THE APPROXIMATION PROPERTY FOR WEIGHTED FUNCTION SPACES

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I. Some remarks on Grothendieck's approximation property

The approximation property for locally convex (Hausdorff topological vector) spaces was defined by A. GROTHENDIECK in his "thèse" [17]:

1. Definition: A locally convex space E has the approximation property (a. p.), iff (= if and only if) the identity id_E of E can be approximated, uniformly on every precompact subset of E, by continuous linear operators from E into E of finite rank (i. e. with finite dimensional range).

If we denote by $\mathfrak{L}_c(E)$ the locally convex space of continuous linear operators from E into E, endowed with the topology of uniform convergence on precompact subsets of E, then it is required that id_E belongs to $\overline{E\otimes E'}^{\mathfrak{L}_c(E)}$.

(E' is the [continuous] dual of E, and the space of continuous linear operators T: E ⟶ E of finite rank is easily seen to be [algebraically] isomorphic to the tensor product E⊗E'.)

The a. p. is quite important in connection with tensor products, with compact and nuclear operators etc., as was shown by Grothendieck. The first examples of locally convex (in fact, of separable reflexive Banach) spaces without a. p. were given by P. ENFLO [10] in 1972. By refinements of his method, due to DAVIE [8] and FIGIEL [11], it is known today that, for every p with $2<p\leq\infty$, there exist closed subspaces of the sequence space l^p without a. p. (Whether this holds true for $1\leq p<2$, too, is an open problem.) And Hogbe-Nlend [18] concluded that, as a consequence of Enflo's counterexample, there are Fréchet-Schwartz- and (DF)-(S)-spaces without a. p. The only classes of spaces for which the a. p. is known, are Hilbert spaces, separable Banach spaces (or a larger class of locally convex spaces) with a Schauder basis, and nuclear (resp. co-nuclear) spaces. Also every complex commutative C^*-algebra is known to possess the a. p. (by the theorem of Gelfand and Naĭmark).

On the other hand, one can prove that many of the "classical" spaces arising in the applications have the a. p., e. g. the spaces $L^p(\mu)$, $1 \leq p \leq \infty$, $C(K)$, K compact, and many others more. We would like to point out that it seems to be open whether the following Banach spaces have the a. p.:

(1) the sup-norm algebra $H^\infty(D)$ of bounded analytic functions on the open unit disk of the complex plane,

(2) $CB^{(n)}(\mathbb{R})$, $1 \leq n < \infty$, i. e. the space of functions f on the real line which are n times continuously differentiable and such that f, together with all derivatives of order 1 up to n, is bounded on $\mathbb{R}$, equipped with the norm

$$\|f\|_{(n)} = \sup_{\nu=0,1,\dots,n} \; \sup_{x \in \mathbb{R}} |f^{(\nu)}(x)|,$$

(3) $\mathfrak{L}(l^2)$ = bounded linear operators on the separable Hilbert space l^2 with the operator norm.

For many purposes, it turns out that an equivalence of the a. p., due to L. SCHWARTZ [22], is quite helpful. It makes use of the notion of Schwartz's ε-product for two locally convex spaces E and F (cf. [22]).

2. Definition. F'_c is the dual of F with the topology of uniform convergence on precompact subsets of F. With this notation in mind, we define:

$$E \varepsilon F = \mathfrak{L}_e(F'_c, E),$$

i. e. the locally convex space of continuous linear operators from F'_c into E, equipped with the topology of uniform convergence on equicontinuous subsets of F'.

If E and F are assumed quasi-complete, $E \varepsilon F \cong F \varepsilon E$, and we have another topological isomorphism of the ε-product with a space of bilinear forms on $E' \times F'$ (see Schwartz [22]). Furthermore, $E \varepsilon F$ is complete, if both E and F are complete spaces.

Now, in a canonical way, $E \otimes F$ can be identified with a linear subspace of $E \varepsilon F$, and the induced topology is the so-called ε-topology of Grothendieck. -Thus we arrive at the equivalent condition for the a.p. given by Schwartz [22]:

3. Theorem. The quasi-complete locally convex space E has the a.p. if and only if $E \otimes F$ is dense in $E \varepsilon F$ for all complete locally convex spaces F (or for all Banach spaces F).

So, if E is complete, it has the a.p., if $E \varepsilon F = E \hat{\otimes}_{\varepsilon} F$ for every complete locally convex (or for every Banach) space F. (Here $E \hat{\otimes}_{\varepsilon} F$ denotes the completion of $E \otimes_{\varepsilon} F$, that is, of the tensor product with the ε-topology).

In fact, 3. is a slight refinement [1] of Schwartz's original theorem (see [2] and [5]). -Theorem 3 will be used later on to give a proof of the a.p. for some function spaces. All the function spaces we deal with here are subspaces of the so-called weighted spaces which we are going to introduce next.

