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ITERATIONS AND FIXPOINTS

BENNO FUCHSSTEINER

Starting with a decreasing map $\varphi: X \to X$ on a partially ordered set X we construct a map It φ which intuitively can be understood as the iteration (countable or transfinite) of φ. The main properties which It φ inherits from φ are investigated. As application of the main result some fixpoint theorems are proved. Besides, our method yields constructive proofs for results which are usually demonstrated with the help of the axiom of choice.

I. The iteration theorem. We consider a partially ordered set $(X, \leqq)$ and a decreasing map $\varphi: X \to X$. For $Y \subset X$ we denote by $\min(Y)$ the set of minimal elements in Y. A totally ordered subset $Y \subset X$ is said to be a *φ-chain* if:

$$(C1)\ y \leqq \varphi(x) \text{ for all } x, y \in Y \text{ with } x \neq y \text{ and } y \leqq x$$

$$(C2)^+\ \varphi(Y \backslash \min(Y)) \subset Y.$$

$(X, \leqq)$ is assumed to be *φ-complete*, which means that every nonempty φ-chain has an infimum. A φ-chain Y is called a *strong φ-chain* if:

$$(C2)^*\ \varphi(\inf(\bar{Y})) \in Y \text{ for all nonempty } \bar{Y} \subset Y \text{ with } \inf(\bar{Y}) \neq \inf(Y).$$

Condition $(C2)^*$ obviously implies $(C2)^+$. Elementary examples for strong φ-chains are $\{x\}$ and $\{x, \varphi(x)\}$. A strong φ-chain Y is said to be a *complete φ-chain* if $\varphi(\inf(Y)) \in Y$.

REMARK 1. (i) A totally ordered subset $Y \subset X$ is a complete φ-chain iff (C1) and

$$(C2)\ \varphi(\inf(\bar{Y})) \in Y \quad \text{for all nonempty} \quad \bar{Y} \subset Y.$$

(ii) The intersection of a family of strong (complete) φ-chains is a strong (complete) φ-chain.

LEMMA 1. *Every strong φ-chain Y is contained in a complete φ-chain. The complete φ-chains containing Y have a minimum I_Y with respect to $\subset$.*

Proof. Let M be the set of strong φ-chains containing Y. We define $\bar{\varphi}: M \to M$ by $\bar{\varphi}(\tilde{Y}) = \tilde{Y} \cup \{\varphi(\inf(\tilde{Y}))\}$. According to Zermelo's fixpoint-lemma [4, Theorem I.2.5] $\bar{\varphi}$ has a fixpoint, which is by definition a complete φ-chain. I_Y is obviously given by $\cap\{\tilde{Y} \in M \mid \bar{\varphi}(\tilde{Y}) = \tilde{Y}\}$.*

A φ-invariant subset $Y \subset X$ is called a *φ-subset* if the infimum of every strong φ-chain in Y is an element of Y. A map $F: X \to X$ is called *φ-absorbing* if:

(A 1) $F \circ \varphi = F$

(A 2) if F is constant on a nonempty strong φ-chain Y then F is constant on $Y \cup \{\inf(Y)\}$.

ITERATION THEOREM. *There is a decreasing map* $\operatorname{It}\varphi: X \to X$ *such that*:

(i) $\operatorname{It}\varphi$ *maps onto the φ-fixpoints of* X

(ii) $\operatorname{It}\varphi \circ \operatorname{It}\varphi = \operatorname{It}\varphi$

(iii) *every φ-subset is* $\operatorname{It}\varphi$*-invariant*

(iv) *for every φ-absorbing map F we have* $F \circ \operatorname{It}\varphi = F$

(v) *for every φ-absorbing map F with* $F(x) \leqq x \Rightarrow F(x) \leqq \varphi(x)$ *we have* $F(x) \leqq x \Rightarrow F(x) \leqq \operatorname{It}\varphi(x)$.

(vi) $\operatorname{It}\varphi$ *is φ-absorbing.*

$\operatorname{It}\varphi$ *is uniquely determined by* φ.

Proof. We define $\operatorname{It}\varphi(x) \doteqdot \inf(I_x)$ where $I_x \supset \{x\}$ is the minimal complete φ-chain given by Lemma 1. We claim that $\operatorname{It}\varphi$ is constant on I_x.

