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RESEARCH ARTICLE

ON EXPOSED SEMIGROUP HOMOMORPHISMS

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In the main theorem of this paper the existence of weakly exposed semigroup homomorphisms is proved. This theorem is effectively equivalent to the axiom of choice and generalizes some well known theorems of functional analysis like the Hahn-Banach theorem, the Krein-Milman theorem, Bauer's minimum principle and a result of T. Husain and I. Tweddle.

1. THE MAIN THEOREM

Let $S = (S,+,\leq)$ be a preordered abelian semigroup with neutral element 0. In a preordered semigroup the semigroup structure has to be compatible with the preorder relation $\leq$, that means inequalities can be added. Furthermore, we consider a complete vector lattive $(R,\leq)$. The supremum and infimum in R is denoted by $\sup(\ )$ and $\inf(\ )$. If $\sup(A) \in A$ we write $\max(A)$ instead of $\sup(A)$. For simplicity an element $-\infty = \inf(R)$ is adjoined to R. Finally, $\bar{R}$ stands for $R \bigcup \{-\infty\}$. Addition is extended to $\bar{R}$ in the obvious way and we define $\lambda(-\infty) = -\infty$, if $0 < \lambda \in \mathbf{R}$ and $0(-\infty) = 0$.

A map $p : S \to \bar{R}$ is called subadditive {superadditive} if $p(0) = 0$ and $p(s+t) \leq \{\geq\}\ p(s) + p(t)$ for all $t, s \in S$. An additive map is one which is sub- and superadditive. Let $f, g \in \bar{R}^S$ then $\bigvee(f,g)$ stands for the set of subadditive maps p

between f and g(i.e. $f(s) \le p(s) \le g(s)$ for all $s \in S$) whereas $\bigwedge(f,g)$ {Add(f,g)} stands for the set of superadditive {additive} maps between f and g. This notation is chosen because $\bigvee(f,g)$ is a sup-semilattice and $\bigwedge(f,g)$ is an inf-semilattice. Supremum and infimum in $\bigwedge(f,g)$ and $\bigvee(f,g)$ are denoted by $\bigwedge$ and $\bigvee$ respectively.

For emphasizing which semigroup we mean we shall write sometimes $\bigwedge_S(f,g)$, $\bigvee_S(f,g)$ and $\mathrm{Add}_S(f,g)$. A map $f : S \to \bar{\mathbb{R}}$ is called monotone if $s \ge t \Rightarrow f(s) \ge f(t)$, and f is called $\mathbb{N}$-homogeneous if $f(ns) = n\, f(s)$ for all $n \in \mathbb{N}$ and $s \in S$.

First we gather some elementary facts about sub- and superadditive maps.

<u>REMARK</u> <u>1.1</u>: For every subadditive p there is a unique maximal subadditive and $\mathbb{N}$-homogeneous $\bar{p} \le p$. $\bar{p}$ is given by $\bar{p}(s) = \inf\{ \frac{1}{m} p(m\, s) \mid 1 \le m \in \mathbb{N}\}$. For superadditive $q \le p$ one has $q \le \bar{p}$.

If π is subadditive and q superadditive such that

(1) $\quad \pi(s) + q(t) \le \pi(s+t) \qquad \forall\, s, t \in S$,

then we write $q \preceq \pi$. If in addition π is $\mathbb{N}$-homogeneous with $\pi \ne q$ then we write $q \prec \pi$.

<u>REMARK</u> <u>1.2</u>: Let p be subadditive and let $q \le p$ be superadditive. Then there is a unique maximal subadditive p_q with $q \preceq p_q \le p$. p_q is given by $p_q(s) = \inf\{p(s+t)-q(t) \mid t \in S, q(t) \ne -\infty\}$.