II. Weighted spaces of vector-valued functions

In this section we look at weighted spaces of functions with values in a locally convex space E. These spaces were introduced in the scalar case

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[1]) Let us remark in this connection that (for instance for a proof of the equivalence of the a.p. for Banach spaces with the approximation of compact operators by operators of finite rank) it seems important to notice that, for a Banach space E, the a.p. of E is even equivalent to the density of $E \otimes F'$ in $E \quad F'$ for all Banach spaces F. In fact, E has the a.p. if $E \otimes E_K'$ is only dense in $E \quad E_K'$ for all absolutely convex compact sets K, see Séminaire Schwartz [23]. (Here the notation E_K is used with its usual meaning.) On the other hand, even for any quasi-complete locally convex space E, the density of $E \otimes E'$ in $E \varepsilon E'_c$ is already enough to imply the a.p. for E, cf. [22].

by L. NACHBIN (see [19]). And in the vector-valued case J. B. PROLLA studied spaces of this type independently in [20].

Let X be a completely regular (Hausdorff) space. A real-valued non-negative upper semicontinuous function v on X will be called a weight (or weight function) on X. Let $V \neq \emptyset$ be a system of weights on X. This system is called a Nachbin family on X, if for any $v_1, v_2 \in V$ and for any $\lambda \geq 0$ there exists a $v \in V$ such that $\lambda v_1, \lambda v_2 \leq v$ (pointwise on X). So a Nachbin family is directed upward in a certain sense. (There is no loss of generality in assuming to have a Nachbin family of weights : If we are given a fixed non-empty system of weights on X, we can take the sup of any two weights and the positive multiples of each weight as weight functions, too, without changing the weight conditions. In this way, we obtain a Nachbin family "equivalent" to the original system.)

We fix X, a Nachbin family V on X, and a locally convex space E to define our weighted spaces of continuous E-valued functions as follows:

4. Definition:

$CV(X, E) := \{f \in C(X, E)$ = space of continuous E-valued functions on X; $(vf)(X)$ is bounded in E for every $v \in V\}$.

$CV^p(X, E) := \{f \in C(X, E);\ (vf)(X)$ is even precompact in E for every $v \in V\}$.

$CV_o(X, E) := \{f \in C(X, E);\ vf$ vanishes at infinity for every $v \in V\}$,

where vf is the function from X into E defined by $(vf)(x) := v(x)f(x)$ for every $x \in X$, and where a function $g: X \longrightarrow E$ is said to vanish at infinity, if for every continuous seminorm p on E and for every $\epsilon > 0$ there is a compact subset $K = K(p, \epsilon)$ of X such that $p(g(x)) < \epsilon$ for all $x \in X \setminus K$.

One of the reasons to admit only upper semicontinuous weights is that such functions are always bounded on compact subsets of X. Using this property, it is easy to show that the following inclusions hold among the linear spaces defined above:

$$CV_o(X, E) \subset CV^p(X, E) \subset CV(X, E).$$

Let us introduce a locally convex topology on CV(X, E) and on its subspaces $CV^p(X,E)$ and $CV_o(X,E)$ by defining a system of semi-norms: Let $\{p_\alpha;\ \alpha\in A\}$ be a (directed) system of semi-norms which gives the topology of E, and put for $\alpha\in A$, $v\in V$:

$$bv_\alpha(f) = \sup_{x\in X} p_\alpha(v(x)f(x)) = \sup_{x\in X} v(x)p_\alpha(f(x)),\ f\in CV(X,E).$$

So we endow CV(X, E) with the "weighted" topology induced by $\{bv_\alpha;\ v\in V,\ \alpha\in A\}$.

To assure that this topology is Hausdorff, we assume from now on $V>0$, i.e. that for every $x\in X$ there is at least one $v\in V$ with $v(x)>0$. - It is easy to see that $CV_o(X,E)$ and $CV^p(X,E)$ are always closed subspaces of CV(X, E) in the weighted topology.

Some examples of weighted spaces are given in the diagram below (see [2]). Remark that, for a linear subspace U of $C(X) = C(X, \mathbb{C})$, the set U^+ of all non-negative real-valued functions in U is always a Nachbin family.

5. Examples. X, E as usual.

Nachbin family V	CV(X, E)	$CV_o(X,E)$	Remarks
$W=W(X):=\{\lambda\chi_K;\lambda\geq 0,$ K compact subset of X$\}$	(C(X, E), co)		χ_K=characteristic function of K, co=topology of uniform convergence on compact subsets of X
Z=Z(X):=positive constants on X (or $CB^+(X)$, instead)	CB(X, E)	$C_o(X,E)$	CB = continuous and bounded.
	both with the topology of uniform convergence on X		C_o=continuous and vanishing at infinity
$C_o^+(X)$, X locally compact	$(CB(X,E),\beta)$		β="strict" topology of BEURLING and BUCK [7], co$\leq\beta\leq$uniform topology
$C^+(X)$, X locally compact and σ-compact	$(C_c(X,E), i)$ for any normed space E		C_c=continuous with compact support, i=(canonical) inductive limit topology

To give a nice sufficient condition for completeness of weighted spaces, we need some more terminology. For two Nachbin families V_1, V_2 on X, we shall write $V_1 \leq V_2$, iff for every $v_1 \in V_1$ there exists a $v_2 \in V_2$ such that $v_1 \leq v_2$. If this relation holds, (e.g.) $CV_2(X,E)$ is continuously embedded in $CV_1(X,E)$.

It is reasonable in many applications to assume that $W \leq V$, i.e. that $CV(X,E)$ and $CV_o(X,E)$ are continuously embedded in $CW(X,E) = (C(X,E), co)$. If we do assume this, we have, of course, tied up the weighted topology with the (weaker) topology co. To make this assumption more meaningful, we should then assume that X contains "enough" compact subsets (in the following sense).

