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Lattices and Choquet's Theorem

BENNO FUCHSSTEINER

Fachbereich Mathematik, Gesamthochschule, D-479, Paderborn

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In this paper we observe that lattices having a special topology (band topology) together with monotone sequences of isotone functions seem to have an important place in analysis. The main result states the existence of certain open ideals having an additional property with respect to the given sequences.

Examples for lattices of this kind are among others:

(1) The compact convex subsets of a compact convex set in a locally convex vector space, and

(2) the subadditive functionals on an abelian semigroup dominated by a fixed subadditive functional.

As application of our main result we do prove that any compact convex set X in a locally convex vector space is the countable convex hull of any countable family of compact convex subsets containing all extreme points of X. The same is proved for p-convex sets. As a second application, we find that given a sequence (p_n) of subadditive functionals on an abelian semigroup S and a subadditive functional π such that every extreme point of $A = \{\nu \mid \nu \text{ additive on } S, \nu \leqslant \pi\}$ is dominated by some p_n then every element of A is dominated by a countable convex-combination of the p_n.

The second result leads to a new proof of Choquet's theorem for cones of functions and to several generalisations of this result.

I. SOME REMARKS ABOUT LATTICES

Let $(X, \leqslant)$ be a complete lattice. The greatest element of X is denoted by I, the smallest by Θ. The bands of X are the sets of the following kind [10, p. 12]:

$$[b] = \{x \in X \mid x \leqslant b\}.$$

A set $\mathscr{F}$ is called an ideal if $z_1, z_2 \in \mathscr{F}$ implies $z_1 \vee z_2 \in \mathscr{F}$ and $z' \leqslant z \in \mathscr{F}$ implies $z' \in \mathscr{F}$. For $a \leqslant b$ and $a \neq b$, we write $a < b$.

We want to introduce some new notions. We call a topology on X a *band topology* if for $a < b$ there is a band $[\alpha]$ which is a neighborhood of $[a]$ and which does not contain b. We do say b is *extreme* to a if $b \vee x \geqslant a$ implies always $x \geqslant a$. In general, this relation is not symmetric. However, for Boolean lattices, this means the same as saying that b and a are disjoint. For $A, B \subset X$, we call

$$\operatorname{Ex}(A, B) = \{a \in A \mid b \text{ is extreme to } a \text{ for all } b \in B\}$$

the *extreme set* of A with respect to B.

Now, consider some band topology on X some quasicompact $A \subset X$ and a countable subset $B = \{b_n \mid n \in \mathbb{N}\}$ of X such that $\vee \operatorname{Ex}(A, B) = T > \Theta$. Furthermore we assume that we have a monotone sequence (φ_n) of isotone functions $X \to X$, (i.e., $\varphi_{n+1} \geqslant \varphi_n \geqslant \mathrm{id}_X$ and $a \geqslant b \Rightarrow \varphi_n(a) \geqslant \varphi_n(b) \forall n \in \mathbb{N}$), such that for all $n \in \mathbb{N}$, $\varphi_n(x) \geqslant T$ implies $x \geqslant T$. Finally, an ideal $\mathscr{F}$ is called an *φ-ideal* if $\varphi_n\{\mathscr{F}\} \subset \mathscr{F} \forall n \in \mathbb{N}$.

THEOREM 1. *There exists an open φ-ideal $\mathscr{F}$ containing B such that*

$$A \backslash \mathscr{F} \neq \varnothing.$$

Proof. By induction we show that we can choose a sequence (q_n) in X with $\operatorname{Ex}(A, B)\backslash[q_n] \neq \varnothing$ such that for all n, $[q_{n+1}]$ is a neighborhood of $[\varphi_n(q_n \vee b_n)]$.

