

An Application of Second Order Approximations for the Green's Function

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In a previous paper¹ there was developed an approximation theory for the Green's functions which gives approximations consistent with the conservation laws of the Hamiltonian. We have chosen a concept of reduction which is different and which leads to other approximations for the particle-hole Green's function than the usual concept of reduction. The particle-hole Green's function is the function

$$\langle \Psi_0 | \tau \{ a_1^+(0) a_2(0) a_3^+(t) a_4(t) \} | \Psi_0 \rangle$$

where $|\Psi_0\rangle$ is the real ground state and τ stands for the time ordered products of the operators.

Now, in this paper we want to give an example of an application of this theory. We present a second order approximation since the first order approximation is the well known Random Phase Approximation.

Introduction

We have considered the space \mathfrak{L} of all linear operators in the space of the many fermion systems. In \mathfrak{L} there was a positive definite hermitian metric defined. So we could define an operator $G(z)$ in \mathfrak{L} by its matrix elements which are the Green's functions. If $A, B \in \mathfrak{L}$ then

$$\begin{aligned} (A | G(z) | B) &= G_{AB}(z) \stackrel{\text{def}}{=} \langle \Psi_0 | A^+ (H - E_0 - z)^{-1} B | \Psi_0 \rangle \\ &\quad + \langle \Psi_0 | B (H - E_0 + z)^{-1} A^+ | \Psi_0 \rangle. \end{aligned}$$

H is the Hamiltonian

$$H = H_0 + V \quad \text{and} \quad H | \Psi_0 \rangle = | \Psi_0 \rangle E_0.$$

¹ Beck, F., Fuchssteiner, B.: Approximation of Green's function consistent with their symmetries and conservation laws. Z. Physik 239, 276–288 (1970).

The unperturbed Hamiltonian H_0 is

$$H_0 = \sum_i H_i a_i^+ a_i$$

and the perturbation V is given by

$$V = \frac{1}{2} \sum_{i, j, k, l} V_{ijkl} a_j^+ a_i^+ a_k a_l.$$

$G_{AB}(z)$ is the Fourier transform of the usual function $G_{AB}(t)$. The particle hole Green's function is the restriction of $G(z)$ onto the subspace $P\mathfrak{L} \subset \mathfrak{L}$, where $P\mathfrak{L}$ is a subspace of the space of all particle hole operators i.e. the linear space which is generated by all:

$$a_i^+ a_k,$$

where i denotes a particle with regard to the unperturbed ground state $|\Phi_0\rangle$ and k a hole or vice versa.

We assume that $P = \tilde{P}$, where the tilde denotes hermitian conjugation in \mathfrak{L} . The metric in \mathfrak{L} induces a metric in $P\mathfrak{L}$ such that for all $a_i^+ a_k$,

$$(a_i^+ a_k | a_r^+ a_s) = \delta_{ir} \delta_{ks}.$$

Let us repeat shortly some notions which are important for the reduction of graphs. The notation of the graphs contributing to the Green's function is the same as in Thouless².

To the Green's function $(a_a^+ a_b | G(t) | a_c^+ a_d)$ contribute all graphs which are linked with the external lines in Fig. 1. The external inter-

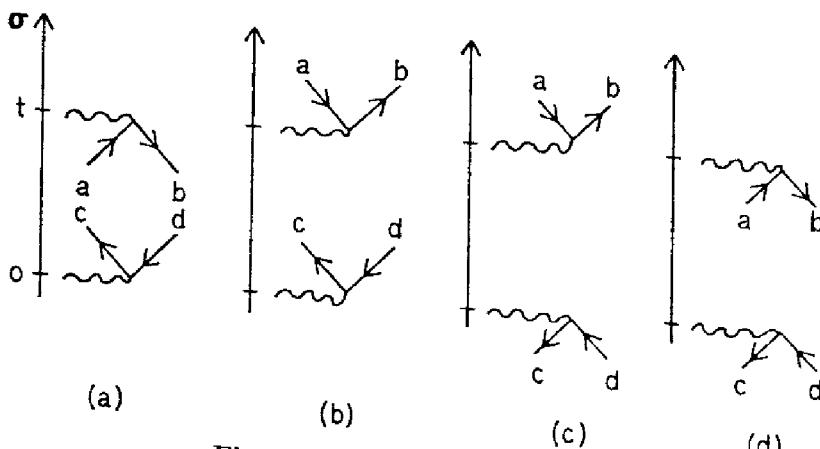


Fig. 1. General types of graphs

² Thouless, D. J.: The quantum mechanics of many-body-systems. New York London: Academic Press 1961.

action line which binds a pair of external lines is called junction*. We consider graphs depending on the time variable t .

