally we will state a theorem due to H. BUCHWALTER [11], 2.7.a): Let E and F be Fréchet spaces one of which has the a.p. Then $(E \epsilon F)'_c = (E \check{\otimes}_\epsilon F)'_c = E'_c \hat{\otimes}_\pi F'_c$ holds [5]. Hence our results have applications to π-tensor product representations of the $'_c$-duals of spaces of vector-valued functions, too.

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[5] There is a similar theorem due to Buchwalter, loc. cit., 2.7 b) — for a proof of the slightly different statement we give here see [8], 4.1: Let both E and F be (F)-spaces [or Montel (DF)-spaces]. Then $(E \hat{\otimes}_\pi F)'_c = E'_c \epsilon F'_c$.

This can be used to show (in an obvious way) that for such spaces E and F, if they both have the a.p., their π-tensor product also possesses the a.p.

III. Functions of several variables and the slice-product

Ever since DIEUDONNE's famous result of 1937 that for two compact spaces K_1 and K_2 the tensor product $C(K_1) \otimes C(K_2)$ is (identified with) a dense subspace of the sup-norm algebra $C(K_1 \times K_2)$ has the tensor product been recognized as an important aid in dealing with functions of several variables.

The next step in the development to our theorem below is due to L. NACHBIN [21], where a "weighted Dieudonné theorem for density in tensor products" was proven. In the following, let X_1 and X_2 be completely regular spaces and $V_1 > 0$ resp. $V_2 > 0$ Nachbin families on X_1 resp. X_2. We denote by $v_1 \otimes v_2$, as usual, the function defined on $X_3 := X_1 \times X_2$ by $(v_1 \otimes v_2)(x_1, x_2) = v_1(x_1) v_2(x_2)$, and by $V_3 = V_1 \otimes V_2$ the set of all $v_1 \otimes v_2$, $v_1 \in V_1$ and $v_2 \in V_2$. It is not hard to see that V_3 is a Nachbin family on X_3 with $V_3 > 0$. Nachbin proved that under the linear mapping

$$T : \sum_{i=1}^{n} f_i \otimes g_i \longrightarrow ((x_1, x_2) \longrightarrow \sum_{i=1}^{n} f_i(x_1) g_i(x_2))$$

for $n \in \mathbb{N}$, $f_i \in CV_1(X_1)$, $g_i \in CV_2(X_2)$; $i = 1, \ldots, n$; $(x_1, x_2) \in X_3$, the tensor product $CV_1(X_1) \otimes CV_2(X_2)$ is identified with a linear subspace of $CV_3(X_3)$, and that the subspace $C(V_1)_o(X_1) \otimes C(V_2)_o(X_2)$ is (after the same identification) dense in $C(V_3)_o(X_3)$.

The first to look at the topology induced on the tensor products by the space $CV_3(X_3)$ was W. H. SUMMERS [25]. In [5], we gave a simpler argument to the following effect:

33. Lemma: It is always true that $CV_3(X_3)$ induces the ε-topology on the spaces $CV_1(X_1) \otimes CV_2(X_2)$ and $C(V_1)_o(X_1) \otimes C(V_2)_o(X_2)$.

Proof: It is clearly enough to look at the first case only. The following notation is convenient: For $i = 1, 2$ set

$$b(v_i)_1 := \{ f \in CV_i(X_i);\ \sup_{x_i \in X_i} v_i(x_i) |f(x_i)| \leq 1 \} \quad \text{and} \quad v_i \Delta_i := \{ v_i(x_i) \delta_{x_i} ;\ x_i \in X_i \}.$$

