AN INTRODUCTION TO LOCALLY CONVEX INDUCTIVE LIMITS

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CONTENTS

PREFACE

Locally convex inductive limits arise in great profusion throughout many fields of functional analysis and its applications; e.g., in distribution theory, partial differential equations and convolution equations, Fourier analysis, complex analysis in one and several (or an infinite number of) variables, spectral theory and the holomorphic functional calculus, measure theory on topological spaces, as well as (of course) the structure theory of abstract locally convex topological vector spaces. All the general texts on topological linear spaces contain the basic definitions and the "standard" theory, at least for strict inductive limits. (We recommend the books of Bourbaki [13], Edwards [22], Floret and Wloka [28], Horváth [35], Jarchow [36], Köthe [38], Pérez Carreras and Bonet [46], Schaefer [52] and Valdivia [59].)

The locally convex structure of general inductive limits easily becomes very intricate, and even specialists may have a hard time studying certain more complicated inductive limits. However, most of the inductive limits encountered in applications are countable, and many occur in the context of function or sequence spaces. Their very special structural properties immediately exclude various pathologies (which are possible and not even too unnatural otherwise). The important methods developed for countable strict and (weakly) compact inductive limits, as well as a good knowledge of the special properties of the classes of (LF)- and (LB)-spaces, usually suffice for many practical purposes. The fact that a large part of this general theory appears to remain not as well-known as it should deserve may derive from a certain lack of good introductory texts which "advertise", say, the powerful methods for (weakly) compact inductive limits and (LB)-spaces (and illustrate them in some detail by discussing concrete examples) instead of "hiding" the material among the usual wealth of details on special, complicated classes of abstract locally convex spaces. (See, however, Floret-Wloka [28].)

But functional analytic applications of "reasonable" locally convex inductive limits have also sometimes been restricted by a widespread belief that, even in good cases, inductive limit topologies and their continuous seminorms were very complicated and not easy to handle and that it was not possible to describe an inductive limit topology and a basis for the continuous seminorms in settings which are general enough to include a substantial part of the applications while at the same time, and above all, being "useful" for providing convergence results and direct estimates in terms of the seminorms. A short glance at various books provides some justification for the belief since most texts neglect this latter aspect, and it often occurs, for example, that $\mathcal{D}(\Omega)$ and $C_c(X)$ are defined, but the well-known natural bases of the continuous seminorms for their inductive limit topologies are not exhibited. (The exceptions include Horváth [35] and Pérez Carreras - Bonet [46].) However, part of the "classical" work on the inductive limits which arise in distribution theory and convolution equations, as well as more recent work on spaces of germs of holomorphic functions and weighted inductive limits, demonstrates that, working just a little bit harder, it is indeed possible to deal with inductive limit topologies in many important and quite general settings and that it is actually often not even too difficult to derive natural, sufficiently general and useful characterizations of the continuous seminorms for inductive limit topologies.

This introduction presents both the general theory on special classes of "good" locally convex inductive limits and important results on descriptions of concrete bases for the continuous seminorms of inductive limit topologies. In the presentation, we hope to provide a good balance between abstract theory and interesting examples. We will now briefly sketch the content of these notes.

The main results on the important special types of inductive limits are given in the first three sections. Section 1. concentrates on (countable) strict inductive limits, Section 2. on (countable) weakly compact, compact and nuclear inductive limits while Section 3. deals with (LF)- and (LB)-spaces.

The discussion starts with the "usual" arrangement of first stating the basic definitions, then the theorems and finally some "illustrative examples". We have actually worked out some of the examples in more detail than usual - $\mathcal{D}(\Omega)$ and $C_c(X)$ for strict inductive limits and $H(K)$ for (compact or even) nuclear inductive limits, including references to related work which deserves further study.

Beginning with the end of Section 2., we gradually switch to a different arrangement of the material. These notes mainly are not intended for specialists in the field (although we have reason to hope that even these would find novelty in our selection of some topics and in some of the details), but for interested analysts who might want to apply some of the results and methods to specific examples which come up in a different context. Hence we look at (echelon and) co-echelon spaces and weighted inductive limits as general classes of "important objects" with natural inductive limit topologies which are interesting for their own sake, and we then search for "appropriate" theorems and methods which would help us in our study of the "given" spaces. — Section 4. concentrates on the problem of projective descriptions for weighted inductive limits of spaces of continuous and holomorphic functions and the consequences for the special case of echelon and co-echelon spaces.

At the "appropriate" places along the way, we have also collected (or recalled) various relevant facts from the (deeper parts of the) theory of abstract locally convex spaces in order to make the notes accessible to a larger audience and to provide the reader with the necessary background. (By now, it should be apparent why we do not start with this material, but with the inductive limits.) In the Appendix to Section 3., the topic of strong regularity conditions is treated for possible use in more subtle applications (and here, as in Section 4., we also report on very recent work by various authors).

Of course, the selection of the material has been strongly influenced by the author's own interests, but we have also learned much from the surveys [24] and [27] of K. Floret and the corresponding part of the forthcoming monograph [46] of J. Bonet and P. Pérez Carreras. We state most of the results without proofs (or with at most sketches of proofs), but we try to make the material transparent by explaining some of the main ideas and various connections.

At the ICPAM Autumn School on Functional Analysis at Nice, France, the course of five lectures (of 45 minutes) which I gave during the week of September 8 - 12, 1986 essentially covered (or at least touched) most of the contents of the present four sections, but it was much more "streamlined", leaving out many details and cutting the background material very short. (As a completely new feature, the Appendix to Section 3. appears in these notes; it was already prepared at Nice, but was not presented there because of lack of time.)

Part of the notes may serve as a (very) first introduction to the subject, and I would then recommend omitting (roughly speaking) Numbers 1.4, 1.8, 1.10 - 1.12 and 1.14; 2.1, 2.6.(a) and 2.10; 3.5, 3.15 - 3.18, as well as the Appendix to Section 3.; 4.8 - 4.11 and 4.14 - 4.16. On the other hand, part of the discussion of $C_c(X)$ in Section 1., the short survey on $H(K)$ in infinite dimensional holomorphy, some "background" results on special locally convex spaces, the Appendix to Section 3., and the report on the author's very recent joint work with J. Bonet at the end of Section 4. would be appropriate topics for a more detailed study in advanced seminars.

"Historical" note. During the Spring Semester of 1978, the author gave a series of talks on locally convex inductive limits, weighted inductive limits and topological tensor products at the University of Maryland, College Park, USA. For the purposes of another series of lectures at the Universidade Federal do Rio de Janeiro, Brazil, August - September 1980, the author adapted, rearranged and expanded his notes on the subject to informal "Lecture Notes on Locally Convex Inductive Limits" (subtitled: "A prejudiced account of part of the general theory and of some examples and applications from weighted spaces and holomorphy in infinite dimensions") which were never completed, but of which a few copies were distributed. The present introductory notes, while based on some of the former text, are actually quite different.

(E.g., the treatment of the general theory was much shorter in 1980, and some of the results included here were only obtained in the last years. On the other hand, the Rio notes of 1980 contained much material from [3], [5], and from what was later to become [7], with full proofs. At that time, according to the author's previous interests in topological tensor products, the setting of "weighted inductive limits" was presented in greater generality, including vector-valued functions and decreasing sequences of systems of weights, and we also treated ε-tensor products and ε-products of inductive limits in some detail.)

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Note on references and general notation. In preparing these notes, I found it next to impossible to give the name(s) of the original author(s) of each "classical" theorem, or the names of all those mathematicians that contributed to major, but nowadays "well-known" results and methods in this area. However, reasonable efforts have been made to mention most of the names (at least once). Also, with the references already containing more than 60 numbers, not each important paper on the subject could be listed here, and we sometimes cannot quote the original article in which some "classical" theorem was first proved, but rather refer to some other source where it is stated, surveyed or just mentioned.

By $\mathbb{N}$, we will always denote the set of all natural numbers, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Finally, $\mathbb{R}_+$ is the set $\{x \in \mathbb{R};\ x \geq 0\}$.

0. Notation; some preliminary remarks

By the term locally convex space (abbreviated as "l.c. space"), we will denote a Hausdorff (real or complex) topological vector space whose topology can be defined by a system of seminorms (in the canonical way). When we deal with l.c. inductive systems (also called l.c. inductive nets or spectra) and their limits, we always assume the following situation:

> E is a (real or complex) linear space, A denotes a directed index set (under an order $\leq$), $(E_\alpha,\tau_\alpha)_{\alpha \in A}$ a family of l.c. spaces E_α (all real or all complex) with topologies τ_α, linear subspaces of E with $E = \bigcup_{\alpha \in A} E_\alpha$, and if $\alpha \leq \beta$, E_α is a linear subspace of E_β such that the canonical inclusion mapping $i_{\alpha\beta}$: $(E_\alpha,\tau_\alpha) \to (E_\beta,\tau_\beta)$ is continuous; i.e., the topology τ_β of the larger space E_β induces a topology on E_α which is weaker than the initial topology τ_α of the smaller space. (Here, $i_{\alpha\alpha}$ is of course the identity mapping of E_α.) Then $(E_\alpha,\tau_\alpha)_{\alpha \in A}$ is called an injective inductive system of l.c. spaces.

The space E is now endowed with the strongest l.c. topology τ (equivalently, this is the strongest topology on E given by a system of seminorms) which makes all the natural injections i_α: $(E_\alpha,\tau_\alpha) \to E$ continuous, and (E,τ) is termed the l.c. inductive limit of the system $(E_\alpha,\tau_\alpha)_{\alpha \in A}$. We write $(E,\tau) = \operatorname*{ind}_{A \ni \alpha \to} (E_\alpha,\tau_\alpha)$.

Moreover, in all that follows, we will tacitly assume that the l.c. inductive limit topology τ on E is again Hausdorff.

A few remarks are in order at this point. In the literature, the terms l.c. inductive system and l.c. inductive limit are often used in a broader sense; viz., for directed systems $(E_\alpha, \tau_\alpha)_{\alpha \in A}$ of l.c. spaces and general, not necessarily injective, continuous linear "linking maps" $i_{\alpha\beta}: (E_\alpha, \tau_\alpha) \to (E_\beta, \tau_\beta)$, $\alpha \leq \beta$, between these spaces. The inductive limit space E and the mappings $i_\alpha: (E_\alpha, \tau_\alpha) \to E$ are then "constructed" in a certain way. To simplify our discussion, to avoid some more complicated terminology and to help the intuition, we restrict our attention to the situation described above. It is usually enough to study only injective l.c. inductive systems. From now on, we will drop the word "injective" from our terminology. Furthermore, we often assume that the topology τ_α of the l.c. space (E_α, τ_α) is fixed, and thus we may suppress τ_α as well as the inductive limit topology τ. E.g., we consequently write, for short, $(E_\alpha)_\alpha$ and $E = \operatorname{ind}_{\alpha \to} E_\alpha$.

Even though we suppose that each of the l.c. spaces E_α in an inductive system carries a Hausdorff topology, it is possible that the inductive limit topology τ of $E = \operatorname{ind}_{\alpha \to} E_\alpha$ is not Hausdorff. It is easy to find examples for this phenomenon (even the inductive limit of a sequence of nuclear Fréchet spaces need not be Hausdorff), and non-separated l.c. inductive limits actually occur in natural questions of analysis (e.g., see [27]). However, in applications, the fact that an inductive limit topology is Hausdorff can usually be checked quite easily by considering a continuous embedding of all E_α into a fixed (Hausdorff) l.c. space F. (E.g., inductive limits $E = \operatorname{ind}_{\alpha \to} E_\alpha$ of sequence or function spaces on a fixed set X are always separated if all the spaces E_α have topologies which are stronger than pointwise convergence on X.) And if one wants to take care of non-Hausdorff inductive limit topologies as well, then this introduces many difficulties which we would rather like to avoid in the present notes. That is the main reason why we will not discuss non-separated l.c. inductive limits in the sequel.

By definition of the l.c. inductive limit topology, an absolutely convex set U in $E = \operatorname{ind}_{\alpha \to} E_\alpha$ is a neighborhood of 0 if and only if

$U \cap E_\alpha$ is a neighborhood of 0 in E_α for each $\alpha \in A$, and a basis of neighborhoods of 0 for E is given by the system of all sets of the form

$$U = \Gamma\Big(\bigcup_{\alpha \in A} U_\alpha\Big) \qquad ,$$

where each U_α is a neighborhood of 0 in E_α and where we let Γ denote the absolutely convex hull. Similarly, a seminorm p on $E = \operatorname{ind}_{\alpha \to} E_\alpha$ is continuous if and only if each restriction $p_{|E_\alpha}$ is continuous on E_α. However, this last equivalence is trivial and does not yield a useful description of the continuous seminorms for the inductive limit topology. When dealing with concrete inductive limits and their topologies, it is often essential for direct estimates and computations to derive a nontrivial characterization (if possible, in terms of a formula) for a basis of the continuous seminorms. Until quite recently, this aspect has often been neglected, but it is one of the aims of these notes to demonstrate that, in many important cases arising in the applications, it is indeed possible to write down bases for the continuous seminorms of l.c. inductive limit topologies explicitly.

If the index set A of a l.c. inductive system or limit is $\mathbb{N}$ with its natural ordering $\leq$, then we will speak of a l.c. inductive sequence $(E_n)_n$ and a countable l.c. inductive limit $E = \operatorname{ind}_{n \to} E_n$. Sometimes a given uncountable l.c. inductive system $(F_\alpha)_\alpha$ can be replaced by an "equivalent" l.c. inductive sequence $(E_n)_n$; that is, a sequence for which

$$\operatorname{ind}_{\alpha \to} F_\alpha = \operatorname{ind}_{n \to} E_n \qquad \text{algebraically and topologically.}$$

Right from the start, we should point out that countability of a l.c. inductive limit makes a big difference. Nearly all positive results in these lecture notes will only arise in the case of countable inductive limits, and, indeed, it is practically impossible to develop any deep and widely applicable general theory for uncountable l.c. inductive limits. (Some of the counterexamples are mentioned as we go along.)

1. Definition. A countable l.c. inductive limit $E = \operatorname{ind}_{n \to} E_n$ of Fréchet (resp., Banach) spaces E_n is called <u>(LF)-</u> (resp., <u>(LB)-</u>) <u>space</u>.

(In some of the older literature, these terms are used in a more restricted sense, but, at that time, inductive limits were often not treated in the present generality anyway.)

Some of the most important examples of l.c. inductive limits belong to one of the classes introduced in Definition 1. When dealing with (LF)- or even (LB)-spaces, we will already know that certain general facts are always true, but some "pathologies" may still occur. Most of our discussions will take place in the setting of (LB)-spaces.

At the end of this preliminary section, we must return to the very definition of l.c. inductive limits $(E,\tau) = \operatorname{ind}_{\alpha \to} (E_\alpha,\tau_\alpha)$. We remember that the l.c. inductive limit topology τ was defined as the finest l.c. topology making all the injections $i_\alpha : (E_\alpha,\tau_\alpha) \to E$ continuous. Now, it is known (and easy to verify) that the category in which projective limits are taken is not important: E.g., for a projective system $(F_i)_{i \in I}$ of l.c. spaces F_i, the projective limit topology τ of $F = \operatorname{proj}_{\leftarrow i} F_i$ always is the weakest l.c. topology, but also the weakest vector space topology or even the weakest topology which makes all the canonical mappings $\pi_i : F \to F_i$ continuous. In other words, the weakest topology making all π_i continuous is again locally convex, together with all the topologies of the F_i. But the situation is completely different for inductive limits: In general, the "topological vector space inductive limit topology" of an inductive system $(E_\alpha)_\alpha$ of l.c. spaces; i.e., the strongest vector space topology which makes all the injections $i_\alpha : E_\alpha \to E$ continuous, need not be locally convex, and the "inductive topology" of an inductive system $(E_\alpha)_\alpha$ of l.c. spaces; i.e., the finest topology making all i_α continuous, need not be a linear topology. The last pathology is not even remedied (in general) if we restrict our attention to countable l.c. inductive limits, and it accounts for part of the trouble. (If $E = \text{ind}_{n\to} E_n$ does carry the finest topology making all $i_n: E_n \to E$ continuous, then a set C in E is closed if and only if $C \cap E_n$ is closed in E_n for each $n \in \mathbb{N}$, and the topology of E is necessarily Hausdorff along with all the topologies of the spaces E_n.) But at least the first pathology cannot occur for countable l.c. inductive limits (cf. Bourbaki [13]).

2. *Proposition.* Any countable l.c. inductive limit $E = \text{ind}_{n\to} E_n$ (of l.c. spaces E_n) also carries the finest vector space topology which makes all the canonical mappings $i_n: E_n \to E$ continuous.

In spite of the differences mentioned above, let us agree from now on to simply speak of inductive systems $(E_\alpha)_\alpha$, sequences $(E_n)_n$ and limits E (of l.c. spaces E_α resp. E_n) whenever we actually mean l.c. inductive systems, sequences and limits. Since we are only interested in such inductive limits, no confusion will arise.

1. Strict inductive limits

Historically speaking, inductive limits of l.c. spaces appeared in the theory of topological vector spaces for the first time when some of the common spaces of distribution theory were topologized in the natural way, and the corresponding spaces $\mathcal{D}(\Omega)$, $\mathcal{D}^m(\Omega)$ and $C_c(X)$ were strict inductive limits. The main positive results on countable strict inductive limits (subsumed in our Theorem 3. below) are due to Dieudonné-Schwartz and Köthe (cf. Horváth [35] or Köthe [38]).

1. *Definition.* An inductive system $(E_\alpha,\tau_\alpha)_{\alpha\in A}$ of l.c. spaces or its limit $(E,\tau) = \text{ind}_{\alpha\to} (E_\alpha,\tau_\alpha)$ is said to be

(a) strict if, for each $\alpha < \beta$, $\tau_{\beta|E_\alpha} = \tau_\alpha$; i.e., $i_{\alpha\beta}: E_\alpha \to E_\beta$ is a topological isomorphism into,

(b) <u>hyperstrict</u> if, for each $\alpha \in A$, $\tau_{|E_\alpha} = \tau_\alpha$; i.e., $i_\alpha: E_\alpha \to E$ is a topological isomorphism into.

Note that hyperstrict ⇒ strict and that a hyperstrict inductive limit clearly has a Hausdorff topology. But Y. Kōmura [40] gave examples of (uncountable) strict l.c. inductive systems with (a) a non-separated limit topology and (b) a Hausdorff inductive limit which fails to be hyperstrict.

We now introduce the notion of regularity for an inductive system since this allows us to formulate Theorem 3. in a more convenient way. This notion is quite important and will be studied in more detail later on.

2. *Definition.* An inductive system $(E_\alpha,\tau_\alpha)_{\alpha\in A}$ of l.c. spaces or its limit $(E,\tau) = \operatorname{ind}_{\alpha\to} (E_\alpha,\tau_\alpha)$ is called <u>regular</u> if, for each bounded set B in (E,τ), there exists $\alpha = \alpha(B) \in A$ such that $B \subset E_\alpha$ and B is τ_α-bounded.

We remark that a set B which is contained and bounded in some (E_α,τ_α) clearly is bounded in (E,τ) as well. Regularity of the inductive system means that all bounded subsets of (E,τ) arise in this way. It is a desirable property of inductive systems. E.g., a regular inductive limit $(E,\tau) = \operatorname{ind}_{\alpha\to} (E_\alpha,\tau_\alpha)$ always carries a Hausdorff topology: The closure $\overline{\{0\}}$ of $\{0\}$ in (E,τ) is a bounded linear subspace of (E,τ), hence contained and bounded in some (E_α,τ_α) by regularity. But each (E_α,τ_α) is Hausdorff, and thus $\overline{\{0\}} = \{0\}$.

Here is the main result on countable strict inductive limits.

3. *Theorem.* (a) The inductive limit of a strict inductive sequence $(E_n)_{n\in\mathbb{N}}$ of l.c. spaces (i.e., of an inductive sequence $(E_n,\tau_n)_{n\in\mathbb{N}}$ with $\tau_{n+1|E_n} = \tau_n$ for each $n \in \mathbb{N}$) is even hyperstrict.

(b) If, in (a), each E_n is closed in (E_{n+1},τ_{n+1}), then E_n is also closed in (E,τ) for $n = 1,2,\ldots$ In this situation, the inductive sequence $(E_n)_n$ is always regular, and (E,τ) is (quasi-) complete if and only if all the spaces (E_n,τ_n) are (quasi-) complete.

In particular, a countable strict inductive limit $E = \operatorname{ind}_{n\to} E_n$ of complete l.c. spaces is always hyperstrict, regular and complete.

4. *Example.* For a fixed nonnegative integer m and Ω open in $\mathbb{R}^N$, $\mathcal{D}^m(\Omega)$ denotes the space of all those continuous functions f on Ω whose partial derivatives $\partial^\alpha f$ exist and are continuous on Ω for each multi-index $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}_0^N$ of order $|\alpha| = \alpha_1 + \dots + \alpha_N \leq m$ and whose support

$$\operatorname{supp} f = \overline{\{x \in \Omega;\ \ f(x) \neq 0\}}$$

is a compact subset of Ω. $\mathcal{D}^0(\Omega)$ is just the space of all continuous functions on Ω with compact supports.

Now fix a compact set $K \subset \Omega$ and let $\mathcal{D}_K^m$ denote the subspace of $\mathcal{D}^m(\Omega)$ of all the functions f with $\operatorname{supp} f \subset K$. On $\mathcal{D}_K^m$, the uniform topology of order m; i.e., the topology of uniform convergence on Ω of the functions and of all their partial derivatives of order $\leq m$, and the compact-open topology of order m; i.e. the topology of uniform convergence on compact subsets of Ω for functions and partial derivatives up to order m, coincide and turn $\mathcal{D}_K^m$ into a Banach space with the norm

$$||f||_m = \sup_{0 \leq |\alpha| \leq m} \ \sup_{x \in \Omega} |\partial^\alpha f(x)| \quad .$$

By definition, we have $\mathcal{D}^m(\Omega) = \bigcup_{K \in \mathcal{K}} \mathcal{D}_K^m$, where $\mathcal{K} = \mathcal{K}(\Omega)$ denotes the system of all compact subsets of Ω (directed with respect to inclusion). It is natural to topologize $\mathcal{D}^m(\Omega)$ as the l.c. inductive limit of the spaces $\mathcal{D}_K^m$, $K \in \mathcal{K}$.

We have pointed out in Section 0. that it is important to know if there is an equivalent countable inductive system. Putting

$$K_n := \{x \in \mathbb{R}^N;\ \|x\| \le n \ \text{ and } \ d(x, \mathbb{R}^N \setminus \Omega) \ge \tfrac{1}{n}\},\ n = 1,2,\ldots,$$

where $\|\cdot\|$ denotes the Euclidean norm and d the Euclidean distance, we see that K_n is contained in the interior $\mathring{K}_{n+1}$ of K_{n+1} for each $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} \mathring{K}_n$. Hence each compact subset K of Ω is contained in some K_n, and $(K_n)_{n\in\mathbb{N}}$ is a countable basis for $\mathcal{K}$. It is now clear that

$$\mathcal{D}^m(\Omega) = \operatorname*{ind}_{n\to} \mathcal{D}^m_{K_n} \qquad ;$$

i.e., $\mathcal{D}^m(\Omega)$ is an (LB)-space. But $\mathcal{D}^m(\Omega)$ clearly is also a strict inductive limit, and so Theorem 3. tells us that $\mathcal{D}^m(\Omega)$ is hyperstrict (which could easily have been seen directly), regular and complete. In particular, the regularity, together with the hyperstrictness, implies that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}^m(\Omega)$ converges to 0 if and only if there exists a compact set $K \subset \Omega$ with $\bigcup_{n\in\mathbb{N}} \operatorname{supp} f_n \subset K$ and $\partial^\alpha f_n \to 0$ uniformly on K for arbitrary $|\alpha| \le m$. (Note that uniform convergence $\partial^\alpha f \to 0$ for arbitrary $|\alpha| \le m$ is only sufficient to imply $f_n \to 0$ in $\mathcal{D}^m(\Omega)$ if the union of the supports of all f_n is contained in a fixed compact set $K \subset \Omega$.)

