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DECOMPOSITION THEOREMS

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The countable-decomposition theorem for linear functionals has become a useful tool in the theory of representing measures (see [4-7]). The original proof of this theorem was based on a rather involved study of extreme points in the state space of a convex cone. Recently M. Neumann [9] gave an independent proof using a refined form of Simons convergence lemma and Choquet's theorem. In this paper a (relatively) short proof of an extension (to a more abstract situation) of the countable-decomposition theorem is given. Furthermore a decomposition criterion is obtained which even works in the case when not all states are decomposable. All the work is based on a complete characterization of those states which are partially decomposable with respect to a given sequence of sublinear functionals.

PRELIMINARIES

For making this paper self-contained we gather first some of the material which will be used in the sequel. $F = (F,+,\leq)$ denotes a preordered convex cone, i.e. $\leq$ is reflexive and transitive and

$$f_i \leq g_i, \quad o \leq \lambda_i \in \mathbb{R} \quad (i = 1,2) \Rightarrow \lambda_1 f_1 + \lambda_2 f_2 \leq \lambda_1 g_1 + \lambda_2 g_2$$

Functionals are maps $p : F \to \bar{\mathbb{R}}$ where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. $o \cdot (-\infty)$ is defined to be o and the other algebraic operations are extended to $\bar{\mathbb{R}}$ in the obvious way. In the set of functionals we consider the pointwise order on F, this order relation is also denoted by $\leq$. Linear (sublinear, superlinear) means positive-homogeneous (i.e. $p(\lambda f) = \lambda p(f)$ $\forall \lambda > o, f \in F$) and additive (subadditive, superadditive). A functional p is called order-preserving if $f \geq g \Rightarrow p(f) \geq p(g)$.

SANDWICH THEOREM ([3]): *Let* p *be a sublinear and orderpreserving functional and let* $\delta \le p$ *be superlinear. Then there is a linear order-preserving* μ *with* $\delta \le \mu \le p$.

As usual, a subset $\Phi \subset F$ is called *downwards directed* if for $f, g \in \Phi$ there is always some $h \in \Phi$ with $h \le f$ and $h \le g$.

LEMMA 1: *Let* p *be a sublinear order-preserving functional and let* $\Phi \subset F$ *be downwards directed. Then there is a linear order-preserving* $\mu \le p$ *such that* $\inf_{f\in\Phi} \mu(f) = \inf_{f\in\Phi} p(f)$.

PROOF: Let $\alpha = \inf_{f\in\Phi} p(f)$ and define a superlinear $\delta \le p$ by $\delta(g) = \sup\{\lambda\alpha \mid \lambda > 0, \exists\, f \in \Phi \text{ with } \lambda f \le g\}$. From the sandwich theorem we get a linear order-preserving μ with $\delta \le \mu \le p$. μ has the desired property because of $\inf_{f\in\Phi} \delta(f) = \alpha$. ■

SUM THEOREM (cf. [3] or [8]) : *Let* μ *be a linear functional and let* p_n *be a sequence of order-preserving sublinear functionals such that for all* $f \in F$ *the sum* $\sum_{n=1}^{\infty} p_n(f)$ *converges in* $\mathbb{R}$ *and is* $\ge \mu(f)$. *Then there are order-preserving linear functionals* $\mu_n \le p_n$ *such that* $f \to \liminf_{m\to\infty} \sum_{n=1}^{m} \mu_n(f)$ *is linear and* $\ge \mu$.

PROOF: By the sandwich theorem there is a linear order-preserving $\bar{\mu}$ with $\mu \le \bar{\mu} \le \sum_{n=1}^{\infty} p_n$. Now, we prove the theorem for

$F_{\bar{\mu}} = \{f \in F \mid \bar{\mu}(f) > -\infty\}$ instead of F. The full result is then obtained by putting $\mu_k(\varphi) = -\infty$ for all $k = 1,2,\ldots$ and $\varphi \in F \setminus F_{\bar{\mu}}$. Let $\bar{F}$ be the cone of sequences $[f_n]$ in $F_{\bar{\mu}}$ for which there is some k (depending on $[f_n]$) such that $f_k, f_{k+1}, f_{k+2}, \ldots$ do have a common upper bound.