Remember from 6. that $\Gamma(v_i\Delta_i)$ is always weak *- or $'_c$-dense in $b(v_i)_1^\circ$. It is then easy to verify the following equality for all $v_1\in V_1$, $v_2\in V_2$ and $f = \sum_{i=1}^{n} f_i\otimes g_i \in CV_1(X_1)\otimes CV_2(X_2)$:

$$\sup\{ |(\mu\otimes\nu)(f)| ; (\mu,\nu)\in b(v_1)_1^\circ \times b(v_2)_1^\circ \}$$

$$= \sup\{ |\sum_{i=1}^{n} \mu(f_i)\nu(g_i)| ; (\mu,\nu)\in b(v_1)_1^\circ \times b(v_2)_1^\circ \}$$

$$= \sup\{ |\sum_{i=1}^{n} \mu(f_i)\nu(g_i)| ; (\mu,\nu)\in \Gamma(v_1\Delta_1)\times\Gamma(v_2\Delta_2) \}$$

$$\stackrel{(1)}{=} \sup\{ |\sum_{i=1}^{n} v_1(x_1)\delta_{x_1}(f_i)\, v_2(x_2)\delta_{x_2}(g_i)| ; x_i\in X_i,\ i = 1, 2 \}$$

$$= \sup\{ v_1(x_1)v_2(x_2) |\sum_{i=1}^{n} f_i(x_1)g_i(x_2)| ; (x_1,x_2)\in X_1\times X_2 \}$$

$$= \sup\{ v_1\otimes v_2)(x_3) |f(x_3)| ; x_3\in X_3 \} = b(v_1\otimes v_2)(f) .$$

(For (1) use the same trick as in the proof of 7. (2) twice.) Next remark that, because of the properties of Nachbin families, a basis for the equicontinuous subsets of $(CV_i(X_i))'$ is given by the sets $b(v_i)_1^\circ$, $v_i\in V_i$. So, by definition of the ϵ-topology as the topology of biequicontinuous convergence, the equality we have just derived proves everything we want. ▭

After Nachbin's weighted Dieudonné theorem, our next theorem is an immediate consequence of 33.:

<u>34. Theorem</u> ([5]) : $C(V_3)_o(X_3) = C(V_1)_o(X_1)\check{\otimes}_\epsilon C(V_2)_o(X_2)$ <u>iff</u> $C(V_3)_o(X_3)$ <u>is complete.</u>

As a simple example, take $V_i = W(X_i)$ (= the Nachbin family W on X_i) for i = 1, 2. It is then obvious that $V_3 = V_1\otimes V_2$ induces the co-topology on $C(V_3)_o(X_3)$. So we get a theorem previously obtained by Buchwalter [11]:

<u>35. Corollary:</u> $(C(X_3), co) = (C(X_1), co)\check{\otimes}_\epsilon (C(X_2), co)$ <u>iff</u> X_3 <u>is a</u> $k_{\mathbb{R}}$<u>-space.</u>

It is true that X_3 $k_{\mathbb{R}}$-space implies X_1 and X_2 $k_{\mathbb{R}}$-spaces, but the converse does not hold. The product $X_3 = X_1 \times X_2$ is a $k_{\mathbb{R}}$-space under the following conditions, however:
X_2, X_2 $k_{\mathbb{R}}$-spaces with one of them even locally compact or X_1 and X_2 both metrizable or both hemicompact $k_{\mathbb{R}}$-spaces (see [11], (2.2)).

The tensor product representation in 34. is certainly true if $Z(X_i) \leq V_i$ for i = 1 and 2 (where $Z(X_i)$ = Nachbin family Z on X_i) or if $W(X_i) \leq V_i$, i = 1, 2, and $X_3 = k_{\mathbb{R}}$-space, because we then have $Z(X_3) \leq V_3$ resp. $W(X_3) \leq V_3$, too. So 34. generalizes a theorem of Summers [25], where a great number of interesting examples is given.

In the case of spaces of type CV(X) there are again more difficulties: In general, $CV_1(X_1) \otimes CV_2(X_2)$ is not dense in $CV_3(X_3)$. And at the time, when [5] was written, no theorem was available that gave a description of the closure of the tensor product. Recently, G. KLEINSTÜCK [18] discovered a new access to our result below. Here we will only sketch the argument we gave in [5].

