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Sandwich theorems and lattice Semigroups*

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I. INTRODUCTION

The main purpose of this paper is to study sandwich theorems for preordered abelian semigroups and to give conditions such that the set of additive order-preserving functionals on such a semigroup is a lattice semigroup and to show how preordered semigroups can be applied. The main theorem of part II is a generalization of a theorem of R. Kaufman. We prove that whenever an order-preserving subadditive mapping δ from an abelian semigroup $S[-\infty, +\infty[$ dominates ($\delta \geq \omega$), a superadditive map $\omega: S \to [-\infty, +\infty[$ then there exists an additive order-preserving $[-\infty, +\infty[$-valued functional which lies between ω and δ. Of course, such a theorem leads in the usual manner to extensions for additive functionals on subsemigroups. As an application we prove a result of H. Dinges which is a generalization of the well-known extension theorem of G. Aumann; in our proof we can drop the regularity condition which was necessary in the original proof. Furthermore we generalize a theorem recently found by H. König to semigroups.

In part III we study the inverse, where we have $\omega \geq \delta$ and we give conditions such that there is an order-preserving additive functional $S \to [-\infty, +\infty[$ which lies between δ and ω. A similar problem is to search for conditions such that the set of order-preserving additive mappings $S \to [-\infty, +\infty[$ is a lattice semigroup. Of course these results are closely related to D. A. Edwards' interpolation theorem. Actually we give a rather general semigroup version of a theorem of L. Asimow and A. J. Ellis which they used for proving Edwards theorem. As an application we obtain a slight generalization of the Cartier–Fell–Meyer theorem and we prove some characterisations of Choquet-Simplexes.

II. Sandwich and Extension Theorems

In the following $(S, +, \leqslant)$ always denotes a preordered abelian semigroup, i.e., the relation $\leqslant$ is reflexive, transitive, and compatible with the semigroup structure ($a \leqslant b$, $c \leqslant d \Rightarrow a + c \leqslant b + d$). Every subsemigroup T of a semigroup S gives a *natural T-preorder* in S by

$$a \leqslant b \Leftrightarrow \exists d \in T \colon a + d = b \qquad (a, b \in S).$$

This relation is clearly compatible with the semigroup structure. Every compatible preorder in an abelian group is of this kind, however this is not true for semigroups.

We are interested in $(S, \leqslant)^*$ the set of order-preserving homomorphisms from $S \to \bar{\mathbb{R}} = [-\infty, +\infty[$ and $(S, \leqslant)^{\#}$ the set of order-preserving subadditive mappings $S \to \bar{\mathbb{R}}$. As usual a mapping p is called *(super)–subadditive* if $p(s_1 + s_2)(\geqslant) \leqslant p(s_1) + p(s_2)$ for all $s_1, s_2 \in S$. If the preorder under consideration is the equality we write S^* and $S^{\#}$. $(S, \leqslant)^*$ and $(S, \leqslant)^{\#}$ endowed with the pointwise order on S (which we also denote by $\leqslant$) are ordered abelian semigroups and they admit a scalar multiplication by $\mathbb{R}^+$. $a \to \hat{a}$ denotes the canonical mapping $S \to [(S, \leqslant)^{\#}]^*$.

If S does not have a neutral element, we adjoin a neutral element $\mathbf{0}$ and extend $\leqslant$ in the trivial way by assuming that $\mathbf{0}$ is only comparable with itself. Thus the proofs become simpler, but all theorems remain valid for semigroups without neutral element.

$p \in (S, \leqslant)^{\#}$ is called *homogeneous* if $p(ns) = np(s)\ \forall s \in S\ \forall n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. For $q \in (S, \leqslant)^{\#}$ there exists a maximal homogeneous $(S, \leqslant)^{\#} \ni \bar{q} \leqslant q$ defined by: $\bar{q}(s) = \inf\{(1/n)\, q(ns) \mid 0 \neq n \in \mathbb{N}\}$. If $p \in (S, \leqslant)^{\#}$ is homogeneous then either $p(0) = 0$ or $p = -\infty$ (i.e., $p(s) = -\infty\ \forall s \in S$).

Theorem 1. *If ω is superadditive $\leqslant \delta \in (S, \leqslant)^{\#}$ then there exists a $\mu \in (S, \leqslant)^*$ such that $\omega \leqslant \mu \leqslant \delta$.*

Proof. Let $\mathfrak{P} \subset \{p \in (S, \leqslant)^{\#} \mid \omega \leqslant p \leqslant \delta\}$ be a maximal chain with respect to the pointwise order on S. Since

$$\mu = \inf_{p \in \mathfrak{P}} (p)$$

is order-preserving and subadditive it is a minimal element of $\{p \in (S, \leqslant)^{\#} \mid \omega \leqslant p \leqslant \delta\}$, this implies in particular that μ is homogeneous. We prove that μ is additive. If there exist $t_1, t_2 \in S$ such that $\mu(t_1 + t_2) < \mu(t_1) + \mu(t_2)$, then there are $\xi, \eta \in \mathbb{R}$ with