One can explicitly determine a basis for the continuous seminorms of the inductive limit topology on $\mathcal{D}^m(\Omega)$, but, since the development is completely analogous, we will only note this in the "limiting cases" $\mathcal{D}(\Omega)$ and $C_c(X)$ below.

5. *Example*. **With Ω as in Example 4., $\mathcal{D}(\Omega)$ denotes the space of all C^∞-functions on Ω with compact supports; i.e., $\mathcal{D}(\Omega) = \bigcap_{m\in\mathbb{N}_0} \mathcal{D}^m(\Omega)$.**

$\mathcal{D}(\mathbb{R}^N)$ will be abbreviated as $\mathcal{D}$. To topologize $\mathcal{D}(\Omega)$, we put, as before

$$\mathcal{D}_K := \{f \in \mathcal{D}(\Omega);\ \ \text{supp } f \subset K\}$$

for arbitrary compact $K \subset \Omega$; that is, $\mathcal{D}_K = \bigcap_{m \in \mathbb{N}_0} \mathcal{D}_K^m$, and $\mathcal{D}_K$ is endowed with the corresponding projective topology, defined by the sequence $(\|\cdot\|_m)_{m \in \mathbb{N}_0}$ of norms. This is the restriction of both the "uniform topology of order ∞" and the "compact-open topology of order ∞", and $\mathcal{D}_K$ is a Fréchet space. Taking $\mathcal{D}(\Omega)$ to be the inductive limit $\operatorname{ind}_{\mathcal{K} \ni K \to} \mathcal{D}_K$, $\mathcal{K} = \mathcal{K}(\Omega)$, or, equivalently, $\operatorname{ind}_{n \to} \mathcal{D}_{K_n}$ for a fundamental sequence of compact subsets of Ω (cf. Example 4.), it is easily verified that $\mathcal{D}(\Omega)$ always is a countable strict inductive limit, hence hyperstrict, regular, and a complete (LF)-space in view of Theorem 3. T. Shirai [56] showed that $\mathcal{D}(\Omega)$ does not carry the finest topology which makes all the injections $\mathcal{D}_{K_n} \to \mathcal{D}(\Omega)$ continuous (cf. our discussion at the end of Section 0.).

In order to give an explicit description of the continuous seminorms for $\mathcal{D}(\Omega)$, we first recall the following definition from general topology: A family $\mathcal{A}$ of subsets of a topological space X is called locally finite if for every $x \in X$, there is a neighborhood V of x such that $V \cap A = \emptyset$ except for finitely many $A \in \mathcal{A}$. If $\mathcal{A}$ is locally finite, then each compact subset of X meets only finitely many $A \in \mathcal{A}$.

Next, consider a family $\mathbf{V} = (v_\alpha)_{\alpha \in \mathbb{N}_0^N}$ of nonnegative continuous functions v_α on Ω such that $(\text{supp } v_\alpha)_\alpha$ is locally finite. Then

$$p_{\mathbf{V}}(f) := \sup_{\alpha \in \mathbb{N}_0^N} \sup_{x \in \Omega} v_\alpha(x) |\partial^\alpha f(x)| \qquad \text{for} \qquad f \in \mathcal{D}(\Omega)$$

denotes a well-defined (continuous) seminorm on $\mathcal{D}(\Omega)$.

6. *Proposition.* The system $(p_{\mathbb{V}})_{\mathbb{V}}$ of seminorms, where $\mathbb{V}$ runs through all possible families $\mathbb{V} = (v_\alpha)_{\alpha \in \mathbb{N}_0^N}$ of nonnegative continuous functions on Ω with $(\text{supp } v_\alpha)_\alpha$ locally finite, induces the (inductive limit) topology of $\mathcal{D}(\Omega)$.

The proof uses an argument involving C^∞-partitions of unity and can be found in Horváth's book [35].

Even though the treatment of $\mathcal{D}^m(\Omega)$ $(m \in \mathbb{N}_0)$ and $\mathcal{D}(\Omega)$ is quite similar, there are some essential differences between these spaces – mainly reflected in the structural fact that $\mathcal{D}^m(\Omega)$ is an (LB)-, but $\mathcal{D}(\Omega)$ "only" an (LF)-space.

$\mathcal{D} = \mathcal{D}(\mathbb{R}^N)$ is the famous "test function space" of Laurent Schwartz; its dual is exactly the space of all distributions. Thus, $\mathcal{D}$ is one of the most prominent spaces in analysis!

We finally turn to the treatment of a "more general version" of $\mathcal{D}^0(\Omega)$.

7. *Example.* Let X denote a locally compact space and $C_c(X)$ the space of all real or complex-valued functions f on X with $\text{supp } f = \overline{\{x \in X;\ f(x) \neq 0\}}$ compact. The definition of the inductive limit topology on $C_c(X)$ follows the "usual" pattern: For any compact $K \subset X$, $C_K := \{f \in C_c(X);\ \text{supp } f \subset K\}$ is endowed with the topology of uniform convergence on X (or K) so that C_K becomes a Banach space, and $C_c(X)$ is topologized as $\operatorname{ind}_{\mathcal{K} \ni K \to} C_K$, where $\mathcal{K} = \mathcal{K}(X)$ denotes the system of all compact sets in X. Again, it is obvious that $C_c(X)$ induces on each C_K the canonical topology of that space, and hence the inductive limit is always hyperstrict. However, $C_c(X)$ is (equivalent to) a countable inductive limit if and only if X has a countable basis for $\mathcal{K}$ which, since X is assumed to be locally compact, amounts to σ-compactness (or "countability at infinity"); i.e., X is a countable union of compact sets.

If X is not σ-compact (and not even paracompact), then various kinds of "pathologies" may occur: For any uncountable discrete space

X, the topology of $C_c(X)$ is strictly weaker than the finest linear topology making all injections $C_K \to C_c(X)$ continuous (see Köhn [37]). And the following delicate results were obtained by A. Douady [19] long ago:

8. *Proposition.* (a) Let Y denote a non-compact σ-compact locally compact space and βY its Stone-Čech compactification. If ω belongs to $\beta Y \setminus Y$ and $X := \beta Y \setminus \{\omega\}$, then the inductive limit topology of $C_c(X)$ coincides with the topology of uniform convergence on X.

(b) There is a σ-compact locally compact space Y such that, if X is defined as in (a) for a suitable point $\omega \in \beta Y \setminus Y$, then there exists (even) a convex compact subset A of $C_c(X)$ which is not contained in any space C_K, $K \in \mathcal{K}$. Hence regularity fails badly for this space $C_c(X)$.

(c) On the other hand, if X is a separable locally compact space, then each convex compact subset of $C_c(X)$ is contained in some C_K. Similarly, if X is any locally compact space, then each convex compact subset of $C_c(X)$ which only contains nonnegative functions must be contained in some C_K.

(d) Now take $Y := \mathbb{N}$ and construct X as in (a). Then the first part of (c) tells us that each convex compact subset of $C_c(X)$ is contained in C_K for some compact set K in X. But there exist compact sets A with $A \not\subset C_K$ for each compact subset K of X. Hence the closed convex hull of any such A cannot be compact which implies that $C_c(X)$ is not quasicomplete. (In fact, one can prove that $C_c(X)$ is not even sequentially complete.)

Theorem 3. applies if X is σ-compact. But, using this result and the characterization of paracompact locally compact spaces as the direct topological sums of σ-compact locally compact spaces (e.g., see Dugundji [20]), it is easy to show the following.

9. *Proposition.* Whenever the locally compact space X is paracompact, $C_c(X)$ is complete and a regular inductive limit.

We now turn to the task of explicitly describing a basis for the continuous seminorms on $C_c(X)$, at least for σ-compact X. Since this prepares our discussion of "projective descriptions for weighted inductive limits" later on, we will go into some more detail here. For the next part, let X denote an arbitrary completely regular Hausdorff space.

10. Definition. $C(X)$ denotes the space of all continuous real- or complex-valued functions on X, and $C^+(X)$ describes the subset of all $f \in C(X)$ with $f \geq 0$ which, for the present purpose, we will abbreviate by V. We put

$$CV(X) := \{f \in C(X);\ \text{for each } v \in V,\ p_v(f) := \sup_{x \in X} v(x)|f(x)| < \infty\},$$

endowed with the ("weighted") complete l.c. topology given by the system $(p_v)_{v \in V}$ of seminorms, and

$$CV_0(X) := \{f \in C(X);\ \text{for each } v \in V,\ vf \text{ vanishes at infinity (on } X\text{); i.e. for each } \varepsilon > 0, \text{ there is a compact } K \subset X \text{ with } v(x)|f(x)| < \varepsilon \text{ for all } x \in X \setminus K\},$$

a closed linear subspace of $CV(X)$, equipped with the induced weighted topology.

It is easy to see that, for any locally compact space X, $C_c(X)$ is continuously embedded and dense in $CV_0(X)$.

11. Proposition. Algebraically, $CV(X)$ equals the space $C_b(X)$ of all those $f \in C(X)$ for which $\operatorname{supp} f$ is a bounding subset of X (i.e., each $g \in C(X)$ is bounded on $\operatorname{supp} f$). Similarly, $CV_0(X)$ equals the intersection of $C_b(X)$ with the space $C_0(X)$ of all the continuous functions on X which vanish at infinity.

It follows from a theorem of Hager and Johnson (e.g., see Buchwalter [15]) that, for $f \in C(X)$, $\operatorname{supp} f$ is bounding if and only if it is pseudocompact; i.e., even each $g \in C(\operatorname{supp} f)$ must be bounded.

Proposition 11., as well as some of the material below, is taken from Summers [57]; also see [2]. Using this result, Summers deduced part (a) of the following:

12. Proposition. (a) If $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and $X = \beta\mathbb{N} \setminus \{\omega\}$, then X is a non-compact extremally disconnected pseudocompact locally compact space for which $C_c(X)$ is a proper subspace of $CV_o(X) = C_o(X)$.

On the other hand, if X is locally compact and if every σ-compact subset of X is contained in an open and closed σ-compact subset of X (in particular, if X is locally compact and paracompact), then $C_c(X)$ algebraically equals $CV_o(X)$.

(b) The algebraic equality $C_c(X) = CV(X)$ holds if and only if, for arbitrary $f \in C(X)$, supp f pseudocompact already implies supp f compact. Completely regular spaces with this property were labelled "Ψ-compact" by Mandelker [40], and, in particular, each paracompact space is Ψ-compact.

Also note that, trivially, $CV_o(X)$ is a proper subspace of $CV(X)$ for each non-compact pseudocompact X.

Returning to the inductive limit topology of $C_c(X)$, Summers showed that, for an uncountable discrete space X, $C_c(X) = CV_o(X) = CV(X)$ holds algebraically, but the inductive limit topology is strictly stronger than the weighted topology, and the two topologies even yield different duals. Thus X locally compact and paracompact does not imply $C_c(X) = CV_o(X)$ as topological spaces.

On the other hand, let X denote a non-compact locally compact and pseudocompact space. Then $CV(X)$ resp. $CV_o(X)$ is just the Banach space $C(X)$ resp. $C_o(X)$ with the topology of uniform convergence on X (and $CV_o(X) \neq CV(X)$). Hence, in this situation, $C_c(X) = CV_o(X)$ holds algebraically and topologically if and only if (i) $C_c(X) = C_o(X)$ as sets and (ii) the inductive limit topology coincides with the topology of uniform convergence on X.

By another result of Summers, (i) and (ii) are true e.g. for $X =$ the space of all ordinals less than the first uncountable ordinal (with the order topology). There are actually cases where (ii) holds, but (i) fails: E.g., if $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ and $X = \beta\mathbb{N}\setminus\{\omega\}$, we can apply both 8.(a) and 12.(a).

It remains open to find necessary and sufficient conditions for the topological vector space equality $C_c(X) = CV_0(X)$ (or for the property that $C_c(X)$ is a topological subspace of $CV_0(X)$). But, using a similar partition of unity argument (originally due to L. Schwartz [54]) as in the proof of 6. (e.g., see [2]), one can prove:

13. Proposition. If X is locally compact and σ-compact, then

$$C_c(X) = CV_0(X) = CV(X) \text{ algebraically and topologically;}$$

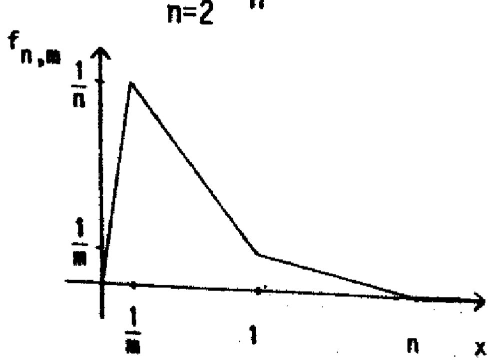
in particular, the inductive limit topology of $C_c(X)$ is given by the system $(p_v)_{v \in C^+(X)}$ of weighted sup-seminorms.

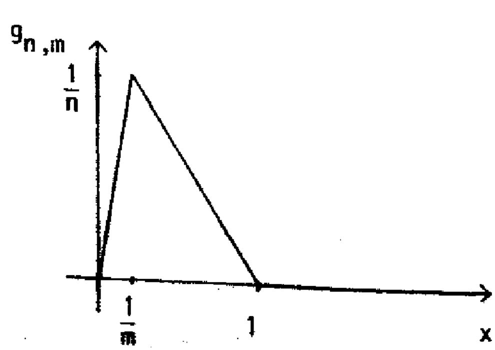
At this point, and in view of what was said in Section 0., we feel that it would be a pity not to include the following nice proof, due to G. A. Edgar [21], of:

14. Proposition. The l.c. inductive limit topology τ of $C_c(\mathbb{R})$ is strictly weaker than the finest topology τ' which makes all injections $C_{[-n,n]} \to C_c(\mathbb{R})$ continuous, $n = 1,2,\ldots$

<u>Proof</u>. A set $F \subset C_c(\mathbb{R})$ is τ'-closed if and only if $F \cap C_{[-n,n]}$ is uniformly closed for each $n \in \mathbb{N}$. We will exhibit a set F with the last property which cannot be τ-closed.

To do this, let $f_{n,m} \in C_c(\mathbb{R})$ $(n,m = 2,3,\ldots)$ denote the piecewise linear function with $\operatorname{supp} f_{n,m} = [0,n]$ whose graph contains $(\frac{1}{m}, \frac{1}{n})$ and $(1, \frac{1}{m})$, and put $F_n := \{f_{n,m};\ m = 2,3,\ldots\} \subset C_{[-n,n]}$ as well as $F := \bigcup_{n=2}^{\infty} F_n$.





To show that $F \cap C_{[-n,n]} = \bigcup_{k=2}^{n} F_k$ is closed in $C_{[-n,n]}$ for each n, it obviously suffices to prove that F_n is uniformly closed in $C_{[-n,n]}$, $n = 2,3,\ldots$ If not, we could find a sequence $(g_i)_{i \in \mathbb{N}}$ of pairwise distinct elements in F_n which converges uniformly on $[-n,n]$. By the Arzelà - Ascoli theorem, $(g_i)_i$ would then be equicontinuous on $[-n,n]$. But we have $(g_i)_i = (f_{n,m_i})_i$ for a sequence $(m_i)_{i \in \mathbb{N}}$ of pairwise distinct natural numbers, and such a sequence certainly fails to be equicontinuous at 0.

It remains to prove that F is not τ-closed, and, in fact, we will verify $0 \in \bar{F}^{\tau}$. Let U denote an arbitrary τ-neighborhood of 0; since τ is locally convex, it contains a convex τ-neighborhood V of 0. Now, by definition of the inductive limit topology, $V \cap C_{[-n,n]}$ is a neighborhood of 0 in $C_{[-n,n]}$ for each $n \in \mathbb{N}$, and hence we can find $m(n) \in \mathbb{N}$, $m(n) \geq 2$, such that

$$V \cap C_{[-n,n]} \supset \{ f \in C_{[-n,n]} \ ; \ \sup_{x \in \mathbb{R}} |f(x)| \leq \frac{2}{m(n)} \} .$$

We fix $m_1 := m(1)$.

Finally, we let $g_{n,m} \in C_c(\mathbb{R})$ denote the piecewise linear function with $\operatorname{supp} g_{n,m} = [0,1]$ whose graph contains the point $(\frac{1}{m}, \frac{1}{n})$, $n,m = 2,3,\ldots$ Then obviously

$$2\, g_{m_1,m(m_1)} \in V \cap C_{[-1,1]}$$

as well as

$$\sup_{x \in \mathbb{R}} |f_{m_1,m(m_1)} - g_{m_1,m(m_1)}| \leq \frac{1}{m(m_1)} ,$$

and hence

$$2(f_{m_1,m(m_1)} - g_{m_1,m(m_1)}) \in V \cap C_{[-m_1,m_1]} .$$

By convexity of V, we can conclude $f_{m_1,m(m_1)} \in F \cap V \subset F \cap U$, as desired. (Incidentally, the characterization of the continuous seminorms for the inductive limit topology of $C_c(\mathbb{R})$ given in Proposition 13. provides another, very easy method of verifying that $0 \in \bar{F}^{\tau} \ldots$)□

We remark that the importance of the space $C_c(X)$ derives from the fact that Radon measures on the locally compact space X are defined as the continuous linear functionals on $C_c(X)$ (see N. Bourbaki [14]); any information on $C_c(X)$ (and its topology) immediately also provides information on the Radon measures on X.

If X is only a completely regular Hausdorff space, but not locally compact, it can happen that $C_c(X)$ is reduced to $\{0\}$. Hence, if one wants to define measures on such a space X as continuous linear functionals on a space of continuous functions on X, then a larger space must be considered; viz., the space $CB(X)$ of all bounded continuous functions on X, endowed with topologies weaker than uniform convergence on X, but stronger than the compact-open topology co. In topological measure theory (e.g., see Fremlin-Garling-Haydon [29], Sentilles [55] or R. F. Wheeler [60]), various such "strict" topologies $(\beta_o, \beta, \beta_1, \beta_e)$ are introduced which yield different types of measures (tight, τ-additive, σ-additive) on X. These strict topologies can also be defined as l.c. inductive limit topologies, but of a much more complicated nature. For the topological vector space properties of strict topologies (and, above all, the (gDF)-property), we also refer to Ruess [51].

Unfortunately, several textbooks on topological vector spaces only discuss the strict inductive limits explicitly. In some cases, it really takes "an expert" to find out that certain other types of inductive limits are also treated (but rather implicitly) among the usual "wealth" of material on special types of abstract l.c. spaces. And yet, at least for weakly compact, compact, and nuclear inductive limits, it is possible to develop a "general theory" with interesting theorems and many applications, and we turn to survey this development in the next section. (For an introduction, and as a good text in view of many interesting applications, we recommend Floret-Wloka [28], and Wloka [61].)

2. Weakly compact, compact and nuclear inductive limits

Before we start defining the special types of l.c. inductive spectra mentioned in the title of this section, we shortly review some basic facts on the duality of inductive and projective limits; e.g., see Floret-Wloka [28] or Köthe [38].

For any inductive system $(E_\alpha)_{\alpha \in A}$ of l.c. spaces, $(E'_\alpha)_{\alpha \in A}$ is a projective system, and we have

$$(\underset{\alpha \rightarrow}{\text{ind}}\ E_\alpha)' = \underset{\leftarrow \alpha}{\text{proj}}\ E'_\alpha \quad \text{algebraically.}$$

The corresponding duality on the side of projective limits requires to pass to reduced projective spectra: A projective system $(E_i)_{i \in I}$ with canonical mappings $(\pi_i)_i$, or its limit $E = \underset{\leftarrow i}{\text{proj}}\ E_i$, is called <u>reduced</u> if $\pi_i(E)$ is dense in E_i for each $i \in I$. In this case, all the transposed mappings ${}^t\pi_i : E'_i \rightarrow E'$ are injective, and $(E'_i)_{i \in I}$ is an injective inductive system. If $(E_i)_{i \in I}$ denotes a reduced projective spectrum, we obtain

$$(\underset{\leftarrow i}{\text{proj}}\ E_i)' = \underset{i \rightarrow}{\text{ind}}\ E'_i \quad \text{algebraically.}$$

Considering topological dualities, the situation (say, for the strong topologies) is more complicated (see the discussion of Köthe [38] and part of our discussion in Section 3.), but we do get the following:

1. Proposition. If $(E_\alpha)_{\alpha \in A}$ is a regular inductive system of l.c. spaces, then

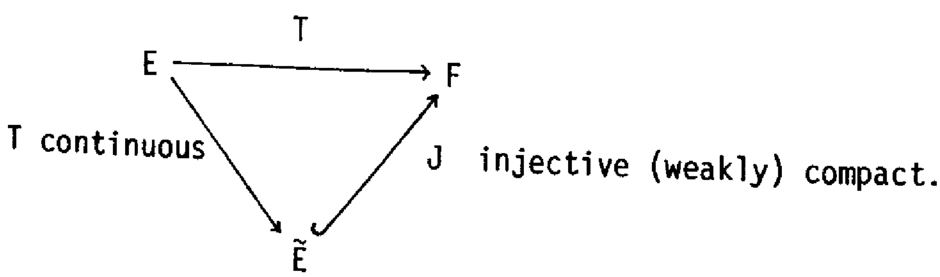
$$(\underset{\alpha \rightarrow}{\text{ind}}\ E_\alpha)'_b = \underset{\leftarrow \alpha}{\text{proj}}\ (E_\alpha)'_b$$

holds algebraically and topologically.

We will now see that the topological dualities of inductive and projective limits work fine for the corresponding strong topologies whenever we are dealing with countable inductive limits that are at least weakly compact.

2. *Definition.* A l.c. inductive system $(E_\alpha)_{\alpha \in A}$ or its limit $E = \operatorname{ind}_{\alpha \to} E_\alpha$ is called weakly compact, compact or nuclear, respectively, if, for each $\alpha \in A$, there exists $\beta \geq \alpha$ such that the canonical injection $i_{\alpha\beta}: E_\alpha \to E_\beta$ is a weakly compact, compact or nuclear operator, respectively.

At this point, factorizations of linear mappings enter the picture for the first time: If E, F denote l.c. spaces and $T: E \to F$ is a (weakly) compact linear mapping (i.e., there is some neighborhood U of 0 in E which T maps onto a relatively (weakly) compact subset of E), then there exists a Banach space $\tilde{E}$, $TE \subset \tilde{E} \subset F$, with canonical injection $J: \tilde{E} \to F$ continuous, such that T factors through $\tilde{E}$ as follows:



(In fact, it suffices to choose an absolutely convex (weakly) compact subset B of F containing the image of some (absolutely convex) neighborhood U of 0 in E and to take $\tilde{E} := F_B$ = the linear span of B in F, endowed with the Minkowski- (or gauge) functional of B.)

It follows that, for any (weakly) compact (injective) inductive system $(E_\alpha)_{\alpha \in A}$, there always exists an equivalent (weakly) compact (injective) system $(\tilde{E}_\alpha)_{\alpha \in A}$ of Banach spaces; i.e.

$$\operatorname{ind}_{\alpha \to} E_\alpha = \operatorname{ind}_{\alpha \to} \tilde{E}_\alpha$$

(also note that $(\tilde{E}_\alpha)_\alpha$ is indexed by the same set as $(E_\alpha)_\alpha$).