In $\bar{F}$ we consider the order relation:

$$[f_n] \leq [g_n] \Leftrightarrow f_n \leq g_n \quad \forall n \in N$$

And we define a sublinear order-preserving functional π on $\bar{F}$ and a superlinear $\delta \leq \pi$ by:

$$\pi([f_n]) = \limsup_{m \to \infty} \sum_{n=1}^{m} p_n(f_n), \quad \delta([f_n]) = \begin{cases} \bar{\mu}(f) \text{ if } f = f_k = f_n \ \forall\, n, k \in N \\ -\infty \text{ otherwise} \end{cases}$$

By the sandwich theorem and Zorn's lemma there is a maximal linear order-preserving ν with $\delta \leq \nu \leq \pi$.
Define $\Delta_k([f_n]) = (o,o,\ldots,o,f_k,o,o,\ldots)$ (everywhere o except f_k at place k) and

$$\rho([f_n]) = \liminf_{m \to \infty} \sum_{k=1}^{m} \nu\, \Delta_k([f_n]).$$

Then ρ is superlinear. Considering the following inequalities we obtain $\rho \geq \nu$

(1) $$\liminf_{m \to \infty} \sum_{k=1}^{m} \nu\, \Delta_k\, ([f_n]) +$$

$$+ \limsup_{m \to \infty} \nu((o,o,\ldots,o f_{m+1}, f_{m+2}, f_{m+3} \ldots)) \geq \nu\, ([f_n])$$

(2) $$\limsup_{m \to \infty} \nu((o,\ldots,o,f_{m+1},f_{m+2},\ldots)) \leq$$

$$\leq \limsup_{m \to \infty} \pi\, ((o,\ldots,o,\, f_{m+1}, f_{m+2},\ldots)) \leq$$

$$\leq \limsup_{m \to \infty} (\limsup_{n \to \infty} \sum_{k=m}^{k=n} p_k(f_k)) \leq$$

$$\leq \limsup_{m \to \infty} \sum_{k=m}^{\infty} p_k(f) = o,$$

(where f is a common upper bound of $f_k, f_{k+1}, \ldots$ for a suitable k). Because of $\rho \le \pi$ the sandwich theorem provides a linear order-preserving $\bar{\nu}$ with $\nu \le \rho \le \bar{\nu} \le \pi$. Therefore the maximality of ν implies $\nu = \rho = \bar{\nu}$. Now, we define $\mu_k(f) = \nu\, \Delta_k\, ([f])$ (where $[f] = (f,f,f,\ldots)$) and we obtain the desired result. ■

<u>FINITE DECOMPOSITION THEOREM</u>: *Let μ be a linear functional and let $p_1,\ldots,p_n$ be sublinear such that $\mu(f) \le \max(p_1(f),\ldots,p_n(f))$ for all $f \in F$. Then there are $\lambda_1,\ldots,\lambda_n \ge 0$ and linear $\mu_1,\ldots,\mu_n$ with $\sum_{k=1}^{n} \lambda_k = 1$ and $\mu_k \le p_k$, $k = 1,\ldots,n$ such that*

$$\mu \le \sum_{k=1}^{n} \lambda_k\, \mu_k \; .$$

PROOF: We may assume $\mu(0) = 0$, otherwise $\mu(f) = -\infty \quad \forall f \in F$ and the theorem is trivial. On the cone

$\bar{F} = \mathbb{R}^{\{p_1,\ldots,p_n\}}$ we consider the sublinear $p(g) = \sup(g(p_1),\ldots g(p_n))$ and the superlinear $\delta(g) = \sup\{\, \mu(f) \mid f \in F$ with $\hat{f} \le g \,\}$ where $\hat{f}$ denotes the function $p_i \to p_i(f)$, $i = 1,\ldots,n$. The order in $\bar{F}$ shall be the pointwise order on $\{p_1,\ldots,p_n\}$. By the sandwich theorem there is a linear order-preserving ν on $\bar{F}$ with $\delta \le \nu \le p$. Let ε_i be the function $p_i \to 1$ and $p_k \to 0$ for $k \neq i$.