$\xi < \mu(t_1)$, $\eta < \mu(t_2)$, $\mu(t_1 + t_2) < \xi + \eta$. The functionals μ_ξ, μ_η defined by

$$\mu_\xi(t) = \inf\{\xi m + \mu(\tilde{t}) \mid m \in \mathbb{N},\ t \leqslant mt_1 + \tilde{t}\}$$
$$\mu_\eta(t) = \inf\{\eta m + \mu(\tilde{t}) \mid m \in \mathbb{N},\ t \leqslant mt_2 + \tilde{t}\}$$

are order-preserving subadditive and $\lneqq \mu$. Since μ is minimal neither $\omega \leqslant \mu_\xi$ nor $\omega \leqslant \mu_\eta$ can be true. So there are $s_1, s_2, y_1, y_2 \in S$ and $0 \neq m_1, m_2 \in \mathbb{N}$ such that

$$s_1 \leqslant m_1 t_1 + y_1, \quad \xi m_1 + \mu(y_1) < \omega(s_1)$$
$$s_2 \leqslant m_2 t_2 + y_2, \quad \eta m_2 + \mu(y_2) < \omega(s_2).$$

Multiplication by m_1, m_2 and addition gives the strict inequality

$$(\xi + \eta)\, m_1 m_2 + m_2 \mu(y_1) + m_1 \mu(y_2) < m_2 \omega(s_1) + m_1 \omega(s_2)$$

Using the inequalities for ξ, η and the fact that μ is subadditive and ω superadditive we get

$$\mu(m_1 m_2(t_1 + t_2) + m_2 y_1 + m_1 y_2) < \omega(m_2 s_1 + m_1 s_2).$$

Since μ is order-preserving $\geqslant \omega$ this is a contradiction to

$$m_2 s_1 + m_1 s_2 \leqslant m_1 m_2 (t_1 + t_2) + m_2 y_1 + m_1 y_2. \quad \blacksquare$$

Corollary 1.1. *If T is a subsemigroup of S, $\mu \in (T, \leqslant)^*$ and $\delta \in (S, \leqslant)^\#$ with $\mu \leqslant_T \delta$ (i.e., $\mu(t) \leqslant \delta(t)\ \forall t \in T$), then there exists a $\nu \in (S, \leqslant)^*$ such that $\mu \leqslant_T \nu \leqslant \delta$.*

Proof. Application of Theorem 1 to the superadditive functional ω defined by $\omega(s) = \{\mu(s)$ if $s \in T$, $-\infty$ otherwise$\}$. $\blacksquare$

Corollary 1.2. *If $p \in (S, \leqslant)^\#$ is homogeneous then*

$$p(s) = \sup\{\mu(s) \mid (S, \leqslant)^* \ni \mu \leqslant p\}.$$

Proof. $T(x) = \{nx \mid n \in \mathbb{N}\}$ is a subsemigroup of S for any $x \in S$, and p is additive on $T(x)$. Corollary 1 gives a $\mu_x \in (S, \leqslant)^*$. $p \leqslant_{T(x)} \mu_x \leqslant p$. $\blacksquare$

Corollary 1.3. *Let T be a subsemigroup of S and $\mu \in (T, \leqslant)^*$, $\delta \in (S, \leqslant)^\#$ such that $\mu \leqslant_T \delta$, then there exists an extension $\tilde{\mu} \in (S, \leqslant)^*$ of μ such that $\tilde{\mu} \leqslant \delta$ if and only if:*

$$(*) \qquad [t_1 \leqslant t_2 + s \Rightarrow \mu(t_1) \leqslant \mu(t_2) + \delta(s)] \qquad \forall t_1, t_2 \in T, \quad s \in S.$$

Proof. Define $\eta \in (S, \leqslant)^{\#}$ by $\eta(s) = \inf\{\mu(t) + \delta(\tilde{s}) \mid t + \tilde{s} \geqslant s, t \in T, \tilde{s} \in S\}$. $(*)$ implies that η and μ are equal on T, so Corollary 1 gives an extension. The only if part is trivial. ∎

Theorem 1 generalizes a result of R. Kaufman [12], who proved the same theorem without considering preorder relations. An equivalent to Kaufman's theorem can be found in a paper by P. Kranz [16]. If S is an abelian group our theorem is in fact a consequence of Kaufmans result, but this is not true in general for semigroups. Sandwich and extension theorems for ordered vector spaces have been studied by many authors, many results can be found in the work of S. Simons [18, 19]. A good survey of the literature connected with the Hahn–Banach theorem (before 1969) can be found in the references of B. Rodriguez–Salinas Palero [20] (125 references!).

If S is a group the Hahn–Banach extension theorems follows immediately from Corollary 3. If in Corollary 1 the semigroups under consideration are real vector spaces and $-\infty \neq \mu$ then the functional ν is automatically an extension of μ and ν is linear if

$$\lim_{\mathbb{R} \ni \lambda \to 0} \delta(\lambda s) = 0 \ \forall s \in S,$$

this follows from the density of the rationals in $\mathbb{R}$.