Hence, from now on, we will suppose without loss of generality that

(*) all the spaces E_α in any (weakly) compact or nuclear inductive system $(E_\alpha)_\alpha$ are Banach.

Indeed, most (weakly) compact inductive limits have a natural "representation" of this type right from the start. Furthermore, we remark that each (weakly) compact inductive limit must be <u>ultrabornological</u> (that is, an inductive limit of Banach spaces), and a countable (weakly) compact inductive limit is even an (LB)-space (which was not true in the case of strict inductive limits).

Next, it is well-known that weakly compact linear mappings always factor through reflexive Banach spaces, compact operators through separable reflexive Banach spaces, and nuclear ones even through separable Hilbert spaces. Thus, we could as well assume much more than (*) without loss of generality. However, in many interesting applications, compact or nuclear sequences of non-reflexive Banach spaces arise, and it may not be trivial to find a "natural" equivalent sequence of reflexive Banach or separable Hilbert spaces. Hence we prefer to take (*) as our only general assumption and not to restrict the generality any further.

Factorization theorems for continuous linear mappings between Banach spaces apply to the study of inductive limits in many other ways, and the theory of operator ideals has interesting consequences for inductive and projective limits. We only mention one such result which will be needed in the discussion of an example below (cf. Pietsch [47]):

Since each nuclear map is absolutely summing while the composition of two absolutely summing operators is nuclear, an inductive system $(E_\alpha)_\alpha$ of Banach spaces is nuclear if and only if, for each $\alpha \in A$, there exists $\beta > \alpha$ such that $i_{\alpha\beta}: E_\alpha \to E_\beta$ is absolutely summing.

(We recall that a $\sigma(E',E)$-compact subset M of the dual unit ball E'_1 of a normed space E is said to be <u>norm-determining</u> (or <u>essential</u>) if

$$\|x\|_E = \sup_{x' \in M} |x'(x)| \qquad \text{for every} \quad x \in E.$$

E.g., if $C(K)$ denotes the Banach space of all scalar continuous functions on some compact space K, and, for each $x \in K$, δ_x denotes point evaluation at x, then $M = \{\delta_x ; x \in K\}$ is norm-determining for any normed subspace of $C(K)$.

A linear mapping T from E into another normed space F is called absolutely summing if and only if there exists a positive Radon measure μ on $(M, \sigma(E',E))$ such that

$$\|Tx\|_F \le \int_M |x'(x)| \, d\mu(x') \qquad \text{for all} \quad x \in E.$$

E.g., a linear operator T on a normed subspace E of some $C(K)$ into a normed space F is absolutely summing if and only if there is a positive Radon measure μ on K with

$$\|Tf\|_F \le \int_K |f(x)| \, d\mu(x) \qquad \text{for all functions} \quad f \in E.)$$

We return to Definition 2. and demonstrate that, as in the case of strict inductive limits, this definition is not very helpful for uncountable inductive limits. Indeed, as Theorem 3. shows, uncountable (weakly) compact or nuclear inductive limits are nearly useless at all! The theorem has a long history, and many people (e.g., Raikov, Hogbe-Nlend, Jarchow) have contributed to weaker forms of the result. M. Valdivia (cf. [27]) finally proved the present strong version.

3. *Proposition.* Let E denote an arbitrary ultrabornological space (that is, a l.c. space which is just the limit of some inductive spectrum of Banach spaces).

(a) There exists a nuclear inductive system $(E_\alpha)_\alpha$ of (separable) Hilbert spaces with $E = \operatorname{ind}_{\alpha \to} E_\alpha$.

(b) More generally, if F is any separable infinite dimensional Banach space, there also exists a nuclear net $(E_\alpha)_\alpha$ such that $E = \underset{\alpha\to}{\text{ind}}\, E_\alpha$ and all the spaces E_α are isometric to F.

Valdivia also gave rather general conditions on l.c. spaces F which imply that each Banach space E can be represented as the limit of an inductive spectrum $(E_\alpha)_\alpha$ of spaces E_α which are all topologically isomorphic to F. E.g., F can here be taken to be an arbitrary Fréchet-Schwartz space with a continuous norm (like $\mathcal{D}_K$) or a strong dual of such a space.

However, again just as in the case of strict inductive limits, countability of the inductive system changes the picture completely, and the good properties of (weakly) compact and nuclear inductive sequences justify Definition 2.

Motivated by applications to spaces of holomorphic functions and analytic functionals, Sebastião e Silva was the first to introduce compact inductive sequences; they were then studied systematically by Raikov and some other Russian mathematicians. Later on, Makarov and Komatsu [39] noticed that the larger class of weakly compact inductive sequences still shares several of the good properties of compact inductive sequences. (Some of the material was already implicit in Grothendieck's work.) For a good survey and complete proofs, we refer to Floret [24].

While we concentrate on inductive limits in this report, it should be pointed out that (weakly) compact and nuclear projective sequences are equally important. In fact, the proof of Theorem 4. below makes use of the dual projective sequences as well. Note that a linear mapping T between Banach spaces E and F is (weakly) compact if and only if its transpose ${}^tT$ enjoys the same property (and that the transpose of a nuclear operator is nuclear as well); moreover, if $T: E \to F$ is weakly compact, then ${}^{tt}T(E'') \subset F$.

4. Theorem. (a) A weakly compact (injective) inductive sequence $(E_n)_{n\in\mathbb{N}}$ is regular, and its limit $E = \underset{n\to}{\text{ind}}\, E_n$ is a complete reflexive

(LB)-space. The dual projective sequence $((E_n)'_b)_n$ is again weakly compact, its limit is a reflexive Fréchet space, and we have the duality

$$E'_b = (\operatorname{ind}_{n\rightarrow} E_n)'_b = \operatorname{proj}_{\leftarrow n} (E_n)'_b;$$

E is the strong dual of this reflexive Fréchet space.

(b) In addition to the properties of (a), a countable compact (injective) inductive limit $E = \operatorname{ind}_{n\rightarrow} E_n$ is a separable Schwartz space (and hence Montel); all bounded subsets of E are metrizable. The strong dual E'_b is a Fréchet-Schwartz space (abbreviated: "(FS)-space"), and E is the strong dual of this (FS)-space.

Moreover, in this case, E does actually carry the finest topology which makes all injections $i_n: E_n \rightarrow E$ continuous, and hence a subset $A \subset E$ is closed if and only if $A \cap E_n$ is closed in E_n for each $n \in \mathbb{N}$.

(c) A countable nuclear (injective) inductive limit E is a nuclear (and even s-nuclear) space, the strong dual of the nuclear Fréchet space (abbreviated: "(FN)-space") E'_b.

For obvious reasons, countable compact inductive limits are termed (DFS)-spaces, (LS)-spaces or Silva spaces; the limits of Theorem 4.(c) are called (DFN)- or (LN)-spaces. Similarly, one finds the terms (DFS*) or (LS_w) for countable weakly compact inductive limits in parts of the literature (and limits of weakly compact projective sequences are called $(F\bar{S}_w)$ in Floret [24]).

Note that Theorem 4.(a) applies to arbitrary (injective) inductive sequences of reflexive Banach spaces. And while there is some overlap between 4.(a) above and Theorem 1.3, an inductive sequence will never be both strict and compact – except for trivial cases! Part of the value of (DFS)-spaces comes from the fact mentioned at the end of 4.(b), and this does not hold for countable strict inductive limits, even in the most interesting cases (see the remark concerning Example 1.5 and Proposition 1.14).

5. *Example.* In the sequel, we fix a non-void compact subset K of $\mathbb{C}^N$ $(N \geq 1)$. By $H(K)$ we denote the space of all "germs of holomorphic functions" on K.

To describe this space, we first consider all complex-valued functions f which are defined and holomorphic on some open set $U = U_f$ containing K (i.e., the domain depends on the function f). Next, we identify any two such functions f and g if they coincide on some open set containing K. Hence germs of holomorphic functions on K are equivalence classes of functions "around K" (modulo the equivalence relation which we just described). And $H(K)$ is the linear space – it is even an algebra! – of all those equivalence classes under the canonically defined algebraic operations.

(One gets used to treat holomorphic germs as if they were holomorphic functions "around the compact set". If K is "big enough" so that any holomorphic function which "lives" on a "reasonably small" open set $U \supset\supset K$ and vanishes on K must be identically 0 on all of U [by the identity theorem for holomorphic functions], we may even think of the elements of $H(K)$ just as functions on K. But if K is e.g. reduced to a single point, it is impossible to consider holomorphic germs as being defined on K alone. In that case, one must take open sets $U \supset\supset K$ into account; that is why we speak of functions "around K".)

After making the (by now) canonical identifications, we certainly obtain

$$H(K) = \bigcup_{U \supset\supset K} H(U) = \bigcup_{U \supset\supset K} H^\infty(U) = \bigcup_{U \supset\supset K} A(U),$$

where the union is taken over all open sets U containing K or, which is certainly enough, only over all those open sets U for which each connected component meets K, and where $H(U)$ denotes the space of all holomorphic functions on U, $H^\infty(U)$ the space of all the bounded holomorphic functions on U and

$$A(U) := \{f \in C(\bar{U});\ f_{|U} \text{ is holomorphic}\}.$$

From now on, we will only consider open sets U such that each connected component V of U satisfies $V \cap K \neq \emptyset$. Then, by the definition of $H(K)$ and by the identity theorem for holomorphic functions, the canonical mappings $H(U) \to H(K)$ are all one to one.

Each $H(U)$ is canonically endowed with the compact-open topology co, and $H^\infty(U)$ as well as $A(U)$ is a Banach space under the sup-norm over U resp. $\bar{U}$. Since we have

$$A(U) \to H^\infty(U) \to (H(U),\text{co})$$

with continuous injections and since each open $U \supset\supset K$ contains an open set V with $K \subset\subset V \subset \bar{V} \subset\subset U$ (so that the restriction mapping $(H(U),\text{co}) \to A(V)$ is well-defined and continuous), the inductive spectra $(H(U),\text{co})_U$, $(H^\infty(U))_U$ and $(A(U))_U$ are equivalent, and we define the natural topology of $H(K)$ by putting

$$H(K) := \operatorname*{ind}_{U \to} (H(U),\text{co}) = \operatorname*{ind}_{U \to} H^\infty(U) = \operatorname*{ind}_{U \to} A(U).$$

Moreover, K has a countable decreasing basis $(U_n)_{n \in \mathbb{N}}$ of open neighborhoods (satisfying $U_n \supset\supset \overline{U_{n+1}}$ for each n), and hence

$$H(K) := \operatorname*{ind}_{n \to} (H(U_n),\text{co}) = \operatorname*{ind}_{n \to} H^\infty(U_n) = \operatorname*{ind}_{n \to} A(U_n)$$

is a countable injective inductive limit of Banach spaces.

Finally, fix $n \in \mathbb{N}$. Then Montel's theorem clearly implies that even the identity mapping $H^\infty(U_n) \to (H(U_n),\text{co})$ is compact. But we can actually do better: Since $U_n \supset\supset \overline{U_{n+1}}$, an easy estimate (say, involving only the Cauchy integral formula for polydisks) yields

$$\|f\|_{\overline{U_{n+1}}} = \sup_{z\in\overline{U_{n+1}}} |f(z)| \le C_n \int_{U_n} |f(w)|d\mu_N(w)$$

$$\le C_n \int_{\overline{U_n}} |f(w)|d\mu_N(w)$$

for arbitrary $f \in A(U_n)$, where μ_N denotes Lebesgue measure on $\mathbb{C}^N = \mathbb{R}^{2N}$ and $C_n > 0$ is a constant (depending only on the relative size of $\overline{U_{n+1}}$ in U_n). Hence the restriction mapping $A(U_n) \to A(U_{n+1})$ is absolutely summing.

At this point, we know that $H(K)$ always is a countable nuclear inductive limit; that is, a (DFN)-space, the strong dual of the nuclear Fréchet space $H(K)_b'$ (in the terminology introduced after Theorem 4.).

(For $N = 1$, the so-called "Köthe duality" explicitly describes the strong dual $H(K)_b'$ of $H(K)$ as the space $H_0(\Omega)$ of all those holomorphic functions on the (open) complement Ω of K in the Riemann sphere which vanish at the point ∞, endowed with the compact-open topology. This duality has a natural interpretation, see Köthe [38]. We note that things are much more complicated for $N > 1$.)

For general compact set $K \subset \mathbb{C}^N$, an explicit description of a basis for the continuous seminorms of the inductive limit topology on $H(K)$ was achieved only quite recently by J. Mujica [43], and it is rather involved.

In order to state Mujica's theorem, we introduce the following notation: If f is holomorphic on an open set $U \subset \mathbb{C}^N$ and $x \in U$, let $P^n f(x)$ denote the n-th homogeneous polynomial in the Taylor expansion of f at x. (Then $P^n f(x)$ is a mapping from $\mathbb{C}^N$ into $\mathbb{C}$.) For $X \subset U$ and $A \subset \mathbb{C}^N$, we let

$$\|P^n f\|_{X,A} := \sup_{a\in A} \sup_{x\in X} |P^n f(x)(a)|.$$

6. *Proposition.* (a) The inductive limit topology of $H(K)$ is generated jointly by the seminorms of the following types (*) and (**):

$$(*)\quad p(f) = \sum_{n=0}^{\infty} (\varepsilon_n)^n \|P^n f\|_{K,L},\ f \in H(K),$$

where L varies among all compact subsets of $\mathbb{C}^N$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ varies among all sequences of nonnegative numbers decreasing to 0;

$$(**)\quad q(f) = \sup_{k\in\mathbb{N}} \sup_{1\le n\le n_k} 2^n \Big| \sum_{m=0}^{n} P^m f(x_k)(a_k) - \sum_{m=0}^{n} P^m f(y_k)(b_k)\Big|,\ f \in H(K),$$

where $(n_k)_{k\in\mathbb{N}}$ varies among all sequences of positive integers, $(x_k)_{k\in\mathbb{N}_0}$ and $(y_k)_{k\in\mathbb{N}_0}$ vary among all sequences in K, and $(a_k)_{k\in\mathbb{N}_0}$ as well as $(b_k)_{k\in\mathbb{N}_0}$ vary among all null sequences in $\mathbb{C}^N$ such that $x_k + a_k = y_k + b_k$ for every k.

(b) If K is locally connected, then the seminorms of type (*) alone suffice to generate the topology of $H(K)$ (but there are even examples of compact sets $K \subset \mathbb{C}$ where the seminorms of type (**) are really needed).

For locally connected compact sets K, Mujica (in a previous paper) also derived a characterization of the continuous linear functionals on $H(K)$ (e.g., cf. [6]).

In infinite dimensional holomorphy (see Dineen's book [17]), one similarly defines spaces $H(K)$ of germs of holomorphic functions on a non-void compact subset K of a complex locally convex space E; the study of these spaces has interesting applications to spaces of holomorphic functions on open subsets of E (by taking suitable projective limits). In general, if E is infinite dimensional, $\operatorname{ind}_{K\subset\subset U\to} (H(U),co)$

and $\operatorname{ind}_{K \subset\subset U \to} H^\infty(U)$ define different topologies τ_0 and τ_ω on $H(K)$. If E is metrizable, then we are back to the case of countable inductive limits, and this case has been treated most of the time. E.g., Mujica's theorem 6. also holds for compact subsets K of (Riemann domains over) a complex Fréchet space E, and in the same paper [43], Mujica proves $\tau_0 = \tau_\omega$ on $H(K)$ for each compact subset K of (a Riemann domain over) a Fréchet-Schwartz space E. (The setting of locally connected compact sets K in a metrizable Schwartz space E also is the one in which Mujica's above mentioned characterization of "analytic functionals"; i.e., continuous linear functionals on $H(K)$, was originally stated.)

Actually, the attention had first been focused on τ_ω and on a study of topological vector space properties of $(H(K),\tau_\omega)$ in dependence on properties of the "underlying" space E; e.g., see [6] for a survey of the main results obtained by 1978. In his 1975 thesis, Mujica had shown that, for compact subsets K of complex metrizable l.c. spaces E, $(H(K),\tau_\omega)$ is always regular, and it took much longer until Dineen proved that this space is even complete. (A more recent, much easier proof of the latter fact is again due to Mujica, see [42] and Section 3. below.) Among other results, the author and R. Meise showed that, for a non-void compact set K in a metrizable l.c. space E, $(H(K),\tau_\omega)$ is a (DFS)-space (resp., a (DFN)-space) if and only if E is Schwartz (resp., nuclear). We again refer to [6] and [17] for more details on this interesting area.

The rest of the section is devoted to a study of Köthe's echelon and co-echelon spaces of arbitrary order p ($1 \le p \le \infty$ or $p = 0$). This setting will serve as an example of the wide applicability of the duality between countable weakly compact projective and inductive limits, and many results will directly follow from Theorem 4. But we will also see that the "general theory" (as it has been developed so far) does not allow to treat the interesting classical "limiting" cases $p = 0$, $p = 1$ and $p = \infty$, and several more delicate questions will arise along the way.

The partial failure of the general theory of l.c. inductive limits is a quite common phenomenon if one is interested in concrete examples and applications, even if they are only "slightly unusual". In many cases, the subsequent different treatment of special inductive limit spaces has led to new results which give rise to an enlargement of the "general theory". – Part of what we are going to report on right now can also serve as a first motivation for the developments in later sections.

7. *Definition.* Let $A = (a_n)_{n \in \mathbb{N}}$ denote a Köthe matrix on a general index set I; i.e., an increasing sequence of strictly positive functions a_n on I, and let $1 \leq p < \infty$. Then the echelon space $\lambda_p = \lambda_p(A)$ of order p is defined as follows:

$$\lambda_p(A) := \{x = (x(i))_{i \in I} \in \mathbb{R}^I \text{ or } \mathbb{C}^I;\ \text{for each } n \in \mathbb{N},$$
$$(a_n(i)x(i))_{i \in I} \text{ is p-absolutely summable on } I;$$
$$\text{that is, } q_{p,n}(x) := (\sum_{i \in I} (a_n(i)|x(i)|)^p)^{1/p} < \infty\}.$$

We also put

$$\lambda_\infty(A) := \{x = (x(i))_{i \in I};\ \text{for each } n \in \mathbb{N},\ q_{\infty,n}(x) :=$$
$$= \sup_{i \in I} a_n(i)|x(i)| < \infty\} \text{ and}$$

$$\lambda_0(A) := \{x = (x(i))_{i \in I};\ \text{for each } n \in \mathbb{N},\ (a_n(i)x(i))_{i \in I}$$
$$\text{tends to } 0 \text{ on } I; \text{ i.e., for each } \varepsilon > 0, \text{ there is a finite subset } J = J(\varepsilon) \text{ of } I \text{ with } a_n(i)|x(i)| < \varepsilon$$
$$\text{for all } i \in I \setminus J\}.$$

Endowed with the sequence $(q_{p,n})_{n \in \mathbb{N}}$ of norms, where we take $q_{0,n} := q_{\infty,n}$, each λ_p is a Fréchet space, $1 \leq p \leq \infty$ or $p = 0$.

To restate the definition in a different form, which will be useful to us in the sequel, let $l_p(a_n)$ denote the corresponding "a_n-

diagonal transform" of the Banach space $l_p = l_p(I)$ of all p-absolutely summable sequences on I, $1 \le p \le \infty$, and $l_o(a_n) = c_o(a_n)$ the a_n-diagonal transform of $c_o = c_o(I)$ (for $p = 0$), $n = 1,2,\ldots$. That is, formally $l_p(a_n) = \lambda_p(C_n)$ for the constant Köthe matrix C_n on I consisting of the single function a_n. Then we have $\lambda_p = \lambda_p(A) = \underset{\leftarrow n}{\operatorname{proj}}\, l_p(a_n)$ algebraically and topologically.

8. *Definition.* Let $V = (v_n)_n$ denote a decreasing sequence of strictly positive functions v_n on an index set I and $1 \le p \le \infty$ or $p = 0$. In the notation introduced above, we put

$$k_p = k_p(V) := \underset{n \rightarrow}{\operatorname{ind}}\, l_p(v_p)$$

(where again $l_o(v_n) = c_o(v_n)$ for arbitrary $n \in \mathbb{N}$). k_p will here be called the <u>co-echelon space</u> of order p (with its natural inductive limit topology).

Now fix a Köthe matrix $A = (a_n)_{n \in \mathbb{N}}$ on some index set I and take the decreasing sequence $V = (v_n)_n$ on I to be the one defined by $v_n = \frac{1}{a_n}$, $n = 1,2,\ldots$. For the moment, we will also fix p with $1 \le p < \infty$ or $p = 0$. Then $\lambda_p = \lambda_p(A)$ is dense in each $l_p(a_n)$, $n \in \mathbb{N}$, and hence the projective limit $\lambda_p = \underset{\leftarrow n}{\operatorname{proj}}\, l_p(a_n)$ is reduced. By the duality of projective and inductive limits (as recalled at the beginning of this section), we thus get algebraically

$$\lambda_p' = \lambda_p(A)' = k_q(V) = k_q, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad q = 1 \quad \text{for} \quad p = 0 .$$

Similarly, for $1 \le p < \infty$ or $p = 0$, the duality of inductive and projective limits yields the canonical algebraic isomorphism

$$k_p' = k_p(V)' = \lambda_q(A) = \lambda_q, \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad q = 1 \quad \text{for} \quad p = 0 .$$

If $1 < p < \infty$, the spaces $l_p(a_n)$ and $l_p(v_n)$ are all reflexive so that the projective sequence $(l_p(a_n))_n$ and the inductive sequence

$(l_p(v_n))_n$ must be weakly compact. Theorem 4. then clearly implies the following.

9. Proposition. Let $A=(a_n)_n$ denote any Köthe matrix and $V=(v_n)_n$ any decreasing sequence of strictly positive functions on an index set I; also fix $1<p<\infty$.

(a) $\lambda_p=\lambda_p(A)$ is a reflexive Fréchet space, and $k_p=k_p(V)$ is a (regular) complete reflexive (LB)-space.

(b) With $\frac{1}{p}+\frac{1}{q}=1$, the following topological vector space dualities hold:

$$(\lambda_p(A))'_b = k_q(V), \quad \text{where } V = (v_n)_n \text{ for } v_n = \frac{1}{a_n}, \text{ and}$$

$$(k_p(V))'_b = \lambda_q(A), \quad \text{where } A = (a_n)_n \text{ for } a_n = \frac{1}{v_n}.$$

In the "limiting" cases $p=0$, $p=1$ and $p=\infty$, our "general theory" does not apply (unless A or V is of a special form), but, from a direct approach (which can also be used to prove the assertions of Proposition 9.), it follows that some of the results of 9. – of course, apart from reflexivity! – remain true while others may actually fail, see [8] (and Köthe's [38] for the famous "Grothendieck-Köthe counterexample" in part (b)):

10. Proposition. (a) $k_1=k_1(V)$ and $k_\infty=k_\infty(V)$ are always (regular) complete (LB)-spaces, and the following topological isomorphisms hold:

$$(\lambda_o(A))'_b = k_1(V) \quad \text{for } V = (v_n)_n,\ v_n = \frac{1}{a_n},\ n = 1,2,\dots,$$

$$(k_o(V))'_b = \lambda_1(A) \quad \text{and} \quad (k_1(V))'_b = \lambda_\infty(A) \quad \text{for}$$

$$A = (a_n),\ a_n = \frac{1}{v_n},\ n = 1,2,\dots$$

(b) If $A=(a_n)_n$ denotes the Köthe matrix on $\mathbb{N}\times\mathbb{N}$ defined by

$$a_n(i,j) = \begin{cases} j, & i \le n \\ 1, & i \ge n+1 \end{cases}$$

and $V=(v_n)_n$, $v_n=\frac{1}{a_n}$, $n=1,2,\ldots$, then $(\lambda_1(A)'=k_\infty(V)$ clearly holds algebraically, but) the strong topology of $(\lambda_1(A))_b'$ is strictly weaker than the inductive limit topology of $k_\infty(V)$. Moreover, for the same decreasing sequence $V=(v_n)_n$, the inductive sequence $(c_o(v_n))_n$ is not regular, and $k_o(V)=\operatorname{ind}_{n\to} c_o(v_n)$ is incomplete.