By identifying a space of functions of two variables with the corresponding space of (vector-valued) functions of the first variable taking their values in the space of (scalar) functions of the second variable - this is done in the canonical way - we obtain:

36. Lemma ([5]): The mapping $I : f \longrightarrow ((x_1, x_2) \longrightarrow [f(x_1)](x_2))$ defines a topological isomorphism of the space $C(V_1)^p(X_1, CV_2(X_2))$ onto the topological linear subspace

$C(V_3)^p(X_3) := \{ f \in CV_3(X_3) ; \{ v_1(x_1) f(x_1, .) ; x_1 \in X_1 \} \subset CV_2(X_2)$ is relatively compact and $\{ v_2(x_2) f(., x_2) ; x_2 \in X_2 \} \subset CV_1(X_1)$ is relatively compact for each $v_1 \in V_1$, $v_2 \in V_2 \}$ of $CV_3(X_3)$,
if $Z(X_i) \leq V_i$ or if $W(X_i) \leq V_i$ (i = 1, 2) and $X_3 = k_{\mathbb{R}}$-space.

Remark that, by 11., $C(V_1)^p(X_1, CV_2(X_2)) = CV_2(X_2) \varepsilon CV_1(X_1) = CV_1(X_1) \varepsilon CV_2(X_2) = C(V_2)^p(X_2, CV_1(X_1))$ in our case, and hence the following is proven:

37. Theorem ([5]) : Under the conditions of 36., $CV_1(X_1) \varepsilon\, CV_2(X_2) = CV_2(X_2) \varepsilon\, CV_1(X_1) = C(V_3)^p(X_3)$, where we can write $\otimes_\varepsilon$, if for V_1 (and X_1) or V_2 (and X_2) (1) and (2) in 14. hold.

Kleinstück's method reveals some interesting connections and indicates that the description of the closure of $CV_1(X_1) \otimes CV_2(X_2)$ in $CV_3(X_3)$ will in general (i. e. not under the restrictions made in 37.) look quite similar. (See the note added in proof.)

The question of general conditions for $C(V_3)^p(X_3) = CV_3(X_3)$ is not an easy one, as the example $V_1 = Z(X_1)$, $V_2 = Z(X_2)$ already shows. Namely, $CB(X_1 \times X_2) = CB(X_1) \otimes_\varepsilon CB(X_2)$ (with the sup-norm topologies) if and only if $X_1 \times X_2$ is pseudocompact. And the last condition implies X_1 and X_2 pseudocompact, but the converse is false. (The theorem we have just mentioned is due to GLICKSBERG and TAMANO, see e. g. Buchwalter [11], (5.7)).

We turn to subspaces of weighted spaces. Assume that, in the following, Y_1 resp. Y_2 is a topological linear subspace of $CV_1(X_1)$ resp. $CV_2(X_2)$ [or $C(V_1)_o(X_1)$ resp. $C(V_2)_o(X_2)$]

38. Definition: The slice-product $Y_1 \# Y_2$ of Y_1 and Y_2 is the topological linear subspace

$$\{ f \in C(V_3)^p(X_3) \ [\text{ or } C(V_3)_o(X_3)\ ] \ ; f(.,x_2) \in Y_1 \text{ and } f(x_1,.) \in Y_2 \text{ for all } (x_1,x_2) \in X_1 \times X_2 \}$$

of $CV_3(X_3)$.

Remark that for $f \in CV_3(X_3)$ [or $C(V_3)_o(X_3)$] $f(.,x_2) \in CV_1(X_1)$ [or $C(V_1)_o(X_1)$] and $f(x_1,.) \in CV_2(X_2)$ [or $C(V_2)_o(X_2)$] holds for all $(x_1,x_2) \in X_3$, if only $V_1 > 0$ and $V_2 > 0$.

Let us assume, additionally, for the rest of this article that $Z(X_i) \le V_i$ or $W(X_i) \le V_i$ ($i = 1, 2$) and X_3 $k_{\mathbb{R}}$-space and that Y_i is closed in $CV_i(X_i)$ [or $C(V_i)_o(X_i)$] for $i = 1, 2$ too. It is then not very hard to derive:

<u>39. Theorem</u> ([5]) : <u>Under our conditions,</u> $Y_1 \# Y_2 \cong Y_1 \varepsilon Y_2$; <u>so also equal to</u> $Y_1 \check{\otimes}_\varepsilon Y_2$, <u>if</u> Y_1 <u>or</u> Y_2 <u>has the a. p.</u>

Actually the proof of 39. is nothing else but looking at various kinds of topological isomorphisms established up to now and applying lemma 6. as well as note [2)] in 16. twice.