Regularly ordered semigroups have been studied by G. Aumann [4] and H. Dinges [8]. In the following we shall derive their results as applications of Theorem 1, it seems interesting to note that the regularity condition on the order relation is not necessary in both cases. As a second application we give a generalization of a theorem recently found by H. König [15].

Applications

If $Z \supset \{0\}$, $Y \supset \{0\}$ are subsemigroups of $(S, +, \leqslant)$ then the "*Überholerhalbgruppe*" $Y_Z^{>}$ and the "*Unterholerhalbgruppe*" $Y_Z^{<}$ are

$$Y_Z^{>} = \{s \in S \mid \exists y \in Y, z \in Z, 0 \neq m \in \mathbb{N}: y \geqslant ms + z\}$$
$$Y_Z^{<} = \{s \in S \mid \exists y \in Y, z \in Z, 0 \neq m \in \mathbb{N}: y \leqslant ms + z\}.$$

THEOREM (Dinges [8, p. 463])*. *Let* $U = (Y + Z)_Y^{<} \cap Y_{Y+Z}^{>}$, $p\colon S \to \overline{\mathbb{R}}$ *be superadditive with* $p(z) > -\infty$ *for* $z \in Z$, *and* μ *be an* $\mathbb{R}$*-valued additive functional on* Y, *then* μ *can be extended to a* $\nu \in (U, \leqslant)^{*}$ *such that*

$$(*)\ \nu(u) \geqslant \nu(\tilde{u}) + p(z) \quad \textit{whenever} \quad u, \tilde{u} \in U, \quad z \in Z, \quad u \geqslant \tilde{u} + z$$

* The statement of the theorem has been slightly modified following J. Horváth.

if and only if

$$(**)\ \mu(y) \geqslant \mu(\tilde{y}) + p(\tilde{z}) \qquad \textit{whenever} \quad y, \tilde{y} \in Y, \quad \tilde{z} \in Z, \quad y \geqslant \tilde{y} + \tilde{z}.$$

Proof. The only if part is trivial. We define

$$\omega(s) = \inf \left\{ \frac{1}{n} \left[\mu(y) - \mu(\tilde{y}) - p(z) \right] \,\middle|\, 0 \neq n \in \mathbb{N}, \right.$$
$$\left. y, \tilde{y} \in Y, z \in Z, y \geqslant ns + \tilde{y} + z \right\}$$

$$\delta(s) = \sup \left\{ \frac{1}{n} \left[\mu(\tilde{y}) - \mu(y) + p(z) \right] \,\middle|\, 0 \neq n \in \mathbb{N}, \right.$$
$$\left. y, \tilde{y} \in Y, z \in Z, y + ns \geqslant z + \tilde{y} \right\}.$$

Then $\omega \in (Y^{\geqslant}_{Y+Z}, \leqslant)^{\#}$ and $\delta\colon (Y + Z)^{\leqslant}_{Y} \to \mathbb{R}$ is superadditive. Let $s \in U$ and take arbitrary elements $y_1, y_2, \tilde{y}_1, \tilde{y}_2 \in Y$, $z_1, z_2 \in Z_1$, $0 \neq n_1, n_2 \in \mathbb{N}$ such that

$$y_1 + n_1 s \geqslant z_1 + \tilde{y}_1 \qquad \text{and} \qquad y_2 \geqslant n_2 s + \tilde{y}_2 + z_2 .$$

By an elementary calculation we get $n_1 y_2 \geqslant n_1 n_2 s + n_1 \tilde{y}_2 + n_1 z_2$ and

$$n_1 \tilde{y}_2 + n_2 y_1 + n_1 n_2 s + n_1 z_2 \geqslant n_2 z_1 + n_2 \tilde{y}_1 + n_1 \tilde{y}_2 + n_1 z_2 .$$

This implies $n_1 y_2 + n_2 y_1 \geqslant n_2 z_1 + n_2 \tilde{y}_1 + n_1 \tilde{y}_2 + n_1 z_2$ and from $(**)$ it follows $n_1[\mu(y_2) - \mu(\tilde{y}_2) - p(z_2)] \geqslant n_2[\mu(\tilde{y}_1) - \mu(y_1) + p(z_1)]$, which implies $\delta \leqslant_U \omega$. From this together with the inequalities $\omega \leqslant_Y \mu \leqslant_Y \delta$ we obtain $\omega =_Y \mu =_Y \delta$. Now, by Theorem 1 there exists a $\nu \in (U, \leqslant)^*$ such that $\delta \leqslant_U \nu \leqslant_U \omega$. ν is clearly an extension of μ. And the desired inequality $(*)$ follows from $\delta \geqslant_U p$ and the fact that $u, \tilde{u} \in U$, $z \in Z$ and $u \geqslant \tilde{u} + z$ implies $z \in U$. ∎

G. Aumann considered the case $Z = S$ (i.e., $p(s) > -\infty\ \forall s \in S$), then we can extend μ on $U = Y_S^{>}$.

Now, we proceed to H. König's Maximumsatz. Consider in $\Omega \subset S^{\#}$ a Hausdorff topology such that Ω is compact and

$$\hat{S} = \{\hat{s}\colon \Omega \to \bar{\mathbb{R}} \mid s \in S\}$$

consists of upper-semicontinuous functions on Ω. Let $\mu \in S^*$ such that $\mu \leqslant \sup_{\omega \in \Omega} (\omega)$.