We will explain in the next section why $k_\infty=k_\infty(V)$ is always (regular and) complete. The following problems which naturally arise from 10.(b) will also be answered later on:

Question 1. What is a necessary and sufficient condition (in terms of $V=(v_n)_n$) for $k_o(V)$ to be (regular or) complete?

Question 2. What is a necessary and sufficient condition (in terms of $A=(a_n)_n$) for the topological equality $(\lambda_1(A))_b'=k_\infty(V)$, $V=(v_n)$ with $v_n=\frac{1}{a_n}$?

Since $\lambda_1(A)$ and $k_\infty(V)$ are the "most classical" Köthe echelon and co-echelon spaces, Question 2. is particularly important, but its solution has only been obtained very recently (and it is quite complicated).

We next turn to the problem of explicitly describing the continuous seminorms for the inductive limit topology of k_p, at least if $1\le p<\infty$. To state a precise form of this description (as given in [8]), we start with:

11. Definition. Fix a decreasing sequence $V=(v_n)_n$ on an index set I and let $A=(a_n)_n$ be defined by $a_n=\frac{1}{v_n}$, $n=1,2,\ldots$. By $\bar{V}=\bar{V}(V)$, we denote the system

$$\lambda_\infty(A)_+ = \{\bar{v}=(\bar{v}(i))_{i\in I}\in(\mathbb{R}_+)^I;\ \text{for each } n\in\mathbb{N},$$
$$\sup_{i\in I} a_n(i)\bar{v}(i) = \sup_{i\in I}\frac{\bar{v}(i)}{v_n(i)}<\infty\}.$$

Next, if $1 \leq p < \infty$, put

$$K_p = K_p(\bar{V}) := \{x = (x(i))_{i \in I} \in \mathbb{R}^I \text{ or } \mathbb{C}^I;\ \text{for each } \bar{v} \in \bar{V},$$
$$r_{p,\bar{v}}(x) := (\sum_{i \in I} (\bar{v}(i)|x(i)|)^p)^{1/p} < \infty\}$$

as well as

$$K_\infty = K_\infty(V) := \{x = (x(i))_i;\ \text{for each } \bar{v} \in \bar{V},$$
$$r_{\infty,\bar{v}} := \sup_{i \in I} \bar{v}(i)|x(i)| < \infty\} \text{ and}$$
$$K_o = K_o(\bar{V}) := \{x = (x(i))_i;\ \text{for each } \bar{v} \in \bar{V},$$
$$(\bar{v}(i)x(i))_{i \in I} \text{ tends to } 0 \text{ on } I\}.$$

Endowed with the system $(r_{p,\bar{v}})_{\bar{v} \in \bar{V}}$ of seminorms, where we take $r_{o,\bar{v}} := r_{\infty,v}$, $K_p = K_p(\bar{V})$ is a complete l.c. space for $1 \leq p \leq \infty$ or $p = 0$.

We note that, even though we assume that all v_n are strictly positive, there are cases where $\bar{V}$ does not contain any strictly positive function on I. But at least if I is countable, there always are strictly positive $\bar{v} \in \bar{V}$, and then we can actually restrict our attention to such functions $\bar{v}$.

It is quite easy to verify that $k_p = k_p(V)$ continuously embeds into $K_p = K_p(\bar{V})$, $\bar{V} = \bar{V}(V)$, for arbitrary p, but much more can be said for $1 \leq p < \infty$.

12. Proposition. For $1 \leq p < \infty$, we actually have $k_p(V) = K_p(\bar{V})$ algebraically and topologically.

In particular, the inductive limit topology of $k_p = k_p(V)$ is given by the system $(r_{p,\bar{v}})_{\bar{v} \in \bar{V}}$, $\bar{V} = \bar{V}(V)$, of seminorms.

For the proof of Proposition 12., we refer to [8]. — Incidentally, the completeness of $k_1 = k_1(V)$ (mentioned in 10.(a)) can be deduced from 12. (but it also follows from the duality $(\lambda_o(A))'_b = k_1(V)$).

Since $K_0 = K_0(\bar{V})$ always is complete, 10.(b) provides us with an example of a sequence $V = (v_n)_n$ for which the topological vector space equality $k_0(V) = K_0(\bar{V})$, $\bar{V} = \bar{V}(V)$, cannot hold. To see that, similarly, $k_\infty(V) = K_\infty(\bar{V})$ is not true in general, we consider the following proposition (which is again taken from [8]).

13. Proposition. For $\bar{V} = \lambda_\infty(A)_+$, we have $(\lambda_1(A))'_b = K_\infty(\bar{V})$ algebraically and topologically.

In view of 13. and of the algebraic equality $(\lambda_1(A))' = k_\infty(V)$, $V = (\frac{1}{a_n})_n$, we realize that $k_\infty(V) = K_\infty(\bar{V})$, $\bar{V} = \bar{V}(V)$, always holds algebraically, but, for the Köthe matrix $A = (a_n)_n$ (on $\mathbb{N} \times \mathbb{N}$) in the Köthe-Grothendieck counterexample 10.(b), the topology of $k_\infty(V)$, $V = (\frac{1}{a_n})_n$, must be strictly finer than the "weighted" topology of $K_\infty(\bar{V})$, $\bar{V} = \bar{V}(V)$.

By the way, Proposition 9.(b), the duality $(\lambda_0(A))'_b = k_1(V)$, $V = (v_n)_n$ with $v_n = \frac{1}{a_n}$, of 10.(a) and Proposition 12. combine to yield

$$(*) \quad (\lambda_p(A))'_b = K_q(\bar{V}) \text{ algebraically and topologically,}$$

$$\bar{V} = \lambda_\infty(A)_+,$$

for $1 < p < \infty$ or $p = 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, where we take $q = 1$ for $p = 0$. (On the other hand, a direct proof of (*) is also possible and this, taken together with 9.(b) and 10.(a), conversely implies 12.)

Returning to Proposition 12., the failure of this result for $p = 0$ and $p = \infty$ naturally leads to the following problem.

Question 3. What is the exact relationship between the spaces $k_0(V)$ and $K_0(\bar{V})$ as well as $k_\infty(V)$ and $K_\infty(\bar{V})$, $\bar{V} = \bar{V}(V)$?

Of course, Question 3. is closely related to the previous Questions 1. and 2. E.g., in view of Proposition 13., Question 2. actually asks for a necessary and sufficient condition (in terms of $V = (v_n)_n$) for the topological equality $k_\infty(V) = K_\infty(\bar{V})$, $\bar{V} = \bar{V}(V)$.

Similarly, since it will turn out that $k_0(V)$ always is a dense topological subspace of $K_0(\bar{V})$, Question 1. is equivalent to asking for a necessary and sufficient condition for the algebraic equality $k_0(V) = K_0(\bar{V})$. Hence a "good form" of Question 3. is even more comprehensive than Questions 1. and 2. together.

In one of the next sections, we will give a complete answer to Question 3., and we will even treat (and solve) the corresponding problems in the more general context of "weighted inductive limits".

To finish this section, we return to applications of Theorem 4. to echelon and co-echelon spaces. While, for $1 < p < \infty$ and a Köthe matrix $A = (a_n)_n$ on some index set I or a decreasing sequence $V = (v_n)_n$ on I, the projective sequence $(l_p(a_n))_{n \in \mathbb{N}}$ as well as the inductive sequence $(l_p(v_n))_{n \in \mathbb{N}}$ always are weakly compact, the situation is different for $p = 0, 1$ or ∞: Each weakly compact subset of l_1 (over an arbitrary index set I) is compact, and hence a weakly compact projective sequence $(l_1(a_n))_n$ or a weakly compact inductive sequence $(l_1(v_n))_n$ must already be compact. By duality, it turns out that, similarly, any of the sequences $(c_0(a_n))_n$, $(c_0(v_n))_n$, $(l_\infty(a_n))_n$ or $(l_\infty(v_n))_n$ can only be weakly compact if it is already compact. The following easily established classical result completely clarifies under which conditions $\lambda_p(A)$ is an (FS)- and $k_p(V)$ a (DFS)-space (for arbitrary p).

14. Proposition. Let $A = (a_n)_n$ denote a Köthe matrix on some index set I and $V = (\frac{1}{a_n})_n$ the corresponding decreasing sequence of strictly positive functions on I. Then, for $1 \le p \le \infty$ or $p = 0$, the following assertions are equivalent:

(1) For each $n \in \mathbb{N}$, there is $m > n$ such that $\frac{a_n}{a_m} = \frac{v_m}{v_n}$ tends to 0 on I,

(2) $\lambda_p = \lambda_p(A)$ is an (FS)-space,

(3) $k_p = k_p(V)$ is a (DFS)-space,

(4) $k_0 = k_\infty$ holds algebraically,

(5) $k_0 = k_0(V)$ is (semi-)reflexive or a (semi-) Montel space.

If condition 14.(1) holds, then clearly $(\lambda_1(A))'_b = k_\infty(V)$ topologically, but it will turn out that a much weaker hypothesis already suffices. Also note that each of the assertions:

(i) $\lambda_0 = \lambda_\infty$ algebraically,

(ii) $K_0(\bar{V}) = K_\infty(\bar{V})$ algebraically,

(iii) $k_\infty = k_\infty(V)$ is (semi-)reflexive or a (semi-)Montel space,

(iv) $K_0 = K_0(\bar{V})$ or $K_\infty = K_\infty(\bar{V})$ is (semi-)reflexive

characterizes the Fréchet-Montel spaces λ_p and the (semi-)Montel spaces K_p, $1 \le p \le \infty$ or $p = 0$, and hence does not suffice for the conclusions of 14.

We finally list the well-known classification of the nuclear projective limits λ_p and the nuclear inductive limits k_p; i.e., of the echelon and co-echelon spaces to which Theorem 4.(c) applies. (See [32].)

15. Proposition. For A and V as in 14. and $1 \le p \le \infty$ or $p = 0$, the following assertions are equivalent:

(1) For each $n \in \mathbb{N}$, there exists $m > n$ such that $\frac{a_n}{a_m} = \frac{v_m}{v_n}$ is (absolutely) summable on I,

(2) $\lambda_p = \lambda_p(A)$ is an (FN)-space,

(3) $k_p = k_p(V)$ is a (DFN)-space.

If 15.(1) holds, then, in fact, all spaces $\lambda_p = \lambda_p(A)$ (as well as all spaces $k_p = k_p(V)$), coincide algebraically and topologically for arbitrary orders p, $1 \le p \le \infty$ or $p = 0$. (And, conversely, the algebraic identity $k_\infty = k_1$ alone obviously is already sufficient to imply condition (1) of Proposition 15.)

3. (LF)- and (LB)-spaces

At the beginning of this section, we state Grothendieck's Theorems A. and B. (from [32]); they probably are the most useful general results on (LF)-spaces. Next, we discuss the regularity of (LF)- and (LB)-spaces and then turn to completeness questions, mainly concentrating on the case of (LB)-spaces.

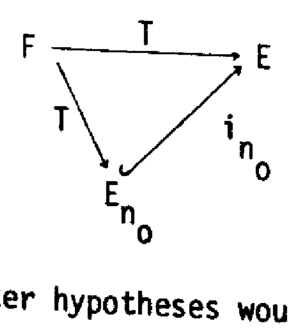
The first result is Grothendieck's closed graph and open mapping theorem for (LF)-spaces.

1. Theorem. (a) Every linear mapping $T: F \to E$ of an ultrabornological space F into an (LF)-space E which has (sequentially) closed graph must already be continuous.

(b) Every continuous linear mapping of an (LF)-space onto an ultrabornological space is open; i.e., a topological homomorphism.

This theorem was extended by de Wilde (e.g., see Köthe [38,II]); he was able to replace the class of the (LF)-spaces by the larger class of all "webbed" spaces in Theorem 1. Grothendieck's famous "factorization theorem", which we will state next, was generalized by de Wilde in a similar, but more technical way.

2. Theorem. Let T denote a continuous linear mapping of a Fréchet space F into an (LF)-space $E = \operatorname{ind}_{n \to} E_n$ (where we again assume that $E = \bigcup_{n \in \mathbb{N}} E_n$ and that the topology of E is Hausdorff). Then there exists $n_o \in \mathbb{N}$ such that $TF \subset E_{n_o}$ and T is continuous as an operator from F into E_{n_o}. Hence T factors continuously as follows:



We note that weaker hypotheses would already suffice in 2.: Let E denote a l.c. space, E_n, F Fréchet spaces, $i_n: E_n \to E$ injective continuous linear mappings and $T: F \to E$ a linear map with sequentially closed graph and $TF \subset \bigcup_n i_n(E_n)$. Then there exists an index $n_o \in \mathbb{N}$ such that T factors continuously through E_{n_o} and i_{n_o}.

A nice application of this general form of Theorem 2. to holomorphic continuations (and domains of holomorphy) was given by Edwards [22].

Returning to the form of the factorization theorem stated as Theorem 2. above, we remark that this result also holds under different hypotheses: The assumptions "F Fréchet" and "$E = \text{ind}_{n\to} E_n$ separated (LF)-space" can be replaced by "F metrizable l.c. space" and "$E = \text{ind}_{n\to} E_n$ countable regular l.c. inductive limit" (cf. Floret [25]).

But the consequences of 2. for the regularity of (LF)-spaces are particularly interesting in our context.

3. *Definition.* (a) An absolutely convex bounded set B in a l.c. space E is called <u>completing</u> (or a <u>Banach ball</u> or <u>Banach disk</u>) if the canonically associated normed space E_B (= linear span of B, endowed with the Minkowski functional of B as norm) is complete (and hence a Banach space).

(b) A l.c. space E is said to be <u>Mackey-complete</u> (or "bornologically complete") if each bounded subset of E is contained in a Banach ball.

It is well-known (e.g., see Köthe [38]) that a sequentially complete closed absolutely convex bounded set in an arbitrary locally convex space always is completing, and hence every sequentially complete l.c. space is Mackey-complete. (In the last section, when we sketched the proof of the result that each weakly compact linear mapping between l.c. spaces factors through a Banach space, we implicitly used that also each weakly compact set in any l.c. space is completing.)

4. *Corollary.* Each Banach ball in an (LF)-space $E = \text{ind}_{n\to} E_n$ (as in Theorem 2.) is already contained, bounded and completing in one of the generating Fréchet spaces E_n. In particular, for (LF)-spaces, the following implications hold:

$$\begin{aligned} &\text{complete} \Rightarrow \text{quasi-complete} \Rightarrow \text{sequentially complete} \\ &\Rightarrow \text{Mackey-complete} \Leftrightarrow \text{regular}. \end{aligned}$$

A proof of the first part of 4. follows quite simply by applying 2. to the embedding $E_B \to E$. Hence Mackey-completeness of an (LF)-space implies regularity, and the converse of this is obvious.

The implications stated at the end of Corollary 4. are all that is known for general (LF)-spaces; the following problem of Grothendieck is still open.

Problem 1. Does the regularity of an (LF)-space already imply its completeness?

The situation is a little bit better for (LB)-spaces: Namely, the general theory of (DF)-spaces, due to Grothendieck [31], in particular applies to (LB)-spaces, and hence an (LB)-space is complete if and only if it is quasi-complete. But it also remains open whether each regular (LB)-space is complete, and this is the author's favorite problem!

We are now going to shortly summarize some facts from Grothendieck's theory of (DF)-spaces which are relevant to our discussion in the sequel.

5. *Definition.* A l.c. space E is said to be a <u>(DF)-space</u> if

(i) it has a fundamental sequence of bounded sets, and

(ii) it is <u>σ-quasibarrelled</u> in the sense that,

for each sequence $(U_n)_{n\in\mathbb{N}}$ of closed absolutely convex 0-neighborhoods in E such that $U := \bigcap_{n\in\mathbb{N}} U_n$ absorbs every bounded set, this intersection U must again be a 0-neighborhood in E.

The strong dual of any metrizable l.c. space is a (DF)-space; each normed space also enjoys the (DF)-property. The strong dual of every (DF)-space is Fréchet.

6. *Proposition.* (a) A (DF)-space is complete if and only if it is quasi-complete.

(b) A countable inductive limit $E = \operatorname{ind}_{n\to} E_n$ of (DF)-spaces E_n is again a (DF)-space.

Moreover, if B is a bounded subset of such a countable inductive limit E, then there exist an index $n = n(B) \in \mathbb{N}$ and a bounded set $B_n = B_n(B)$ in E_n such that

$B \subset \overline{B_n}^{E}$ (where the closure is taken in the inductive limit topology of E).

The first part of (b) implies that each (LB)-space is (DF), and hence, by (a), in particular each (LB)-space is complete if and only if it is quasi-complete (as we had asserted before). Also, in view of part (i) in Definition 5., any (LB)-space does have a fundamental sequence of bounded sets, but since we have seen that there exist non-regular (LB)-spaces (cf. 2.10.(b)), it is possible that no sequence which only consists of multiples of the unit balls of the generating Banach spaces will form a basis for the bounded subsets of the (LB)-space. On the other hand, the second part of 6.(b) asserts that any countable inductive limit $E = \operatorname{ind}_{n \to} E_n$ of (DF)-spaces is "almost regular" in a very precise sense. (Non-regularity can only come from the fact that there exists a bounded subset B_n of some E_n whose closure in the inductive limit topology is [bounded in $E = \operatorname{ind}_{n \to} E_n$, but] not a bounded subset of E_m for each $m \geq n$).

We will return to 6.(b) in a moment, but first would like to finish our summary of some relevant properties of (DF)-spaces by pointing out that the topology of a (DF)-space E with a fundamental sequence $(B_n)_n$ of closed absolutely convex bounded subsets B_n is <u>localized</u> to this sequence $(B_n)_n$ in the sense that an absolutely convex set U in E is a 0-neighborhood if and only if $U \cap B_n$ is a 0-neighborhood in B_n (with the induced topology) for each $n \in \mathbb{N}$. Keeping the fundamental sequence of bounded sets and this localization property, various people extended the class of (DF)-spaces to what is now called "(gDF)-spaces" (for: "generalized (DF)-spaces"). (gDF)-spaces share many of the important properties of (DF)-spaces, and the strict topologies of Buck, Fremlin-Garling-Haydon, Sentilles and Wheeler (cf. the end of Section 1.) yield important examples of (gDF)-spaces which are not already (DF). (See Jarchow's book [36] and the survey [51] of Ruess.)

(Of course, this extension of the class of the (DF)-spaces is closely related to another generalization: One can generalize l.c. inductive limits by introducing "generalized inductive limits" as Garling [30] did [also see the book [46] of Bonet - Pérez Carreras], and, say, much of the theory of strict inductive limits carries over to the more general context.

In this exposition where our [main] aim is not "generality", but where we intend to give an introduction to various aspects of l.c. inductive limits relevant in view of the applications and the examples, we naturally have to stick to the classical setting. Also, we have only treated (DF)-spaces [very shortly] here since, at this moment, we are interested in their applications to (LB)-spaces, and the latter really are (DF)-spaces in Grothendieck's classical sense.)

Returning to 6.(b), it is now easy to prove the following useful criterion for regularity of (LB)-spaces.

7. *Corollary.* If $E = \operatorname{ind}_{n\to} E_n$ is some (Hausdorff) (LB)-space (or just the separated countable inductive limit of normed spaces E_n), let B_n denote the closed unit ball of E_n for $n = 1,2,\dots$. If each B_n is even closed in the inductive limit topology (a fortiori, if it is closed in any weaker Hausdorff topology), then $E = \operatorname{ind}_{n\to} E_n$ is regular.

Proof. For an arbitrary bounded set $B \subset E$, 6.(b) yields an index $n \in \mathbb{N}$ and a number $\lambda > 0$ with $B \subset \overline{\lambda B_n}^E$. But our assumption implies $\overline{\lambda B_n}^E = \lambda \bar{B}_n^E = \lambda B_n$; hence the assertion.□

At this point, we are able to understand why the co-echelon spaces $k_\infty = k_\infty(V) = \operatorname{ind}_{n\to} l_\infty(v_n)$ are regular for each decreasing sequence $V = (v_n)_{n\in\mathbb{N}}$ of strictly positive functions and why this is not true for $k_0 = k_0(V) = \operatorname{ind}_{n\to} c_0(v_n)$ in general: The unit ball B_n of $l_\infty(v_n)$ is completely described by the inequality $\sup_{i\in I} v_n(i)|x(i)| \le 1$, and hence it is even closed under the topology of pointwise convergence on I (which is clearly weaker than the inductive limit topology of k_∞), while the unit ball C_n of $c_o(v_n)$ will obviously not be closed under pointwise convergence in general!

We would now like to define the "weighted inductive limits" $\mathcal{V}C(X)$ and $\mathcal{V}_oC(X)$ ([5]), which generalize k_∞ and k_o, and will deduce the regularity of $\mathcal{V}C(X)$ in exactly the same way as before.

8. *Definition.* Let X denote a completely regular Hausdorff space and $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ a decreasing sequence of strictly positive upper semi-continuous functions (<u>"weights"</u>) on X. For each $n\in\mathbb{N}$, we put:

$$Cv_n(X) := \{f \in C(X);\ \|f\|_n := \sup_{x\in X} v_n(x)|f(x)| < \infty\} \quad \text{and}$$

$$C(v_n)_o(X) := \{f \in C(X);\ v_nf \text{ vanishes at infinity on } X;\ \text{i.e., for each } \varepsilon > 0, \text{ there is a compact } K \subset X \text{ with } v_n(x)|f(x)| < \varepsilon \text{ for all } x \in X \setminus K\}.$$

$Cv_n(X)$ is normed by $\|\cdot\|_n$, and $C(v_n)_o(X)$ is a closed linear subspace of $Cv_n(X)$ which will be equipped with the induced norm.

We define

$$\mathcal{V}C(X) := \operatorname*{ind}_{n\to} Cv_n(X) \quad \text{and} \quad \mathcal{V}_oC(X) := \operatorname*{ind}_{n\to} C(v_n)_o(X);$$

these are the <u>weighted inductive limits</u> of spaces of continuous functions (which we will study in more detail later on).

If X is some index set I with the discrete topology, then all functions on X are continuous, and hence each strictly positive v on X is a weight. In this case, $\mathcal{V}C(X)$ is nothing but the co-echelon space $k_\infty = k_\infty(\mathcal{V})$, and $\mathcal{V}_oC(X)$ reduces to $k_o = k_o(\mathcal{V})$.

In any case, the topologies of all the normed spaces $Cv_n(X)$, and hence also the inductive limit topologies of $\mathcal{V}C(X)$ and $\mathcal{V}_oC(X)$, are stronger than pointwise convergence on X. Since, moreover, the unit ball B_n of each $Cv_n(X)$ is closed in $C(X)$ with respect to the topology of pointwise convergence on X, we can conclude from 7. that $\mathcal{V}C(X)$ always is a regular inductive limit.