Let q_1 be equal to Θ. Now, assume that we have chosen the $q_n (n \leqslant m)$ up to a number m. By definition of the extreme set, we obtain $\operatorname{Ex}(A, B)\backslash[q_m \vee b_m] = \operatorname{Ex}(A, B)\backslash[q_m] \neq \varnothing$. Therefore, $[q_m \vee b_m]$ does not contain T. This implies $T \notin [\varphi_m(q_m \vee b_m)]$. Now, using the properties of our band topology, we can find a suitable q_{m+1}. This completes the induction argument. Let S_n be the open kernel of $[q_n]$; then we have $[q_n] \subset S_{n+1} \forall n \in \mathbb{N}$. This means that $\mathscr{F} = \bigcup_{n \in \mathbb{N}} [q_n] = \bigcup_{n \in \mathbb{N}} S_n$ is an open ideal containing B. For $n' \in \mathbb{N}$ and $b \in \mathscr{F}$, there is some $m' \in \mathbb{N}$ with $m' \geqslant n'$ such that $b \in [q_{m'}]$. Therefore, $\varphi_{n'}(b) \leqslant \varphi_{m'}(q_{m'} \vee b_{m'}) \in [q_{m'+1}] \subset \mathscr{F}$. So $\mathscr{F}$ has to be an φ-ideal. ∎

II. p-CONVEX SETS

Let E be a locally p-convex Hausdorff vector space $(0 < p \leqslant 1)$. That means $0 \in E$ has a neighborhood-base consisting of p-convex sets, where $A \subset E$ is called p-convex if $a, b \in A$ implies $\alpha a + \beta b \in A$

for all α, $\beta \geqslant 0$ such that $\alpha^p + \beta^p = 1$. For $p = 1$, this gives the usual convexity notion, $a \in A$ is called *extreme-point* of A if for all $a_1, a_2 \in A$ and $1 > \alpha, \beta > 0$ such that $\alpha a_1 + \beta a_2 = a$ and $\alpha^p + \beta^p = 1$, we have $a = a_1 = a_2$. Every nonempty compact set in E has at least one extreme point ([3, 4]). A subset F of A is called a *face of A* if for all $a_1, a_2 \in A$, $\alpha a_1 + \beta a_2 \in F$ with $1 > \alpha$, $\beta > 0$ and $\alpha^p + \beta^p = 1$ implies $a_1, a_2 \in F$.

Now, let K be a nonempty compact p-convex subset of E and $(X, \subset)$ the complete lattice of compact p-convex subsets of K. We consider in X the topology which has the sets $W_S = \{L \in X \mid L \subset S\}$ as a base of its open sets, where S can be any open p-convex subset of E. Since $L \in X$ is the intersection of all open p-convex subsets of E containing L [4, p. 173], this has to be a band topology. For nonempty $L_1, L_2 \in X$ the join is given by:

$$L_1 \vee L_2 = \{\alpha a + \beta b \mid \alpha, \beta \geqslant 0, a \in L_1, b \in L_2, \alpha^p + \beta^p = 1\}.$$

Let $B = \{K_n \mid n \in \mathbb{N}\}$ be a countable set of compact p-convex subsets of K such that every extreme point of K is contained in some K_n, and let $A = \{\langle x\rangle \mid x \in K\}$, where $\langle x\rangle$ denotes the closed p-convex hull of $\{x\}$. Of course, for $p = 1$ we have $\langle x\rangle = \{x\}$ and for $0 < p < 1$ $\langle x\rangle$ is equal to $\{\lambda x \mid 0 \leqslant \lambda \leqslant 1\}$ since $2^{1-1/p}\tilde{x} \in \langle x\rangle$ whenever $\tilde{x} \in \langle x\rangle$. It is easily seen that A is quasicompact. If $\Gamma \subset K$, then we denote by $F(\Gamma)$ the maximal face of K disjoint to Γ, that is, the set of all $x \in K$ such that $x = \alpha z + \beta y$ with $z \in \Gamma$, $y \in K$; $\alpha, \beta \geqslant 0$ and $\alpha^p + \beta^p = 1$ implies $\alpha = 0$.