THEOREM [König, 15]. *There exists a probability measure σ on Ω such that*

$$\mu(s) \leqslant \int_\Omega \hat{s}(\omega)\, d\sigma(\omega)\ \forall s \in S.$$

Proof. The result is trivial if $\mu = -\infty$, therefore we can assume $\mu(0) = 0$ and we consider in USC(Ω) (upper semicontinuous functions $\Omega \to \overline{\mathbb{R}}$) the pointwise order on Ω. Let $\delta \in (\mathrm{USC}(\Omega), \leqslant)^{\#}$ be given by

$$\delta(f) = \sup_{\omega\in\Omega} f(\omega).$$

Then $\delta\ {}_S\!\geqslant \hat{\mu}$ where $\hat{\mu} \in (\hat{S}, \leqslant)^*$ is defined by $\hat{\mu}(\hat{s}) = \mu(s)$. There exists a $\nu \in (\mathrm{USC}(\Omega), \leqslant)^*$ such that

$$\hat{\mu} \leqslant_S \nu \leqslant_{\mathrm{USC}(\Omega)} \delta.$$

By the Riesz representation theorem the restriction of ν to the continuous functions is a probability measure σ. And

$$\nu(\hat{s}) \leqslant \int_\Omega \hat{s}(\omega)\, d\sigma(\omega)\ \forall s \in S$$

since ν is order-preserving. Therefore has σ the desired properties. ∎

For the special case $\Omega = \{p_1, \dots, p_n\}$ the measure σ has to be a convex-combination of Dirac measures.

COROLLARY. *If $\mu \in S^*$ with $\mu \leqslant \max(p_1, \dots, p_n)$, where $p_1, \dots, p_n \in S^{\#}$ then μ is dominated by a convex-combination of the p_i, i.e., there exist nonnegative real numbers $\lambda_1, \dots, \lambda_n$ such that*

$$\sum_{i=1}^n \lambda_i = 1 \quad \text{and} \quad \mu \leqslant \sum_{i=1}^n \lambda_i p_i .$$

III. LATTICE SEMIGROUPS

We call $(S, +, \leqslant)$ an ***L-semigroup*** if $\leqslant$ is an order relation (antisymmetric preorder) such that any two elements $a, b \in S$ do have a *glb* $a \wedge b$ in S, such that the distributive law is valid for $\wedge$, that is, $(b + a) \wedge (b + c) = b + c \wedge a$ for all $a, b, c \in S$. An L-semigroup is called a ***CL-semigroup*** if it is conditionally complete (in the sense that every nonempty subset of S has a *glb*) such that the unrestricted distributive law is valid for the *glb* [Birkhoff 6, p. 200].

$$\bigwedge\{b + a \mid a \in \Sigma\} = b + \bigwedge(\Sigma) \quad \forall \varnothing \neq \Sigma \subset S.$$

Every bounded set Q in a *CL*-semigroup S (i.e., $\exists\, b \in S\ \forall\, q \in Q$: $b \geqslant q$) has a *lub* which we denote by $\bigvee(Q)$ being equal to

$$\bigwedge\{s \in S \mid q \leqslant s\ \forall q \in Q\}.$$

$(S, \leqslant)$ has the *decomposition property* (DCP) if for all $a, b, c \in S$ such that $a \leqslant b + c$ there exist $\tilde{b} \leqslant b$, $\tilde{c} \leqslant c$ such that $a = \tilde{b} + \tilde{c}$. $(S, \leqslant)$ has the *semiinterpolation property* (SIP) if for all $a, b, c, d \in S$ such that $a \leqslant b + c$, $a \leqslant b + d$ there exists an $s \in S$ such that $s \leqslant d$, $s \leqslant c$, $a \leqslant b + s$. Every *L*-semigroup and every preordered group has the SIP. S has the *finite sum property* (FSP) if for $s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in S$ with $s_1 + s_2 = \tilde{s}_1 + \tilde{s}_2$ there exist $u_{ij} \in S$ $(i = 1, 2; j = 1, 2)$ such that $s_i = u_{i1} + u_{i2}$ and $\tilde{s}_i = u_{1i} + u_{2i}$ $(i = 1, 2)$.

Of course this property implies that whenever $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} \tilde{s}_j$ there exist $u_{ij} \in S$ such that $s_i = \sum_{j=1}^{n} u_{ij}$ and $\tilde{s}_j = \sum_{i=1}^{m} u_{ij}$. This is proved by a simple inductive argument. For any abelian semigroup S the FSP implies the DCP for the natural S-preorder. If the cancellation law [Chevalley 7, p. 42] is valid in S then S has the FSP if and only if it has the DCP with respect to the natural S-preorder [Alfsen 1, p. 85].