Definition 1. The exchange of a pair of particle-hole lines is a cut of the lines at the point $\sigma = \sigma_0$ followed by a connection of different lines resulting in a junction (Fig. 2).

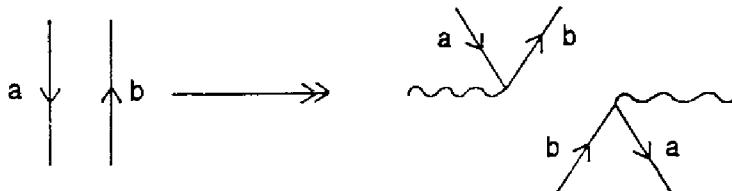


Fig. 2. The exchange of two lines

Definition 2. By distinction of pairs we understand a distinction of all pairs of lines for which the following conditions are fulfilled.

- a) Both lines run through the same time interval $\Delta\sigma$.
- b) The lines must not end up in a junction which connects them both.
- c) The exchange of a pair converts a connected graph into a disconnected one where each part has two junctions.
- d) To each pair (ik) corresponds an operator $\in P\Omega$, i.e. the operator $a_i^+ a_k \in P\Omega$ where i and k are the indices of the two lines.

A pair of lines in the Fourier transform of a graph i.e. a graph depending on z , is distinguished if it is distinguished in the graph depending on t .

Definition 3. A reduction of a graph is the exchange of a distinguished pair.

Definition 4. A reduction of first kind leaves all distinguished pairs distinguished except for the exchanged pair.

Definition 5. An irreducible graph is a graph which cannot be reduced by a reduction of the first kind.

Definition 6. An irreducible graph belongs to the class $[\gamma_n]$ if exactly n different decompositions in irreducible graphs are possible. A decomposition is an arbitrary number of reductions. Two decompositions are equal if the set of reduced pairs is equal.

* In the following we shall only draw graphs linked with the external lines in Fig. 1 a. But, of course, with each graph like Fig. 1 a the graphs corresponding to Fig. 1 b, c, d, also contribute to the various matrix elements.

With this concept of reduction an exact equation for $G(z)$ was derived

$$G(z) = G_0(z) + \Gamma(z) G(z), \quad (1)$$

$$\Gamma(z) = \sum_0^{\infty} (1-n) \gamma^n(z). \quad (2)$$

$G_0(z)$ is the Green's function of the unperturbed system.

The matrix elements $(a_c^+ a_d | \gamma^n(z) | a_b^+ a_a)$ are given by the contribution of all graphs like Fig. 3a, which would be $\in [\gamma_n]$ if we closed the lines (a, b) with a junction i.e. the graph in Fig. 3b has to be a graph $\in [\gamma_n]$.

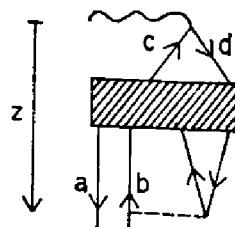


Fig. 3a. A graph contributing to the matrix elements of γ^n

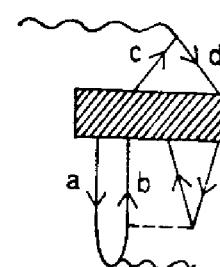


Fig. 3b. A graph $\in [\gamma^n]$

The contributions of these graphs are calculated according to the following rules.

Rule 1. *Each vertex like Fig. 4 gives a factor $(1/2) V_{abcd}$.*



Fig. 4. An interaction line

Rule 2. *There is a factor -1 for each closed loop, for each hole line, and for the z line. The lines (a, b) in Fig. 3a which are not closed do not count in this rule.*

Rule 3. *Each interval between successive vertices produces a factor equal to the reciprocal of the sum of unperturbed energies of the hole lines which are crossing the horizontal section between the two vertices, minus the sum of the unperturbed energies of the particle lines which are crossing the section.*

In the third rule the z line running from the junction to the last interaction line of the pair which is not closed counts as a particle line with the energy z if the z line is running up and it counts as a hole line if it is running down. In rule 3 the pair (a, b) does only count as far as to the point of the last interaction line.