LEMMA 1. *For every $x \in K$ there are $\beta > 0$, $\alpha \geqslant 0$, $y \in K$, and $z \in \bigcup_{n\in\mathbb{N}} K_n$ such that $x = \alpha y + \beta z$ and $\alpha^p + \beta^p = 1$.*

Proof. We define $L_n \in X$ by

$$L_n = \Big\{\sum_{m\leqslant n} \alpha_m x_m + \beta y \mid \beta \geqslant 0, \alpha_m \geqslant 0, 1 - \beta^p = \sum_{m\leqslant n} \alpha_m{}^p \geqslant \frac{1}{n}, \\ x_m \in K_m, y \in K\Big\}$$

and B to be $\{L_n \mid n \in \mathbb{N}\}$. $F = K \backslash (\bigcup_{n\in\mathbb{N}} L_n)$ is a face equal to $F(\bigcup_{n\in\mathbb{N}} K_n)$, and by the definition of the extreme set we get $\operatorname{Ex}(A, B) \supset \{\langle x\rangle \mid x \in F\}$. We claim that $\vee \operatorname{Ex}(A, B) \underset{\text{def}}{=} T = \varnothing$, which proves our Lemma since it implies $F = \varnothing$. Now, we assume $T \supsetneqq \varnothing$ and define a sequence (φ_n) with the desired properties by

$$\varphi_n(L) = \{\alpha a + \beta b \mid a \in L, b \in T, \alpha \geqslant 1/n, \beta \geqslant 0, \alpha^p + \beta^p = 1\}.$$

Theorem 1 gives us an open φ-ideal $\mathscr{F}$ which contains B and which has a nonempty complement with respect to A. This means that $H = \{x \in K \mid \langle x \rangle \in \mathscr{F}\}$ is an open p-convex subset of K which contains $\bigcup_{n \in \mathbb{N}} L_n$ and has a nonempty complement $\Sigma = K \backslash H$. And for any $\alpha > 0$, $\beta \geqslant 0$ with $\alpha^p + \beta^p = 1$, we have $\alpha H + \beta F \subset H$ since $\mathscr{F}$ is a φ-ideal. This implies that Σ is a nonempty face of K because of $\Sigma \subset F$. Now, the compactness of Σ gives us an extreme point of Σ which has to be an extreme point of K not contained in $\bigcup_{n \in \mathbb{N}} K_n$. This is a contradiction to the fact that every extreme point of K is contained in $\bigcup_{n \in \mathbb{N}} K_n$. ∎

THEOREM 2. *Let K be a compact p-convex subset of a locally p-convex Hausdorff vector space and $\{K_n \mid n \in \mathbb{N}\}$ a countable set of compact p-convex subsets of K such that every extreme point of K is contained in some K_n; then every $x \in K$ may be represented as*

$$x = \sum_{n \in \mathbb{N}} \alpha_n z_n ,$$

where $\alpha_n \geqslant 0$, $z_n \in K_n$, and $\sum_{n \in \mathbb{N}} \alpha_n{}^p = 1$; i.e., every point $x \in K$ is a countable p-convex-combination of points in $\bigcup_{n \in \mathbb{N}} K_n$.

Proof. We consider in $\mathbb{R}^{\mathbb{N}}$ the pointwise order relation and the topology of pointwise convergence and we define $| f | = \sum_{n \in \mathbb{N}} | f(n) |^p$ for $f \in \mathbb{R}^{\mathbb{N}}$. Let Y be the compact set of nonnegative elements f of $\mathbb{R}^{\mathbb{N}}$ such that $| f | \leqslant 1$. We consider the map

$$Y \times T \underset{\text{def}}{=} Y \times K \times \Big(\prod_{n \in \mathbb{N}} K_n\Big) \mapsto K$$

given by:

$$(f, z, x_1, x_2, x_3, \ldots) \mapsto (1 - | f |)^{1/p} z + \sum_{n \in \mathbb{N}} f(n)\, x_n .$$

If $x \in K$, then we call $f \in Y$ a *representation* for x if the image of $\{f\} \times T$ contains x. Lemma 1 gives us a representation f with $| f | > 0$. Since the representations for x are a compact subset of Y, we can always find a maximal representation f with respect to the order of $\mathbb{R}^{\mathbb{N}}$. Our theorem is proved if we can show that $| f | = 1$. If $(f, z, x_1, x_2, x_3, \ldots) \mapsto x$, then we take a representation g of z such that $| g | > 0$. One easily calculates that $h \in Y$ given by $h(n) = [(f(n))^p + (g(n))^p(1 - | f |)]^{1/p}$ is again a representation of x. h is strictly greater than f if $| f | < 1$. That contradicts to the maximality of f. ∎

For $p = 1$, this theorem was already proved in [6] using the Hahn–Banach theorem countably many times. There it was also shown that this theorem is closely related to Choquet's theorem.