The subsemigroup of $(S, \leqslant)^{\#}$ consisting of the homogeneous elements and $(S, \leqslant)_{\#} = \{\mu: S \to \overline{\mathbb{R}} \mid \mu$ order-preserving and superadditive$\}$ are examples for *CL*-semigroups. The *lub* $\bigvee_{(S,\leqslant)^{\#}}(B) = p$ of bounded sets $B \subset (S, \leqslant)^{\#}$ is $p(s) = \sup_{\delta \in B} \delta(s)$, the pointwise supremum of B on S.

From this we can calculate the *glb* $\bigwedge_{(S,\leqslant)^{\#}}(\Sigma)$ for nonempty sets Σ. Let us show that the unrestricted distributive law is valid if the elements of Σ and b are homogeneous. Obviously

$$b + \bigwedge\nolimits_{(S,\leqslant)^{\#}}(\Sigma) \leqslant \bigwedge\nolimits_{(S,\leqslant)^{\#}}(b + \Sigma)$$

and from $S^* \ni h \leqslant b + \delta\ \forall \delta \in \Sigma$ it follows from Theorem 1 that there is a $\mu \in (S, \leqslant)^*$ such that $h - b \leqslant \mu \leqslant \delta\ \forall \delta \in \Sigma$, and from this it follows

$$h \leqslant b + \bigwedge\nolimits_{(S,\leqslant)^{\#}}(\Sigma).$$

Corollary 1.2 gives now the desired result. The *glb*

$$\bigwedge\nolimits_{(S,\leqslant)_{\#}}(A)$$

of nonempty sets $A \subset (S, \leqslant)_{\#}$ is equal to the pointwise infimum of A on S, the validity of the unrestricted distributive law follows immediately from this fact.

Now, we shall give formulas for the *lub* in $(S, \leqslant)_{\#}$ and *glb* in $(S, \leqslant)^{\#}$. Let $f: S \to \bar{\mathbb{R}}$, and consider the functions

$$f^{\wedge}(s) \underset{\text{def}}{=} \inf \left\{ \sum_{i=1}^{n} f(s_i) \mid n \in \mathbb{N},\ s \leqslant \sum_{i=1}^{n} s_i \right\} \tag{1}$$

$$f^{\vee}(s) \underset{\text{def}}{=} \sup \left\{ \sum_{i=1}^{n} f(s_i) \mid n \in \mathbb{N},\ \sum_{i=1}^{n} s_i \leqslant s \right\}. \tag{2}$$

Obviously $f \geqslant f^{\wedge} \in (S, \leqslant)^{\#}$ and whenever $f \geqslant \varphi \in (S, \leqslant)^{\#}$, then $\varphi \leqslant f^{\wedge}$. If $f \leqslant \omega$ is bounded by an element $\omega \in (S, \leqslant)_{\#}$ then $f \leqslant f^{\vee} \in (S, \leqslant)_{\#}$. Thus whenever $\varnothing \neq Q \subset (S, \leqslant)^{\#}$ and $g(s) = \inf\{q(s) \mid q \in Q\}$ then $g^{\wedge} = \bigwedge_{(S,\leqslant)^{\#}}(Q)$ is the *glb* of Q, and for any bounded set $P \subset (S, \leqslant)_{\#}$, $h^{\vee} = \bigvee_{(S,\leqslant)_{\#}}(P)$ is the *lub* of P in $(S, \leqslant)_{\#}$ where $h(s) = \sup\{p(s) \mid p \in P\}$. The following remark is easily checked by a straightforward calculation.

Remark. If S has the FSP and $(S, \leqslant)$ the DCP then $\omega^{\wedge}$ is additive whenever $\omega \in (S, \leqslant)_{\#}$. In this case it follows that we have

$$\omega^{\wedge}(s) = \inf \left\{ \sum_{i=1}^{n} \omega(s_i) \mid n \in \mathbb{N},\ s = \sum_{i=1}^{n} s_i \right\}.$$

If S has the FSP and $(S, (\leqslant)^{-1})$ the DCP, where $(\leqslant)^{-1}$ is the inverse of $\leqslant$ then $\delta^{\vee}$ is additive whenever $\delta \in (S, \leqslant)^{\#}$ is dominated by an $\omega \in (S, \leqslant)_{\#}$. We have in this case

$$\delta^{\vee}(s) = \sup \left\{ \sum_{i=1}^{n} \delta(s_i) \mid n \in \mathbb{N},\ \sum_{i=1}^{n} s_i = s \right\}.$$

As a consequence of this remark we obtain:

THEOREM 2.

(i) *If S has the* FSP *and $(S, \leqslant)$ the* DCP *then $(S, \leqslant)^*$ is a CL-semigroup such that for any nonempty $A \subset (S, \leqslant)^*$*

$$\bigwedge_{(S,\leqslant)^{\#}}(A) = \bigwedge_{(S,\leqslant)^{*}}(A) = \bigwedge_{S^{\#}}(A) = \bigwedge_{S^{*}}(A).$$

(ii) *If S has the* FSP *and* $(S, (\leqslant)^{-1})$ *the* DCP *then* $(S, \leqslant)^*$ *is a CL-semigroup such that for any $A \subset (S, \leqslant)^*$ bounded by an $\omega \in (S, \leqslant)_{\#}$ we have*

$$\bigvee_{(S,\leqslant)^*} (A) = \bigvee_{(S,\leqslant)_{\#}} (A) = \bigvee_{S_{\#}} (A) = \bigvee_{S^*} (A).$$

(iii) *If S has the* FSP *and* $(S, \leqslant)$ *or* $(S, (\leqslant)^{-1})$ *the* DCP *then* $(S, \leqslant)^*$ *interposes* $[(S, \leqslant)_{\#}, (S, \leqslant)^{\#}]$, *i.e., whenever* $(S, \leqslant)_{\#} \ni \omega \geqslant \delta \in (S, \leqslant)^{\#}$ *then there exists a* $\mu \in (S, \leqslant)^*$ *such that* $\omega \geqslant \mu \geqslant \delta$.