Approximation of $\Gamma(z)$

In this chapter we shall approximate $\Gamma(z)$ by all graphs of first and second order in V which are contributing to the sum (2). These graphs contributing to $(a_2^+ a_1 | \Gamma(z) | a_3^+ a_4)$ are shown in Fig. 5*. All these

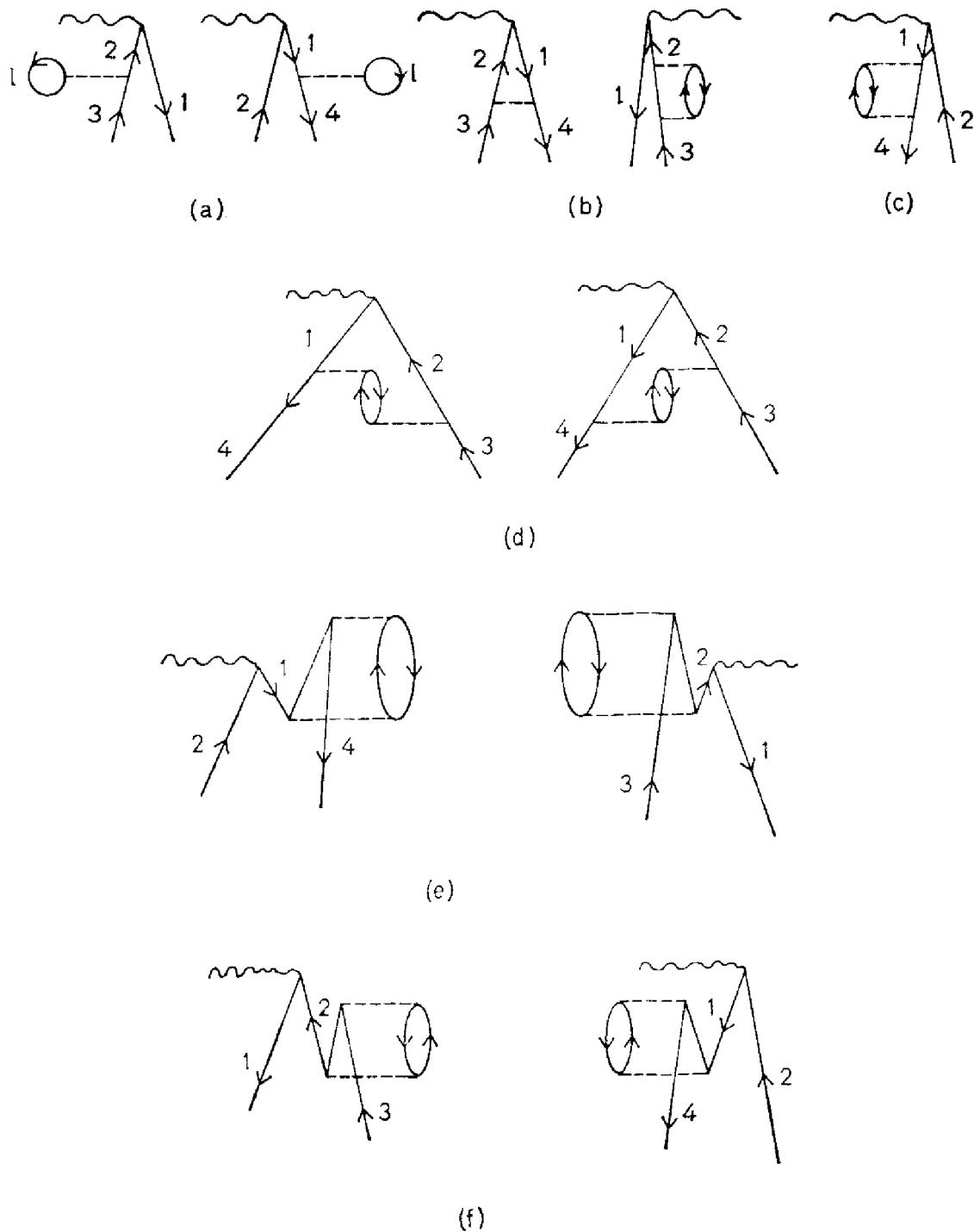


Fig. 5. Graphs contributing to the second order approximation

* We did neither draw the exchange terms of these graphs nor the graphs corresponding to Fig. 1 b, c, d. But of course these graphs do contribute to our approximation.

graphs are elements of $[\gamma_0]$. According to the graph rules we get the second order approximation