For later purposes, we note that $Cv_n(X)$ carries a topology which is even finer than the compact-even topology co whenever,

(*) for each compact $K \subset X$, we have $\inf_{x \in K} v_n(x) > 0$

(which certainly holds for continuous functions v_n). It is easy to see that, in the presence of (*), $Cv_n(X)$ is complete (and hence $Cv_n(X)$ and $C(v_n)_o(X)$ are Banach spaces) if

(**) X is a $k_{\mathbb{R}}$-space; i.e., any function $f\colon X \to \mathbb{R}$ whose restriction to each compact subset of X is continuous must already belong to $C(X)$.

(Note that $(C(X), co)$ is complete if and only if X is a $k_{\mathbb{R}}$-space.) Clearly, each locally compact space X and each metrizable X is a $k_{\mathbb{R}}$-space.

By what we have just said, if (*) holds for each $n \in \mathbb{N}$ and if (**) X is a $k_{\mathbb{R}}$-space, then all spaces $Cv_n(X)$ and all spaces $C(v_n)_o(X)$ are complete, and hence $\mathcal{V}C(X)$ as well as $\mathcal{V}_oC(X)$ are (LB)-spaces.

Turning from regularity to completeness, Corollary 4. clearly suggests that proving the regularity of an (LF)-space (and even more so for an (LB)-space) can be considered as a "first step" towards a proof of the completeness. Hence we would now like to sketch the ideas which are involved in showing the completeness of the co-echelon spaces $k_\infty = k_\infty(V)$, and we want to state a useful criterion for completeness of (LB)-spaces patterned along the lines of Corollary 7. But this first requires to look at two natural inductive limit constructions which arise with any given l.c. space E.

We have already mentioned how an absolutely convex bounded set B in a l.c. space E leads to the construction of a normed space E_B, continuously embedded in E. If $\mathcal{B} = \mathcal{B}(E)$ denotes the system of all absolutely convex bounded subsets of the fixed space E, then $(E_B)_{B \in \mathcal{B}}$, together with the canonical injections $i_{BC}\colon E_B \to E_C$ for $B \subset C$, is an injective inductive net of normed spaces. Its limit $\operatorname{ind}_{\mathcal{B} \ni B \to} E_B$ equals E

algebraically, but the inductive limit topology τ^x may be strictly stronger than the initial topology τ of E.

We note that the inductive limit $(E,\tau^x) = \underset{B\to}{\text{ind}}\, E_B$ is countable (i.e., may be reduced to a countable "sublimit" $\underset{n\to}{\text{ind}}\, E_{B_n}$) if and only if E has a fundamental sequence of bounded subsets.

The following are well-known facts on τ^x (e.g., see Köthe [38]):

(i) τ^x is the finest l.c. topology on E which has the same bounded sets as τ. The system of all those absolutely convex subsets U of E which absorb all τ-bounded sets yields a neighborhood base of 0 for τ^x. The dual of (E,τ^x) is just the set of all "<u>locally bounded</u>" linear forms on E (i.e., the space of all those linear functionals which are bounded on each bounded subset of E).

(ii) (E,τ^x) is bornological (as an inductive limit of normed spaces), and (E,τ) is bornological if and only if $\tau = \tau^x$.

Hence (E,τ^x) is called the <u>bornological space associated with</u> (E,τ).

We would like to add that, obviously, the inductive limit $(E,\tau^x) = \underset{B\to}{\text{ind}}\, E_B$ is always regular. Moreover, by (ii) above, a l.c. space is bornological if and only if it is an inductive limit of normed spaces. Similarly, one can check that a l.c. space is a countable inductive limit of normed spaces if and only if it is a bornological (DF)-space.

We have used the term "ultrabornological" to denote l.c. spaces which are inductive limits of Banach spaces. Now, clearly, each Mackey-complete bornological space must even be ultrabornological. Let us mention in passing (again, see Köthe [38,II]) that one can similarly give a construction of "the <u>ultrabornological space</u> (E,τ^u) <u>associated with</u> any l.c. space (E,τ)" as inductive limit

$\underset{\hat{\mathcal{B}}\ni B\to}{\text{ind}}\, E_B$, where $\hat{\mathcal{B}} = \hat{\mathcal{B}}(E)$ denotes the system of all Banach balls in E, or, equivalently, as

$\operatorname{ind}_{\hat{K} \ni K \to} E_K$, where $\hat{K} = \hat{K}(E)$ is the system of all absolutely convex compact sets in E.

More interesting in our context is a similar method of construction for a l.c. inductive limit topology on the dual E' of an arbitrary l.c. space E: For any neighborhood U of 0 in E, the (absolute) polar U° is an absolutely convex weak*-compact (equicontinuous, and hence $\beta(E',E)$-bounded) set in E'; E'_{U° denotes the associated Banach space (which is continuously embedded in E'_b). Let $\mathcal{U} = \mathcal{U}(E)$ denote the system of all (absolutely convex closed) neighborhoods of 0 in E. Then $(E'_{U^\circ})_{U \in \mathcal{U}}$ is an injective inductive net of Banach spaces (with respect to reversing inclusions of sets $U \in \mathcal{U}$ and taking canonical injections of subspaces of E'), and its inductive limit $\operatorname{ind}_{U \to} E'_{U^\circ}$ coincides with E' algebraically; the inductive limit topology is stronger than the strong topology $\beta(E',E)$.

9. *Definition.* For any l.c. space E, E'_i denotes E', endowed with the inductive limit topology of $\operatorname{ind}_{U \to} E'_{U^\circ}$.

Note that $\operatorname{ind}_{U \to} E'_{U^\circ}$ is countable (that is, reduces to a countable "sublimit" $\operatorname{ind}_{n \to} E'_{U_n^\circ}$) if and only if E has a countable basis of 0-neighborhoods, and hence precisely if E is metrizable. The topology of E'_i must be strictly stronger than $\beta(E',E)$ if E'_b is not (ultra-) bornological.

If E is a quasibarrelled l.c. space, then each bounded subset of E'_i, being a fortiori bounded in E'_b, is equicontinuous and thus contained in U° for some $U \in \mathcal{U}$; in this case $\operatorname{ind}_{U \to} E'_{U^\circ}$ obviously is regular. In fact, for a quasibarrelled space E, it is now clear that E'_b and E'_i have the same bounded sets; viz., the equicontinuous subsets of E'; hence E'_i is exactly the bornological (or even the ultrabornological) space associated with E'_b, and a quasibarrelled space E satisfies $E'_b = E'_i$ topologically if and only if E'_b is bornological.

Locally convex spaces E with $E_b' = E_i'$ were called "reinforced regular" by Berezanskii, and a closer inspection (e.g., see Floret [26]) shows that even all σ-quasibarrelled spaces E with E_b' bornological, and hence all (DF)-spaces E, must satisfy $E_b' = E_i'$.

If a Fréchet space E is represented as the reduced projective limit $\operatorname{proj}_{\leftarrow n} E_n$ of a decreasing projective sequence $(E_n)_{n\in\mathbb{N}}$ of Banach spaces, then the limit of the dual (injective) inductive sequence $((E_n)_b')_n$ is exactly E_i'. (A simple argument shows this; in fact, a similar statement holds, say, for reduced filtrating projective nets of normed spaces. But the restricted case above is especially interesting in view of our discussion in Section 2., and it already explains why the topological duality even of countable reduced projective limits and of their dual inductive limits is rather complicated for the strong topologies.)

In particular, the inductive dual $(\lambda_1(A))_i'$ of an arbitrary echelon space $\lambda_1(A) = \operatorname{proj}_{\leftarrow n} l_1(a_n)$ is precisely the co-echelon space $k_\infty(V) = \operatorname{ind}_{n\rightarrow} l_\infty(v_n)$, $V = (v_n)_n$, $v_n = \frac{1}{a_n}$, $n = 1,2,\dots$. Now we see that there actually exist Fréchet spaces E for which $E_b' \neq E_i'$, and hence E_b' is not bornological; e.g., one can take the Grothendieck-Köthe counter-example $\lambda_1(A)$ from Proposition 2.10.(b)!

Grothendieck [31] (also, see Horváth's book [35]) showed that the strong dual of a metrizable l.c. space is bornological if and only if it is barrelled. Since a l.c. space E is said to be <u>distinguished</u> if E_b' is barrelled (sometimes this definition is rephrased in a different, but equivalent form), we arrive at:

10. Proposition. A metrizable l.c. space E is distinguished if and only if $E_b' = E_i'$.

Using Proposition 10., it becomes obvious that our Question 2. of Section 2. also exactly amounts to asking for a necessary and sufficient condition (in terms of $A = (a_n)_n$) for distinguishedness of the echelon space $\lambda_1(A)$.

Here is another result of Grothendieck [31] which is of immediate interest for our study of completeness in (LB)-spaces.

11. Proposition. The inductive dual E_i' of any metrizable l.c. space E is complete.

Grothendieck proved this result by showing that E_i', as the bornological space associated with E_b', has a basis of 0-neighborhoods which are $\beta(E',E)$-closed. Since E_b' is complete, the closed neighborhood condition implies completeness of E_i'. (Moreover, the proof also shows that the inductive topology of E_i' exactly equals $\beta(E',E'')$.) – A different proof is possible: It is implicit in Grothendieck's article that the bornological and barrelled topologies associated with the strong dual E_b' of an arbitrary metrizable l.c. space always coincide. But a barrelled topology associated with a complete l.c. space is known to be itself complete. (E.g., see Schmets [53].)

Of course, by Proposition 11., $k_\infty = k_\infty(V) = (\lambda_1(A))_i'$ must always be complete, a fact which had been stated in Proposition 2.10.(b) and which we had promised to explain at that time.

For echelon and co-echelon spaces, there is a natural duality which allows an application of 11. for a completeness proof of the corresponding inductive limit spaces. In general, however, one finds many cases where such a duality is not apparent, and then 11. cannot be applied directly. But the following result, due to J. Mujica [42], the hypotheses of which are stronger, but patterned along the lines of the condition in Corollary 7., may help.

12. Theorem. Let E denote the (injective) inductive limit of an increasing sequence of Banach spaces E_n.

(a) *If* there exists a Hausdorff l.c. topology τ on E such that the closed unit ball B_n of each E_n, $n=1,2,\ldots$, is τ-compact, then there exists a Fréchet space Y with $Y_i' = E$ algebraically and topologically, and hence E is complete.

(In fact, one can take Y to be the Fréchet space of all those linear forms on E whose restrictions to each B_n are τ-continuous, equipped with the topology of uniform convergence on all the sets B_n.)

(b) If, in addition, E has a basis of τ-closed absolutely convex 0-neighborhoods, then E actually is the strong dual of Y, and Y is a distinguished Fréchet space.

The hypothesis of (b) is satisfied, in particular, if, for each neighborhood U of 0 in E, there exists a sequence of τ-closed absolutely convex 0-neighborhoods V_n in E with $V_n \cap B_n \subset U$ for $n = 1,2,\dots$.

The main purpose of Mujica's Completeness Theorem 12. was to provide an easier proof for the completeness of the space $(H(K),\tau_\omega)$ of germs of holomorphic functions on a compact set K in a complex metrizable l.c. space E (cf. [42] and the discussion after Proposition 2.6), but there are many other applications of 12., one of which we will now present (see [10]).

13. *Definition.* For an arbitrary locally compact Hausdorff space X, let A(X) denote a linear subspace of the space C(X) of all continuous (real- or complex-valued) functions on X with the following two properties:

(i) A(X) is closed in C(X) with respect to the compact-open topology co, and

(ii) (A(X),co) is a semi-Montel space (i.e., every bounded subset is relatively compact).

(Note that, for any σ-compact locally compact X, (C(X),co) is a Fréchet space, and then property (ii) clearly implies (i).)

Next (cf. Definition 8.), let $V = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions v_n on X and put, for every $n \in \mathbb{N}$,

$$Av_n(X) := \{f \in A(X);\ \|f\|_n = \sup_{x\in X} v_n(x)|f(x)| < \infty\} \text{ and}$$

$$A(v_n)_0(X) := \{f \in A(X);\ v_n f \text{ vanishes at infinity on } X\}.$$

Clearly, $Av_n(X) = Cv_n(X) \cap A(X)$ and $A(v_n)_0(X) = C(v_n)_0(X) \cap A(X)$ with the induced norm, and since, under our present hypotheses, the topology induced by $\|\cdot\|_n$ is stronger than co, it follows from (i) that $Av_n(X)$ (resp., $A(v_n)_o(X)$) is closed in $Cv_n(X)$ (resp., $C(v_n)_o(X)$) and hence a Banach space. We define

$$\mathcal{V}A(X) := \underset{n \to}{\operatorname{ind}} Av_n(X) \quad \text{and} \quad \mathcal{V}_oA(X) := \underset{n \to}{\operatorname{ind}} A(v_n)_o(X);$$

these are the <u>weighted inductive limits</u> of spaces of "A-functions". In our setting, $\mathcal{V}A(X)$ and $\mathcal{V}_oA(X)$ are always (LB)-spaces.

Of course, in 13., we may take X to be some index set I with the discrete topology and $A(X) = C(X)$; then, clearly, $A(X) = k_\infty(\mathcal{V})$ and $\mathcal{V}_oA(X) = k_o(\mathcal{V})$. But there are more interesting cases: E.g., if X is an open subset of $\mathbb{C}^N$, $N \leq 1$, we may let $A(X)$ denote the space of all holomorphic functions on X. Or, if X is an open subset of $\mathbb{R}^N$, $N > 1$, the space of all harmonic functions on X also enjoys properties (i) and (ii). There are various other possibilities like spaces of solutions of homogeneous hypoelliptic linear partial differential operators or systems, some spaces of harmonic functions in abstract potential theory etc. (see [4]).

Up to this point, we have not made use of property (ii) in Definition 13., but this condition enters the proof of the following corollary to 12. in a crucial way.

14. Corollary. Under our general hypotheses, the (LB)-spaces $\mathcal{V}A(X)$ are always complete.

Proof. Letting τ denote the compact-open topology co on X, it is quite obvious that the unit ball B_n of each space $Av_n(X)$ is τ-closed (in $A(X)$). But since B_n clearly is a bounded subset of $(A(X),co)$, it must also be τ-relatively compact by property (ii) in Definition 13. Hence the assumptions of Mujica's Completeness Theorem 12. are satisfied, and we can conclude.□

In his recent communication [44], Mujica used a refinement of his idea of proof of 12. to relax the hypotheses of Theorem 12.(a) as follows:

15. *Theorem.* Let $E = \operatorname{ind}_{n\to} E_n$ denote an arbitrary ("injective") (LB)-space; i.e., E is not a priori supposed to be separated. We assume that, for each $n \in \mathbb{N}$, there exists a Hausdorff l.c. topology τ_n on E_n such that

(i) the inclusion mapping $(E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$ is continuous, and

(ii) the closed unit ball B_n of E_n is τ_n-compact, $n = 1,2,\ldots$.

Then, again, $E = Y_i'$ holds algebraically and topologically for a suitable Fréchet space Y, and hence $E = \operatorname{ind}_{n\to} E_n$ is Hausdorff, regular and complete.

We note that 15. formally includes a completeness proof for countable weakly compact inductive limits $E = \operatorname{ind}_{n\to} E_n$ (by taking $\tau_n = \sigma(E_{m(n)}, E'_{m(n)})$ for a suitable increasing sequence $(m(n))_n$ of indices $m(n) \geq n$).

And, as a corollary of 15., Mujica also deduced the following: Let $E = \operatorname{ind}_{n\to} G_n'$ denote the inductive limit of a sequence of dual Banach spaces such that the inclusion mappings $G_n' \to G_{n+1}'$ are dual (i.e., transposed) mappings. Then $E = G_i'$ holds for $G := \operatorname{proj}_{\leftarrow n} G_n$, and hence E is Hausdorff, regular and complete. — In particular, this directly implies the completeness of all co-echelon spaces $k_1 = k_1(V) = \operatorname{ind}_{n\to} l_1(v_n)$ of order 1, another result stated in Proposition 2.10.(a).

Now, it appears that even Theorem 15. does not apply to (LB)-spaces of type $\mathcal{V}C(X)$ (cf. Definition 8.) in general; that is, say, whenever the general hypotheses of Definition 13. on X and $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ hold. (Note that $(C(X), co)$ only "rarely" is a semi-Montel space, and switching to the topology of pointwise convergence or to weak topologies does not really seem to help.) Hence the following question is still open.

Problem 2. Is it true that, for arbitrary locally compact Hausdorff spaces X and arbitrary decreasing sequences $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions v_n on X, the regular (LB)-spaces $\mathcal{V}C(X)$ are always complete?

We remark that, under relatively "mild" hypotheses on $\mathcal{V}=(v_n)_n$, completeness can indeed be proved, but we conjecture that the above problem has a positive solution in full generality.

Finishing our discussion of completeness (which concentrated mainly on the setting of (LB)-spaces), we mention in passing that Raikov gave an interesting completeness theorem for inductive limits (cf. Floret [24]) which (at least formally) does not require the inductive limit to be an (LB)-space.

We finally turn to an interesting special case of the general "subspace problem" for l.c. inductive limits. After introducing the weighted inductive limits $\mathcal{V}C(X)$ and $\mathcal{V}_0C(X)$ of spaces of continuous functions and the corresponding inductive limits $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$ of spaces of A-functions, it is natural to ask:

Problem 3. If X is a locally compact space, $\mathcal{V}=(v_n)_{n\in\mathbb{N}}$ a decreasing sequence of strictly positive continuous functions on X and $A(X)$ a linear subspace of $C(X)$ with the properties (i) and (ii) of Definition 13., must then $\mathcal{V}A(X)$ (resp., $\mathcal{V}_0A(X)$) be a topological subspace of $\mathcal{V}C(X)$ (resp., $\mathcal{V}_0C(X)$)?

In fact, it takes a closer look to realize that the latter property really does not (directly) follow from the definition of the inductive limit topologies! $\mathcal{V}A(X)$ (resp., $\mathcal{V}_0A(X)$) carries the strongest l.c. topology which makes all the injections $Av_n(X)\to\mathcal{V}A(X)$ (resp., $A(v_n)_0(X)\to\mathcal{V}_0A(X)$) continuous while the inductive limit topology of $\mathcal{V}C(X)$ (resp., $\mathcal{V}_0C(X)$) is "only" the finest l.c. one which makes the injections $Cv_n(X)\to\mathcal{V}C(X)$ (resp., $C(v_n)_0(X)\to\mathcal{V}_0C(X)$) (of much larger spaces) continuous. That is, by definition we clearly have continuous injections $\mathcal{V}A(X)\to\mathcal{V}C(X)$ and $\mathcal{V}_0A(X)\to\mathcal{V}_0C(X)$, but a priori, there is absolutely no need that these injections really were topological isomorphisms into!

Looking at this type of problem from a more general point of view, we fix an (LF)-space $E = \operatorname{ind}_{n\rightarrow} E_n$. It is a bad and disturbing fact – which actually led to many errors in the literature – that a closed linear subspace F of E need not be an (LF)-space in the induced topology: That is, if F_n denotes the closed linear subspace $F \cap E_n$ of E_n with the induced topology, then $\operatorname{ind}_{n\rightarrow} F_n$ clearly equals F algebraically, but, for the reasons detailed in the special case above, the inductive limit topology may be strictly stronger than the topology which E induces. Even worse, the two topologies can have different duals, and all this can happen "in a very natural way"!

16. Definition. Let $(E,\tau) = \operatorname{ind}_{\alpha\rightarrow} (E_\alpha,\tau_\alpha)$ be an injective l.c. inductive limit and F a linear subspace of E. We put $F_\alpha := F \cap E_\alpha$ with the topology (induced by) τ_α.

(a) F is termed <u>stepwise closed</u> if F_α is closed in E_α for each α.

(b) F is called a <u>limit subspace</u> if the inductive limit topology τ of $E = \operatorname{ind}_{\alpha\rightarrow} (E_\alpha,\tau_\alpha)$ induces the inductive limit topology $\tilde{\tau}$ of $\operatorname{ind}_{\alpha\rightarrow} (F_\alpha,\tau_\alpha)$ on F.

(c) F is called <u>well-located</u> (in E) if the restriction of the inductive limit topology τ of E to F and the topology $\tilde{\tau}$ of $\operatorname{ind}_{\alpha\rightarrow} (F_\alpha,\tau_\alpha)$ yield the same dual of F; i.e., $(F,\tau)' = (F,\tilde{\tau})'$.

Clearly, each closed linear subspace F of $(E,\tau) = \operatorname{ind}_{\alpha\rightarrow} (E_\alpha,\tau_\alpha)$ is stepwise closed, and each limit subspace must be well-located. In the other direction, the following are some general results. (Proposition 17.(c) is due to de Wilde, cf. [27].)

17. Proposition. (a) In a countable weakly compact inductive limit $E = \operatorname{ind}_{n\rightarrow} E_n$, each stepwise closed linear subspace is closed and well-located.

(b) In a countable compact inductive limit $E = \operatorname{ind}_{n\to} E_n$ (where, by 2.4.(b), the inductive limit topology is just the finest topology which makes all injections $i_n: E_n \to E$ continuous), each stepwise closed subspace must even be a (closed) limit subspace.

(c) In countable inductive limits of metrizable Schwartz spaces (such as $\mathcal{D}(\Omega)$), well-located subspaces are always limit subspaces.

But the work of Ehrenpreis, Hörmander and Malgrange made it transparent that the question of solving a linear partial differential equation (or a convolution equation), say, in the space $\mathcal{D}'(\Omega)$ is closely related to the well-locatedness of the range of the transposed continuous linear operator (cf. Floret [27]). Due to this fact, there are (actually a vast amount of) closed subspaces of $\mathcal{D}(\Omega)$ which are not well-located!

In the literature, the problem of limit subspaces and well-located subspaces has been treated quite thoroughly, with special emphasis on strict inductive limits and many applications to convolution operators on distribution spaces. At this point, we only refer to Dostal [18], to the last part of Floret's survey article [27], and to the (many) references quoted there.

We proceed to show that an open mapping lemma, due to A. Baernstein [1], yields a positive solution to Problem 3. under an additional, but quite natural hypothesis on the sequence $V = (v_n)_n$ of weights. The corresponding result also allows a refinement of Proposition 17.(b), and it actually is just this refinement which is necessary to make the application to our Problem 3. work.

18. Theorem. Let F denote a l.c. semi-Montel space and E a (DF)-space. (In fact, it would suffice to assume that E_b' is a Fréchet space and that all null sequences in E_b' are equicontinuous). Suppose that $T: F \to E$ is a continuous linear (but not necessarily surjective) mapping such that

(*) for each bounded subset B of E, $T^{-1}(B)$ is bounded in F.

Then T^{-1} exists as a continuous linear mapping from TF onto F, and hence T is open and even a topological isomorphism into.

Concerning the hypotheses on E given in parentheses, we point out that part (ii) in the Definition 5. of a (DF)-space E (i.e., σ-quasibarrelledness) exactly means that every bounded subset of E_b' which is the union of countably many equicontinuous sets must also be equicontinuous. Next, we note that, by condition (*), clearly $T^{-1}(0) = \{0\}$, and hence T has to be injective.

Baernstein's original proof of his "Open Mapping Lemma" 18. deduced the result by applying Pták's Open Mapping Theorem to the transposed map ${}^tT\colon E_b' \to F_b'$ (and using various "rather standard" results from the duality theory of l.c. spaces). Other proofs of 18. (which only require that E is a (gDF)-space) are possible (and were given by B. Ernst and W. Ruess).

19. Corollary. Let $E = \operatorname{ind}_{n\to} E_n$ denote a countable regular (injective) inductive limit of (DF)-spaces. For a linear subspace F of E, put $F_n := F \cap E_n$ with the induced topology, $n = 1,2,\ldots$, and equip F with the inductive limit topology of $\operatorname{ind}_{n\to} F_n$.