6. Definition: X is called a $k_{\mathbb{R}}$-space (notation introduced by H. BUCHWALTER), iff a function f: $X \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) with the property $f_{|K}$ continuous for every $K \subset X$ compact is already continuous on X.

It is a simple consequence that, on a $k_{\mathbb{R}}$-space X, a function f with values in any completely regular space E is continuous if and only if $f_{|K}$ is continuous as a function from K into E for every compact subset K of X. A theorem, due to S. WARNER [28], says that a completely regular X is a $k_{\mathbb{R}}$-space, if and only if $(C(X), co)$ is complete. Examples of $k_{\mathbb{R}}$-spaces include the class of k-spaces (of Kelley), so for instance all locally compact or all metrizable spaces.

Then it is easy to deduce (see e.g. [2]):

7. Proposition: $CV(X,E)$ (and hence a fortiori $CV^p(X,E)$ resp. $CV_o(X,E)$) is complete if either $Z \leq V$ or if $W \leq V$ and X = $k_{\mathbb{R}}$-space.

We do not want to give a more general completeness theorem which could be derived in a similar way (see e.g. Prolla [20] and A. GOULLET DE RUGY [15] for spaces of type $CV_o(X, \mathbb{R})$). Proposition 7. is enough for most practical purposes.

To finish this section, let us assume (for simplicity) from now on that we deal with complex scalars all the time. Thus we define (for instance):

$$CV(X) := CV(X, \mathbb{C}) = CV^p(X, \mathbb{C}) \text{ and } CV_o(X) := CV_o(X, \mathbb{C}).$$

These are the original weighted spaces of Nachbin [19].

III. Connection between vector-valued functions and the ε-product of a space of scalar functions; application to the a. p.

The key to the relation between the a. p. for subspaces of weighted spaces and (weighted) spaces of vector-valued functions is the following theorem the proof of which will be sketched in [4]:

8. Theorem ([3]). Assume that $Z \leq V$ or that $W \leq V$ and $X = \underline{k}_R$-space. Let E be a quasi-complete locally convex space. Then there are natural topological isomorphisms as follows:

$$CV(X)\varepsilon E \cong CV^p(X, E), \quad CV_o(X)\varepsilon E \cong CV_o(X, E).$$

When the isomorphic spaces are identified (which we will do from now on), $E \otimes_\varepsilon CV(X)$ resp. $E \otimes_\varepsilon CV_o(X)$ corresponds to the space of functions $f \in CV(X, E)$ resp. $CV_o(X, E)$ such that f(X) is contained in some finite dimensional linear subspace of E. (This part of the theorem holds even without the assumptions on X, V, and E above.)

By the way, let us remark that a similar, but more complicated description of the ε-products (or, rather, of a related space which we are used to call $\tilde{\varepsilon}$-product) is known, even if E is not quasi-complete and if none of the assumptions on V and X mentioned in 8. holds. For more information on this see [3].

To apply theorem 8. to our situation in which we would like to look at subspaces of CV(X) instead it is important to observe that the ε-product behaves nicely with respect to subspaces: If E, F_1, F_2 are quasi-complete locally convex spaces, F_1 being a topological linear subspace of F_2, we obviously get $F_1 \varepsilon E \subset F_2 \varepsilon E$ topologically. Taking this and the nature of the topological isomorphism in 8. into account, we obtain a refinement:

9. Theorem ([3]). Assume the conditions of 8. and let Y be a closed topological linear subspace of CV(X) (resp. $CV_o(X)$). Then there is a natural topological isomorphism of $Y \varepsilon E$ with the topological linear subspace

$\{f \in CV^p(X,E)$ (resp. $CV_o(X,E)$); $e' \circ f \in Y$ for each $e' \in E'\}$ of $CV(X,E)$. [$(e' \circ f(x) = e'(f(x))$ for all $x \in X$.] And $Y \otimes_\varepsilon E$ "is" the subset of all f out of this space with the property that $f(X)$ is contained in a finite dimensional linear subspace of E.

As a direct consequence of 9. and 3., one arrives at (under the conditions of 9.):

10. Theorem ([3]). The following conditions are equivalent:

(1) Y (as a topological linear subspace of $CV(X)$ resp. $CV_o(X)$) has the a. p.

(2) For all complete locally convex (or all Banach) spaces E, the space of functions "with finite dimensional ranges" is dense in
$\{f \in CV^p(X,E)$ resp. $CV_o(X,E)$; $e' \circ f \in Y$ for every $e' \in E'\}$
(under the weighted topology).

So the a. p. of Y is equivalent to the approximation of vector-valued functions by functions with values in finite dimensional subspaces. The definition of the a. p. requires to approximate precompact subsets of Y (a space of scalar functions) uniformly - and in a linear way - by subsets in finite dimensional subspaces of Y. On the other hand, by 10. (2), it is "enough" to approximate each (single, arbitrary, fixed) vector-valued function (in "a" certain space) by functions of finite dimensional range.