$$\begin{aligned}
 & (a_2^+ a_1 | \Gamma(z) | a_3^+ a_4) \\
 &= \sum_{l \text{ all holes}} \frac{\delta_{23} V_{4l,11} - \delta_{14} V_{2l,13}}{(z+H_1-H_2)} - \frac{S(1,2) V_{42,13}}{(z+H_1-H_2)} \\
 &+ \sum_{abc} \left\{ \frac{S(1a4, cb2) \delta_{14} V_{2acb} V_{c b 3 a}}{(z+H_1-H_2)(z+H_1+H_a-H_c-H_b)} \right. \\
 &+ \left. \frac{S(1bc, 3a2) \delta_{23} V_{cb1a} V_{4acb}}{(z+H_1-H_2)(z+H_c+H_b-H_a-H_2)} \right\} \\
 &+ \sum_{ab} S(1a4, 3b2) \left\{ \frac{V_{4ab1} V_{b23a}}{(z+H_1-H_2)(z+H_a+H_4-H_b-H_2)} \right. \\
 &+ \left. \frac{V_{4ab1} V_{b23a}}{(z+H_1-H_2)(z+H_b+H_1-H_a-H_3)} \right\} \\
 &+ \sum_{ab} S(1a3, 4b2) \left\{ \frac{V_{4ab1} V_{b23a}}{(z+H_1-H_2)(H_a+H_3-H_b-H_2)} \right. \\
 &+ \left. \frac{V_{4ab1} V_{b23a}}{(z+H_1-H_2)(H_b+H_1-H_a-H_3)} \right\} \\
 &+ \sum_{abc} \frac{\delta_{14} S(1bc, b23) V_{2acb} V_{c b 3 a}}{(H_2+H_a-H_b-H_c)} \\
 &\cdot \left\{ \frac{1}{(H_b+H_c-H_3-H_a)} + \frac{1}{(z+H_1-H_2)} \right\} \\
 &+ \sum_{abc} \frac{\delta_{23} S(1a4, cb2) V_{cb1a} V_{4acb}}{(H_c+H_b-H_1-H_a)} \\
 &\cdot \left\{ \frac{1}{(H_4+H_a-H_b-H_c)} + \frac{1}{(z+H_1-H_2)} \right\}.
 \end{aligned} \tag{3}$$

$S(a, b)$ and $S(abc, def)$ is defined by:

$$S(a, b) = \begin{cases} 1 & \text{if } a \text{ denotes a hole and } b \text{ denotes a particle} \\ -1 & \text{if } a \text{ denotes a particle and } b \text{ denotes a hole} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$S(abc, def) = \begin{cases} 1 & \text{if } a, b, c \text{ denote holes and } d, e, f \text{ particles} \\ -1 & \text{if } a, b, c \text{ denote particle and } d, e, f \text{ holes} \\ 0 & \text{otherwise} \end{cases}$$

The lines of Eq. (3) are in the same order as the graphs in Fig. 5. The approximation (3) is a consistant approximation in the sense of¹. Let us assume that $|\Psi_0\rangle$ and $|\Phi_0\rangle$ are eigenstates of an operator $T=T^+$ for which the conservation law $[H, T]=0$ holds. We denote by $\varphi(T)$ an hermitian operator in \mathfrak{L} such that $\varphi(T) A=[T, A]$ for all $A\in\mathfrak{L}$. Let Π_m be the projection operator on an invariant subspace $\mathfrak{L}_m\subset\mathfrak{L}$ with regard to $\varphi(T)$, i.e. $\varphi(T) A\in\mathfrak{L}_m$ for all $A\in\mathfrak{L}_m$.

Now, if $[P, \Pi_m]=0$ where P is the projection onto the subspace $P\mathfrak{L}$ then it holds the conservation law

$$[\Gamma(z), \Pi_m]=0. \quad (5)$$

Since $G(z)$ is given by (1) we have

$$[G(z), \Pi_m]=0. \quad (6)$$

If (5) and (6) are valid for all invariant subspaces A_m with regard to $\varphi(T)$ then

$$\begin{aligned} [\Gamma(z), \varphi(T)] &= 0 \\ [G(z), \varphi(T)] &= 0. \end{aligned} \quad (7)$$

The validity of the conservation law $[G(z), \Pi_m]=0$ for the approximated $G(z)$ depends only on $P\mathfrak{L}$ not on the order of the approximation. That means if the conservation law is fulfilled for the first order approximation (Random Phase Approximation) then it holds for all higher order approximations.

The poles of the Green's function in the z -plane give us the excitation energy of the considered fermion system. The poles of $G(z)$ which are different from the poles of $G_0(z)$ are identical with the zeros of the determinant

$$\text{Det}_{P\mathfrak{L}}(I-\Gamma(z))=0 \quad (8)$$

where the determinant is taken over an orthonormal basis in $P\mathfrak{L}$. This procedure is well known from the theorie of systems of linear equations. If $P\Pi_m=\Pi_mP=0$ then the validity of Eq. (5) and (6) makes it sure that no matrix elements $(A|\Gamma(z)|B)$ and $(A|G(z)|B)$ resp. contribute to the zeros of $\Gamma(z)$ and the poles of $G(z)$ resp. if A or $B\in\mathfrak{L}_m$ generates an excitation of T_m of the eigenvalue of T .