III. Application to Semigroups

Let $(S, +)$ be an abelian semigroup and π a subadditive functional on S; i.e., a mapping $S \mapsto [-\infty, +\infty[$ such that: $\pi(ns) = n\pi(s)$ and $\pi(s + t) \leqslant \pi(s) + \pi(t)\ \forall s, t \in S, n \in \mathbb{N}$. Let $(X, \leqslant)$ be the complete lattice of all subadditive functionals $\leqslant \pi$, where $\leqslant$ is the pointwise order on S. Then the join is defined by $(a \vee b)(s) = \max(a(s), b(s))$ for $a, b \in X$ and $s \in S$. The smallest element Θ of X is the functional $S \mapsto -\infty$. We take the weakest topology on X such that the point-evaluations given by the elements of S are continuous mappings on $[-\infty, +\infty[$. This is a band topology and X is a compact space. We call this topology the *weak S-topology*. By $\hat{S}$ we denote the functions on X given by S.

The set of additive functionals in X we denote by A. A is compact with respect to the weak S-topology since every ultrafilter on A converges to an element of A.

Let us briefly recall the notion of the Choquet-boundary. If Ω is a nonempty compact space and Ψ a family of upper semicontinuous functions on Ω separating the points of Ω, then a probability measure σ on Ω is called a *representing measure* for $x \in \Omega$ if for all $f \in \Psi$,

$$\int_\Omega f\, d\sigma \geqslant f(x).$$

The *Choquet-boundary* $\mathrm{Ch}(\Omega)$ (with respect to Ψ) is the set of those $x \in \Omega$ such that the Dirac-measure δ_x (the measure with support $\{x\}$) is the only representing measure for x. The Choquet-boundary is not empty ([5] or [1, p. 46]).

Now, let K be a nonempty compact subset of A. $\mu \in K$ is called an *extreme point of* K if for all $\nu_1, \nu_2 \in K$ and $1 > \alpha > 0$ such that $\alpha\nu_1(s) + (1 - \alpha)\nu_2(s) \geqslant \mu(s)\quad \forall s \in S$, we do have $\nu_1 = \nu_2 = \mu$. Clearly, every element of the Choquet-boundary $\mathrm{Ch}(K)$ with respect to $\hat{S}$ is an extreme point. That proves that K has always extreme points.

Let x be an extreme point of A; then any representing measure σ on A for x with respect to $\hat{S}$ has a support consisting of only one point, otherwise σ would be equal to a nontrivial convex-combination of probability measures on A, their integrals would be additive functionals on S, giving a contradiction to the fact that x is an extreme point.

So the set of extreme points of A is equal to the Choquet-boundary of A.

We have proved ([7, Corollary to Königs theorem]) that A has the following property:

(*) If $\mu \in A$ and $a, b \in X$ such that $\mu \leqslant a \vee b$, then there is a $1 \geqslant \alpha \geqslant 0$ such that $\mu \leqslant \alpha a + (1 - \alpha)b$.

Now, we consider a sequence (p_n) in X such that for every extreme point x of A there is some n with $x \leqslant p_n$.