Now, put $\lambda_i = \nu(\varepsilon_i)$ then $\lambda_i \ge 0$ (since ν is order-preserving) and $\sum_{i=1}^{n} \lambda_i = \nu(1) = 1$ (since $\nu(-1) \le p(-1) = -1$ and $\nu(1) \le p(1) \le 1$). And we obtain

$$\mu(f) \le \delta(\hat{f}) \le \nu(\hat{f}) = \nu\Big(\sum_{i=1}^{n} p_i(f)\,\varepsilon_i\Big) \le \inf_{k\in\mathbb{N}} \nu\Big(\sum_{i=1}^{n} \max(p_i(f),-k)\varepsilon_i\Big) =$$

$$= \inf_{k\in\mathbb{N}} \Big\{ \sum_{i=1}^{n} \nu(\varepsilon_i)\, \max(p_i(f),-k) \Big\} = \sum_{i=1}^{n} \lambda_i\, p_i(f) \; .$$

Application of the sum theorem to $\mu \le \sum_{i=1}^{n} \lambda_i p_i$ gives the desired result. Here the sum theorem is applied in the case of the trivial preorder given by $=$.

COUNTABLE DECOMPOSITION

Let $I \in F$ with $I > o$, where $I > o$ means $I \geq o$ but not $I \leq o$. (F,I) is called order - unit cone if for every $f \in F$ there is an $n \in \mathbb{N}$ such that $f \leq nI$. By S_I we denote the sublinear functional

$$f \to \inf\{r \in \mathbb{R} \mid rI \in F,\ f \leq rI\}.$$

S_I is called the order - unit functional. We say that (F,I) contains the constants if $F \supset \{rI \mid r \in \mathbb{R}\}$. Obviously we have then $S_I(I) = -S_I(-I) = 1$, or equivalently $S_I(rI) = r \quad \forall r \in \mathbb{R}$. Furthermore

$$p(f + rI) = p(f) + r \quad \forall f \in F,\ r \in \mathbb{R}$$

for any sublinear $p \leq S_I$. This is an easy consequence of the sublinearity of p and the linearity of S_I on the constants $\mathbb{R}I$.

Of course, subtraction is not defined in F, but we shall write $f - h \in F$ if there is a $g \in F$ with $h + g = f$.

If not otherwise mentioned we consider from now on the following:

SITUATION: (F,I) is an order - unit cone containing the constants. $S = S_I$ is the order-unit functional on F and $p_n \leq S$ is a sequence of sublinear order-preserving functionals.

REMARK: This situation is rather general. Let for example G be a cone and let $A \in G$ with $G \supset \{rA \mid r \in \mathbb{R}\}$. If π is sublinear on G with $\pi(rA) = r \quad \forall r \in \mathbb{R}$ then

$$f \leq g \Leftrightarrow \exists h \in G \text{ with } \pi(h) \leq o \text{ and } g + h = f$$

is a preorder on G such that $A = I$ is an order-unit with $\pi = S_I$. And every sublinear functional $p \leq \pi$ is order - preserving.

We need a simple convergence lemma.

LEMMA 2: Let $\lambda_n \geq o$ with $\sum_{n \in \mathbb{N}} \lambda_n = 1$, then $\sum_{n \in \mathbb{N}} \lambda_n p_n(f)$ converges in $\bar{\mathbb{R}}$ for all $f \in F$.

PROOF: For $r = S(f)$ we have

$$\sum_{n=1}^{m} \lambda_n p_n(f) = \sum_{n=1}^{m} \lambda_n p_n(f - (r + \tfrac{1}{n})I) + \sum_{n=1}^{m} \lambda_n (r + \tfrac{1}{n}) .$$

Now, the convergence (in $\bar{\mathbb{R}}$) of the sum follows from the fact that $\lambda_n p_n(f - (r + \frac{1}{n})I)$ is $\leq o$ for all n. ∎

Of course, this lemma holds for any sequence $[\pi_n]$ of linear or sublinear $\pi_n \leq S$.