For an easier calculation we assume in the following that

$$\begin{aligned} \frac{1}{z_0+H_4-H_b-H_a-H_c} - \frac{1}{H_3+H_b-H_a-H_c} &\simeq 0 \\ \frac{1}{z_0+H_c+H_b-H_a-H_3} - \frac{1}{H_c+H_b-H_a-H_4} &\simeq 0 \end{aligned} \quad (9)$$

where z_0 is the zero of (8). This approximation holds if the excitation energies of the system do not differ very much from the excitation energies of the unperturbed system. With this assumption Eq. (3) has the form

$$\begin{aligned}
 & (a_2^+ a_1 | \Gamma(z) | a_3^+ a_4) \\
 &= \sum_{\text{1 all holes}} \frac{\delta_{23} V_{4111} - \delta_{14} V_{2113}}{(z_0 + H_1 - H_2)} + \frac{S(1, 2) V_{4213}}{(z_0 + H_1 - H_2)} \\
 &+ \sum_{abc} \left\{ \frac{S(1a4, cb2) \delta_{14} V_{2acb} V_{cb3a}}{(z_0 + H_1 - H_2)(z_0 + H_1 + H_a - H_c - H_b)} \right. \\
 &+ \left. \frac{S(1bc, 3a2) \delta_{23} V_{cb1a} V_{4acb}}{(z_0 + H_1 - H_2)(z_0 + H_c + H_b - H_a - H_2)} \right\} \\
 &+ \sum_{ab} \frac{S(1a4, 3b2) V_{4ab1}}{(z_0 + H_1 - H_2)} \\
 &\cdot \left\{ \frac{V_{2b3a}}{(z_0 + H_a + H_4 - H_b - H_2)} + \frac{V_{b23a}}{(z_0 + H_b + H_1 - H_a - H_4)} \right\} \\
 &+ \sum_{ab} \frac{S(1a3, 4b2) V_{4ab1}}{(z_0 + H_1 - H_2)} \\
 &\cdot \left\{ \frac{V_{b23a}}{(H_a + H_3 - H_b - H_2)} + \frac{V_{b23a}}{(H_b + H_1 - H_a - H_4)} \right\} \\
 &+ \sum_{abc} \frac{\delta_{14} S(1bc, a23) V_{2acb} V_{cb3a}}{(z_0 + H_1 - H_2)(z_0 + H_4 + H_b - H_c - H_a)} \\
 &+ \sum_{abc} \frac{\delta_{23} S(1a4, cb2) V_{cb1a} V_{4acb}}{(z_0 + H_1 - H_2)(z_0 + H_b + H_c - H_a - H_3)}. \tag{10}
 \end{aligned}$$

In the following we shall apply Eq. (10) for the excitation energies of the ${}^4\text{He}$ states. As the unperturbed Hamiltonian we take a shell model Hamiltonian

$$H_{SH} = \frac{A^4 \hbar^2}{2m} \mathbf{r}^2 + \frac{\mathbf{p}^2}{2m} \tag{11}$$

where m is the mass of a nucleon. \mathbf{r} and \mathbf{p} resp. are coordinate and momentum resp. of the particle. The excitation energy of the unperturbed system is then³

$$\varepsilon = \frac{\hbar^2 c^2}{mc^2 A^2}.$$

³ Elliot, J. P., Lane, A. M.: The nuclear shell-model. Handbuch der Physik, Bd. XXXIX. Berlin-Göttingen-Heidelberg: Springer 1957.

For the ${}^4\text{He}$ we assume⁴ $A = 1.31 \text{ fm}$ then we get

$$\varepsilon = 24 \text{ MeV}.$$

For the perturbation interaction we choose

$$\begin{aligned} V(r, \sigma_1, \sigma_2, \tau_1, \tau_2) \\ = V(r) \{ 0.33 + 0.03 \sigma_1 \cdot \sigma_2 - 0.11 \tau_1 \cdot \tau_2 - 0.11 \sigma_1 \cdot \sigma_2 \tau_1 \cdot \tau_2 \} \end{aligned}$$

where σ is the spin-operator and τ is the isospin-operator. This interaction is the so called Soper-mixture. $V(r)$ is assumed to be a Gauß potential

$$V(r) = V_0 e^{-\frac{r^2}{2A^2}}.$$

We have calculated the zeros of $\text{Det}(I - \Gamma(z))$ for total angular momentum $J=0$ and total isospin $T=0.1$. For $P\Omega$ we have taken the space of all particle-hole operators which are scattering particles between the first two shells. $|\phi_0\rangle$ is the state containing four particles in the first shell. Our calculation shows that there are excitation energies which do not differ very much from the excitation energies of the unperturbed system. Therefor one has to investigate if these excitations are given by a center of mass motion.