Both directions of the equivalence in 10. seem to be important: (1) ⇒ (2) requires to know the a. p. for certain function spaces and gives applications tothe tensor product (see [4]). Here we deal only with the converse implication (2) ⇒ (1) which allows to prove the a. p. for many weighted spaces and for some subspaces. This will be sketched in the remaining sections. In the next section we intend to apply a vector-valued "generalized Stone-Weierstrass theorem" due to J. B. Prolla [21] to the present situation.

weighted spaces of type $CV_o(X)$ instead of $C(K)$ for a compact K - J. B. Prolla [21] proved the following "generalized Stone-Weierstrass theorem":

12. Theorem. Fix a subalgebra A of $C(X)$, and assume that for all $a \in A$ and all $v \in V$ the restriction $a|_{\text{supp } v}$ is bounded. (This is clearly the case, if e.g. $A \subset CB(X)$ or if $V \subset C_c(X)$. The condition implies that $CV_o(X)$ is a module over the algebra A [with respect to pointwise multiplication].)
Let W be a vector subspace of $CV_o(X, E)$, E a given locally convex space. Suppose W is an A-module with respect to pointwise multiplication. (So it is an A-submodule of $CV_o(X, E)$.)
Then $f \in CV_o(X, E)$ belongs to $\overline{W}^{CV_o(X, E)}$ if and only if for each $K \in \mathcal{K}_A$ we have $f|_K \in \overline{W|_K}^{C(V|_K)_o(K, E)}$.

(Notice that the restriction $V|_K$ of the Nachbin family V to K is a Nachbin family on K such that the weighted space $C(V|_K)_o(K, E)$ is well-defined. This space has the natural "restriction topology" of $CV_o(X, E)$.)

To enlighten the conditions of the theorem (somewhat), we remark that, in Bishop's original theorem, one approximates by the elements of a certain subalgebra, and that this is crucial (in a certain sense). On the other hand, in general neither $CV_o(X)$ nor $C(X, E)$, E locally convex, has the structure of an algebra. Therefore it is natural to assume that we deal with a submodule W of $CV_o(X, E)$ over a certain algebra A instead. For a more detailed discussion of questions of this kind, of the so - called "weighted approximation problem" - we look only at the "bounded case" of this problem here - and of the results similar to 12. see e.g. Nachbin [19], Prolla [21], and W. H. SUMMERS [25], [26].

After all the preparations we made, now we come to our main result. Namely by a combination of 10. and 12., we can prove:

13. Theorem. Let Y be a closed subspace of $CV_o(X)$, X locally compact and $0 < V \subset C^+(X)$. Let A be a subalgebra of $C(X)$ such that for all $a \in A$ and all $v \in V$ the restriction function $a|_{\text{supp } v}$ is bounded. Assume that Y is an A-module with respect to pointwise multiplication.

Then Y (with the topology induced by $CV_o(X)$) has the a. p., if only for every $K \in \mathcal{K}_A$ the restriction space $Y|_K$, equipped with the (natural restriction) topology of $C(V|_K)_o(K)$, has the a. p.

Proof. By 10. (2) we must show: For any (fixed) complete locally convex space E and any (fixed) $f \in CV_o(X, E)$ such that $e' \circ f \in Y$ for every $e' \in E'$, f belongs to $\overline{Y \otimes E}^{CV_o(X,E)}$. But apply 12. to the A-module $W = Y \otimes E \subset CV_o(X, E)$: We must now verify that $f|_K \in \overline{Y|_K \otimes E}^{C(V|_K)_o(K,E)}$ for $K \in \mathcal{K}_A$. However, $f|_K$ is an element of $C(V|_K)_o(K, E)$ with the (crucial) property $e' \circ f|_K \in Y|_K$ for every $e' \in E'$. We know that $Y|_K$ has the a. p., and if this space is closed in $C(V|_K)_o(K)$, a direct application of (1)⇒(2) in 10. yields the required approximation. Even if we do not assume that $Y|_K$ is closed in $C(V|_K)_o(K)$, we can proceed in the same way and use a slight strengthening of 3., 9., and 10. (instead of 10. itself) in the last step.□

(In fact, in 13. one does not really need the a. p. for the spaces $Y|_K$, but a slightly weaker property which we are used to call Schwartz's a. p.- approximation of the identity on absolutely convex compact sets only [instead of precompact ones]. A revision of the proof of 13. makes this clear.)

As a first application of this interesting theorem, we derive immediately:

14. Corollary. Under our assumptions ($0 < V \subset C^+(X)$ and X locally compact), $CV_o(X)$ has the a. p.

Proof. Take $Y = CV_o(X)$. This is a module over $A = CB(X)$. As we mentioned earlier, $\mathcal{K}_A$ consists of the one point sets only in this case. So every $Y|_K$, $K \in \mathcal{K}_A$, is a one dimensional space, and we are done by 13.□

It should be remarked that weaker assumptions than those in 14. are already sufficient for the a. p. of the spaces $CV_o(X)$. We return to this remark in the next section. - An application of entirely different type is given next.

15. Corollary. Let K be a compact subset of the complex plane and G_i ($i = 1, \ldots, n$; $n \in \mathbb{N}$) - for simplicity only - open connected sets with the

properties $G_i \subset K$, $\overset{\circ}{\overline{G_i}} = G_i$, and $\overline{G_i} \cap \overline{G_j} = \emptyset$ $(i \neq j)$. Then the algebra

$$A = \{f \in C(K);\ f|_{G_i} \text{ analytic for } i = 1, \ldots, n\},$$

equipped with the sup-norm on K, has the a. p.