A linear $\mu \leq S$ is said to be <u>decomposable</u> (with respect to $(p_n)_{n \in \mathbb{N}}$) if there are $\lambda_n \geq o$ and linear $\mu_n \leq p_n$ with $\sum_{n \in \mathbb{N}} \lambda_n = 1$ such that $\mu \leq \sum_{n \in \mathbb{N}} \lambda_n \mu_n$. μ is said to be <u>partially decomposable</u> if there are $\varepsilon > 0$, n and linear $\nu, \bar{\nu}$ with $\varepsilon \leq 1$, $\nu \leq p_n$, $\bar{\nu} \leq S$ such that $\mu \leq \varepsilon \nu + (1-\varepsilon) \bar{\nu}$. In the last definition the emphasis is on the fact that ε is strictly positive. $t \in \mathbb{R}_+^{\mathbb{N}}$ with $|t| = \sum_{n \in \mathbb{N}} t(n) \leq 1$ is called a <u>representation</u> of μ (with respect to $(p_n)_{n \in \mathbb{N}}$) if

$$\mu(f) \leq \sum_{k \in \mathbb{N}} t(k) p_k(f) + (1 - |t|) S(f) \qquad \forall f \in F$$

In the set of representations we consider the pointwise order on $\mathbb{N}$. Thenfor every $\mu \leq S$ there is a maximal representation (consequence of Zorn's Lemma or the compactness of the set of represenations with respect to the weak*-topology given by c_o).

<u>PARTIAL DECOMPOSITION THEOREM</u>: <u>Let μ be linear $\leq S$. Then the following are equivalent</u>:

(i) μ <u>is partially decomposable</u>

(ii) <u>For every decreasing sequence</u> f_m <u>in</u> F <u>with</u>

$f_{m+1} - f_m \in F$ <u>and</u> $\inf_{m\in\mathbb{N}} \mu(f_m) > -\infty$ <u>we have</u>

$$\sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} p_n(f_m) > -\infty$$

PROOF: (i) $\Rightarrow$ (ii): is trivial.
(ii) $\rightarrow$ (i) : Put $\pi_n(f) = \max(p_1(f), p_2(f),\dots,p_n(f))$ then π_n is an increasing sequence with $\pi_n \le S$. Assume that for every n there is a $\bar{g}_n \in F$ with

$$\mu(\bar{g}_n) > \frac{1}{n} \pi_n(\bar{g}_n) + (1 - \frac{1}{n}) S(\bar{g}_n) .$$

We replace $\bar{g}_n$ by

$$g_n = \bar{g}_n - \{S(\bar{g}_n) + \varepsilon_n\} I,$$

where

$$\varepsilon_n = \mu(\bar{g}_n) - S(\bar{g}_n) - \frac{1}{n} \{ \pi_n(\bar{g}_n) - S(\bar{g}_n) \} > 0 .$$

This is an element of F because of $S(\bar{g}_n) \ge \mu(\bar{g}_n) > -\infty$.
Then $g_n \le o$, $S(g_n) = -\varepsilon_n < 0$ and

$$o > -\varepsilon_n \ge \mu(\bar{g}_n) - S(\bar{g}_n) - \varepsilon_n = \mu(g_n) = \frac{1}{n} \{ \pi_n(\bar{g}_n) - S(\bar{g}_n) \} =$$

$$= \frac{1}{n} \{ \pi_n(g_n) - S(g_n) \} > \frac{1}{n} \pi_n(g_n) .$$

Hence we have found the inequality

$$o \ge \mu(g_n) > \frac{1}{n} \pi_n(g_n) ,$$

and multiplication of g_n with a suitable positive constant gives an $h_n \le o$ with $o \ge \mu(h_n) \ge -\frac{1}{n^2}$ and $-\frac{1}{n^2} > \frac{1}{n} \pi_n(h_n)$,

i.e. $-\frac{1}{n} > \pi_n(h_n)$. Since $[\pi_n]$ is increasing we have in addition

$$-\frac{1}{n} > \pi_k(h_n) \quad \forall\, n \ge k .$$

Now, we define $f_n = \sum_{k=1}^{n} h_k$ and obtain:

$$\inf_{n\in\mathbb{N}} \mu(f_n) \geq - \sum_{n=1}^{\infty} \frac{1}{n^2} = - \frac{\pi^2}{6} ,$$

$$\sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} \pi_n(f_m) \leq \sup_{n\in\mathbb{N}} \left(- \sum_{m=n}^{\infty} \frac{1}{m} \right) = - \infty$$