Slice products were, at least in the case of compact spaces X_1 and X_2 and of sup-norms, well-known for a long time (see e. g. EIFLER [13]). <u>The problem whether at least the slice product of uniform algebras was always equal to the (completed) tensor product was finally solved in the negative,</u> as a consequence of ENFLO's counterexample to the approximation problem, by H. MILNE [20]. In this article, Milne proved that <u>every complex Banach space is isometrically isomorphic to a complemented subspace of a suitable uniform algebra.</u> As a consequence of Enflo's example, it is then not hard to establish the existence of two uniform algebras A and B with $A \# B \neq A \check{\otimes}_\varepsilon B$.

For practical purposes, we would like to reformulate 39. in a similar manner as we did with 16. in section 2. So we make the following definition (resembling, in a certain sense, 28.) :

<u>40. Definition:</u> Let P be a linear property that a function f in $C(V_3)^P(X_3)$ resp. $C(V_3)_o(X_3)$ may have (or not). Let P_i be a linear property that functions in $CV_i(X_i)$ resp. $C(V_i)_o(X_i)$ may have, $i = 1, 2$.
Define $P(V_3)^P(X_3)$ resp. $P(V_3)_o(X_3) := \{ f \in C(V_3)^P(X_3)$ resp. $C(V_3)_o(X_3)$; f has property P $\}$ and $P_iV_i(X_i)$ resp.
$P_i(V_i)_o(X_i) := \{ f \in CV_i(X_i)$ resp. $C(V_i)_o(X_i)$; f has $P_i \}$ ($i = 1, 2$) .
By our assumption on P, P_1, and P_2, these spaces are linear subspaces of $CV_3(X_3)$ and $CV_1(X_1)$, $CV_2(X_2)$, respectively. We equip them with the induced weighted topologies.
<u>We say that P is a slice property with respect to</u> P_1 <u>and</u> P_2, <u>if a function f in</u> $C(V_3)^P(X_3)$ <u>resp.</u> $C(V_3)_o(X_3)$, $X_3 = X_1 \times X_2$ <u>and</u> $V_3 = V_1 \otimes V_2$, <u>has</u>

property P if and only if the partial functions $f(.,x_2)$ *and* $f(x_1,.)$ *have properties* P_1 *and* P_2, *respectively, for all* $(x_1,x_2) \in X_3$.

So if P is a slice property w.r.t. P_1 and P_2, we get in our terminology:

$P(V_3)^p(X_3)$ resp. $P(V_3)_o(X_3) = P_1V_1(X_1) \# P_2V_2(X_2)$ resp. $P_1(V_1)_o(X_1) \# P_2(V_2)_o(X_2)$.

Remark that in this case $P_1V_1(X_1)$ resp. $P_1(V_1)_o(X_1)$ and $P_2V_2(X_2)$ resp. $P_2(V_2)_o(X_2)$ closed in $CV_1(X_1)$ resp. $CV_2(X_2)$ imply $P(V_3)^p(X_3)$ resp. $P(V_3)_o(X_3)$ closed in $CV_3(X_3)$, as one can easily see.

Now the reformulation of 39. we intended to give reads as follows:

41. Theorem: Suppose property P as in 40. is a slice property with respect to P_1 *and* P_2. *Suppose moreover that* $P_iV_i(X_i)$ *resp.* $P_i(V_i)_o(X_i)$ *is a closed subspace of* $CV_i(X_i)$ *resp.* $C(V_i)_o(X_i)$ *for i = 1 and 2. Then* $P(V_3)^p(X_3) = P_1V_1(X_1) \varepsilon P_2V_2(X_2)$ *as well as* $P(V_3)_o(X_3) = P_1(V_1)_o(X_1) \varepsilon P_2(V_2)_o(X_2)$, *and we can replace the* ε*-product by the* ε*-tensor product if at least one of the spaces* $P_1V_1(X_1)$, $P_2V_2(X_2)$ *resp.* $P_1(V_1)_o(X_1)$, $P_2(V_2)_o(X_2)$ *has the a.p.*

Among other things, 41. allows to give *tensor product representations for spaces of holomorphic functions of several variables or for spaces of functions with "mixed" dependence,* i.e. for instance holomorphic in the first (set of) variable(s) and only continuous in the second (set of) variable(s). We do not deal here with the last type of examples, but will state the results for holomorphic functions after another (rather trivial) example.