LEMMA 2. *For every $\mu \in A$, there are $n \in \mathbb{N}$ and α with $1 \geqslant \alpha > 0$ such that $\mu \leqslant \alpha p_n + (1 - \alpha)\pi$.*

Proof. We use the same arguments we gave in the proof of Lemma 1. We define q_n to be $\bigvee \{(1/n)p_m + (1 - 1/n)\pi \mid m \leqslant n\}$ and $B = \{q_n \mid n \in \mathbb{N}\}$ and F to be the set of $\mu \in A$ such that there are no $1 \geqslant \alpha > 0$ and $n \in \mathbb{N}$ with $\mu \leqslant \alpha p_n + (1 - \alpha)\pi$. As a consequence of (*), we obtain that μ is an element of F if and only if there are no $1 \geqslant \beta > 0$ and $n \in \mathbb{N}$ with $\mu \leqslant \beta q_n + (1 - \beta)\pi$. From this and (*) we get $F \subset \mathrm{Ex}(A, B)$. Furthermore, the definition of (q_n) implies that F is the complement of $\Gamma = \{x \in A \mid x \leqslant q_n \text{ for some } n\}$ in A. We claim that $\bigvee \mathrm{Ex}(A, B) \underset{\mathrm{def}}{=} T = \Theta$, which proves our Lemma. We assume $T > \Theta$ and define a sequence (φ_n) by:

$$\varphi_n(x) = \bigvee \{\alpha x + (1 - \alpha)T \mid 1 \geqslant \alpha \geqslant 1/n\} \ \forall x \in X \quad (\text{where } 0 \cdot (-\infty) = 0).$$

Theorem 1 gives us an open φ-ideal $\mathscr{F}$ containing B such that $\Sigma = A \backslash \mathscr{F}$ is a nonempty subset of F since $\mathscr{F} \supset \Gamma$. For all $1 \geqslant \alpha > 0$, we have $\alpha\mathscr{F} + (1 - \alpha)F \subset \mathscr{F}$ since $\mathscr{F}$ is an φ-ideal, that implies $\alpha\mathscr{F} + (1 - \alpha)\Sigma \subset \mathscr{F}$, i.e., every extreme point of Σ has to be an extreme point of A. Such an extreme point exists since Σ is nonempty and compact. Because of $\Sigma \subset F$, this gives a contradiction to the fact that for every extreme point ν of A we have $\nu \leqslant p_n$ for some n. ∎

Another property of A is the following ([7, Theorem 3]):

(**) If $\mu \in A$ and $a, b \in X$, $0 \leqslant \lambda \leqslant 1$ with $\mu \leqslant \lambda a + (1 - \lambda)b$, then there are $\mu_1, \mu_2 \in A$ such that $\mu_1 \leqslant a$, $\mu_2 \leqslant b$, and $\mu \leqslant \lambda\mu_1 + (1 - \lambda)\mu_2$.

From this we obtain

THEOREM 3. ***Let S be an abelian semigroup and π a subadditive functional: $S \to [-\infty, +\infty[$ and $\{p_n \mid n \in \mathbb{N}\}$ a family of subadditive functionals on S such that for any extreme point ν of $A = \{\mu \mid \mu$ additive*** *on S, $\mu \leqslant \pi\}$ there is some $p_n \geqslant \nu$. Then for any μ of A we have $\lambda_n \geqslant 0$ with $\sum_{n\in\mathbb{N}} \lambda_n = 1$ and $\mu_n \in A$ with $\mu_n \leqslant p_n$ such that*

$$\mu \leqslant \sum_{n\in\mathbb{N}} \lambda_n \mu_n ,$$

i.e., every $\mu \in A$ is dominated by a countable convex-combination of the p_n.

Proof. Y its order and its topology is the same as in the proof of Theorem 2, where $|f| = \sum_{n\in\mathbb{N}} |f(n)|$. For every $s \in S$, we define an upper semicontinuous map:

$$\rho_s : Y \to [-\infty, +\infty[\text{ by } f \to (1 - |f|)\,\pi(s) + \sum_{n\in\mathbb{N}} f(n)\, p_n(s).$$

$f \in Y$ is called a representation for $\mu \in A$ if $\mu(s) \leqslant \rho_s(f)\,\forall s \in S$. Since Y is compact and since the ρ_s are upper semicontinuous, we can find a maximal representation f (with respect to the order of Y) for $\mu \in A$. Property (**) gives us the existence of $\tilde{\mu}$, $\tilde{\mu}_n \in A$ with $\mu_n \leqslant p_n \forall n \in \mathbb{N}$ such that $\mu \leqslant (1 - |f|)\tilde{\mu} + \sum_{n\in\mathbb{N}} f(n)\, \mu_n$. By Lemma 2 there is a representation g for $\tilde{\mu}$ with $|g| > 0$. Now, if $|f| < 1$, then $f + (1 - |f|)g$ is a representation for μ which is strictly greater than f. That contradicts to the maximality of f. Therefore $|f| = 1$, which proves the theorem. ∎