COROLLARY 2.1. *If S has the* FSP *then S^* is a CL-semigroup such that* $\bigwedge_{S^*}(A) = \bigwedge_{S^{\#}}(A)$ *for* $\varnothing \neq A \subset S^*$ *and* $\bigvee_{S^*}(B) = \bigvee_{S^{\#}}(B)$ *for any bounded* $B \subset S^*$. *Furthermore S^* interposes* $[S_{\#}, S^{\#}]$.

This generalizes a result of L. Asimow and A. J. Ellis [5, p. 304] which leads to D. A. Edwards' interpolation theorem [9] and to T. Andô's theorem [2]. The following is a partial converse of Theorem 2.

THEOREM 3. *If* $\mu \in (S, \leqslant)^*$, $q_1, q_2 \in (S, \leqslant)^{\#}$ *and* $\mu \leqslant q_1 + q_2$ *then there are* $\mu_1, \mu_2 \in (S, \leqslant)^*$, $\mu_1 \leqslant q_1$, $\mu_2 \leqslant q_2$ *such that* $\mu \leqslant \mu_1 + \mu_2$. *If* $(S, \leqslant)$ *has the* SIP *then* μ_1 *and* μ_2 *can be chosen such that* $\mu = \mu_1 + \mu_2$.

Proof. Consider in $S \oplus S$ the preorder

$$(s_1, s_2) \leqslant (\tilde{s}_1, \tilde{s}_2) \Leftrightarrow (s_1 \leqslant \tilde{s}_1 \text{ and } s_2 \leqslant \tilde{s}_2)$$

and let ΔS be the diagonal subsemigroup $\{(s, s) \mid s \in S\}$ of $S \oplus S$. Define $\tilde{\mu} \in (\Delta S, \leqslant)^*$, $\delta \in (S \oplus S, \leqslant)^{\#}$ by $\tilde{\mu}(s, s) = \mu(s)$ and

$$\delta(s_1, s_2) = q_1(s_1) + q_2(s_2).$$

Since $\delta_{\Delta S} \geqslant \tilde{\mu}$ there is a $\nu \in (S \oplus S, \leqslant)^*$ such that $\tilde{\mu} \leqslant_{\Delta S} \nu \leqslant \delta$. If $(S, \leqslant)$ has the SIP and $(s_0, s_0) \leqslant (s_1, s_1) + (s_2, s_3)$ then there exists an s_4 such that $(s_4, s_4) \leqslant (s_2, s_3)$ and $(s_0, s_0) \leqslant (s_1, s_1) + (s_4, s_4)$; this implies $\tilde{\mu}(s_0, s_0) \leqslant \tilde{\mu}(s_1, s_1) + \delta(s_2, s_3)$, and using Corollary 1.3 we may assume $\tilde{\mu} =_{\Delta S} \nu$ if $(S, \leqslant)$ has the SIP. By taking μ_1, μ_2 defined by $\mu_1(s) = \nu(s, 0)$ and $\mu_2(s) = \nu(0, s)$ the theorem is proved. ∎

COROLLARY 3.1. $(S, \leqslant)^*$ *has the* DCP *whenever* $(S, \leqslant)$ *has the* SIP.

Since every L-semigroup has the SIP, $(S, \leqslant)^*$ has the DCP for every L-semigroup.

Applications

Combining H. König's theorem and Theorem 3 we obtain the following result, which is also due to H. König if S is a vector space without order structure.

THEOREM. *Let* $\mu \in (S, \leqslant)^*$ *and* $q_1, \ldots, q_n \in (S, \leqslant)^\#$ *such that* $\mu \leqslant \max(q_1, \ldots, q_n)$ *then there are* $\mu_1, \ldots, \mu_n \in (S, \leqslant)^*$ *with* $\mu_i \leqslant q_i$ *for* $i = 1, \ldots, n$ *such that* μ *is dominated by a convex combination of the* μ_i. *If* $(S, \leqslant)$ *has the* SIP *we can chose the* μ_i *such that* μ *is equal to a convex combination of the* μ_i.

For the next application consider the following definition.