Proof of the claim. Let $y_0 \in I_x$. Then $\{y \in I_x \mid y \leqq y_0\}$ and $\{y \in I_x \mid y \geqq y_0\} \cup I_\infty$ are complete φ-chains $\supset \{y_0\}$ and $\supset \{x\}$ respectively. Hence from minimality of I_x and I_∞ it follows $I_x = \{y \in I_x \mid y \geqq y_0\} \cup I_\infty$.

This implies $\operatorname{It}\varphi(y_0) = \inf(I_\infty) = \inf(I_x) = \operatorname{It}\varphi(x)$.

(i) and (ii): $\operatorname{It}\varphi(x) = \inf(I_x)$ is a φ-fixpoint because of $\inf(I_x) \geqq \varphi(\inf(I_x)) \in I_x$. Now, let $x = \varphi(x)$ then $I_x = \{x\}$ (minimality). This implies $\operatorname{It}\varphi(x) = \inf(I_x) = x$. Hence $\operatorname{It}\varphi$ maps onto the fixpoints and is idempotent.

(iii): Let $x \in Y$ where Y is a φ-subset. Then $\bar{I}_x \doteqdot \{y \in I_x \mid \{\xi \in I_x \mid y \leqq \xi\} \subset Y\}$ is a complete φ-chain. We have $x \in \bar{I}_x$ because $\xi \leqq x\ \forall \xi \in I_x$. Hence $\bar{I}_x = I_x$ (minimality) and $\operatorname{It}\varphi(x) = \inf(I_x) \in \bar{I}_x \subset Y$.

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* The fixpoint-lemma is applied with respect to the following order in M: $Y_1 < Y_2 \Leftrightarrow Y_1$ is ideal of Y_2, i.e. $Y_1 \subset Y_2$ and $y \leqq x\, \forall y \in Y_2 \setminus Y_1,\ x \in Y_1$.

(iv): Consider $\bar{I}_x = \{y \in I_x \mid F(x) = F(\xi) \forall \xi \in I_x \text{ with } \xi \geqq y\}$ where F is φ-absorbing. $\bar{I}_x$ is a complete φ-chain $\supset \{x\}$. Hence $I_x = \bar{I}_x$ and $\mathrm{It}\,\varphi(x) \in \bar{I}_x$. This implies $F \circ \mathrm{It}\,\varphi(x) = F(x)$.

(v): Let F be as in (v) and $F(x) \leqq x$. Then $\bar{I}_x = \{y \in I_x \mid y \geqq F(y) = F(x)\}$ is a complete φ-chain $\supset \{x\}$. This gives $I_x = \bar{I}_x$ and $F(x) = F(\inf(I_x)) \leqq \inf(I_x) = \mathrm{It}\,\varphi(x)$.

(vi): (A 1) is a consequence of $\varphi(x) \in I_x$ and the fact that $\mathrm{It}\,\varphi$ is constant on I_x. Now, let $x \in Y$ where Y is a strong φ-chain on which $\mathrm{It}\,\varphi$ is constant. For $\bar{x} = \inf(Y)$ we show that $I_x \cap I_{\bar{x}} \neq \emptyset$. This gives the desired result because for $y \in I_x \cap I_{\bar{x}}$ we have $\mathrm{It}\,\varphi(x) = \mathrm{It}\,\varphi(y) = \mathrm{It}\,\varphi(\bar{x}) = \mathrm{It}\,\varphi(\inf Y)$. Assume therefore that $I_x \cap I_{\bar{x}} = \emptyset$. The set $Y \cup I_{\bar{x}}$ is a complete φ-chain $\supset \{x\}$. Hence $Y \cup I_{\bar{x}} \geqq I_x$ and I_x must be a subset of Y. This gives $\bar{x} \lneqq \inf(I_x)$ ($\neq$ because of $\bar{x} \notin I_x$). Since $Y \supset I_x$ is totally ordered there is some $y \in Y$ with $y \gneqq \inf(I_x)$. From this it follows $\mathrm{It}\,\varphi(y) \leqq y \gneqq \inf(I_x) = \mathrm{It}\,\varphi(x)$ which is in contradiction to the fact that $\mathrm{It}\,\varphi$ is constant on Y.

Proof of uniqueness. Let F be φ-absorbing and decreasing then according to (v) we have $F(x) \leqq \mathrm{It}\,\varphi(x)$ for all x. Thus $\mathrm{It}\,\varphi$ is the maximum of all decreasing φ-absorbing maps. Hence $\mathrm{It}\,\varphi$ is unique.