Let E be a real vector space, $\{p_n \mid n \in \mathbb{N}\}$ a bounded family of sublinear functionals on E, and $A = \{\mu \mid \mu \text{ linear}, \mu \leqslant \sup_{n\in\mathbb{N}}(p_n)\}$. By observing that an additive functional $\mu_n \leqslant p_n$ with $\mu_n(0) = 0$ has to be linear and that $\nu \leqslant \tilde{\nu}$ for linear functionals implies $\nu = \tilde{\nu}$, we obtain:

Corollary 1. *If every extreme point of A is dominated by some p_n, then any $\mu \in A$ is a countable convex-combination of linear functionals $\mu_n \leqslant p_n$ $(n \in \mathbb{N})$.*

IV. Choquet's Theorem

Consider the same situation as in the last chapter:

S an abelian semigroup, π subadditive on S, and $A = \{\mu \mid \mu \text{ additive on } S, \mu \leqslant \pi\}$ equipped with the weak S-topology.

Let us first introduce some notation. Let $C(A)$ be the $[-\infty, +\infty[$-valued continuous functions on A. Of course, $\hat{S}$ is a subset of $C(A)$. If $p \leqslant \pi$ is subadditive on S, then

$$\hat{p}(f) \underset{\text{def}}{=} \sup\{f(a) \mid a \in A, a \leqslant p\} \qquad \forall f \in C(A)$$

is subadditive on $C(A)$. If P is a set of subadditive functionals on S, then $\hat{P}$ denotes $\{\hat{p} \mid p \in P\}$. For a subsemigroup T of $C(A)$ $\hat{p}_{/T}$ denotes the restriction of $\hat{p}$ to T and $\hat{P}_{/T}$ shall be $\{\hat{p}_{/T} \mid \hat{p} \in \hat{P}\}$. A functional ν on T is called order-preserving if $t_1, t_2 \in T$ with $t_1(\mu) \leqslant t_2(\mu) \forall \mu \in A$ implies $\nu(t_1) \leqslant \nu(t_2)$, A_T is the set of order-preserving additive functionals ν on T such that $\nu(t) \leqslant \hat{\pi}(t) \forall t \in T$. Of course, A_X contains $\hat{A}_{/T}$. A_T is compact under the weak T-topology and it is ordered under $\leqslant$, where $\nu_1 \leqslant \nu_2$ means $\nu_1(t) \leqslant \nu_2(t) \forall \in T$. For any $\nu \in A_T$, Zorn's lemma gives us a $\tilde{\nu} \in A_T$ being *maximal* with respect to $\leqslant$ such that $\nu \leqslant \tilde{\nu}$. ∂A_T denotes the set of extreme points of A_T and $\mathrm{Ch}_L(\Omega)$ the Choquet-boundary of a compact $\Omega \subset A_T$ with respect to $L \subset T$. In the last section we showed $\partial A_T = \mathrm{Ch}_T(A_T)$.

Now, let M be the cone stable for the pointwise-maximum operation generated in $C(A)$ by $\hat{S}$ and the constant functions. We can state the following remarks.

Remark 1. $\partial A_M = (\widehat{\partial A})_{/M}$.