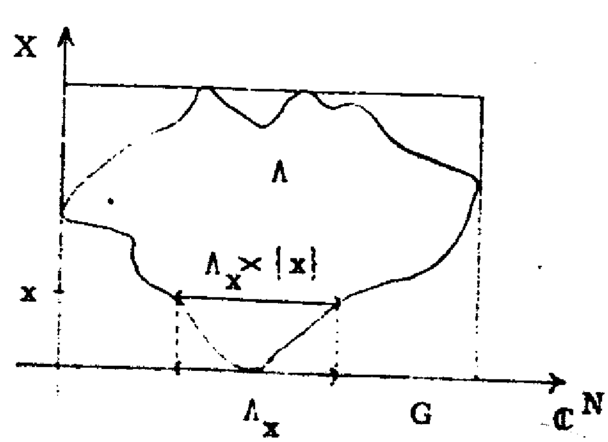
Proof. Here X = K compact, $(C(K), \|\cdot\|) = CV_o(X)$ for e. g. $V = CB^+(X)$. Take now Y = A. This is an A-module, and all our assumptions are satisfied in this situation. It turns out that the maximal antisymmetric subsets of K with respect to A consist of $\overline{G_i}$ $(i = 1, \ldots, n)$ and one point sets in $K \setminus (\bigcup_{i=1}^{n} \overline{G_i})$. There is of course no problem with restrictions of A to sets of the last type. On the other hand, one can easily establish that for $i = 1, \ldots, n$:

$$A|_{G_i} = A(\overline{G_i}) = \{f \in C(\overline{G_i});\ f \text{ analytic on the interior } G_i = \overset{\circ}{\overline{G_i}} \text{ of } \overline{G_i}\}$$

and that the restriction topology is the sup-norm topology on $\overline{G_i}$. But, by a (rather difficult) result of EIFLER [9] and DAVIE (cf. GAMELIN [12]), we know that all sup-norm algebras $A(\overline{G_i})$ possess the a. p. So apply 13. □

This example can be extended in several ways, as the preceding proof suggests. (We could for instance allow countably many G_i in 15. under some extra conditions.) We do not deal with such extensions here. We have just wanted to mention some applications of 13. in this expository paper and do therefore intend to present simple, but typical examples only.

For the next application let X be locally compact, $\emptyset \neq G$ an open subset of $\mathbb{C}^N$ $(N \geq 1)$. For $\emptyset \neq \Lambda$ open in $G \times X$ and $x \in X$ arbitrary define the "slice" $\Lambda_x := \{z \in G;\ (z, x) \in \Lambda\}$. This is always an open subset of G.



We then consider the following locally convex space

$$CH(\Lambda) = \{f \in C(\Lambda);\ f(\cdot, x) \text{ holomorphic on } \Lambda_x \text{ for each } x \in X$$
$$\text{(or for each } x \in X \text{ such that } \Lambda_x \neq \emptyset)\},$$

equipped with the topology co.

The idea of studying spaces of this type is due to B. GRAMSCH (cf. [16]; notice also the connection to recent work mentioned in [6]).

16. Corollary. $CH(\Lambda)$ has the a. p., too.

Proof. For $V = C_c^+(\Lambda)$, $CV_o(\Lambda) = (C(\Lambda), co)$, and $Y = CH(\Lambda)$ is a closed subspace. As algebra A take

$$A = \{f \in C(\Lambda);\ f \text{ is constant on each set } \Lambda_x \times \{x\},\ \text{(i.e.}$$
$$f(z', x) = f(z'', x) \text{ for all } z', z'' \in \Lambda_x)\}.$$

It is trivial that Y is an A-module, and an obvious argument proves that each maximal antisymmetric subset of Λ with respect to A has the form $\Lambda_x \times \{x\}$ for some $x \in X$. Every set K of type $\Lambda_x \times \{x\}$ is closed in Λ, and of course, by a simple isomorphism, we can identify $Y_{|K}$ (with its topology) with a topological linear subspace of $(H(\Lambda_x), co)$, the space of holomorphic functions on Λ_x. $(H(\Lambda_x), co)$ is nuclear, hence the subspace $Y_{|K}$ is nuclear, too. This yields the a. p. of $Y_{|K}$ for any $K \in \mathcal{K}_A$, which is enough by 13. □

Of course, 16. rather indicates a whole class of examples, for which 13. proves the a. p. (Compare [6], and remark on the other hand that we did not really make use of weight functions in 16.)

In application 16. we were lucky enough to find that any $Y_{|K}$ $(K \in \mathcal{K}_A)$ had the a. p. as a subspace of a nuclear space. In other (similar) cases it might not be so easy. $Y_{|K}$ could for instance always be a topological linear subspace of a well-known function space with a. p. To be able to conclude that $Y_{|K}$ has the a. p., however, we do need some more information. How this difficulty can sometimes be overcome in practice is demonstrated in the last example of this section.