From now on $\mathrm{It}\,\varphi$ is called the *iteration* of φ. The reason for choosing this name will become obvious in the next chapter. It should be mentioned that the proof of the iteration theorem does not make use of Zorn's lemma or the axiom of choice. The crucial tool was Zermelo's fixpoint-lemma (see also [15]) which can be proved constructively.

II. Examples and applications.

II.1. *Monotony and a theorem of Tarski.* First we show that $\mathrm{It}\,\varphi$ inherits monotony from φ.

LEMMA 2. *Let φ be monotone.*
(i) *$\mathrm{It}\,\varphi(x)$ is the maximum of the φ-fixpoints $\leqq x$.*
(ii) *$\mathrm{It}\,\varphi$ is monotone.*

Proof. (i) Let x_0 be a fixpoint $\leqq x$. We show $x_0 \leqq \mathrm{It}\,\varphi(x)$. Obviously the constant map $F: X \to X$ given by $F(y) = x_0$ is φ-absorbing. $x_0 \leqq y$ implies $\varphi(x_0) = x_0 \leqq \varphi(y)$ because φ is monotone. Hence F fulfills (v) of the iteration theorem and we have $x_0 = F(x) \leqq \mathrm{It}\,\varphi(x)$.

(ii) Let $y \leqq x$ then $\mathrm{It}\,\varphi(y)$ is a φ-fixpoint $\leqq x$. By (i) we know $\mathrm{It}\,\varphi(y) \leqq \mathrm{It}\,\varphi(x)$. Thus $\mathrm{It}\,\varphi$ must be monotone.

EXAMPLE 1. Let L be a complete lattice and $\mu: L \to L$ a

monotone map. μ is decreasing on $X \doteqdot \{x \in L \mid \mu(x) \leqq x\}$ which is not empty because $\sup(L) \in X$.

X contains $\mathrm{Fix}(\mu)$ (set of μ-fixpoints).

X is μ-complete because the infimum of every chain in X is again element of X. This yields that $\mathrm{Fix}(\mu) \neq \varnothing$ (see [9]) because $\mathrm{Fix}(\mu)$ is the image of X under $\mathrm{It}\,\mu_{|X}$.

Now, let $Y \subset \mathrm{Fix}(\mu)$ then by Lemma 2(i) $\mathrm{It}\,\mu_{|X}(\inf(Y))$ is the maximum of all fixpoints $\leqq \inf(Y)$. Hence $\mathrm{It}\,\mu_{|X}(\inf(Y))$ is the infimum of Y in $\mathrm{Fix}(\mu)$. So we have proved:

THEOREM (Tarski [14]). *Let $\mu: L \to L$ be monotone on the complete lattice L, then $\mathrm{Fix}(\mu)$ is a nonempty complete lattice.*

II.2. *Contracting and condensing maps.* The next example is rather trivial. Nevertheless it shows why the name iteration was chosen for $\mathrm{It}\,\varphi$. Let (Z, d) be a complete and bounded metric space. We call $T: Z \to Z$ a *generalized contraction* if there is a $q < 1$ such that $\delta(T(A)) \leqq q\delta(A)$ for all T-invariant $A \subset Z$, where $\delta(\ )$ denotes the diameter. On $X = \{A \subset Z \mid A$ closed, nonempty, T-invariant$\}$ we define a $\subset$-decreasing map φ_T by $\varphi_T(A) =$ closure $(T(A))$. Then X is φ_T-complete because a φ_T-chain either contains only finitely many elements or is the basis of a Cauchy filter. $\mathrm{It}\,\varphi_T(X)$ is a singleton because as φ_T-fixpoints it has diameter 0. Furthermore $\mathrm{It}\,\varphi_T(X)$ contains (Lemma 2(i)) the closure of the set of T-fixpoints. Hence the iteration theorem yields:

THEOREM (Banach and others). *Every generalized contraction on Z has a unique fixpoint.*

Generalized contractions are for example the quasicontractions

$$d(T(x), T(y)) \leqq q \max\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\} \quad \forall x, y \in Z$$

recently considered [3] by Lj. B. Ćirić.

The contraction property in the last example was mainly used to assure that X is φ_T-complete. But if (Z, d) is compact then X is φ_T-complete for any map T. Therefore the same argument leads to:

THEOREM (M. Edelstein [5]). *Every condensing map T (i.e. a map with $\delta(T(A)) < \delta(A)$ for every T-invariant $A \subset Z$, with $\delta(A) > 0$) on a compact metric space has a unique fixpoint.*

For the next example let us consider a continuous map $T: S \to S$, where S is a nonempty closed bounded convex subset of a separated locally convex vector space. T is said to be *limit-compact* if every closed $M \subset S$ with $\overline{\text{co}}\ T(M) \supset M$ has to be compact. ($\overline{\text{co}}$ denotes the closed convex hull).