The Spurious States

In shell model calculation we have always trouble with the motion of the center of mass, since we have assumed a fixed origin of the shell model potential V_{SH} . For the center of mass motion⁵ we have the Hamiltonian

$$H_S = \frac{NA^4 \hbar^2}{2m} \mathbf{R}^2 + \frac{\mathbf{P}^2}{2mN}$$

where N is the operator of the number of particles. $\mathbf{R} = \sum_i \mathbf{r}_i/N$ is the coordinate of the center of mass and $\mathbf{P} = \sum_i \mathbf{p}_i$ is the momentum of the center of mass motion. There hold the conservation laws

$$[H, H_S] = 0, \quad [H_{SH}, H_S] = 0.$$

All states which differ from the ground state in the eigenvalue of H_S are unphysical states. But it is often difficult to eliminate these states

⁴ Elton, L. R. B.: Nuclear sizes. Oxford: University Press 1961.

⁵ Baranger, E., Chong Wan Lee: Spurious states arising from the centre-of-mass motion of a nucleus. Nucl. Phys. **22**, 157–163 (1961).

since H_s is not a one particle operator, i.e. a representation of H_s in the terms of second quantization has terms $a_i^+ a_k^+ a_r a_s$. Therefore we have generally $[\varphi(H_s), P] \neq 0$ if $P\mathfrak{L}$ consists only of one particle operators. That means that we must investigate the spectrum of $\varphi(H_s)$. From⁵ we see that the eigenvectors $\in \mathfrak{L}$ of $\varphi(H_s)$ are given by products of A^- and A^+ .

$$A^+ = \left(P + \frac{iNm}{\hbar} \varepsilon R \right), \quad A^- = \left(P - \frac{iNm}{\hbar} \varepsilon R \right).$$

It holds

$$[H_s, A^+] = \varepsilon A^+, \quad [H_s, A^-] = -\varepsilon A^-.$$

A^- and A^+ are elements of a representation of the rotation group with $J=1$. For \mathfrak{L} we take an orthogonal decomposition $\mathfrak{L} = \mathfrak{L}_1 \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2$ where \mathfrak{L}_1 is generated by the A_i^+ and \mathfrak{L}_1 is generated by A_i^- ($i=1, 2, 3$).

\mathfrak{L}_1 and \mathfrak{L}_1 are invariant subspaces with regard to $\varphi(H_s)$ since $\varphi(H_s)\alpha = \varepsilon\alpha$ if $\alpha \in \mathfrak{L}_1$ and $\varphi(H_s)\beta = -\varepsilon\beta$ if $\beta \in \mathfrak{L}_1$.

If Π and $\tilde{\Pi}$ are the projections onto \mathfrak{L}_1 and \mathfrak{L}_1 resp. then it holds that

$$P\Pi = \Pi P = 0, \quad P\tilde{\Pi} = \tilde{\Pi} P = 0$$

because P is a projection onto a subspace with $J=0$ and $\Pi, \tilde{\Pi}$ are projections onto subspaces with $J=1$. Of course most desirable would be if $[P, \varphi(H_s)] = 0$ but since H_s is not a one-particle operator this is impossible. So we have seen that there are no single spurious excitations contributing to the poles of $G(z)$ but there may be contributions from multiple spurious excitations. This result is not surprising since one already knew that single spurious excitation would not contribute to the poles calculated with the Random-phase-approximation.

Calculation of ${}^4\text{He}$ Excitation Energies

In order to have an easy calculation we shall consider as particles only the states in the $1p$ -shell. Then for the reason of parity conservation the graphs of Fig. 6 are zero. That means that the second line of Eqs. (3) and (10) vanishes. In Eq. (10) each index stands for the following set of indices

$\begin{bmatrix} x \\ l_x \\ m_x \\ s_x \\ t_x \end{bmatrix}$ denoting the different shells
 $\begin{bmatrix} l_x \\ m_x \end{bmatrix}$ denoting the orbit angular momentum
 m_x denoting the third component of orbital angular momentum
 s_x denoting the third component of the spin
 t_x denoting the third component of the isospin.