Here X is again locally compact. $\emptyset \neq \Lambda$ will be a closed subset of $\mathbb{C} \times X$, and the notation $\Lambda_x = \{z \in \mathbb{C};\ (z, x) \in \Lambda\}$ for $x \in X$ is introduced in an analogous manner as above. In this case Λ_x is always a closed subset of $\mathbb{C}$. We assume that for each $x \in X$ the set Λ_x is even compact and define

$CA(\Lambda) := \{f \in C(\Lambda);\ f(\cdot, x) \text{ analytic on } \mathring{\Lambda}_x \text{ for every } x \in X \text{ (such that } \mathring{\Lambda}_x \neq \emptyset)\}$,

with the topology co of compact convergence on Λ.

Furthermore, we need a certain kind of (geometric) regularity condition (*) on Λ:

(*) For each $x \in X$, Λ_x does not separate the plane, i.e. $\mathbb{C} \setminus \Lambda_x$ is connected.

(Regularity conditions were suggested by the work of Gramsch [16].)

17. Corollary. Under our assumptions, $CA(\Lambda)$ always has the a.p.

Proof. Take again $V = C_c^+(\Lambda)$; so $CV_o(\Lambda) = (C(\Lambda), co)$ and $Y = CA(\Lambda)$ is closed herein (all the Λ_x being compact). The algebra A is chosen as before, i.e.

$A = \{f \in C(\Lambda);\ f \text{ is constant on each } \Lambda_x \times \{x\}\}$.

So Y is a module over A and $\mathcal{K}_A = \{\emptyset \neq K := \Lambda_x \times \{x\};\ x \in X\}$. For $K \in \mathcal{K}_A$, $Y|_K$ can be identified (exactly as in 16.) with a topological linear subspace of the sup-norm algebra (on Λ_x) $A(\Lambda_x) = \{f \in C(\Lambda_x);\ f \text{ analytic on } \mathring{\Lambda}_x\}$.

By Eifler's theorem (from [9]) already mentioned in the proof of 15., we know that $A(\Lambda_x)$ always has the a.p. [At this point, we do not really make use of (*), as Davie's generalization of Eifler's theorem indicates.] Yet, it is not clear a priori whether the subspace $Y|_K$ of $A(\Lambda_x)$ has the a.p., too.

Our regularity condition (*) implies by MERGELYAN's famous theorem on polynomial approximation that the polynomials are dense in $A(\Lambda_x)$. So, by a simple reasoning, $Y|_K$ is a fortiori dense in $A(\Lambda_x)$. Remark now that a locally convex space E possesses the a.p. if only its completion does. As the completion of $Y|_K$ is $A(\Lambda_x)$, we conclude that every $Y|_K$, $K \in \mathcal{K}_A$, hence also Y, has the a.p. □

A careful inspection of the proof of 17. reveals that (*) was only needed to obtain the density of $Y|_K$ in $A(\Lambda_x)$. [We made use, however, of the a.p.

of A(S) for (any) compact S in the complex plane.] Thus a generalization of 17. to certain subsets $\Lambda\subset\mathbb{C}^N\times X$, $N\geq 1$, is possible (using recent results on polynomial approximation in $\mathbb{C}^N$ [due to LIEB, cf. WEINSTOCK [29]] and the proof of the a.p. of A(S) for special compact S in the complex N-space, see BEKKEN [1]).

A final remark seems in order at this place. The main assumptions we had to make in this section (i.e. X locally compact, $V\subset C^+(X)$, and the condition for the bounded case of the weighted approximation problem, namely that for all $a\in A$ and all $v\in V$ the restriction function $a|_{\text{supp } v}$ is bounded) were only imposed in order to be able to apply Prolla's vector-valued generalized Stone-Weierstrass theorem 12. to the tensor product $Y\otimes E$ (in the proof of 13.). So one could hope to generalize 13. considerably, if there was a different method to prove a localization of the a.p. for subspaces of weighted spaces which made use only of scalar functions. If such a method could be developed, we could apply Summers' extremely general solution of the weighted approximation problem (see [25], [26], and [27]) instead of Prolla's result.

V. The a.p. for $CV_o(X)$ and $CV(X)$

In this section we return to corollary 14. and deal with proofs of the a.p. for the weighted spaces $CV_o(X)$ and $CV(X)$ themselves. Except for some technical restrictions, it turns out that all these spaces have the a.p.

Instead of using 9. or 10. and a general Stone-Weierstrass theorem, it is more convenient in the case of $CV_o(X)$ to have a second look at 8. Indeed, a relatively simple partition of unity argument (which we are not going to give here) shows directly:

18. Theorem ([2]). For any Nachbin family $V>0$ on a completely regular space X and for arbitrary locally convex E, $E\otimes CV_o(X)$ (identified with a subspace of $CV_o(X,E)$, cf. the second part of 8.) is dense in $CV_o(X,E)$.

So, combining 3., 8., and 18., we obtain:

19. Theorem ([2]). If either $Z \leq V$ or $W \leq V$ and $X = k_{\mathbb{R}}$-space, then $CV_o(X)$ possesses the a. p.

This generalizes 14. considerably: For instance, every metrizable topological vector space is a $k_{\mathbb{R}}$-space, but only finite dimensional ones are locally compact. And in 19. we also allow upper semicontinuous, but not necessarily continuous weight functions. (There are some famous examples - e.g. in connection with the strict topology - which show that such systems of weights can be of interest, too.)