THEOREM (Sadovski [13]). *Let T be limit-compact and assume that there is some nonempty $A \subset S$ with $\overline{\text{co}}\ T(A) \supset A$. Then T has a fixpoint.*

Proof. On $X = \{Y \subset S \mid \overline{\text{co}}\ Y = Y\}$ we consider the $\subset$-decreasing map $\varphi(Y) \doteqdot Y \cap \overline{\text{co}}\ T(Y)$ and the constant map $F(Y) \doteqdot \overline{\text{co}}(A)$. X is a complete lattice with respect to $\subset$ and therefore φ-complete. The φ-fixpoint $K \doteqdot \operatorname{It} \varphi(S)$ is compact because T is limit-compact. (v) of the iteration theorem gives $\emptyset \neq \overline{\text{co}}(A) = F(S) \subset \operatorname{It} \varphi(S) = K$. So K is nonempty. Finally we get $T(K) \subset K$ from the fact that $\bar{X} = \{Y \in X \mid T(Y) \subset Y\}$ is a φ-subset. Hence the Schauder–Tychonoff Theorem [4, Theorem, V.10.5] provides a fixpoint.

A special case of this theorem is that every continuous Δ-condensing operator on a closed bounded convex set has a fixpoint (Δ is a measure of noncompactness, see [12]).

II.3. *Normal structure.* Throughout this chapter S will be a nonempty weakly compact set with normal structure in a Banach space.

Normal structure means that if M is a convex subset of S which contains more than one element then it holds that

$$r(M) \doteqdot \inf_{z \in M} \sup_{x \in M} \|x - z\| < \delta(M) \doteqdot \sup_{z, x \in M} \|x - z\|.$$

We know that for every nonempty closed convex subset $M \subset S$ the Čebyčev-centre of M with respect to S $\operatorname{Ceb}_S(M) \doteqdot \{x \in S \mid \sup_{z \in M} \|x - z\| \leqq r(M)\}$ is again nonempty closed and convex (see [2], [8]). We shall write $\operatorname{Ceb}_M(M)$ for $\operatorname{Ceb}_S(M) \cap M$ and $\operatorname{Ceb}(M)$ for $\operatorname{Ceb}_M(M)$. $\operatorname{Ceb}_S(M) \supset M \neq \emptyset$, or equivalently $\operatorname{Ceb}(M) = M \neq \emptyset$, happens if and only if M is a singleton. We are interested in nonexpansive maps $T: S \to S$. *Nonexpansive* means that $\|T(x) - T(y)\| \leqq \|x - y\|$ for all $x, y \in S$. It is well known ([6], [8]) that with additional conditions $\operatorname{Ceb}_S(M)$ is invariant under a nonexpansive map. We need this result in a slightly more general form.

LEMMA 3. *Let $f: S \to S$ be nonexpansive and $M \subset S$ such that $\overline{\text{co}} f(M) \supset M = \overline{\text{co}}(M)$. Then $\operatorname{Ceb}_S(M)$ is f-invariant.*

Proof. Let $x \in M$ and $x_0 \in \mathrm{Ceb}_S(M)$ be arbitrary. By assumption for $\epsilon > 0$ there are $\lambda_1, \cdots, \lambda_n \geqq 0$ with $\Sigma \lambda_i = 1$ and $y_1, \cdots, y_n \in M$ such that:

$$\begin{aligned} \|f(x_0) - x\| &\leqq \|f(x_0) - \Sigma \lambda_i f(y_i)\| + \epsilon \leqq \Sigma \lambda_i \|f(x_0) - f(y_i)\| + \epsilon \\ &\leqq \Sigma \lambda_i \|x_0 - y_i\| + \epsilon \quad \text{(because } f \text{ is nonexpansive)} \\ &\leqq r(M) + \epsilon. \end{aligned}$$

This implies $\|f(x_0) - x\| \leqq r(M) \forall x \in M$ and $f(x_0)$ must be in $\mathrm{Ceb}_S(M)$.