Fig. 6. Graphs vanishing because of parity conservation

In $P\mathfrak{L}$ we perform a transformation of the basis

$$|\alpha, \beta, j_\alpha, j_\beta, \tau, \tau_3\rangle = \sum_{\substack{m_\alpha s_\alpha t_\alpha M_\alpha \\ m_\beta s_\beta t_\beta M_\beta}} a_{\begin{bmatrix} \alpha \\ l_\alpha \\ m_\alpha \\ s_\alpha \\ t_\alpha \end{bmatrix}}^+ a_{\begin{bmatrix} \beta \\ l_\beta \\ m_\beta \\ s_\beta \\ t_\beta \end{bmatrix}} \cdot (l_\alpha, m_\alpha, \frac{1}{2}, s_\alpha | j_\alpha M_\alpha) (l_\beta, m_\beta, \frac{1}{2}, s_\beta | j_\beta M_\beta) \cdot (j_\beta, -M_\beta, j_\alpha, M_\alpha | 00) (\frac{1}{2}, t_\beta, \frac{1}{2}, t_\alpha | \tau \tau_3) (-1)^{j_\beta + M_\beta + \frac{1}{2} + t_\beta}.$$

The brackets $(j, m, j', m' | JM)$ are the Clebsch-Gordan coefficients in the standardisation of Fano-Racah⁶. This transformation corresponds to the reduction of the basis vectors of $P\mathfrak{L}$ into irreducible tensorial sets.

Let us choose in the space of all two-particle states the following basis

$$|\alpha, \beta, j_\alpha, j_\beta, J, M, \tau, \tau_3\rangle = \sum_{\substack{m_\alpha s_\alpha t_\alpha M_\alpha \\ m_\beta s_\beta t_\beta M_\beta}} \Psi_{\begin{bmatrix} \alpha \\ l_\alpha \\ m_\alpha \\ s_\alpha \\ t_\alpha \end{bmatrix}} \Psi_{\begin{bmatrix} \beta \\ l_\beta \\ m_\beta \\ s_\beta \\ t_\beta \end{bmatrix}} \cdot (l_\alpha, m_\alpha, \frac{1}{2}, s_\alpha | j_\alpha, M_\alpha) (l_\beta, m_\beta, \frac{1}{2}, s_\beta | j_\beta, M_\beta) \cdot (j_\alpha, M_\alpha, j_\beta, M_\beta | JM) (\frac{1}{2}, t_\alpha, \frac{1}{2}, t_\beta | \tau \tau_3).$$

Where the Ψ are orthonormal wave functions of the shell model states. For getting the matrix elements of V we can apply the Wigner-Eckart Theorem.

$$\begin{aligned} & \langle \alpha, \beta, j_\alpha, j_\beta, J, M, \tau, \tau_3 | V | \alpha', \beta', j_{\alpha'}, j_{\beta'}, J', M', \tau', \tau'_3 \rangle \\ &= V \begin{pmatrix} \alpha & \beta & \alpha' & \beta' & J, \tau \\ j_\alpha & j_\beta & j_{\alpha'} & j_{\beta'} & J, \tau \end{pmatrix} \cdot \frac{\delta_{JJ'} \delta_{MM'} \delta_{\tau\tau'} \delta_{\tau_3\tau'_3}}{\sqrt{2J+1} \sqrt{2\tau+1}}. \end{aligned}$$

Antisymmetrisation of the reduced matrix element gives us

$$\begin{aligned} & \mathfrak{W} \begin{pmatrix} \alpha & \beta & \alpha' & \beta' & J, \tau \\ j_\alpha & j_\beta & j_{\alpha'} & j_{\beta'} & J, \tau \end{pmatrix} \\ &= V \begin{pmatrix} \alpha & \beta & \alpha' & \beta' & J, \tau \\ j_\alpha & j_\beta & j_{\alpha'} & j_{\beta'} & J, \tau \end{pmatrix} + (-1)^{j_{\alpha'} + j_{\beta'} - J - \tau} V \begin{pmatrix} \alpha & \beta & \beta' & \alpha' & J, \tau \\ j_\alpha & j_\beta & j_{\beta'} & j_{\alpha'} & J, \tau \end{pmatrix}. \end{aligned}$$

⁶ Fano, U., Racah, G.: Irreducible tensorial sets. New York: Academic Press 1959.

We recouple \mathfrak{W}

$$W\left(\begin{array}{ccccc} \alpha & \beta & \alpha' & \beta' & \\ j_\alpha & j_\beta & j_{\alpha'} & j_{\beta'} & J, \tau \end{array}\right) = \sum_{\tilde{J} \tilde{\tau}} \sqrt{(2J+1)(2\tilde{J}+1)(2\tau+1)(2\tilde{\tau}+1)} \cdot \left\{ \begin{array}{c} j_{\alpha'} \ j_{\beta'} \ \tilde{J} \\ j_\alpha \ j_\beta \ J \end{array} \right\} \left\{ \begin{array}{c} \frac{1}{2} \ \frac{1}{2} \ \tilde{\tau} \\ \frac{1}{2} \ \frac{1}{2} \ \tau \end{array} \right\} \mathfrak{W}\left(\begin{array}{ccccc} \alpha & \beta & \alpha' & \beta' & \\ j_\alpha & j_\beta & j_{\alpha'} & j_{\beta'} & \tilde{J}, \tilde{\tau} \end{array}\right)$$

where the large brackets denote the 6-j-symbols of Wigner in the standardisation of Fano-Racah⁶.