DEFINITION. For $\Phi \subset S^*$ the *Φ-decomposition preorder* $\leqslant_\Phi$ in S is given by:

$s \leqslant_\Phi \tilde{s} \Leftrightarrow$ whenever there are $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in S$ such that

$$\sum_{i=1}^{n} s_i = s$$

then there are $\tilde{s}_1, \ldots, \tilde{s}_n$ such that

$$\sum_{i=1}^{n} \tilde{s}_i = \tilde{s} \qquad \text{and} \qquad \tilde{s}_i(\varphi) = s(\varphi)\ \forall \varphi \in \Phi.$$

Remark 1. Let S have the FSP; then $\leqslant_\Phi$ is compatible with the semigroup structure in S and S^* is a CL-semigroup containing Φ. Now, let $\{\varphi_1, \ldots, \varphi_n\}$ be a finite bounded subset of the semigroup generated in S^* by Φ. If

$$\sum_{i=1}^{n} s_i \leqslant_\Phi s$$

then there is a decomposition of s:

$$\sum_{i=1}^{n} \tilde{s}_i = s$$

such that $\varphi_k(\tilde{s}_i) = \varphi_k(s_i)$ for all $i, k \leqslant n$. Therefore we obtain by Eq. (2)

$$\sup \left\{ \sum_{i=1}^{n} \varphi_i(s_i) \,\middle|\, \sum_{i=1}^{n} s_i \leqslant_\Phi s \right\} = \sup \left\{ \sum_{i=1}^{n} \varphi_i(s_i) \,\middle|\, \sum_{i=1}^{n} s_i = s \right\}$$
$$= (\varphi_1 \vee \cdots \vee \varphi_n)(s),$$

where $\vee$ denotes the *lub*-operation in S^*. This implies that $(\varphi_1 \vee \cdots \vee \varphi_n) \in (S, \leqslant_\Phi)^*$.

Remark 2. Let Ω be a compact space. The semigroup $M_+(\Omega)$ consisting of the positive Borel measures on Ω, has the Riesz decomposition property and therefore $M_+(\Omega)$ has the FSP. Consider a subset Φ of $B_{\mathbb{R}}(\Omega)$, the real-valued Borel-measurable functions on Ω, and let $\check{\Phi}$ be the max-stable cone generated by Φ in $B_R(\Omega)$. A cone is max-stable if with two functions their maximum belongs also to the cone. Obviously $\Phi \subset (M_+(\Omega), \leqslant_\Phi)^*$. Since the *lub* of two measurable functions in the CL-semigroup $(M_+(\Omega))^*$ is the maximum of the two functions we obtain by Remark 1 the result $\check{\Phi} \subset (M_+(\Omega), \leqslant_\Phi)^*$.

Remark 3. Now, let F be a max-stable cone in $C_{\mathbb{R}}(\Omega)$ (real-valued continuous functions on the compact space Ω) which contains the constants and consider in F the usual pointwise order on Ω. Let $\sigma, \sigma_1, \sigma_2 \in M_+(\Omega)$ be such that for all $f \in F$: $\sigma(f) \geqslant \sigma_1(f) + \sigma_2(f)$. Since F is an L-semigroup there are (Corollary 4.1) $\mu_1, \mu_2 \in (-F, \leqslant)^*$ such that for all $f \in F$: $\mu_1(f) \geqslant \sigma_1(f)$, $\mu_2(f) \geqslant \sigma_2(f)$ and $\sigma(f) = \mu_1(f) + \mu_2(f)$. Now, one can extend μ_1, μ_2 uniquely to order-preserving linear functionals on F–F and then by the Hahn–Banach theorem (Corollary 1.3) to positive measures $\tilde{\mu}_1, \tilde{\mu}_2$ on Ω such that $\sigma = \tilde{\mu}_1 + \tilde{\mu}_2$.[1] This implies that $(M_+(\Omega), (\leqslant_F)^{-1})$ has the DCP, where $\leqslant_F$ stands for:

$$\nu \leqslant_F \tilde{\nu} \Leftrightarrow \nu(f) \leqslant \tilde{\nu}(f)\ \forall f \in F.$$

Using the last two remarks we have proved.

THEOREM. *Let Φ be a subset of $B_{\mathbb{R}}(\Omega)$, $\check{\Phi}$ be the max-stable cone generated by Φ in $B_{\mathbb{R}}(\Omega)$ and F be a max-stable subcone of $C_{\mathbb{R}}(\Omega)$ and let σ, $\nu \in M_+(\Omega)$.*

(i) *If for any decomposition of*

$$\sigma = \sum_{i=1}^{n} \sigma_i$$

into positive measures there exists a decomposition

$$\nu = \sum_{i=1}^{n} \nu_i$$

into positive measures such that for $i = 1,\ldots, n$ $\nu_i(\varphi) = \sigma_i(\varphi)\ \forall \varphi \in \Phi$ then $\sigma(f) \leqslant \nu(f)\ \forall f \in \check{\Phi}$.

[1] We extend first $\mu_1 \leqslant_{G+} \sigma$ to a positive $\tilde{\mu}_1 \leqslant_{G+(\Omega)} \sigma$ on $C_+(\Omega)$, where G_+ and $C_+(\Omega)$ are the positive functions in $F - F$ and $C_{\mathbb{R}}(\Omega)$ respectively. Then we define $\tilde{\mu}_2 = \sigma - \tilde{\mu}_1$ and take the obvious extensions to $C_{\mathbb{R}}(\Omega)$. The conditions of Corollary 1.3 are fulfilled since $F - F$ is a lattice which contains the constants.