If F is a semi-Montel space (in particular, if $(F_n)_n$ is a compact inductive sequence or if all the spaces F_n are semi-Montel), then $F = \operatorname{ind}_{n\to} F_n$ is a topological linear subspace of $E = \operatorname{ind}_{n\to} E_n$ (i.e., F is a limit subspace).

Proof. Let $T\colon F = \operatorname{ind}_{n\to} F_n \to E = \operatorname{ind}_{n\to} E_n$ denote the canonical (continuous linear) inclusion map. By our assumption, F is semi-Montel and E is a (DF)-space. (Note that the regular inductive limit of semi-Montel spaces certainly is semi-Montel, and that a countable inductive limit of (DF)-spaces is again (DF) by 6.(b). Moreover, if $(E_\alpha)_\alpha$ is an arbitrary inductive net and if, for a linear subspace F of $E = \operatorname{ind}_{\alpha\to} E_\alpha$, we take $F_\alpha = F \cap E_\alpha$ with the induced topology, then $(E_\alpha)_\alpha$ regular clearly implies $(F_\alpha)_\alpha$ regular.) Now fix a bounded subset B of $E = \operatorname{ind}_{n\to} E_n$. By regularity, there is $n \in \mathbb{N}$ with $B \subset E_n$ bounded. Hence $B \cap F$ is bounded in

F_n whereby $T^{-1}(B)$ must be bounded in $F = \operatorname{ind}_{n\to} F_n$. We can then apply Theorem 18. to obtain the assertion.□

We remark that, indeed, 19. yields a much sharper result than Proposition 17.(b) since there the "large" inductive limit $E = \operatorname{ind}_{n\to} E_n$ was supposed to be compact (in order to permit the conclusion that each closed linear subspace F is a limit subspace). In 19., it is only necessary to suppose that the "large" inductive limit $E = \operatorname{ind}_{n\to} E_n$ is a regular (LB)-space (which holds for countable compact inductive limits in view of 2.4) and that the "small" inductive limit $F = \operatorname{ind}_{n\to} (F \cap E_n)$ is compact (which would be implied by the assumption of compactness for $E = \operatorname{ind}_{n\to} E_n$, too). It turns out that this improvement of 17.(b) is essential for our application to $VA(X) \subset VC(X)$ since $VA(X)$ may be a compact inductive limit while $VC(X)$ "very rarely" is. (See [5].)

20. Proposition. Let $V = (v_n)_n$ denote a decreasing sequence of strictly positive continuous functions on a locally compact space X and $A(X)$ a linear subspace of $C(X)$ satisfying properties (i) and (ii) of Definition 13. If

(S) for each $n \in \mathbb{N}$, there exists $m > n$ such that $\frac{v_m}{v_n}$ vanishes at infinity on X,

then the following assertions are true:

(a) $VC(X) = V_oC(X)$ (as well as $VA(X) = V_oA(X)$) holds algebraically and topologically.

(b) If, given $n \in \mathbb{N}$, $m > n$ is chosen as in condition (S), then $Cv_m(X)$ and $(C(X),co)$ (and hence $VC(X)$, too) induce the same topology on each bounded subset of $Cv_n(X)$.

(c) With the same choice of n and m, the injection $Av_n(X) \to Av_m(X)$ is compact, and hence $VA(X) = V_oA(X)$ is a (DFS)-space.

Since 20.(a) is obvious, and since 20.(c) clearly follows from part (b), only 20.(b) has to be checked, and this is quite easy. (In fact, there is also the following converse of 20.(b): If v_n and $v_m \le v_n$ are continuous strictly positive functions on a locally compact space X, and if $Cv_m(X)$ induces the compact-open topology co on each bounded subset of $Cv_n(X)$, then $\frac{v_m}{v_n}$ must necessarily vanish at infinity on X.)

For the sequence space case of Proposition 20., see 2.14. – It may also be useful to recall at this point that, by a lemma of Grothendieck (e.g., see Horváth's book [35]), two l.c. topologies which coincide on an absolutely convex subset actually induce the same uniform structures on this set, too.

We are now ready to present an affirmative solution to Problem 3. in the case that the sequence V satisfies condition (S).

21. Corollary. Under the hypotheses of Proposition 20., $VA(X) = V_0A(X)$ is indeed a topological subspace of $VC(X) = V_0C(X)$.

Proof. By our remarks after Definition 8., the inductive limit $VC(X)$ is always regular, and by Proposition 20., $VA(X)$ is a compact inductive limit whenever condition (S) holds. Hence the assertion follows from Corollary 19. □

We note that condition (S) is very natural and that this condition is satisfied in most applications for weighted inductive limits of A-functions (such as holomorphic or harmonic functions). However, the general case of Problem 3. remains open.

Appendix. Strong regularity conditions

In this Appendix to Section 3., we study several "strong regularity conditions" for l.c. inductive limits (which were introduced by various authors, cf. [31], [25], [3], [5], [6] and [45]) and the relations between them (listing interesting results of Neus [45] and – very recently – Cascales-Orihuela [16]). At the end of the appendix, we shortly return to the problem of limit subspaces and well-located subspaces, reporting on the work of Retakh [49], [50] which is closely related to some ideas in our study of strong regularity.

We start with the so-called "bounded retractivity" which appears to be a very natural property and which is suggested by Grothendieck's "strict Mackey convergence condition". Only this first part of the Appendix will actually be needed in Section 4.

1. Definition. An (injective l.c.) inductive system $(E_\alpha, \tau_\alpha)_{\alpha \in A}$, or its limit $(E,\tau) = \operatorname{ind}_{\alpha \to} (E_\alpha, \tau_\alpha)$, is said to be <u>boundedly retractive</u> if, for every bounded subset B of (E,τ), there exists $\alpha = \alpha(B) \in A$ such that

(i) B is contained (and bounded) in (E_α, τ_α) and, moreover,

(ii) $\tau_{|B}$ equals $\tau_{\alpha|B}$;

i.e., B, with the restriction of the inductive limit topology τ, can actually be "retracted" as a bounded subset of one of the generating spaces E_α, together with the restriction of the topology τ_α.

Clearly, countable strict inductive limits $E = \operatorname{ind}_{n \to} E_n$ with E_n closed in E_{n+1} for each n are not only regular (by 1.3.(b)), but even boundedly retractive. And since the topology of each relatively compact set coincides with any weaker Hausdorff topology, it is easily verified that countable compact (injective) inductive limits (see 2.4.(b) for their regularity) are boundedly retractive as well. Thus, the class of all (countable) boundedly retractive inductive limits provides a joint generalization of all "good" strict and all "good" compact inductive limits. Moreover, in the setting of Proposition 3.20 (i.e., in particular, if the sequence V satisfies condition (S)), part (b) of this proposition implies that $VC(X) = V_0C(X)$ is a boundedly retractive inductive limit which, obviously, is neither strict nor compact in general.

On the other hand, boundedly retractive inductive limits are always regular, and all the bounded subsets in a boundedly retractive inductive limit of metrizable l.c. spaces must themselves be metrizable.

Furthermore, since also the uniform structure induced by the inductive limit topology τ of a boundedly retractive inductive limit $(E,\tau) = \underset{\alpha \to}{\text{ind}}\ (E_\alpha, \tau_\alpha)$ coincides on each bounded subset B of E with the uniform structure induced by some τ_α (see the note after 3.20), the boundedly retractive inductive limits of quasi-complete l.c. spaces are quasi-complete as well. In particular, in view of 3.6, this implies that the inductive limit of a boundedly retractive sequence of (quasi-) complete (DF)-spaces must again be a complete (DF)-space. At this point, it is obvious that Problem 2. of Section 3., asking for the completeness of $VC(X)$, does have an affirmative answer whenever the decreasing sequence V satisfies condition (S) of Proposition 3.20 (but a much weaker condition would already suffice as we will see in Section 4.).

The following remark of Mujica (which may be deduced from the last part of 3.6.(b) in a rather straightforward way) shows that, in particular, for inductive sequences of Banach spaces, hypothesis (ii) in Definition 1. already implies (i); i.e., in this case, regularity of the inductive sequence actually follows from assumption (ii) on the topologies.

2. *Proposition.* Let $E = \underset{n \to}{\text{ind}}\ E_n$ denote the (Hausdorff injective) inductive limit of a sequence of complete (DF)-spaces. If, for each $n \in \mathbb{N}$, there exists $m \geq n$ such that E and E_m induce the same topology on each bounded set B in E_n, then $(E_n)_n$ is regular (and hence boundedly retractive).

For general l.c. spaces, Grothendieck [31] introduced the following:

3. *Definition.* A l.c. space (E,τ) satisfies <u>the strict Mackey convergence condition (s.M.c.c.)</u> if, for each bounded set A in E, there exists a closed bounded and absolutely convex subset B of E containing A such that E and E_B induce the same topology on A.

Returning to our previous terminology and taking into account that, on a set A as in Definition 3., the inductive limit topology τ^x of $\underset{B \to}{\text{ind}}\ E_B$ lies between the topologies induced by E_B and E, we see that the s.M.c.c. is stronger than requiring that the (always regular) inductive limit $\underset{B \to}{\text{ind}}\ E_B$ is boundedly retractive and that it actually coincides with this requirement whenever E is bornological (so that $\tau = \tau^x$). Similarly, if E is a quasibarrelled l.c. space, $E_i' = \underset{U \to}{\text{ind}}\ E_{U^\circ}'$ is always regular, and the s.M.c.c. for E_b' is stronger than requiring that $\underset{U \to}{\text{ind}}\ E_{U^\circ}'$ is boundedly retractive and exactly coincides with this statement whenever $E_b' = E_i'$.

As another connection between Definitions 1. and 3., we should perhaps explicitly state that, obviously, an inductive limit of normed spaces is boundedly retractive if and only if it is regular and satisfies the s.M.c.c.

The next definition is again due to Grothendieck [31].

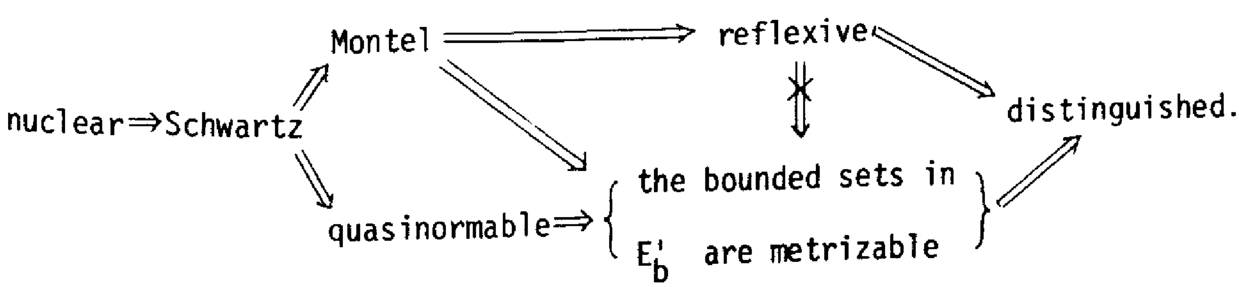
4. *Definition.* A. l.c. space E is called quasinormable if, for every closed absolutely convex 0-neighborhood U in E, there exists a 0-neighborhood V in E such that, for each $\alpha > 0$, we can find a bounded set B in E with $V \subset \alpha U + B$, or equivalently, if, for every equicontinuous subset A of E', there exists a neighborhood V of 0 in E such that E_b' and the topology of uniform convergence on V coincide on A.

By duality, we now obviously obtain:

5. *Proposition.* A quasibarrelled l.c. space E is quasinormable if and only if its strong dual E_b' satisfies the strict Mackey convergence condition.

In order to understand the quasinormability in a better way, let us recall the following facts (most of which are taken from [31], but the first one is actually an improvement of Grothendieck's work, due to Kats and Ruess, cf. [51]): Each (DF)- (and even each (gDF)-) space is quasinormable. A l.c. space E is a Schwartz space if and only if E is quasinormable and each bounded subset of E is precompact. There are Fréchet-Montel spaces E (even such spaces of type $\lambda_1(A)$, see Köthe [38]) which are not Schwartz, and hence cannot be quasinormable.

If E is a metrizable l.c. space, E_b' is of type (DF), and hence the localization property mentioned after Proposition 3.6 holds. Thus, if E additionally is quasinormable, then the s.M.c.c. for E_b' (see Proposition 5. above), asserting (in particular) that the topologies of E_b' and E_i' agree on each bounded subset of E_b', implies the topological equality $E_b' = E_i'$; i.e., in view of Proposition 3.10, it implies that E is distinguished. In fact, due to Grothendieck [31], we even have the following diagram of implications between various properties of Fréchet spaces E (in which, as it also turns out, none of the implications can be reversed):



We now turn to the other strong regularity conditions for inductive limits which we will study and start with a list of definitions.

6. *Definition.* Let $(E,\tau) = \operatorname{ind}_{\alpha \to} (E_\alpha, \tau_\alpha)$ denote a (Hausdorff injective) l.c. inductive limit. The inductive spectrum $(E_\alpha, \tau_\alpha)_{\alpha \in A}$ or its limit (E,τ) is called

(a) <u>strongly boundedly retractive</u> if

(i) $(E_\alpha, \tau_\alpha)_\alpha$ is regular and if,

(ii) for each $\alpha \in A$, there exists $\beta \geq \alpha$ such that (E,τ) and (E_β, τ_β) induce the same topology on each bounded subset B of (E_α, τ_α);

(b) Cauchy regular if, given a bounded set B in (E,τ), there is $\alpha=\alpha(B)\in A$ such that

(i) B is contained and bounded in (E_α,τ_α), and

(ii) a net $(x_i)_{i\in I}\subset B$ is τ-Cauchy if and only if it is Cauchy in (E_α,τ_α);

(c) (convex) compactly regular if, for each (convex) compact subset K of (E,τ), there is $\alpha=\alpha(K)\in A$ such that K is contained and compact in (E_α,τ_α);

(d) sequentially retractive if, for each null sequence $(x_m)_{m\in\mathbb{N}}$ in (E,τ), there exists some $\alpha=\alpha((x_m)_m)\in A$ such that $(x_m)_m$ is contained and a null sequence in (E_α,τ_α).

(In the references [45] resp. [16], one can also find the following variants of (c) resp. (d): $(E_\alpha,\tau_\alpha)_\alpha$ or (E,τ) is said to be

(e) sequentially compactly regular if, given any sequentially compact subset S of (E,τ), there is $\alpha=\alpha(S)\in A$ such that S is contained and sequentially compact in (E_α,τ_α);

(f) precompactly retractive if, given any precompact subset P of (E,τ), there is $\alpha=\alpha(P)\in A$ such that P is contained in (E_α,τ_α) and $\tau_{|P}=\tau_{\alpha|P}$.)

A few (easy) remarks are in order at this point. Countable strict inductive limits $E=\operatorname{ind}_{n\to} E_n$ with E_n closed in E_{n+1} for each $n\in\mathbb{N}$ and countable compact inductive limits are even strongly boundedly retractive, as is $\mathcal{V}C(X)=\mathcal{V}_0C(X)$ in the setting of Proposition 3.20 (i.e., if $\mathcal{V}$ satisfies condition (S)). And, concerning Definition 6.(a), Proposition 2. actually shows that, for an inductive sequence $(E_n)_{n\in\mathbb{N}}$ of complete (DF)-spaces, part (ii) of 6.(a) already implies (i).

Clearly, strongly boundedly retractive inductive limits are boundedly retractive, and an inductive limit of normed spaces is boundedly retractive if and only if it is strongly boundedly retractive. Similarly, boundedly retractive inductive limits must be Cauchy regular (recall the note on uniform structures after 3.20), and an inductive limit of quasi-complete spaces is Cauchy regular if and only if it is boundedly retractive.

Next, boundedly retractive implies both regular and compactly regular. Moreover, since the topology of every compact set coincides with any weaker Hausdorff topology, compact regularity equals "compact retractivity" in the sense that, for each compact K in $E = \operatorname{ind}_{\alpha \to} E_\alpha$, there is $\alpha \in A$ such that the topology of E_α and the (weaker Hausdorff) topology of E coincide on K. Hence compactly regular inductive limits are sequentially retractive. For examples of (uncountable) hyperstrict inductive limits which are not convex compactly regular, resp., which are convex compactly regular, but neither compactly regular nor regular, see 1.8.(b) and (d).

In the case of countable inductive limits, sequential retractivity was introduced and studied by Floret [25]. He proved that sequentially retractive inductive sequences are regular and that sequentially retractive (LF)-spaces are sequentially complete.

(We note that properties 6.(a) and (b) imply regularity, and hence our initial hypothesis that the inductive limit topology is Hausdorff is not needed there. Similarly, by Floret's result which we have just quoted, inductive sequences with the properties of 6.(c) and (d) always have a separated inductive limit.)

For some natural examples of l.c. inductive limits with the properties defined in 6.(a) through (d) and for some applications of these notions, see [3], [5], [6] and [45]. – Part of the preceding discussion can be summed up as follows.

7. *Proposition*. For arbitrary inductive limits $E = \operatorname{ind}_{\alpha \to} E_\alpha$ of Banach spaces, the following conditions are equivalent:

(1) strong bounded retractivity, (2) bounded retractivity,
(3) Cauchy regularity, (4) regularity plus the s.M.c.c.

And any of these conditions implies compact regularity, sequential retractivity and regularity.

In the setting of countable inductive limits, Neus [45] found some "less obvious" equivalences which we will now survey. – We refer to the book [46] of Bonet-Pérez Carreras for a thorough discussion of strong regularity conditions (including results of Neus) and full proofs; also see Valdivia [59].

Up to this moment, all our regularity conditions actually involved the inductive limit topology itself, but note that it may not be so easy to get a hold of this topology! We will now define two other properties of inductive limits which are only formulated in terms of the topologies of the generating spaces in the inductive spectrum.

8. Definition. An injective l.c. inductive spectrum $(E_\alpha,\tau_\alpha)_{\alpha\in A}$ (or its limit (E,τ)) is termed

(a) <u>boundedly stable</u> if, for every set B which is bounded in one of the generating spaces (E_α,τ_α), there exists $\beta=\beta(B)\geq\alpha$ such that, for every $\gamma\geq\beta$, (E_β,τ_β) and (E_γ,τ_γ) induce the same topology on B;

(b) <u>strongly boundedly stable</u> if, for each $\alpha\in A$, there exists $\beta\geq\alpha$ such that, for every $\gamma\geq\beta$, (E_β,τ_β) and (E_γ,τ_γ) induce the same topology on each bounded subset of (E_α,τ_α).

Obviously, not even strongly boundedly stable inductive limits need to be regular (just take any non-regular hyperstrict inductive limit). And, clearly, strongly boundedly stable implies boundedly stable, while the two notions are equivalent for inductive limits of normed spaces.

Next, (strongly) boundedly retractive inductive limits are both regular and (strongly) boundedly stable, and one can hope that, at least for some "good" regular inductive limits, the converse also holds. In fact, say, if $(E_n)_{n\in\mathbb{N}}$ is an (injective) inductive sequence and if, for each $n\in\mathbb{N}$, there is $m\geq n$ such that, for all $k\geq m$, (E_m,τ_m) and

(E_k,τ_k) induce the same topology on each bounded subset B of (E_n,τ_n), then there might also exist $N \geq n$ such that (E_N,τ_N) (and hence all (E_k,τ_k) for $k \geq m$) and the inductive limit (E,τ) induce the same topology on each bounded set B in (E_n,τ_n). There is a result of this type, due to Retakh [49] (and based on the homological theory of Palamodov), which (together with a lemma of Grothendieck and a result of de Wilde) leads Neus [45] to:

9. *Proposition.* Let $(E,\tau) = \operatorname{ind}_{n\to} (E_n,\tau_n)$ denote a countable (injective) inductive limit.

(a) $(E_n,\tau_n)_{n\in\mathbb{N}}$ is sequentially retractive if and only if it is sequentially compactly regular (in the sense of Definition 6.(e)).

(b) If the inductive sequence $(E_n,\tau_n)_n$ is sequentially retractive, then, for each bounded subset B of (E,τ), there exists $n = n(B) \in \mathbb{N}$ such that

(i) B is contained (and bounded) in (E_n,τ_n) and

(ii) (B,τ) and (B,τ_n) have the same convergent sequences.

Hence, if, additionally, the bounded subsets of each space (E_n,τ_n) are metrizable, then $\tau_n|_B = \tau_m|_B$ holds for every $m \geq n$ (and if we also know that the bounded subsets of (E,τ) are metrizable, then we can even conclude $\tau_{n|B} = \tau_{|B}$). In particular, a sequentially retractive inductive sequence $(E_n,\tau_n)_n$ of metrizable spaces is (regular and) boundedly stable (and if the bounded subsets of the limit (E,τ) are metrizable as well, it must even be boundedly retractive.)

(c) A countable strongly boundedly stable inductive limit of (DF)- (or (gDF)-) spaces always satisfies property (ii) in Definition 6.(a) (of strong bounded retractivity). Hence each regular limit of this type (and, in view of Mujica's Proposition 2., in particular each strongly boundedly stable sequence of complete (DF)-spaces) is strongly boundedly retractive.

(d) For countable inductive limits of normed spaces, the following properties are equivalent:

(1) sequentially retractive,

(2) sequentially compactly regular,

(3) compactly regular,

(4) regular plus (strongly) boundedly stable,

(5) (strongly) boundedly retractive.

The article [45] of Neus also contains an example of a boundedly retractive inductive limit of bornological (DF)-spaces which is not strongly boundedly retractive.

It is interesting to observe that, by 7. and 9., all the different strong regularity assumptions for l.c. inductive limits to which various authors were led (following different approaches, and having different examples and applications in mind) really yield the same notion for (LB)-spaces. This notion of, say, bounded retractivity is strictly stronger than regularity and implies completeness of the (LB)-space, but not conversely:

In fact, there is another famous example of Köthe and Grothendieck (again, see [38]); viz., a Fréchet-Montel space E (of type $\lambda_1(A)$) which is not Schwartz and hence not quasinormable. The strong dual E_b' $(= K_\infty(\bar{V}))$ of this space is complete, and it coincides with the inductive dual $E_i' = \operatorname{ind}_{n \to} E'_{U_n^\circ}$, where $(U_n)_{n \in \mathbb{N}}$ denotes a decreasing fundamental sequence of 0-neighborhoods for E (and $E_i' = k_\infty(V)$). Now, E_i' is a (regular and) complete (LB)-space, but since E is not quasinormable, $E_i' = E_b'$ does not have the s.M.c.c. and hence none of the "strong regularity" properties.

For a detailed study of the differences between (Fréchet-Montel and (FS)-spaces as well as between) strong duals of Fréchet-Montel and (DFS)-spaces and some interesting related material, we again refer to Bonet-Perez Carreras [46].

We turn to the work of Retakh [50] on limit and well-located subspaces of l.c. inductive limits (see 3.16 for definitions). It will immediately become clear after the next definition that the ideas in this approach are closely related to part of our preceding discussion.