Now, if we calculate z_0 from the Eq. (10) and from

$$\text{Det}_{P\mathfrak{L}}(I - \Gamma(z)) = 0$$

we get

$$z_0^2 = \varepsilon^2 + \frac{1}{2} \left(P^2 + 2P\varepsilon + 2M^2 - \left[S + \frac{Q}{\varepsilon} \right]^2 \right) \pm \sqrt{\frac{1}{4}(P^2 + 2P\varepsilon + 2M)^2 - \left[S + \frac{Q}{\varepsilon} \right]^2 - M^4 + 2MP\varepsilon} \quad (12)$$

where we used the following abbreviations

$$\begin{aligned} M^2 &= \frac{1}{2} \sum_{\tilde{J} \tilde{\tau}} W\left(\begin{array}{ccccc} 2 & 2 & 1 & 1 & \tilde{J} \tilde{\tau} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) W\left(\begin{array}{ccccc} 1 & 1 & 2 & 2 & \tilde{J} \tilde{\tau} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) \\ P(\tau) &= (2\tau+1)^{-\frac{1}{2}} W\left(\begin{array}{ccccc} 1 & 2 & 1 & 2 & 0, \tau \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) \\ &\quad - \frac{1}{4} W\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0, 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) + \frac{1}{4} W\left(\begin{array}{ccccc} 2 & 1 & 1 & 2 & 0, 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) \\ Q(\tau) &= \frac{1}{2} \sum_{\tilde{J} \tilde{\tau}} (-1)^{\tau+\tilde{\tau}} \left\{ \begin{array}{c} \frac{1}{2} \ \frac{1}{2} \ \tilde{\tau} \\ \frac{1}{2} \ \frac{1}{2} \ \tau \end{array} \right\} \\ &\quad \cdot W\left(\begin{array}{ccccc} 2 & 2 & 1 & 1 & \tilde{J} \tilde{\tau} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) W\left(\begin{array}{ccccc} 2 & 1 & 1 & 2 & \tilde{J} \tilde{\tau} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) \\ S(\tau) &= (2\tau+1)^{-\frac{1}{2}} W\left(\begin{array}{ccccc} 2 & 2 & 1 & 1 & 0, \tau \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{array}\right) \\ \varepsilon &= H_2 - H_1 = 24 \text{ MeV.} \end{aligned} \quad (13)$$

1 denotes the 1-s-shell and 2 denotes the 1-p-shell. Taking from Talmi⁷ the reduced matrix elements and calculating the terms (13) for the chosen

⁷ Talmi, I.: Nuclear spectroscopy with harmonic-oscillator wave-functions. *Helvetica Physica Acta* **25**, 185–234 (1952).

Hamiltonian we get

$$\begin{array}{ll}
 \tau=0 & \tau=1 \\
 M^2 = 0.006 V_0^2 & M^2 = 0.006 V_0^2 \\
 P = -0.15 V_0 & P = -0.25 V_0 \\
 S = -0.015 V_0 & S = -0.065 V_0 \\
 Q = -0.009 V_0^2 & Q = 0.04 V_0^2.
 \end{array}$$

Considering z_0 as a function of V_0 we see that the terms of higher than linear order in V_0 are very small for reasonable V_0 ($|V_0| \leq 50$ MeV). For $V_0 = -40$ MeV we get the following excitation energies.

Table

	z_{01}	z_{02}	z_{RPA}
$\tau=0$	24.2	30.5	30.0 [MeV]
$\tau=1$	24.2	34.0	34.0 [MeV]

z_{RPA} is the result of the Random Phase Approximation for the same Hamiltonian

$$z_{\text{RPA}}^2 = (\varepsilon + P)^2 - S^2.$$

We see that the excitation energies which differ very much from ε are nearly the same as the excitation energies calculated with the Random Phase Approximation. But as a result of the second order approximation we get also excited states whose excitation energies are almost 24 MeV. For the conclusion that these states are not spurious with respect to the center of mass it was indeed necessary to show that single spurious excitation does not contribute to these poles since the energy contribution of a single spurious excitation would be also 24 MeV. However the physical meaning of the second excitation energy shall not be discussed in this paper.

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