THEOREM (Kirk [8]). *Every nonexpansive $T: S \to S$ has a fixpoint.*

Proof. $X = \{Y \subset S \mid \varnothing \neq Y = \overline{\mathrm{co}}(Y), T(Y) \subset Y\}$ is φ-complete for every decreasing φ because it consists of weakly compact sets. We define a decreasing $\varphi_1: X \to X$ by $\varphi_1(Y) = \overline{\mathrm{co}}\, T(Y)$. The Čebyčev centre of $\mathrm{It}\, \varphi_1(Y)$ is T-invariant because of Lemma 3.

Thus $\varphi_2(Y) \doteqdot \mathrm{Ceb}(\mathrm{It}\, \varphi_1(Y))$ is a decreasing map $X \to X$ and $K = \mathrm{It}\, \varphi_2(S)$ must be an element of X with $\mathrm{Ceb}(K) = K$. Hence K is a T-invariant singleton. That means K consists of one T-fixpoint.

We close this section with a theorem of the Belluce–Kirk type. Let $T: S \to S$ be nonexpansive and $\mathcal{F}$ a family of nonexpansive with T commuting maps $S \to S$ such that for every $f \in \mathcal{F}$ there is a $g \in \mathcal{F}$ with $f \circ g = T$.

THE LAST THEOREM. *$\mathcal{F}$ has a common fixpoint.*

Proof. Consider

$$Z^* = \{Y \subset S \mid T(Y) \subset Y\},$$

$$Z = \{Y \in Z^* \mid f(Y) \subset Y \, \forall f \in \mathcal{F}\}$$

and

$$X = \{M \subset S \mid \varnothing \neq M = \overline{\mathrm{co}}\, M, T(M) \subset M, f(M) \subset M \, \forall f \in \mathcal{F}\}.$$

The map $\varphi(Y) \doteqdot T(Y)$ is decreasing monotone on Z^*. Since Z, is a φ-subset of Z^*, $\mathrm{It}\, \varphi$ maps $Z \to Z$. By Kirk's theorem there is a T-fixpoint $x_M \in M$ for every $M \in X$. Thus $\{x_M\}$ is a φ-fixpoint and according to Lemma 2(i) we have $x_M \in \mathrm{It}\, \varphi(M)$ and $\mathrm{It}\, \varphi(M)$ must be nonempty. Furthermore we know $\overline{\mathrm{co}}\, f(\mathrm{It}\, \varphi(M)) \supset f(\mathrm{It}\, \varphi(M)) =$

It $\varphi(M)$ because of $\operatorname{It}\varphi(M) = T(\operatorname{It}\varphi(M)) = f \circ g(\operatorname{It}\varphi(M)) \subset f(\operatorname{It}\varphi(M))$. With the aid of Lemma 3 we conclude that $\Psi\colon M \to \operatorname{Ceb}_M \overline{\operatorname{co}}(\operatorname{It}\varphi(M))$ is a map $X \to X$. Obviously we have $\operatorname{Ceb}(\operatorname{It}\Psi(S)) = \operatorname{It}\Psi(S)$. Hence $\operatorname{It}\Psi(S)$ is a singleton consisting of a fixpoint for $\mathcal{F}$.

COROLLARY (Belluce–Kirk [1]). *Let $\mathcal{F}$ be a finite family of commuting nonexpansive maps $S \to S$. Then $\mathcal{F}$ has a common fixpoint in S.*

Proof. Use the last theorem for $T = f_1 \circ f_2 \circ \cdots \circ f_n$, where $\mathcal{F}$ consists of all compositions of different elements of $\{f_1, \cdots, f_n\}$.

The last theorem is quite general because whenever a family $\tilde{\mathcal{F}}$ of nonexpansive maps has a common fixpoint $x_0 \in S$ then we can find a T and a family $\mathcal{F} \supset \tilde{\mathcal{F}}$ fulfilling the required conditions. For example define $T(x) = x_0 \forall x \in S$ and $\mathcal{F} = \tilde{\mathcal{F}} \cup \{T\}$ then obviously $f \circ T = T \forall f \in \tilde{\mathcal{F}}$.

There are very many other theorems which are simple applications of the iteration theorem. One example: the fixpoint theorem of Brodski–Milman [2]. Lim's generalization of the Belluce–Kirk Theorem ([10], [11]) also can be simplified with the iteration theorem. But it seems that Lim's theorem cannot be done without the axiom of choice. All theorems of this paper are theorems of constructive functional analysis although many of them are usually proved with the aid of Zorn's lemma. This seems worthwhile to mention because the interest in constructive analysis has grown with the discovery of Solovay's model [7].

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GESAMTHOCHSCHULE
D479 PADERBORN
GERMANY