In which concerns upper semicontinuous weight functions, we would like to mention that it is possible to prove the following result by a more technical and somewhat more complicated, but very similar method:
20. Theorem ([2]). For locally compact X and $V>0$, $CV_o(X)$ always has Schwartz's a. p. (i. e. approximation of the identity by continuous linear operators of finite rank is possible uniformly on absolutely convex compact sets). Hence $CV_o(X)$ then has the usual a. p., if it is only quasi-complete.

This seems to be the right place to come back to our remark after theorem 10. In fact, the proof of the a. p. for $CV_o(X)$ we gave in 19. (although already much less complicated and more direct than the proof of 14.) might still look strange and far-fetched at first glance. One could therefore ask whether a proof of the a. p. of $CV_o(X)$ starting right from the definition of the a. p. in 1. is possible (instead of using an involved argument with vector-valued functions and tensor products).

Let me point out that this can indeed be achieved under additional assumptions on V and X. For instance in the book [13] of GARNIR, DE WILDE, SCHMETS, the authors considered weighted spaces of type $CV_o(X)$, too, where X is a locally compact subspace of $\mathbb{R}^n$ $(n \geq 1)$, however, and where all weight functions $v \in V>0$ are supposed continuous. They establish a characterization of precompact subsets of $CV_o(X)$ by an Arzela-Ascoli type theorem and, as a consequence of this and of a partition of unity argument (sic!), they succeed in obtaining the a. p. of $CV_o(X)$. Their method is considerably more direct than the way sketched here. It may also be possible to generalize this method, for instance to any locally compact X. There will be (natural) difficulties, however - e. g. with a generalization of the Arzela-Ascoli type theorem - in case X is no longer locally compact or if $V \not\subset C^+(X)$. So in the end it may be as hard to prove a theorem as general as 19. in their way as it (maybe) was in the reasoning outlined here. On the other hand, the connection we found in 9. will be of theoretical interest, and our method has a lot of different applications, too, as we demonstrated in the last section.

The next part of this section will be devoted to a proof of the a. p. for CV(X). All the proofs of the a. p. of $CV_o(X)$ we mentioned up to this moment relied (in one way or another) heavily on properties of $CV_o(X)$ that CV(X) does not share: For approximation arguments in $CV_o(X)$, it is important that we can restrict functions of type vf, $v \in V$, $f \in CV_o(X)$, to compact subsets of X without losing too much information. This cannot be done in the case of $f \in CV(X)$ in general. So a partition of unity argument in the usual way does not seem to work. And up to now, a generalized Stone-Weierstrass theorem for CV(X, E) is not available. (There are even some theoretical limitations: There can for instance be no "nice" Stone-Weierstrass theorem for the sup-norm algebra CB(X), X not compact, and one would have to look at the Stone-Čech compactification βX of X in this case instead.)

We now sketch a different method to prove the a. p. of weighted spaces. This method reduces the a. p. of either CV(X) or $CV_o(X)$ to the (well-known) a. p. of the sup-norm algebras C(K) and $C_o(Y)$, Y locally compact. One could also say: We make use of the a. p. for (complex) commutative C^*-algebras, but by the Gelfand-Naĭmark theorem, every such algebra is equivalent to some $C_o(Y)$.

So suppose X is completely regular and $C = CV(X)$ or $CV_o(X)$ Hausdorff. For technical reasons, we assume that no weight function in V vanishes identically. We will also need the following conditions:

(1) For every $v \in V$ the restriction $v|_{\text{supp } v}$ is continuous.

(2) For all $f, f' \in C$ and any $v \in V$ there exists a function $g \in C$ such that $g|_{\text{supp } v} = vff'|_{\text{supp } v}$

We could also require instead that only some Nachbin family V′ "equivalent" to V, i.e. with $V \leq V'$ and $V' \leq V$, satisfies (1) and (2), because in this case, CV(X) is topologically isomorphic to CV′(X) and the same holds for $CV_o(X)$ and $CV'_o(X)$.)

Usually, in presence of (1), another condition (2′) is more convenient than (2) for practical purposes:

(2′) For each $f \in C$ and each $v \in V$, one can find an extension $f_v \in CB(X)$ of the function $vf|_{\text{supp } v} \in CB(\text{supp } v)$.

That (2′) implies (2) is a simple consequence of the fact that $C = CV(X)$ or $CV_o(X)$ is a module over CB(X) with respect to pointwise multiplication. Of course, in many cases (1) and (2′) are satisfied, e.g. whenever $V \subset C^+(X)$ holds or if (1) is satisfied and X = normal.

21. Theorem ([3]). If we assume conditions (1) and (2) (or (1) and (2′)), $C = CV(X)$ resp. $CV_o(X)$ has the a.p.

Proof (sketched). The following notation is standard: With the seminorm $bv(f) = \sup_{x \in X} v(x)|f(x)|$, $v \in V$ (and $f \in C$), we put $bv^{-1}(0) := \{f \in C; bv(f) = 0\}$, $C_v := C/_{bv^{-1}(0)}$. C_v is a normed space with the norm $\|\hat{f}\|_v = bv(f)$ for all $\hat{f} \in C_v$, where $f \in \hat{f}$ is chosen arbitrarily. Take $\hat{C}_v :=$ completion of $(C_v, \|\cdot\|_v)$. Then a well-known theorem (which allowed to reduce the approximation problem of Grothendieck to Banach spaces) states that C possesses the a.p., if only all $\hat{C}_v$ do.