This contradicts (ii). So we have proved that there is some $n \in \mathbb{N}$ with $\mu \leq \frac{1}{n} \pi_n + (1 - \frac{1}{n}) S$. By the sum theorem there are linear ν, φ with $\mu \leq \nu + \varphi$, $\nu \leq \frac{1}{n} \pi_n$, $\varphi \leq (1 - \frac{1}{n}) S$. From the finite decomposition theorem we get linear $\nu_1, \ldots, \nu_n$ and positive $\lambda_1, \ldots, \lambda_n$ with $\sum_{k=1}^{n} \lambda_k = \frac{1}{n}$ and $\nu_k \leq p_k$ such that

$\nu \leq \sum_{k=1}^{n} \lambda_k \nu_k$. This obviously implies (i). ■

<u>DECOMPOSITION THEOREM</u>: <u>The following are equivalent:</u>

(i) <u>Every linear</u> $\mu \leq S$ <u>is partially decomposable</u>.

(ii) <u>Every linear</u> $\mu \leq S$ <u>is decomposable</u>.

(iii) <u>For every decreasing sequence</u> f_n <u>in</u> F <u>we have</u>

$$\sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} p_n(f_m) = \inf_{m\in\mathbb{N}} S(f_m) .$$

(iv) <u>For every decreasing sequence</u> f_m <u>in</u> F <u>with</u> $f_{m+1} - f_m \in F$ <u>such that there is a linear</u> $\mu \leq S$ <u>with</u> $\inf_{m\in\mathbb{N}} \mu(f_m) > - \infty$ we have $\sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} p_n(f_m) > - \infty$

PROOF: (i) $\Rightarrow$ (ii) : We take a maximal representation t for μ. If $|t| = 1$ then the decomposition of μ follows via lemma 2 from the sum theorem. Therefore we assume $|t| < 1$. By the sum theorem there are linear $\mu_n \leq p_n$ and $\nu \leq S$ such that

$$\mu \leq \sum_{n\in\mathbb{N}} t(n)\, \mu_n + (1 - |t|)\, \nu .$$

Now, (i) provides a representation $\bar{t}$ for ν with $|\bar{t}| > o$. So, we obtain in contradiction to the maximality of t a representation $\hat{t} = t + (1 - |t|)\,\bar{t}$ for μ which is strictly greater than t.

(ii) $\Rightarrow$ (iii): From lemma 1 we get a linear μ with $\inf_{m\in\mathbb{N}} \mu(f_m) = \inf_{m\in\mathbb{N}} S(f_m)$. Let $\sum_{n\in\mathbb{N}} \lambda_n \mu_n$ be a decomposition of μ

then $\mu \leq \sum_{n\in\mathbb{N}} \lambda_n p_n$ and.

$$\inf_{m\in\mathbb{N}} S(f_m) = \inf_{m\in\mathbb{N}} \mu(f_m) \leq \inf_{m\in\mathbb{N}} \Big(\sum_{n\in\mathbb{N}} \lambda_n p_n(f_m)\Big) \leq$$

$$\leq \sum_{n\in\mathbb{N}} \lambda_n \inf_{m\in\mathbb{N}} p_n(f_m) \leq \sup_{n\in\mathbb{N}} \inf_{m\in\mathbb{N}} p_n(f_m).$$

This together with $p_n \leq S$ gives the desired equality.

(iii) $\Rightarrow$ (iv) is trivial and (iv) $\Rightarrow$ (i) follows from the partial decomposition theorem. ■

As corollaries we derive decomposition theorems for <u>concrete order-unit cones</u>. We consider a convex cone $F(X)$ of real upper-bounded functions on some set X. By $VF(X)$ we denote the max-stable cone generated by $F(X)$; i.e. the set of functions $x \to \max(f_1(x),\ldots,f_k(x))$ where $f_1,\ldots,f_k \in F(X)$. $F(X)$ and $VF(X)$ are equipped with the pointwise order on X. A linear functional μ on $F(X)$ (or $VF(X)$) is called a <u>state</u> if $\mu(f) \leq \sup_{x\in X} f(x)$ for all f.

<u>COROLLARY 1</u>: (cf [5]) <u>If $F(X)$ contains the constant functions on X then the following are equivalent</u>:

(i) <u>For every decreasing sequence f_m in $VF(X)$ we have</u>

(*) $$\sup_{x\in X} \inf_{m\in\mathbb{N}} f_m(x) = \inf_{m\in\mathbb{N}} \sup_{x\in X} f_m(x).$$

(ii) (*) <u>holds for every decreasing sequence f_m in $F(X)$</u>.