Proof. We have $\partial A_M = \mathrm{Ch}_M(A_M)$. This set is equal to $\mathrm{Ch}_{\hat{S}}(A_M)$ since the Choquet-boundary does not change by going from a set of upper semicontinuous functions to the max-stable cone of functions generated by them and the constants ([1, p. 47] or [8, Satz 3]). But $A_{M|\hat{S}}$ (restriction of A_M to $\hat{S}$) is a subset of A, and by the sandwich theorem ([7, Cor. 1.1]), there is for any $\mu \in A$ a $\tilde{\mu} \in A_M$ such that $\mu \leqslant \tilde{\mu}_{/\hat{S}}$. That implies $\mathrm{Ch}_{\hat{S}}(A_M) = \widehat{\mathrm{Ch}_{\hat{S}}(A)}_{/M}$ and finishes the proof since $\partial A = \mathrm{Ch}_{\hat{S}}(A)$. ▮

Remark 2. If μ is a maximal element of A_M, then there exists a unique probability measure m_μ on A such that

$$\mu(h) = \int_A h \, dm_\mu \qquad \forall h \in M.$$

Proof. Let F be the finite-valued functions in M. Then μ has a unique additive extension $\tilde{\mu}$ to $F - F$. This extension is order-preserving since $f_1 - g_1 \geqslant f_2 - g_2$ $(f_1, f_2, g_1, g_2 \in F)$ implies $\mu(f_1) + \mu(g_2) \geqslant \mu(f_2) + \mu(g_1)$, and therefore we have

$$\tilde{\mu}(f_1 - g_1) = \mu(f_1) - \mu(g_1) \geqslant \mu(f_2) - \mu(g_2) = \tilde{\mu}(f_2 - g_2).$$

That means $\tilde{\mu}$ is positive on $G = F - F$ and therefore continuous with respect to uniform convergence. The Stone–Weierstrass theorem tells us that G is dense in the set of finite-valued functions in $C(A)$, so there is a unique probability measure m_μ (positive Borel measure with $m_\mu(A) = 1$) on A such that:

$$\int_A f\, dm_\mu = \mu(f) \qquad \forall f \in F.$$

For $h \in M$ is the set $F_h = \{f \in F \mid f(a) \geqslant h(a) \forall a \in A\}$ downwards directed. From this and the fact that $\tilde{\mu}$ is order-preserving, we obtain:

$$\mu(h) \leqslant \inf\{\tilde{\mu}(f) \mid f \in F_h\} = \int_A h\, dm_\mu .$$

Now, the maximality of μ gives us equality. ∎

THEOREM 4. *For every $\mu \in A$, there is a probability measure m_μ on A with $\int_A \hat{s}\, dm_\mu \geqslant \mu(s) \forall s \in S$ such that for all $f \in C(A)$, $\int_A f\, dm_\mu$ is dominated by a countable convex-combination of $\hat{p}_n(f)$ for any sequence (p_n) of subadditive functionals on S having the property that for every ν in the extreme points ∂A of A (Choquet-boundary of A), there is some n with $p_n(s) \geqslant \nu(s)\ \forall s \in S$.*

Proof. For $\mu \in A$, we choose a maximal element μ_1 in A_M such that $\mu \leqslant \mu_{1/S}$. Let m be the measure given by Remark 2. Then Remark 1 together with Theorem 3 implies that μ_1 is dominated by a countable convex-combination of the $\hat{p}_{n/M}$, i.e.,

$$\mu_1 \leqslant \sum_{n \in \mathbb{N}} \lambda_n \hat{p}_{n/M} . \tag{*}$$

By the sandwich theorem ([7, Cor. 1.1]), there is an additive order-preserving functional δ on $C(A)$ such that $\mu_1 \leqslant \delta_{/M}$ and $\delta \leqslant \sum_{n \in \mathbb{N}} \lambda_n \hat{p}_n$. The maximality of μ_1 implies $\mu_1 = \delta_{/M}$, so the measure given by δ has to be m. That means $m_\mu = m$ has the desired properties since the inequality (*) implies

$$\int_A f\, dm \leqslant \sum_{n \in \mathbb{N}} \lambda_n \hat{p}_n(f) \qquad \forall f \in C(A). \ ∎$$

Let us rephrase this result for a special case. Let Ω be a compact set, Ψ a set of point-separating upper semicontinuous functions on Ω, $\mathrm{Ch}(\Omega)$ be the Choquet-boundary of Ω (which is also the Choquet boundary with respect to the cone S_Ψ generated by Ψ), and let Σ be the least σ-algebra on Ω such that all functions in Ψ are measurable.