10. *Definition.* A countable l.c. inductive limit $(E,\tau) = \operatorname{ind}_{n\to} (E_n,\tau_n)$ of metrizable l.c. spaces satisfies

(a) <u>condition (M)</u> if there exists an increasing sequence of absolutely convex neighborhoods U_n of 0 in (E_n,τ_n) such that, for each $n \in \mathbb{N}$, there is $m \geq n$ with the property that, for every $k \geq m$, τ_m and τ_k induce the same topology on U_n;

(b) <u>condition (M_0)</u> if there exists an increasing sequence of absolutely convex neighborhoods U_n of 0 in (E_n,τ_n) such that, for each $n \in \mathbb{N}$, there is $m \geq n$ with the property that, for every $k \geq m$, for every $f \in E_m'$ and for every $\varepsilon > 0$, there exists $g \in E_k'$ with

$$|f(x) - g(x)| < \varepsilon \quad \text{for all } x \in U_n.$$

As Neus [45] remarks, a result of de Wilde permits the following equivalent reformulation of condition (M_0):
There exists an increasing sequence of absolutely convex 0-neighborhoods U_n in (E_n,τ_n) such that, for each $n \in \mathbb{N}$, there is $m \geq n$ with the property that, for every $k \geq m$, the weak topologies $\sigma(E_m,E_m')$ and $\sigma(E_k,E_k')$ coincide on U_n.

Countable strict and countable compact inductive limits of metrizable l.c. spaces clearly satisfy condition (M), and any countable weakly compact inductive limit of metrizable l.c. spaces satisfies (the second version of) condition (M_0). It is not hard to verify that (M) implies (M_0) (e.g., for the original version of (M_0), this follows from a well-known approximation lemma of Grothendieck, cf. [35]).

Condition (M) also clearly implies bounded stability, and for countable inductive limits of normed spaces, the converse of this implication holds as well. Hence, in view of Proposition 9.(d), a countable regular inductive limit of normed spaces satisfies condition (M) if and only if it is (strongly) boundedly retractive. In a more general setting, Retakh [49] asserts (cf. Proposition 9.(c); as Valdivia [59] notes, the metrizability of the spaces E_n is not needed here):

11. Proposition. Let $(E_n,\tau_n)_{n\in\mathbb{N}}$ denote an (injective) inductive sequence of metrizable l.c. spaces and (E,τ) its limit. Then condition (M) implies (and is obviously always implied by) the following condition (M'):

There exists an increasing sequence of absolutely convex 0-neighborhoods U_n in (E_n,τ_n) such that, for each $n\in\mathbb{N}$, there is $m\geq n$ with the property that τ_m and the inductive limit topology τ coincide on U_n.

In particular, if $E = \text{ind}_{n\to} E_n$ is a countable regular inductive limit of metrizable l.c. spaces with condition (M), then it is already strongly boundedly retractive.

From the point of view of finding a very strong property which is shared by both countable strict and countable compact inductive limits of Fréchet (and not "just" Banach) spaces, condition (M) is better than strong bounded stability, and condition (M') is better than part (ii) of the definition of strong bounded retractivity since it takes into account that compact linear operators between l.c. spaces map suitable 0-neighborhoods (and not "just" all bounded sets) into compact sets. — Retakh's results on the subspace and well-locatedness problems in (LF)-spaces demonstrate the importance of the conditions (M), (M') and (M_0).

Before stating Retakh's main theorem, we remark that, as part of the definition, condition (M) resp. (M_0) on a countable l.c. inductive limit $E = \text{ind}_{n\to} E_n$ always requires that all the spaces E_n are metrizable. Moreover, Retakh states his result without the hypothesis that F is a stepwise closed linear subspace of E; we have added this assumption here in order to ensure that all the quotients E_{n/F_n}, $F_n = F\cap E_n$, are Hausdorff.

12. Theorem. Let $(E,\tau) = \text{ind}_{n\to} (E_n,\tau_n)$ denote a countable (injective) inductive limit and $F\subset E$ a stepwise closed linear subspace. We put, as usual, $F_n := F\cap E_n$ with the induced topology, $n = 1,2,\ldots$

(a) If $\text{ind}_{n\to} E_{n/F_n}$ satisfies condition (M) (resp., (M_0)), then F is a limit subspace of E (resp., well-located in E).

(b) Now assume that $E = \operatorname{ind}_{n\to} E_n$ satisfies condition (M) (resp., (M_0)), or that $E = \operatorname{ind}_{n\to} E_n$ is strict and all the quotients E_{n/F_n} are metrizable. If F is a limit subspace of E (resp., well-located in E), then, conversely, $\operatorname{ind}_{n\to} E_{n/F_n}$ must satisfy condition (M) (resp., (M_0)), too.

We remark that Theorem 12. completely solves the subspace and well-locatedness problem in many important cases (say, if $E = \operatorname{ind}_{n\to} E_n$ is an (LF)-space with condition (M) resp. (M_0)). But, as it sometimes happens with full characterizations, 12. often remains a "theoretical" solution in the sense that the hypothesis is formulated in terms of the quotients E_{n/F_n}, which may not be easily accessible, and therefore it can be hard to verify condition (M) resp. (M_0) for these quotients in many concrete examples.

E.g., for various applications (as in Corollary 3.21), Corollary 3.19 to Baernstein's Lemma – which, of course, only yields a "nice" sufficient condition for an affirmative solution of the subspace problem, but remains far from any characterization – is much easier to handle than Retakh's Theorem 12. (And, indeed, it is non-trivial to deduce some version of Corollary 3.19 from 12.)

At the end of this appendix, it remains to report on (very recent) interesting work of Cascales-Orihuela (cf. [16]) on countable inductive limits of metrizable (or more general) l.c. spaces. – The following definition is due to Fremlin (see Floret [62] for more information).

13. Definition. A l.c. space E is called <u>angelic</u> if, for every relatively countably compact subset B of E,

(i) B must actually be relatively compact and,

(ii) for each $x \in \bar{B}$, there exists a sequence in B which converges to x.

Among other things, Cascales and Orihuela give an affirmative answer to Question 7.6 in Floret [27] whether (LF)-spaces must be angelic. Here is a summary of (part of) their main results.

14. Proposition. (a) There is a class θ of l.c. spaces which contains all metrizable and all dual metric l.c. spaces and which is stable under arbitrary linear subspaces, separated quotients, completions as well as countable topological products and l.c. direct sums, and hence also under countable projective and (Hausdorff) l.c. inductive limits (in particular, each countable injective Hausdorff inductive limit $E = \underset{n \to}{\text{ind}}\, E_n$ of metrizable l.c. spaces E_n belongs to this class θ), such that:

(b) Every space E in the class θ is angelic, and all the precompact subsets of E are metrizable.

(c) Each space $E \in \theta$ is even weakly angelic; i.e., angelic with respect to the topology $\sigma(E,E')$.

(d) If $E = \underset{n \to}{\text{ind}}\, E_n$ is a countable (injective) inductive limit of spaces $E_n \in \theta$, then the following properties of E are equivalent:

(1) sequentially retractive,

(2) sequentially compactly regular,

(3) compactly regular,

(4) precompactly retractive (cf. Definition 6.(f)).

And if, additionally, all E_n are complete, then these properties are also equivalent to "precompact regularity" in the sense that,

(5) for every precompact subset P of E, there exists $n \in \mathbb{N}$ such that P is contained and precompact in E_n.

(e) If E is an inductive limit as in (d), then the following conditions are equivalent:

(1) For every null sequence $(x_m)_{m \in \mathbb{N}}$ in E, there is $n \in \mathbb{N}$ such that $(x_m)_m$ is contained and a $\sigma(E_n,E_n')$-null sequence in E_n;

(2) for every precompact set P in E, there is $n \in \mathbb{N}$ such that P is contained in E_n and $\sigma(E,E')_{|P} = \sigma(E_n,E_n')_{|P}$.

(For a result similar to Proposition 11., but concerning condition (M_0) and weak topologies, and for some related results in the direction of Proposition 14.(e), we again refer to Valdivia [59].)

4. Projective descriptions of weighted inductive limits

In Section 3., the weighted inductive limits $\mathcal{V}C(X)$, $\mathcal{V}_0C(X)$ (resp., $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$) of spaces of continuous functions (resp., spaces of "A-functions", such as holomorphic or harmonic functions) were introduced. Following the program that was outlined in Section 0., we will now set out to give an explicit characterization of a basis for the continuous seminorms of the corresponding inductive limit topologies. In fact, we will study the general problem of "projective descriptions" for weighted inductive limits in some detail and solve it in a number of important cases. Then, restricting the results to the setting of sequence spaces and using the duality theory available in this setting, the promised complete solutions to Questions 1. through 3. of Section 2. will follow. (Actually, the solution to Question 2. involves the main results in the author's (very recent) joint article [11] with José Bonet.)

References on weighted inductive limits and their projective descriptions are [3] - [5], [7], [9] and [10] (as well as Bonet [12], and other articles of Bonet quoted in [10], for the case of vector-valued functions); we especially recommend [5] and the survey [10]. Special attention is given to the sequence space setting in [8] - [11] (as well as in Reiher [48], where the more general cases of "Dubinsky echelon and co-echelon spaces" and "Köthe function spaces" are treated, using the same methods). We also refer to the book [46] of Bonet and Pérez Carreras.

In the general setting of weighted spaces of continuous functions as outlined in Definition 3.8, a seminorm p on $\mathcal{V}C(X) = \operatorname{ind}_{n\to} Cv_n(X)$ resp. $\mathcal{V}_0C(X) = \operatorname{ind}_{n\to} C(v_n)_0(X)$ is continuous if and only if its restriction to each of the normed spaces $Cv_n(X)$ resp. $C(v_n)_0(X)$ is continuous;

i.e., if and only if, for each $n \in \mathbb{N}$, there is a constant $C_n > 0$ such that

$$p(f) \leq C_n \|f\|_n = C_n \sup_{x \in X} v_n(x)|f(x)|$$

for each $f \in Cv_n(X)$ resp. $C(v_n)_o(X)$.

In this way, we get an inequality

$$p(f) \leq \inf_{n \in \mathbb{N}} \sup_{x \in X} C_n v_n(x)|f(x)|,$$

but it would be much better if we could interchange inf and sup here in order to obtain

$$p(f) \leq \sup_{x \in X} (\inf_{n \in \mathbb{N}} C_n v_n(x))|f(x)|$$

since $\inf_n C_n v_n$ can be considered as a new non-negative, upper semicontinuous "weight" $\bar{v}$ on X, associated with the decreasing sequence $V = (v_n)_{n \in \mathbb{N}}$ of strictly positive upper semicontinuous functions v_n on X. Of course, the interchange of inf and sup remains doubtful, but it certainly makes sense to define:

1. Definition. For a decreasing sequence $V = (v_n)_{n \in \mathbb{N}}$ of strictly positive weights v_n on X, put

$$\bar{V} = \bar{V}(V) = \{\bar{v}: X \to \mathbb{R}_+ \text{ upper semicontinuous; for each } n \in \mathbb{N},$$
$$\sup_{x \in X} \frac{\bar{v}(x)}{v_n(x)} < \infty\}.$$

In other words, $\bar{V}$ collects all those weights $\bar{v}$ on X which yield weight conditions weaker than all the weight conditions given by the functions v_n, $n = 1,2,\ldots$, and each $\bar{v} \in \bar{V}$ is dominated by a function of the form $\inf_n C_n v_n$, where C_n denotes a positive constant for each $n \in \mathbb{N}$.

In general, $\bar{V}$ need not contain any strictly positive function on X (for an example, see [8]), but if, say, X is locally compact and σ-compact and all the weights v_n in V are continuous, then one can easily see (cf. [7]) that each $\bar{v} \in \bar{V}$ is even dominated by a strictly positive and continuous function $\tilde{v} \in \bar{V}$.

We now introduce the weighted spaces canonically associated with the "Nachbin family" $\bar{V}$.

2. *Definition.* For the system $\bar{V} = \bar{V}(V)$ associated with a decreasing sequence $V = (v_n)_n$ as in Definition 1., we put

$$C\bar{V}(X) := \{f \in C(X);\ \text{for each } \bar{v} \in \bar{V},\ p_{\bar{v}}(f) := \sup_{x \in X} \bar{v}(x)|f(x)| < \infty\}$$

and

$$C\bar{V}_0(X) := \{f \in C(X);\ \text{for each } \bar{v} \in \bar{V},\ \bar{v}f \text{ vanishes at infinity on } X\}.$$

Endowed with the system $(p_{\bar{v}})_{\bar{v} \in \bar{V}}$ of seminorms, $C\bar{V}(X)$ is a l.c. space, and $C\bar{V}_0(X)$ is a closed subspace which will be equipped with the induced topology.

If (*) for each compact $K \subset X$, $\inf_{x \in K} v_n(x) > 0$, $n = 1,2,\ldots$, then, as we have pointed out after 3.8, the topologies of $VC(X)$ and $V_0C(X)$ are stronger than the compact-open topology co. In fact, since (*) implies that the characteristic function of each compact subset of X belongs to $\bar{V}$, the weighted topology of $C\bar{V}(X)$ is again stronger than co, and hence $C\bar{V}(X)$ and $C\bar{V}_0(X)$ are complete if X is a $k_{\mathbb{R}}$-space (again, see after 3.8).

By the very definition of $\bar{V}$ and of the inductive limit topology, we have continuous injections

$$VC(X) \to C\bar{V}(X) \quad \text{and} \quad V_0C(X) \to C\bar{V}_0(X),$$

but, hopefully, more than this trivial fact holds!

<u>General problem of projective descriptions for weighted inductive limits of continuous functions</u>. Describe the exact relationship between $\mathcal{V}C(X)$ and $C\bar{V}(X)$ as well as between $\mathcal{V}_0C(X)$ and $C\bar{V}_0(X)$.

An algebraic equality, say, $\mathcal{V}C(X) = C\bar{V}(X)$ would at least tell us that not too much is lost by our rather "simple-minded" direct approach. But if (say) $\mathcal{V}_0C(X)$ turned out to be a topological subspace of $C\bar{V}_0(X)$, then we would actually obtain the desired explicit description of a basis of the continuous seminorms for the inductive limit topology of $\mathcal{V}_0C(X)$ by means of "natural" weighted sup-seminorms.

Similarly, in the general setting of Definition 3.13 (with a predetermined linear subspace $A(X)$ of $C(X)$ satisfying conditions (i) and (ii) of that definition), we put

$$A\bar{V}(X) := C\bar{V}(X) \cap A(X) \quad \text{and}$$
$$A\bar{V}_0(X) := C\bar{V}_0(X) \cap A(X)$$

with the induced weighted topology. $\mathcal{V}A(X)$ is continuously injected in $A\bar{V}(X)$, as is $\mathcal{V}_0A(X)$ in $A\bar{V}_0(X)$, but, again, there is some hope that more might be true.

<u>General problem of projective descriptions for weighted inductive limits of spaces of A-functions</u>. Describe the exact relationship between $\mathcal{V}A(X)$ and $A\bar{V}(X)$ as well as $\mathcal{V}_0A(X)$ and $A\bar{V}_0(X)$.

Not all cases of these two very general problems are solved completely, but full characterizations will be achieved below in the most important settings arising from applications and/or considerations from the structure theory of sequence spaces.

From our approach above, using ideas involving growth conditions and weighted spaces, the construction of the system $\bar{V} = \bar{V}(\mathcal{V})$ of associated weights appears to be quite natural, and this approach is probably easier to understand than the corresponding developments for Köthe co-echelon spaces in Section 2. It actually was in the present context that $\bar{V}$ and the weighted spaces $C\bar{V}(X)$, $C\bar{V}_0(X)$, $A\bar{V}(X)$ and $A\bar{V}_0(X)$

were first introduced (in [5] and [7]). Only subsequently, a specialization to sequence spaces (in part of [7]) and an extension to Köthe echelon and co-echelon spaces of arbitrary order (see [8]) led to the setting of Definition 2.11 (and the results following it).

We now come to the "general solution" of the problem of projective descriptions for weighted inductive limits of continuous functions. Quite surprisingly, there is a clear dichotomy between the two different cases; viz., those of o- and O-growth conditions.

3. Theorem. (a) If X is locally compact, then $V_0C(X)$ always is a (dense) topological subspace of $C\bar{V}_0(X)$; hence, the inductive limit topology of $V_0C(X)$ is given by the system $(p_{\bar{v}})_{\bar{v}\in\bar{V}}$ of weighted sup-seminorms (where it suffices to consider only those $p_{\bar{v}}$ for which $\bar{v} = \inf_n C_n v_n$ with suitable positive constants $C_n > 0$), and, in particular, the injection $V_0C(X) \to VC(X)$ is a topological isomorphism into.

If, additionally, $\inf_{x\in K} v_n(x) > 0$ holds for each compact $K \subset X$ and each $n \in \mathbb{N}$ (e.g., if all v_n are continuous), then, algebraically and topologically,

$$C\bar{V}_0(X) = \widehat{V_0C(X)} = \text{the completion of } V_0C(X).$$

However, it is possible that the inductive limit $V_0C(X)$ is incomplete, and if this happens, $V_0C(X)$ is a proper (topological) linear subspace of $C\bar{V}_0(X)$.

(b) $VC(X)$ always equals $C\bar{V}(X)$ algebraically, and the two spaces even have the same bounded subsets; thus, as topological vector spaces,

$$VC(X) = (C\bar{V}(X))_{bor} = \text{the bornological space associated with } C\bar{V}(X).$$

In general, however, the inductive limit topology of $VC(X)$ may be strictly stronger than the weighted topology of $C\bar{V}(X)$.

Concerning the proof of Theorem 3., we first note that, by Proposition 2.10.(b) and Proposition 2.13, if A is the Köthe matrix of the Grothendieck-Köthe counterexample of a non-distinguished echelon space $\lambda_1(A)$ (A is explicitly stated in 2.10.(b)) and if $V = (\frac{1}{a_n})_{n\in\mathbb{N}}$ denotes the associated decreasing sequence, then the space $k_0 = k_0(V)$ is incomplete, and $k_\infty = k_\infty(V)$ has a strictly stronger topology than $K_\infty = K_\infty(\bar{V})$, $\bar{V} = \bar{V}(V)$. Hence counterexamples to the algebraic equality $V_0C(X) = C\bar{V}_0(X)$ and to the topological identity $VC(X) = C\bar{V}(X)$ already arise in the sequence space setting (and, in fact, one counterexample already serves both purposes).

To show the positive part of (b), it suffices to establish that, for each bounded subset of $C\bar{V}(X)$, there exists $n \in \mathbb{N}$ such that B is contained and bounded in the normed space $Cv_n(X)$, and this can be done by contradiction. It remains to prove the positive part of (a), which certainly is the most interesting result contained in Theorem 3. Actually, since $C_c(X)$ is dense in both $C\bar{V}_0(X)$ and $V_0C(X)$, it is enough to show the following (crucial).

4. *Lemma.* $V_0C(X)$ and $C\bar{V}_0(X)$ induce the same topology on $C_c(X)$.

Proof. Since the canonical injection of $V_0C(X)$ into $C\bar{V}_0(X)$ is continuous, we can fix an arbitrary neighborhood U of 0 in $V_0C(X)$ and then have to prove that the intersection of $C_c(X)$ with some 0-neighborhood in $C\bar{V}_0(X)$ is contained in U. By the description of a basis for the 0-neighborhoods in an inductive limit (cf. Section 0.), we may assume without loss of generality that U is of the form

$$\Gamma(\bigcup_{n\in\mathbb{N}} B_n), \text{ where } B_n := \{f \in C(v_n)_0(X);\ \|f\|_n \le \rho_n\}$$

and ρ_n is positive for each $n \in \mathbb{N}$. At this point, we put $\bar{v} := \inf_{n\in\mathbb{N}} \frac{2^n}{\rho_n} v_n \in \bar{V}$ and claim that $\{f \in C_c(X);\ p_{\bar{v}}(f) < 1\} \subset U$.

It remains to establish this claim; we fix $f \in C_c(X)$ with $p_{\bar{v}}(f) = \sup_{x\in X} \bar{v}(x)|f(x)| < 1$. For each $n \in \mathbb{N}$, let F_n denote the closed subset

$$\{x \in X;\ \frac{2^n}{\rho_n} v_n(x)|f(x)| \geq 1\}$$

of X. We observe that, clearly, $\bigcap_{n \in \mathbb{N}} F_n$ must be empty because, for any $x \in \bigcap_n F_n$, $\frac{2^n}{\rho_n} v_n(x)|f(x)| \geq 1$ holds for each n, whereby $\bar{v}(x)|f(x)| = \inf_n \frac{2^n}{\rho_n} v_n(x)|f(x)| \geq 1$, contradicting $p_{\bar{v}}(f) < 1$. Hence, putting $U_n := X \setminus F_n$ for each $n \in \mathbb{N}$, $(U_n)_{n \in \mathbb{N}}$ is an open covering of X, and since supp f is compact, there exists $m \in \mathbb{N}$ so that supp $f \subset \bigcup_{n=1}^{m} U_n$. Now let $(\varphi_n)_{n=1}^m \subset C_c(X)$ be a finite continuous partition of unity on supp f which is subordinate to $(U_n)_{n=1}^m$ (i.e., we have $0 \leq \varphi_n \leq 1$, supp $\varphi_n \subset U_n$ for $n = 1,\dots,m$, as well as $\sum_{n=1}^m \varphi_n \equiv 1$ on supp f and ≤ 1 on all of X). We then take $g_n := 2^n \varphi_n f \in C_c(X) \subset C(V_n)_o(X)$ for $n = 1,\dots,m$ and estimate

$$\sup_{x \in X} v_n(x)|g_n(x)| = \sup_{x \in X} \varphi_n(x)\ 2^n v_n(x)|f(x)|$$

$$= \sup_{x \in U_n} \rho_n \varphi_n(x) \frac{2^n}{\rho_n} v_n(x)|f(x)| \leq \rho_n$$

(because $x \in U_n$ implies $\frac{2^n}{\rho_n} v_n(x)|f(x)| < 1$). Thus, each g_n belongs to B_n, $n = 1,\dots,m$, and hence

$$f = \sum_{n=1}^m \varphi_n f = \sum_{n=1}^m \frac{1}{2^n} g_n \quad \text{is an element of } \Gamma(\bigcup_n B_n) = U,$$

which finishes the proof.□

If $V = (v_n)_n$ is a decreasing sequence of strictly positive continuous functions on a locally compact space X for which condition (S) of Proposition 3.20 holds, we have $VC(X) = V_oC(X)$ algebraically and topologically, and this is a boundedly retractive (LB)-space and hence complete (see the Appendix to Section 3.). Then the two positive assertions of Theorem 3. combine to yield

5. *Corollary.* For any decreasing sequence $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ of strictly positive continuous functions on a locally compact space X which satisfies

(S) for each $n \in \mathbb{N}$, there is $m > n$ such that $\frac{v_m}{v_n}$ vanishes at infinity on X,

we get algebraically and topologically

$$\mathcal{V}C(X) = \mathcal{V}_0 C(X) = C\bar{V}_0(X) = C\bar{V}(X) \quad .$$

If, under the general conditions of 3.(a), $A(X)$ denotes a linear subspace of $C(X)$ (with the properties (i) and (ii) of Definition 3.13), then Theorem 3.(a), together with any positive solution of the subspace problem for $\mathcal{V}_0 A(X)$ in $\mathcal{V}_0 C(X)$, immediately leads to the projective description result that $\mathcal{V}_0 A(X)$ is a topological subspace of $A\bar{V}_0(X)$, as the following diagram illustrates:

$$\begin{array}{ccc} \mathcal{V}_0 C(X) & \underset{\text{top.}}{\subset} & C\bar{V}_0(X) \\ \cup \text{ top.} & & \cup \text{ top.} \\ \mathcal{V}_0 A(X) & \hookrightarrow & A\bar{V}_0(X). \end{array}$$

Therefore, applying Corollary 3.21, we arrive at:

6. *Corollary.* If, under the conditions of Corollary 5., $(A(X),co)$ denotes a closed linear subspace of $(C(X),co)$ with the semi-Montel property, then we also obtain the algebraic and topological identity

$$\mathcal{V}A(X) = \mathcal{V}_0 A(X) = A\bar{V}_0(X) = A\bar{V}(X).$$

We remark that, from the point of view of weighted inductive limits of A-functions, our method of proof of Corollary 6. certainly was quite indirect, first using a partition of unity argument which only worked for the corresponding weighted inductive limits of continuous functions (to establish Theorem 3.) and then, essentially, (through Corollary 3.21) Baernstein's Open Mapping Lemma (to return to spaces of A-functions). But this method and the resulting Corollary 6. are very powerful and cover many important applications!