For both X_1 and X_2 topological vector spaces such that $X_3 = X_1 \times X_2$ is a $k_{\mathbb{R}}$-space (e.g. X_1 and X_2, hence X_3, metrizable), let V_1 resp. V_2 denote special Nachbin families associated with systems $\mathcal{S}_1$ resp. $\mathcal{S}_2$ of (closed and bounded) subsets of X_1 resp. X_2 as after 29. above. We will assume that $\mathcal{S}_i$ satisfies the conditions (i) to (iii) given there for

$i = 1, 2$. Then, of course, the property P of being bilinear (a bilinear functional) on $X_1 \times X_2$ is a slice property with respect to the properties P_1 and P_2 of being linear [in the first and second variable, respectively]. In the usual terminology, $P_i V_i(X_i) = (X_i)'_{\mathcal{S}_i}$, and

$P(V_3)^P(X_3) = \mathcal{B}^P_{\mathcal{S}_1, \mathcal{S}_2}(X_1, X_2) := \{ f \in \mathcal{B}(X_1, X_2) ; f(S_1, x_2)$ relatively compact in $(X_1)'_{\mathcal{S}_1}$ and $f(x_1, S_2)$ relatively compact in $(X_2)'_{\mathcal{S}_2}$ for all (fixed) $(x_1, x_2) \in X_1 \times X_2,\ S_1 \in \mathcal{S}_1,\ S_2 \in \mathcal{S}_2 \}$.

So, by 41. if $(X_1)'_{\mathcal{S}_1}$ or $(X_2)'_{\mathcal{S}_2}$ has the a. p. $\mathcal{B}^P_{\mathcal{S}_1, \mathcal{S}_2}(X_1, X_2) = (X_1)'_{\mathcal{S}_1} \check{\otimes}_\epsilon (X_2)'_{\mathcal{S}_2}$.

The result on holomorphic functions of several variables, on the other hand, makes use of the simple fact that for functions on the product $X_3 = X_1 \times X_2$ of two open subsets X_1 resp. X_2 of $\mathbb{C}^N$ resp. $\mathbb{C}^M$, the property P of being holomorphic on $X_1 \times X_2$ is a slice property w.r.t. P_i = holomorphic on X_i ($i = 1, 2$). We can, for convenience, denote P, P_1, P_2 by H and obtain immediately:

42. Corollary ([5]) : For open subsets $X_1 \subset \mathbb{C}^N$, $X_2 \subset \mathbb{C}^M$ $(N, M \geq 1)$ and any Nachbin families V_i on X_i with $W(X_i) \leq V_i$ ($i = 1, 2$) [where $V_3 = V_1 \otimes V_2$ again]:
$H(V_3)^P(X_3) = HV_1(X_1) \epsilon HV_2(X_2)$ and $H(V_3)_o(X_3) = H(V_1)_o(X_1) \epsilon H(V_2)_o(X_2)$.

For open sets X_1 and X_2 in (complex) topological vector spaces E_1 resp. E_2 with $X_3 = X_1 \times X_2$ a $k_{\mathbb{R}}$-space [as mentioned above, it is sufficient to take E_1 and E_2 metrizable [6]], we could as well treat (continuous and) holomorphic functions on $X_1 \times X_2$ by use of 41. Remember

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[6] By the general theory of such spaces and by note [4], (separated) (LS)-spaces are always hemicompact $k_{\mathbb{R}}$-spaces (i.e. have a countable basis for the compact sets). Hence, using e.g. Buchwalter's result mentioned after 35., we can conclude that both X_1 and X_2 (LS)-spaces implies $X_1 \times X_2 = k_{\mathbb{R}}$-space. And again, a fortiori, open subsets of $X_1 \times X_2$ are then $k_{\mathbb{R}}$-spaces, too.

that, if $W(X_i) \leq V_i$ for $i = 1, 2$, and if P, P_1, and P_2 denote any types of holomorphic functions on X_3, X_1, and X_2, respectively, we only need the following conditions in order to apply 41.:

(i) $P_i V_i(X_i)$ resp. $P_i(V_i)_o(X_i)$ is closed in $CV_i(X_i)$ resp. $C(V_i)_o(X_i)$, $i = 1, 2$, and

(ii) P is a slice property w.r.t. P_1 and P_2, i.e. a function $f \epsilon C(V_3)^P(X_3)$ resp. $C(V_3)_o(X_3)$ is holomorphic in the sense of property P if and only if the partial functions $f(x_1, .)$ and $f(., x_2)$ are holomorphic in the sense of properties P_2 and P_1 for all $(x_1, x_2) \epsilon X_1 \times X_2$.

As we did in 31., we will deal here only with the case P, P_1 and P_2 indicating holomorphic functions in the sense of G-analyticity (= Gâteaux-analyticity) plus continuity. Then (i) is certainly satisfied under our assumptions, and so we have only to assure that G-analyticity is a slice-property w.r.t. (separate) G-analyticity in the first resp. second variables. Now it is trivial that G-analyticity already implies separate G-analyticity. The converse was for instance shown by Schottenloher [23] (using a simple trick that e.g. Pizanelli [22] had given before). For completeness , let me record the proof here.

For a separately G-analytic complex-valued mapping f on $X_3 = X_1 \times X_2$, we first establish that the mapping

$(\lambda_1, \lambda_2) \longrightarrow f(x_1+\lambda_1 a_1, \ x_2+\lambda_2 a_2)$,

with arbitrary (fixed) $x_1, a_1 \epsilon X_1$, $x_2, a_2 \epsilon X_2$, is anlytic on $\{ (\lambda_1, \lambda_2) \epsilon \mathbb{C}^2 \ ; x_1+\lambda_1 a_1 \epsilon X_1, \ x_2+\lambda_2 a_2 \epsilon X_2 \}$. But by the finite dimensional Hartogs theorem, it is enough to prove separate analyticity for this mapping, and this is obvious from the assumption of separate G- analyticity of f. Having proved this fact, we consider the function $\lambda \longrightarrow (\lambda, \lambda)$: It defines, of course, an analytic map of $\{ \lambda \epsilon \mathbb{C} \ ; x_1+\lambda a_1 \epsilon X_1, \ x_2+\lambda a_2 \epsilon X_2 \}$ into $\{ (\lambda_1, \lambda_2) \epsilon \mathbb{C}^2 ; x_1+\lambda_1 a_1 \epsilon X_1, \ x_2+\lambda_2 a_2 \epsilon X_2 \}$. The composition of the two (finite dimensional) analytic mappings we exhibited yields the (analytic) function $\lambda \longrightarrow f(x_1+\lambda a_1, x_2+\lambda a_2) = f(x+\lambda a)$ for arbitrary fixed $x = (x_1, x_2)$, $a = (a_1, a_2) \in X_1 \times X_2$. Hence we have the conclusion: f is G-analytic.

Denoting holomorphy = Gâteaux-analyticity plus continuity by H, as before, we have established:

43. Corollary: Under the more general conditions just given (i.e. X_1, X_2 open subsets of complex topological vector spaces E_1, E_2 with $X_3 = X_1 \times X_2$ $k_{\mathbb{R}}$-space, $W(X_i) \leq V_i$ for $i = 1, 2$, and H = holomorphy), 42. remains true.

For the special case of the co-topologies, 43. is again due to M. Schottenloher [23].

In concluding, let me remark that a generalization of many results in this section is possible (in the same way as we got generalizations of 11. in numbers 20., 24., and 27.). More examples could be given, too (cf. the remarks in section 2.). Finally, a repeated application (and a combination) of the results of section 2. and 3. will yield even more general ϵ-tensor- or ϵ-product representations for vector-valued functions of several variables. (You may sometimes need the associativity of the ϵ-product here ([24])).

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Klaus - D. BIERSTEDT
Arbeitsstelle Mathematik,
Fachbereich 17 der GH Paderborn,
D - 479 Paderborn
Pohlweg,
Postfach 1621
Germany (Fed. Rep.)