(ii) *If* $\sigma(f) \leqslant \nu(f)\ \forall f \in F$ *then for any decomposition*

$$\sigma = \sum_{i=1}^{n} \sigma_i$$

into positive measures there is a decomposition

$$\nu = \sum_{i=1}^{n} \nu_i$$

into positive measures such that: $\nu_i(f) \geqslant \sigma_i(f)\ \forall f \in F,\ i = 1, \ldots, n.$

Specializing this result by taking a compact convex set as Ω, the affine continuous functions on Ω as Φ and as F the sup-norm closure of $\check{\Phi}$ (the convex continuous functions) one obtains the well-known Cartier–Fell–Meyer theorem [Alfsen, 1, p. 23].

In the next application we shall derive some of the classical characterizations for simplexes. Let K be a compact convex subset of a locally convex Hausdorff vector space, $M_+(K)$ the positive Borel-measures on K, $A(K)$ the continuous real-valued affine functions on K, A^* the dual space with respect to the sup-norm and $(A^*)^+ \supset K$ its positive cone which has the compact base K. By $A_L(K)$ we denote the lower-semicontinuous affine functions $K \to \bar{\mathbb{R}}$, by $P_L(K)$ the min-stable cone generated by $A_L(K)$ and by $P(K)$ the continuous functions in $P_L(K)$. For the minimum of the functions $f_1, \ldots, f_n$ we write $\min(f_1, \ldots, f_n)$. In the function semigroups we consider the pointwise order $\leqslant$ on K, in $(A^*)^+$ its natural semigroup order, and in $M_+(K)$ the order $\prec$ defined by $\mu \prec \nu \Leftrightarrow \mu(f) \geqslant \nu(f)\ \forall f \in P(K)$. π shall be the weak*-continuous barycentric map $M_+(K) \to (A^*)^+$.

THEOREM. *The following statements are equivalent*:

(i) A^* *is a vector lattice*;

(ii) $A_L(K)$ *is a L-semigroup*;

(iii) *there exists an order-preserving additive map* φ: $P_L(K) \to A_L(K)$ *such that* $\varphi(a) = a$ *and* $\varphi(a + b) = a + \varphi(b)$ *for all* $a \in A_L(K)$ *and* $b \in P_L(K)$;

(iv) *there exists an additive map* $\tilde{\varphi}$: $(A^*)^+ \to M_+(K)$ *such that* $\tilde{\varphi}(x)(a) = x(a)$ *for all* $x \in (A^*)^+$ *and* $a \in A(K)$;

(v) *For every* $x \in (A^*)^+$ *there exists a unique measure* μ *maximal with respect to* $\prec$ *such that* $\pi(\mu) = x$.

Proof. (i) ⇒ (ii). If A^* is a vector lattice then $(A^*)^+$ has the DCP (Riesz decomposition property) and therefore it has the FSP. Then by Theorem 2 and Eq. (1), $((A^*)^+)^*$ is a CL-semigroup $\supset A_L(K)$ such that for any two $a, b \in A_L(K)$ and $x \in K$

$$\begin{aligned}(a \wedge b)(x) &= \inf\{a(x_1) + b(x_2) \mid x_1, x_2 \in (A^*)^+, x_1 + x_2 = x\} \\ &= \inf\{\mu(\min(a, b)) \mid \mu \in M_+(K), \pi(\mu) = x\}.\end{aligned}$$

$a \wedge b$ must be lower-semicontinuous on K because

$$\{\mu \in M_+(K) \mid \pi(\mu) \in K\}$$

is a compact space and $\min(a, b)$ is lower-semicontinuous on this space and π is continuous.

(ii) ⇒ (iii). We define for $a_1, a_2, \ldots, a_n \in A_L(K)$

$$\varphi(\min(a_1, \ldots, a_n)) = a_1 \wedge \cdots \wedge a_n .$$

(iii) ⇒ (iv). Take for $x \in (A^*)^+$ the restriction of $x \cdot \varphi$ to $P(K)$ then there is a unique order-preserving extension to $P(K) - P(K)$ and by the Stone–Weierstrass theorem $x \cdot \varphi$ can be extended uniquely to a measure on K. Now, define this extension to be $\tilde{\varphi}(x)$.

(iv) ⇒ (v). Since $\tilde{\varphi}$ is additive we obtain from part (i) of our last Theorem (Cartier–Fell–Meyer theorem) that $\mu \prec \tilde{\varphi}(x)$ for any μ such that $\pi(\mu) = x$. So $\tilde{\varphi}(x)$ must be maximal.

(v) ⇒ (i). Since $M_+(K)$ has the FSP the $\prec$-maximal measures are a semigroup[2] with the FSP. By the uniqueness of the maximal measures, $(A^*)^+$ has the FSP and therefore the DCP. Since A^* is directed and $(A^*)^+$ has a compact base [Alfsen, 1, p. 85] A^* is a vector lattice.

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[2] The semigroup structure of the maximal measures is a consequence of part (ii) of our last theorem.

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