For spaces of entire functions, the first general theorem of the type of our Corollary 6. was proved by B. A. Taylor [58], but his proof (using Hörmander's methods from several complex variables, and completely remaining in the context of spaces of holomorphic functions) required stronger hypotheses on $V = (v_n)_n$ and can certainly not be adapted to work in the very general context of 6.

The main applications of 6. lie in what Ehrenpreis [23] calls "Fourier Analysis in Several Complex Variables"; i.e., in the study of convolution equations via duality and Fourier-Laplace transforms (and the so-called "Fundamental Principle" of Ehrenpreis). There Corollary 6. may serve as a handy tool to prove that many (LB)-spaces of test functions and (ultra-) distributions are "analytically uniform" (in the sense of [23]); for concrete new applications of this type, see O. v. Grudzinski [33]. We cannot go into details at this point since that would require several (long) definitions and various (complicated) examples, but we refer to [7] and a forthcoming joint article of the author with R. Meise and B. A. Taylor (on "sufficient" and "weakly sufficient" sets).

After stating the "general" results collected in Theorem 3., two interesting problems remain for weighted inductive limits of continuous functions; viz.:

(1) Exactly when is $V_0C(X)$ equal to $C\bar{V}_0(X)$ algebraically? Or, equivalently, find a necessary and sufficient condition for the completeness of $V_0C(X)$.

(2) When do we actually get $VC(X) = C\bar{V}(X)$ topologically? Or, equivalently, give conditions for $C\bar{V}(X)$ being bornological.

Note that, in the setting of sequence spaces, (1) and (2) essentially reduce to the Questions 1. and 2. of Section 2. (Of course, one could also ask the same type of questions for weighted inductive limits of A-functions, but, in that setting, we cannot give any "reasonable" answers in a more general context than the one of Corollary 6. Hence, from this point on, we will not return to the spaces of A-functions any more.)

We actually have a complete answer to (1).

7. *Theorem.* Let X denote a locally compact space and $V = (v_n)_n$ a decreasing sequence of strictly positive continuous functions on X. Then the following conditions are equivalent:

(1) V is "regularly decreasing"; i.e.,

for each $n \in \mathbb{N}$, there is $m \geq n$ such that, for every subset Y of X,

$$\inf_{y \in Y} \frac{v_m(y)}{v_n(y)} > 0 \quad \text{implies} \quad \inf_{y \in Y} \frac{v_k(y)}{v_n(y)} > 0 \quad \text{for all} \quad k > m.$$

(2) For every $n \in \mathbb{N}$, there exists $m \geq n$ so that, for every subset Y of X with

$$\inf_{y \in Y} \frac{v_m(y)}{v_n(y)} > 0, \quad \text{it is possible to find } \bar{v} \in \bar{V} \text{ with } \bar{v} \geq v_m \text{ on } Y.$$

(3) $VC(X) = \text{ind}_{n \to} Cv_n(X)$ is (strongly) boundedly retractive (and hence complete).

(4) $V_oC(X) = \text{ind}_{n \to} C(v_n)_o(X)$ is a (strongly) boundedly retractive inductive limit.

(5) $V_oC(X)$ is complete.

(6) $\mathcal{V}_0 C(X) = C\bar{V}_0(X)$ holds algebraically (and topologically).

(7) $\mathcal{V}_0 C(X) = \operatorname{ind}_{n\to} C(v_n)_0(X)$ is a regular inductive limit (which, in this case, is also equivalent to the following weaker property: For each bounded set B in $\mathcal{V}_0 C(X)$, there is $n \in \mathbb{N}$ with $B \subset C(v_n)_0(X)$.).

(8) $\mathcal{V}_0 C(X)$ is a closed (topological) subspace of $\mathcal{V}C(X)$, and hence the closure of $C_c(X)$ in $\mathcal{V}C(X)$. (Incidentally, this also turns out to be equivalent to $\mathcal{V}_0 C(X)$ being stepwise closed in $\mathcal{V}C(X)$.)

We provide a rough sketch of proof of Theorem 7.: (1) $\Leftrightarrow$ (2) follows from a direct calculation. For a more explicit formulation of (2) $\Rightarrow$ (3), we note that one actually shows the following: If, given $n \in \mathbb{N}$, $m \geq n$ is chosen as in condition (2), then $Cv_m(X)$ and $C\bar{V}(X)$ (and hence $\mathcal{V}C(X)$, too) induce the same topology on each bounded subset of $Cv_n(X)$. (One should compare this with Proposition 3.20.(b).) In order to show (3) $\Rightarrow$ (4), it is clear that only the second part (ii) of the strongly boundedly retractive property for $\mathcal{V}_0 C(X)$ has to be proved since the regularity then follows from Proposition 2. in the Appendix to Section 3. And this part can easily be checked using the fact that $\mathcal{V}_0 C(X)$ always is a topological subspace of $\mathcal{V}C(X)$ by Theorem 3.(a). Moreover, by this last fact, (5) $\Rightarrow$ (8) becomes obvious, too. Next, (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (7) are trivial while (5) $\Leftrightarrow$ (6) holds by Theorem 3.(a). (So far, all the implications remain true for arbitrary sequences $\mathcal{V} = (v_n)_n$ of strictly positive weights v_n on a locally compact space X if only condition (*) after Definition 2. is satisfied; i.e., the continuity of the weights is not needed in this direction.) – The converse is based on a direct, but technically rather complicated construction which simplifies quite a bit in the case of sequence spaces; we refer to [7] and [8] for details.

The regularly decreasing condition (1) of Theorem 7. is certainly implied by (S) in, say, Corollary 5. above. But it is much weaker than that condition since, e.g., constant sequences of strictly positive weights are regularly decreasing without satisfying (S).

Concerning Question (2) above, we have the following partial (but seemingly also quite technical) result:

8. Proposition. Let $V = (v_n)_n$ denote a decreasing sequence of strictly positive weights on a completely regular Hausdorff space X. We assume that there exists an increasing sequence $J = (X_m)_{m \in \mathbb{N}}$ of subsets of X such that,

(N,J) for every $m \in \mathbb{N}$, there is $n_m \geq m$ with $\inf_{x \in X_m} \frac{v_k(x)}{v_{n_m}(x)} > 0$ for all $k > n_m$, while

(M,J) for each $n \in \mathbb{N}$ and each subset Y of X with $Y \cap (X \setminus X_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there exists $n' = n'(n,Y) > n$ such that $\inf_{y \in Y} \frac{v_{n'}(y)}{v_n(y)} = 0.$

Moreover, we must make the following technical assumption:

(i) X is normal, and (+) for every $m \in \mathbb{N}$, there is $k_m > m$ with $X_m \subset \mathring{X}_{k_m}$, or

(ii) all weights v_n are continuous, and (++) any function $f: X \to \mathbb{R}_+$ with $f_{|X_m}$ continuous for each $m \in \mathbb{N}$ must already belong to $C(X)$.

Under these conditions, we have $VC(X) = C\bar{V}(X)$ topologically, and hence $C\bar{V}(X)$ is bornological. If, in addition, X is a $k_{\mathbb{R}}$-space and, if, in case (i), we also suppose that $\inf_{x \in K} v_n(x) > 0$ for all compact $K \subset X$ and all $n \in \mathbb{N}$, then $VC(X)$ is complete.

As a step towards a better understanding of the relevant conditions (N,J) and (M,J) in this result, we note that any increasing sequence $J = (X_m)_m$ of subsets of X with condition (M,J) must clearly satisfy $X = \bigcup_{m \in \mathbb{N}} X_m$, and condition (N,J) then implies that there exists a strictly positive weight $\bar{v} \in \bar{V}$. Also, since $X = \bigcup_m X_m$,

(++) in the second case of our technical assumption certainly follows from (+) of the first case.

Next, condition (N,J) just means that, for each $m \in \mathbb{N}$, the restrictions to X_m of all the weights v_k for k large enough (viz., $k \geq n_m$) induce the same weight condition. On the other hand, if we consider any set Y which contains points in the complement of each X_m, then condition (M,J) requires that, for each $n \in \mathbb{N}$, there exists $n' > n$ such that the restriction to Y of the weight condition given by $v_{n'}$ is strictly weaker than the weight condition induced by v_n (in the sense that $\inf_{y \in Y} \frac{v_{n'}(y)}{v_n(y)} = 0$).

From this point of view, (M,J) and (N,J) are quite natural conditions. In fact, besides covering another interesting case for the topological equality $VC(X) = C\bar{V}(X)$ as well, Proposition 8. yields the following important

9. *Corollary*. For any regularly decreasing sequence $V = (v_n)_n$ of strictly positive continuous functions v_n on a completely regular Hausdorff space X, the topological identity $VC(X) = C\bar{V}(X)$ holds.

It is now interesting to compare the the two cases of o- and O-growth conditions again, but from a different point of view: We fix a decreasing sequence $V = (v_n)_n$ of strictly positive continuous functions on a locally compact space X and consider only the two equations for topological vector spaces: $VC(X) = C\bar{V}(X)$ and $V_0C(X) = C\bar{V}_0(X)$. Then, in view of Theorem 7. and Corollary 9., $V_0C(X) = C\bar{V}_0(X)$ actually implies $VC(X) = C\bar{V}(X)$! Hence the "full" projective description of weighted inductive limits of continuous functions holds for O-growth conditions under weaker hypotheses than for o-growth conditions.

Actually, in the sequence space case (see below), it turns out that the existence of an increasing sequence $J = (X_m)_{m \in \mathbb{N}}$ of subsets of X satisfying conditions (M,J) and (N,J) is indeed equivalent to $VC(X) = C\bar{V}(X)$ topologically. But the Appendix to Part II of [9] contains an example of a decreasing sequence V of strictly positive continuous functions on a completely regular Hausdorff space X such that $\mathcal{V}C(X) = C\bar{V}(X)$ holds topologically while there does not exist a sequence J with (M,J) and (N,J). Hence the analogy between the sequence and function space cases stops at this point, and it seems extremely hard (if it is possible at all in a satisfactory way) to obtain a full characterization of the topological identity $\mathcal{V}C(X) = C\bar{V}(X)$!

We now leave the function space setting. The rest of this section will be devoted to a study of sequence spaces, and we will use the terminology introduced at the end of Section 2. (Note that we have the following correspondences between the notation of echelon and co-echelon spaces and the one used in the function space case: I index set $\leftrightarrow$ $X = I$ with the discrete topology, $V = (v_n)_n \leftrightarrow \mathcal{V}$, $k_0 = k_0(V) \leftrightarrow \mathcal{V}_0C(X)$, $k_\infty = k_\infty(\bar{V}) \leftrightarrow \mathcal{V}C(X)$, $\bar{V} = \bar{V}(V) \leftrightarrow \bar{V} = \bar{V}(\mathcal{V})$, $K_0 = K_0(\bar{V}) \leftrightarrow C\bar{V}_0(X)$, and $K_\infty = K_\infty(\bar{V}) \leftrightarrow C\bar{V}(X)$.)

Let us first "translate" Theorem 3.; in view of some results of Section 2., we state it in a slightly "expanded" form.

10. Theorem. (a) Algebraically and topologically, $K_0 = K_0(\bar{V})$, $\bar{V} = \bar{V}(V)$, is exactly the completion $\hat{k}_0$ of $k_0 = k_0(V)$, and $k_\infty = k_\infty(V) = (\lambda_1(A))'_i = (\lambda_1)'_i$ is the (always complete) bornological space associated with $K_\infty = K_\infty(\bar{V}) = (\lambda_1)'_b$, where $A = (\frac{1}{v_n})_n$.

The canonical injection $k_0 \to k_\infty$ is a topological isomorphism into. But, in general, k_0 may be incomplete, and the inductive limit topology of k_∞ can be strictly stronger than the topology of K_∞.

(b) We have the dualities $(k_0)'_b = (K_0)'_b = \lambda_1$ and $K_\infty = ((k_0)'_b)'_b = ((K_0)'_b)'_b$, and the following assertions are equivalent:

(1) $(\lambda_1)'_b = k_\infty$,

(2) $K_\infty = k_\infty$ topologically,

(3) K_∞ is barrelled/bornological,

(4) λ_1 is distinguished.

Next, we note that the sequence space case of Corollary 5. (involving condition (S)) was already treated in Proposition 2.14. And, in view of the duality theory of echelon and co-echelon spaces of arbitrary order, as well as Grothendieck's results on quasinormable Fréchet spaces (see the Appendix to Section 3.), Theorem 7. can be strengthened for sequence spaces as follows:

11. *Theorem.* Let $A = (a_n)_n$ denote a Köthe matrix on some index set I, $V = (v_n)_n$, $v_n = \frac{1}{a_n}$, the associated decreasing sequence and $\bar{V} = \bar{V}(V)$. Then, for $1 \leq p \leq \infty$, the following assertions are equivalent:

(1) V is regularly decreasing; i.e., for each $n \in \mathbb{N}$, there is $m \geq n$ such that, for every subset I_0 of I for which $\inf\limits_{i \in I_0} \frac{a_n(i)}{a_m(i)} = \inf\limits_{i \in I_0} \frac{v_m(i)}{v_n(i)} > 0$, we also have $\inf\limits_{i \in I_0} \frac{a_n(i)}{a_k(i)} = \inf\limits_{i \in I_0} \frac{v_k(i)}{v_n(i)} > 0$ for all $k > m$.

(2) $\lambda_p = \lambda_p(A)$ is quasinormable.

(3) $K_p = K_p(\bar{V})$ satisfies the s.M.c.c.

(4) $k_p = k_p(V)$ is a boundedly retractive inductive limit.

(5) $k_o = k_o(V)$ is regular.

(6) k_o is complete.

(6') $k_o = K_o$ holds algebraically.

(6") k_o is closed (or, equivalently, stepwise closed) in k_∞.

We remark that part of Theorem 11. was proved independently by M. Valdivia (e.g., see [59]) and that Reiher [48] extended it to the setting of "Dubinsky echelon spaces" (and to the general framework of Köthe function spaces).

Now, Theorem 11. provides a full solution to Question 1. of Section 2., and, together with Theorem 10.(a), it also completely solves the part of Question 3. concerning k_o and K_o. In order to deal with Question 2. (and the other half of Question 3., for k_∞ and K_∞), we formally define (cf. the first part of Proposition 8.):

12. Definition. A Köthe matrix $A = (a_n)_n$ on an index set I or the associated decreasing sequence $V = (v_n)_n$, $v_n = \frac{1}{a_n}$, satisfies <u>condition (D)</u> if there exists an increasing sequence $J = (I_m)_{m \in \mathbb{N}}$ of subsets of I such that

(N,J) for every $m \in \mathbb{N}$, there is $n_m \geq m$ with $\inf_{i \in I_m} \frac{a_{n_m}(i)}{a_k(i)} = \inf_{i \in I_m} \frac{v_k(i)}{v_{n_m}(i)} > 0$ for all $k > n_m$ while,

(M,J) for each $n \in \mathbb{N}$ and each subset I_o of I with $I_o \cap (I \setminus I_m) \neq \emptyset$ for all $m \in \mathbb{N}$, there exists $n' = n'(n, I_o) \geq n$ with $\inf_{i \in I_o} \frac{a_n(i)}{a_{n'}(i)} = \inf_{i \in I_o} \frac{v_{n'}(i)}{v_n(i)} = 0.$

Condition (D) is weaker than, but closely related to a condition considered by Grothendieck [32] (in connection with the quasinormability of λ_1). The part (M,J) of (D) should also be compared with the following <u>condition</u>:

(M) For each $n \in \mathbb{N}$ and each infinite subset I_o of I, there is $m = m(n, I_o) > n$ such that $\inf_{i \in I_o} \frac{a_n(i)}{a_m(i)} = \inf_{i \in I_o} \frac{v_m(i)}{v_n(i)} = 0.$

Clearly, (M) is weaker than (S), and it is exactly equivalent to $\lambda_1 = \lambda_1(A)$ being Montel (or reflexive); e.g., see Köthe [38]. On the other hand, noting that, in the presence of condition (M), the index set I can at most be countable, and taking $J = (I_m)_{m \in \mathbb{N}}$ to be an increasing sequence of finite subsets of I with $I = \bigcup_m I_m$, it is easy to see that $(M) \Rightarrow (D)$. Similarly, one can deduce that any regularly decreasing sequence $V = (v_n)_n$ satisfies condition (D). - For a more detailed discussion of (D) (and related assumptions), we refer to [9].

Here, then, is the complete solution to Question 2. of Section 2.

13. Theorem. For an echelon space $\lambda_1 = \lambda_1(A)$, the following assertions are equivalent:

(1) The Köthe matrix A satisfies condition (D).

(2) Each bounded subset of $K_\infty = K_\infty(\bar{V}) = (\lambda_1)_b'$ is metrizable.

(3) λ_1 is distinguished.

(3') $(\lambda_1)_b' = K_\infty$ equals k_∞ topologically.

Probably since condition (D) is the weakest of all conditions for Köthe matrices $A = (a_n)_n$ (or associated decreasing sequences $V = (\frac{1}{a_n})_n$) considered so far, the proof of Theorem 13. (see [9], [10] and [11]) turns out to be quite hard. - One direction is easier: Translating conditions (N,J) and (M,J) into topological facts for $K_\infty = K_\infty(\bar{V})$, one soon arrives at $(1) \Rightarrow (2)$. In view of Grothendieck's result mentioned after Proposition 5. in the Appendix to Chapter 3., we have $(2) \Rightarrow (3)$ while $(3) \Leftrightarrow (3')$ was already mentioned in 10.(b).

The converse, however, is not so easily accomplished and requires a characterization of the Fréchet spaces with the following property (introduced by S. Heinrich [34]).

14. Definition. For a l.c. space E, let $\mathcal{U}(E)$ denote the system of all closed absolutely convex 0-neighborhoods in E and $\mathcal{B}(E)$ the collection of all closed absolutely convex bounded subsets of E. Then E is said to satisfy the density condition if, for each function λ: $\mathcal{U}(E) \to \mathbb{R}_+ \setminus \{0\}$ and each $V \in \mathcal{U}(E)$, there exist a finite subset $\mathcal{U}$ of $\mathcal{U}(E)$ and some $B \in \mathcal{B}(E)$ such that

$$\bigcap_{U \in \mathcal{U}} \lambda(U)U \subset B + V.$$

Heinrich notes that every quasinormable (and hence every (DF)-) space satisfies the density condition, but it is possible to find examples of reflexive Fréchet spaces without this property.

Now, taking polars in Definition 14. and using that, for a metrizable l.c. space E, E_b' has the (DF)-property, one easily derives that a metrizable l.c. space with density condition must be distinguished. But since we noted that there are reflexive Fréchet spaces without density condition, the converse does not hold.

The following theorem provides a complete characterization of the metrizable l.c. spaces with the density condition (and, as all the subsequent results, is taken from [11]). - In the statement of this theorem, we use the vector-valued sequences spaces $l_1(E)$ and $l_\infty(E_b')$ (which are defined in the canonical way).

15. Theorem. For a metrizable l.c. space E, the following assertions are equivalent:

(1) E satisfies the density condition,

(2) each bounded subset of E_b' is metrizable.

(3) $l_1(E)$ is distinguished,

(3') $l_\infty(E_b') = (l_1(E))_b'$ is barrelled/bornological,

(3") $l_1(E)$ satisfies the density condition,

(3"') the bounded subsets of $l_\infty(E')$ are metrizable.

At this point, the proof of 13.(3) $\Rightarrow$ (2) amounts to showing that an echelon space λ_1 satisfies the density condition if (and always only if) it is distinguished, and this can be done by use of the equivalence between conditions (1) and (3) of Theorem 15. Then, finally, with a direct construction (and involving a result from [9]), the proof of Theorem 13. can be finished by showing (2) $\Rightarrow$ (1).

To achieve a better understanding, it is now useful to again study echelon spaces $\lambda_p = \lambda_p(A)$ of order p, $1 < p < \infty$ or $p = 0$. There the situation is radically different from the case $p = 1$: We know from Section 2. that λ_0 is always distinguished, and all λ_p for $1 < p < \infty$ are even reflexive. However, the equivalence 13.(1) $\Leftrightarrow$ (2) remains valid, and in view of Definition 14. and Theorem 15., it takes the following form.

16. Theorem. For an echelon space $\lambda_p = \lambda_p(A)$ of order p, $1 \leq p < \infty$ or $p = 0$, the following are equivalent:

(1) A satisfies condition (D),

(2) λ_p satisfies the density condition,

(3) each bounded subset of $(\lambda_p)'_b = K_q$ $(= k_q)$ is metrizable, where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p < \infty$ and $q = 1$ for $p = 0$.

Hence, condition (D) on the Köthe matrix A should rather be regarded as characterizing the density condition for $\lambda_p = \lambda_p(A)$, $1 \leq p < \infty$ or $p = 0$ (and the equivalence of 13.(1) and (3) turns out to be a "coincidence"). (This idea might also help to understand the fact - mentioned after Corollary 9. above - that "(D)" does not characterize the topological identity $VC(X) = C\bar{V}(X)$.)

Finally, applying Theorem 16. with $1 < p < \infty$ (say, $p = 2$), a countable index set I (say, $I = \mathbb{N} \times \mathbb{N}$) and a Köthe matrix A on I which does not satisfy condition (D) (in view of Theorem 13., we may take A as in Proposition 2.10.(b)), we arrive at an example of a separable reflexive (even hilbertisable) Fréchet space E $(= \lambda_p(A))$ such that the strong dual E'_b is again separable, but contains bounded subsets which are not metrizable.

<u>Notes added in proof</u> (February 1988).

1. In the recent article "Dual Density Conditions in (DF)-Spaces" (preprint, Paderborn and Valencia 1987), the author and J. Bonet solve Problem 2. in Section 3. of the present article affirmatively. In fact, it is shown that $VC(X)$ is (not only the bornological, but also) the barrelled space associated with the complete l.c. space $C\bar{V}(X)$, and hence is itself complete, whenever $V = (v_n)_n$ is a decreasing sequence of strictly positive continuous functions on a completely regular $k_{\mathbb{R}}$-space X. – The article also contains some other results connected with our discussion in Section 4.

2. F. Bastin (private communication, Liège 1987) gives various reformulations of condition (D) (cf. Definition 4.12), respectively, of the related conditions (N,J) and (M,J) (see Proposition 4.8). As a consequence, she is able to derive a converse of Proposition 4.8 in a more restricted setting: If X is a locally compact and σ-compact space and if $V = (v_n)_n$ is a decreasing sequence of strictly positive continuous functions on X, then the topological equality $VC(X) = C\bar{V}(X)$ does indeed imply that V satisfies (N,J) and (M,J) for a suitable increasing sequence $J = (X_m)_m$ of closed subsets X_m of X. (As remarked in Section 4., this converse is false if one omits the hypothesis that X is locally compact and σ-compact.)

3. Concerning countable weakly compact projective and inductive limits, M. Valdivia (lecture at the Oberwolfach Conference on Functional Analysis, October 1987) recently proved the following interesting result: A Fréchet space E is totally reflexive (i.e., every separated quotient of E is reflexive) if and only if E is the countable projective limit of reflexive Banach spaces (which is equivalent to requiring that its strong dual E_b' is the inductive limit of a sequence of reflexive Banach spaces).

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