(iii) For every decreasing sequence f_m in $F(X)$ with $f_{m+1} - f_m \in F(X)$ such that there is a state μ with $\inf_{m \in \mathbb{N}} \mu(f_m) > -\infty$ we have $\sup_{x \in X} \inf_{m \in \mathbb{N}} f_m(x) > -\infty$.

(iv) For every state μ on $F(X)$ and for every sequence $Y_n \subset X$ with $\cup \{Y_n | n \in \mathbb{N}\} = X$ there are states μ_n and $\lambda_n \geq 0$ with $\sum_{n \in \mathbb{N}} \lambda_n = 1$ and $\mu_n(f) \leq \sup_{x \in Y_n} f(x) \quad \forall f \in F(X)$ such that $\mu \leq \sum_{n \in \mathbb{N}} \lambda_n \mu_n$.

PROOF: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv): consider S and p_n defined by $S(f) = \sup_{x \in X} f(x)$ and $p_n(f) = \sup_{y \in Y_n} f(y)$. Then (iv) follows from the decomposition theorem.

(iv) $\Rightarrow$ (i): Let $E(X)$ stand for the vector lattice $VF_B(X) - VF_B(X)$ where $VF_B(X)$ are the bounded functions in $VF(X)$. Now, assume

$$\inf_{m \in \mathbb{N}} \sup_{x \in X} f_m(x) = \beta > \alpha = \sup_{x \in X} \inf_{m \in \mathbb{N}} f_m(x)$$

and take γ, δ with $\alpha < \delta < \gamma < \beta$.

By the Stone-Kakutani Theorem [1,p.76] the set Ω of lattice-preserving states is compact under pointwise convergence on $E(X)$ and $(E(X)$, sup-norm) is isometric to a dense subspace of $C(\Omega)$. Therefore we obtain from Dini's lemma a lattice-preserving state $\mu \in \Omega$ with

(3) $$\inf_{m \in \mathbb{N}} \mu(\tilde{f}_m) = \inf_{m \in \mathbb{N}} \sup_{x \in X} \tilde{f}_m(x) = \beta,$$

where $\tilde{f}_m = \max(f_m, \delta)$. We extend μ to a state on $VF(X)$ by putting $\mu(f) = \inf_{n \in \mathbb{N}} \mu(\max(f,-n)) \quad \forall f \in VF(X)$.

And we define $p_n(g) = \sup_{y \in Y_n} g(y) \quad \forall g \in VF(X)$, where $Y_n = \{x \in X | f_n(x) \leq \gamma\}$. By (iv) there must be a decomposition

(4) $$\mu(f) \leq \sum_{n \in \mathbb{N}} \lambda_n p_n(f) \qquad \forall f \in F(X)$$

with $\lambda_n \geq 0$ and $\sum_{n \in \mathbb{N}} \lambda_n = 1$. Since μ is lattice-preserving and every $g \in VF(X)$ is of the form $g = \max(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k \in F(X)$ the inequality (4) must also hold for all $f \in VF(X)$. This together with (3) implies $\gamma = \beta$. Therefore $\alpha \geq \beta$. And $\alpha \leq \beta$ follows immediately from the definition of α and β. ■

The next corollary is closely related to the theory of signed representing measures (cf. [6]).

COROLLARY 2: For a convex cone $F(X)$ of bounded functions (not necessarily containing the constants) the following are equivalent:

(i) For every linear $\mu : F(X) \to \mathbb{R}$ with $\mu(f) \leq \sup_{x \in X} |f(x)| \quad \forall f \in F(X)$ and for every sequence $Y_n \subset X$ with $\cup \{Y_n | n \in \mathbb{N}\} = X$ there are $\lambda_n \geq 0$ and linear μ_n with $\sum_{n \in \mathbb{N}} \lambda_n = 1$ and $\mu_n(f) \leq \sup_{y \in Y_n} |f(y)| \quad \forall f \in F$ such that $\mu \leq \sum_{n \in \mathbb{N}} \lambda_n \mu_n$.