THEOREM 5. (i) *Let $x \in \Omega$ and $A \supset \mathrm{Ch}(\Omega)$ be an F_σ-set. Then there is a probability measure $\sigma_{x,A}$ on Ω such that $\sigma_{x,A}(A) = 1$ and $\int_\Omega \varphi \, d\sigma_{x,A} \geqslant \varphi(x) \; \forall \varphi \in \Psi$.*

(ii) *Let $x \in \Omega$; then there is a positive Σ-measure σ_x on Ω such that $\sigma_x(B) = \sigma_x(\Omega) = 1$ for all $\mathrm{Ch}(\Omega) \subset B \in \Sigma$ and $\int_\Omega \varphi \, d\sigma_x \geqslant \varphi(x) \; \forall \varphi \in \Psi$.*

(iii) *If all $\varphi \in \Psi$ are continuous, then for $x \in \Omega$ there is a probability measure σ_x on Ω such that $\sigma_x(A) = 1$ for all F_σ-sets, $A \supset \mathrm{Ch}(\Omega)$ and $\int_\Omega \varphi \, d\sigma_x \geqslant \varphi(x) \; \forall \varphi \in \Psi$.*

Proof. For compact K we denote by p_K the subadditive functional defined on the upper-semicontinuous functions f on K by $p_K(f) = \sup_{x \in K} f(x)$. Let S_Ψ be the cone of upper semicontinuous functions on Ω generated by Ψ and $X = \{\mu \mid \mu \text{ additive on } S_\Psi, \mu \leqslant p_\Omega\}$. In X we consider the weak S_Ψ-topology. Of course, Ω is a subset of X. We have $\partial X \subset \mathrm{Ch}(\Omega)$, because by the Hahn–Banach theorem, there is for any $x \in X$ a representing measure supported by Ω (compare [7, König's theorem]).

Now, (i) is proved by the following argument: Let

$$\mathrm{Ch}(\Omega) \subset A \subset \bigcup_{n \in \mathbb{N}} K_n ,$$

where the K_n are compact. According to Theorem 3, x is dominated on S_Ψ (and therefore on Ψ) by a countable convex-combination $\sum_{n \in \mathbb{N}} \lambda_n p_{K_n}$ of the p_{K_n}. By the sandwich theorem, there is a representing measure $\sigma_{x,A}$ for x such that $\int_\Omega f \, d\sigma_{x,A} \leqslant \sum_{n \in \mathbb{N}} \lambda_n p_{K_n}(f)$ $\forall f \in C(X)$. This certainly implies $\sigma(\bigcup_{n \in \mathbb{N}} K_n) = \sigma(\Omega) = 1$.

According to Theorem 4, there is for $x \in X$ a representing measure m_x on X for x such that for any sequence of compact subsets K_n of X with $\bigcup_{n \in \mathbb{N}} K_n \supset \mathrm{Ch}(\Omega)$, we have a countable convex-combination of the p_{K_n} with:

$$\int_\Omega f \, dm_x \leqslant \sum_{n \in \mathbb{N}} \lambda_n p_{K_n}(f) \qquad \forall f C(X). \tag{*}$$

That means m_x is supported by any Baire set of X containing $\mathrm{Ch}(\Omega)$. Now, (ii) is proven by the observation that $\Sigma \subset \{B \cap \Omega \mid B \text{ Baire set in } X\}$ and by taking for σ_x the measure defined by $\sigma_x(B \cap \Omega) = m_x(B)$ for all Baire sets B in X.

(iii) follows from (*) and the fact that Ω is a compact set with respect to the topology induced by X if all $\varphi \in \Psi$ are continuous. ∎

Remark. (iii) implies that for all $x \in \Omega$, there is a representing measure σ_x which lives on the Baire sets containing the Choquet-boundary of Ω ([1, Theorem 5.23]). That the maximal measures for continuous Ψ are living on the F_σ-sets containing the Choquet-boundary is well-known if Ψ is a group of continuous functions ([11, p. 30]); for a semigroup, this statement seems to be new.

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