Condition (1) allows to define a mapping $A_v: C_v \longrightarrow CB(\text{supp } v)$ by $A_v(\hat{f}) = vf|_{\text{supp } v}$ for any $f \in \hat{f}$. It is a simple exercise to show that A_v is well-defined and an isometric isomorphism of C_v into CB(supp v), equipped with the sup-norm. Put $S_v = A_v(C_v) \subset CB(\text{supp } v)$ and remark that S_v is a (*-) subalgebra of CB(supp v) by condition (2). But then $\hat{C}_v$ is isometrically isomorphic to the closure $\overline{S_v}$ of S_v in CB(supp v). It is obvious that $\overline{S_v}$ is a complex commutative C^*-algebra. Therefore $\overline{S_v}$, and hence a fortiori $\hat{C}_v$, has the a. p. for every $v \in V$. By the aforementioned theorem, the proof is finished. □

It should be added that the method of the proof used in 21. can be applied to certain subspaces of weighted spaces, too. We formulate the corresponding result in the simplest case and show how the argument above must be modified.

Take X = D = open unit disk of the complex plane and define, for a Nachbin family V on D,

$$HV_o(D) = \{f \in CV_o(D);\ f \text{ analytic on } D\}$$

with the weighted topology induced from $CV_o(D)$.

22. Definition. A continuous weight v without zeros on D is called normal, if

(i) v is radial, e. g. $v(z) = v(|z|)$ for all $z \in D$, and

(ii) v(z) tends to zero as z approaches the boundary of D faster than some power of $(1-|z|)$, but less fast than some other power of $(1-|z|)$, that is:

There exist $k > \varepsilon > 0$ and $r_o < 1$ such that for all $r \geq r_o$, as $r \longrightarrow 1-$,

$$\frac{v(r)}{(1-r)^{\varepsilon}} \searrow 0 \quad \text{and} \quad \frac{v(r)}{(1-r)^{k}} \nearrow \infty .$$

This definition and also the main results used in the proof of the following theorem are due to SHIELDS and WILLIAMS [24].

23. Theorem ([5]). If the Nachbin family V contains only normal weight functions, $HV_o(D)$ has the a. p.

Proof. Take $E = HV_o(D)$, $E_v = E/_{bv^{-1}(0)} = E$ with the norm $bv = \|\cdot\|_v$ $(v \in V)$. E_v is a normed subspace of $H(V_v)_o(D)$, where $V_v = \{\lambda v;\ \lambda > 0\}$, and it contains all the polynomials (cf. (ii) in 22.). By a result of Shields and

Williams [24], the polynomials are dense in $H(V_v)_o(D)$, so we get for the completion $\hat{E}_v$ of E_v : $\hat{E}_v = H(V_v)_o(D)$.

Now another theorem of Shields and Williams [24] proves that, for any fixed normal v, $H(V_v)_o(D)$ is isometrically isomorphic to a closed complemented (i.e. continuously projected) subspace of $C_o(D)$ under the identification $f \longleftrightarrow vf$. Therefore, with $C_o(D)$, every $H(V_v)_o(D)$ has the a.p. (This permanence property of the a.p. is well-known.) Hence we have proved the a.p. for each $\hat{E}_v$, $v \in V$, and can conclude as in the proof of 21. □

Final remark. The space $H^\infty(X)$ of bounded holomorphic functions on $X = D$ with the (induced) strict topology β, i.e. $(H^\infty(D), \beta) = HV_o(D)$ for $V = C_o^+(D)$, is not included in 23. One can prove, however, that this space has the a.p., too. Furthermore, this result remains true for instance, if X is any simply connected region in $\mathbb{C}$ (instead of the disk D). This follows by an easy application of the Riemann mapping theorem (see [3]). (The Riemann mapping theorem allows also to generalize 23., but we leave the details to the interested reader.)

Added in proof: In his recent reprint entitled "Der beschränkte Fall des gewichteten Approximationsproblems für vektorwertige Funktionen" (Paderborn, April 1975), G. KLEINSTÜCK has been able to prove a generalization of Prolla's vector-valued Stone-Weierstrass theorem: Theorem 12. of the present article holds for any completely regular space X and any Nachbin family V on X (again under the assumption of the bounded case, that is if for all $a \in A$ and all $v \in V$ the restriction $a|_{\text{supp } v}$ is bounded). By use of this result, the method of the proof of 13. shows already (as indicated at the end of section IV) that theorem 13. remains true, if $Z \leq V$ or if $W \leq V$ and $X = k_{\mathbb{R}}$-space (instead of X locally compact and $0 < V \subset C^+(X)$). Moreover, in the same reprint, Kleinstück obtained approximation results and a vector-valued Stone-Weierstrass theorem for modules in $CV^p(X, E)$, too. (In the corresponding statements, compactifications of X play a crucial role, of course.) He was then able to derive, as a corollary, that

$CV(X) \otimes E$ is always dense in $CV^p(X, E)$. And exactly as in the deduction of 19. from 18., it follows that $CV(X)$ has the a.p., if either $Z \leq V$ or if $W \leq V$ and $X = k_{\mathbb{R}}$-space.

For more details and the proofs (using information on the dual space of $CV(X)$ resp. $CV_o(X)$ and a reduction of some questions on vector-valued functions to this information) we refer to Kleinstück's interesting paper.

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