(ii) For every sequence $(f_n, r_n) \in F(X) \times \mathbb{R}$ such that $r_n + f_n(x)$ and $r_n - f_n(x)$ are decreasing for all $x \in X$ we have:

$$\sup_{x \in X} \inf_{n \in \mathbb{N}} (|f_n(x)| + r_n) = \inf_{n \in \mathbb{N}} \sup_{x \in X} (|f_n(x)| + r_n)$$

PROOF: In $\bar{F} = F(X) \times \mathbb{R}$ we consider the order-relation

$$(f,r) \leq (g,\bar{r}) \Leftrightarrow \sup_{x \in X} |f(x) - g(x)| \leq \bar{r} - r$$

Then $(f_n, r_n) \in \bar{F}$ is decreasing if and only if the sequences $r_n + f_n(x)$ and $r_n - f_n(x)$ are decreasing for all $x \in X$. Furthermore we consider the sublinear order-preserving functionals

$$\bar{p}_{Y_n}(f,r) = \sup_{y \in Y_n} |f(y)| + r$$

where $Y_n \subset X$. Then $(\bar{F}, I = (o,1))$ is an order-unit cone and $S_I((f,r)) = \sup_{x \in X} |f(x)| + r$. Now, for a decreasing sequence $(f_n, r_n) \in \bar{F}$ condition (ii) is equivalent to:

$$\sup_{n \in \mathbb{N}} \inf_{m \in \mathbb{N}} \bar{p}_{Y_n}(f_m, r_m) = \inf_{m \in \mathbb{N}} S_I(f_m, r_m)$$

for all sequences Y_n with $\cup\{Y_n | n \in \mathbb{N}\} = X$. And the equivalence (i) $\Leftrightarrow$ (ii) is a consequence of the decomposition theorem. ■

The decomposition theorems we have given so far are dealing with the situation that all states are decomposable. The characterisation of decomposability for a single linear functional is much more difficult. The next theorem is the only result we are able to present in this direction.

Again we use the notation $p_{Y_n}(f) = \sup_{y \in Y_n} f(y)$.

THEOREM 1: Let $F(X)$ be a convex cone of upper bounded functions containing the constants. Let μ be an order-preserving state on $F(X)$ and let $Y_n \subset X$ be a sequence with $\cup\{Y_n | n \in \mathbb{N}\} = X$. Furthermore we assume:

(a) $f \in F(X)$, $r \in \mathbb{R} \Rightarrow \max(f,r) \in F(X)$

(b) for every representation t of μ with respect to p_{Y_n} there are order-preserving states ν and $\nu_n \leq p_{Y_n}$

such that $\mu = \sum_{n=1}^{\infty} t(n)\, \nu_n + (1 - |t|)\, \nu$.

Then the following are equivalent:

(i) μ *is decomposable with respect to* p_{Y_n} .

(ii) *For every positive decreasing sequence* f_m *in* $F(X)$ *with* $\sup_{n \in \mathbb{N}} \inf_{m \in \mathbb{N}} \sup_{y \in Y_n} f_m(y) = o$ we have $\inf_{m \in \mathbb{N}} \mu(f_m) = o$

PROOF: (i) ⇒ (ii) is trivial.
(ii) ⇒ (i): Consider a maximal representation t for μ and choose ν and ν_n according to (b). (i) is trivial for $|t| = 1$. Assume therefore $|t| < 1$. Then (ii) implies $\inf_{m \in \mathbb{N}} \nu(f_m) = o$.

This means that (ii) is also valid for ν instead of μ . From (a) together with $F(X) \supset \mathbb{R}$ and (ii) one can easily conclude that (ii) of the partial decomposition theorem holds for ν . So, ν has a representation $\bar{t}$ with $|\bar{t}| > 0$. This implies that $\hat{t} = t + (1 - |t|)\bar{t}$ is in contradiction to the maximality of t a representation strictly greather then t. ■

The condition (b) imposed on μ in the above theorem is quite often fulfilled. For example if μ is maximal or if $F(X)$ is min-stable or if it is a vector space.

This means that states on a vector lattice $\supset \mathbb{R}$ fullfilling Stones condition (cf. [2]) are always decomposable. This fact together with an application of the Riesz representation theorem can be used (in this very special case) to prove the Daniell -Stone theorem.

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