PROOF: Let p_q be defined as above. p_q is subadditive because of the subadditivity of p and the superadditivity of q. From (1) it follows that every π with $q \preceq \pi \le p$ has to be dominated by p_q . So it remains to prove $q \preceq p_q$. For $q(t) = -\infty$ we have trivially $p_q(s) + q(t) \le p_q(s+t)$. Therefore we assume $q(t) \ne -\infty$ and we get via the superadditivity of q the following inequalities :

$$\begin{aligned} p_q(s) + q(t) &= \inf\{p(s+\tilde{t}) - q(\tilde{t}) + q(t) \mid \tilde{t} \in S,\ q(\tilde{t}) \ne -\infty\} \\ &\le \inf\{p(s+t+\tilde{t}) - q(t+\tilde{t}) + q(t) \mid \tilde{t} \in S,\ q(\tilde{t}) \ne -\infty\} \\ &\le \inf\{p(s+t+\tilde{t}) - q(\tilde{t}) \mid \tilde{t} \in S,\ q(\tilde{t}) \ne -\infty\} = p_q(s+t)\ . \blacksquare \end{aligned}$$

As a consequence of 1.1. and 1.2 we obtain:

REMARK 1.3: Let p be subadditive and let $q \leq p$ be superadditive. Then there is a unique maximal and $\mathbb{N}$- homogeneous subadditive $[p,q]$ with $q \preceq [p,q] \leq p$. $[p,q]$ is defined by

(2) $[p,q](s) = \inf\{ \frac{1}{m} (p(ms + t) - q(t)) | 1 \leq m \in \mathbb{N}, t \in S,\ q(t) \neq -\infty \}$.

$[p,q]$ is obviously monotone in p and antitone in q, that means for $p' \geq p$ and $q' \leq q$ we have $[p', q'] \geq [p,q]$.

DEFINITION 1: $\mu \in \bigwedge(f,g)$ is called weakly exposed with respect to f and g (exposed in short) iff for every $\mathbb{N}$- homogeneous p' and for every q' with $f \leq q' \prec p' \leq g$ there is an $s_o \in S$ such that

(*) $$q'(s_o) < \mu(s_o) \geq p'(s_o) .$$

In this definition the emphasis lies on the fact that $q'(s_o)$ is strictly less than $\mu(s_o)$ and that we have the two inequalities for the same s_o. The full force of the property required in this definition becomes obvious in the second chapter. If we had required only $q'(s_o) < \mu(s_o)$ instead of (*) we would have described maximal elements of $Add(f,g)$. But (*) implies a property which is even stronger than being an extreme point of $Add(f,g)$.

LEMMA 1: Let p be subadditive and let q be superadditive with $q \leq p$.

(i) If $\delta \in \bigwedge(q,p)$ is exposed and $\delta \leq \mu \in \bigwedge(q,p)$ then μ is additive. In particular δ itself is additive.

(ii) Let $\delta \in \bigwedge(q,p)$ be exposed and let $\mu \in Add(q,p)$ with $\mu \neq \delta$. Then there is an $s_o \in S$ with $\mu(s_o) < \delta(s_o)$.

(iii) Every exposed $\delta \in \bigwedge(q,p)$ is maximal, i.e. $\delta \leq \mu \in \bigwedge(q,p)$ implies $\delta = \mu$.

(iv) Assume that p is monotone and that $q \preceq p$. Let $T \supset \{0\}$ be a subsemigroup of S, let $\delta \in \bigwedge_T(q,p)$ be monotone and exposed. We fix $s_0 \in S$ and consider the

<u>subsemigroup</u> $T(s_o) = \{t + n\, s_o \mid t \in T,\ n \in \mathbb{N}\}$. <u>Then there is a monotone and exposed extension</u> $\tilde{\delta} \in \bigwedge_{T(s_o)}(q,p)$ <u>of</u> δ.

PROOF: (i) Consider $p' = [p,\mu]$. If $p' \neq \mu$ then by definition 1 there would be an $s_o \in S$ providing the contradiciton $\delta(s_o) > \mu(s_o)$ because δ is exposed. Therefore $p' = \mu$ and μ must also be subadditive.

(ii) Let $p' = \bigvee\{\delta,\mu\}$, $q' = \bigwedge\{\delta,\mu\}$, then $q' \prec p'$. Since δ is exposed, there is an $s_o \in S$ with $\inf\{\delta(s_o), \mu(s_o)\} < \delta(s_o) \geq \sup\{\delta(s_o), \mu(s_o)\}$. Thus we have $\mu(s_o) < \delta(s_o)$.

(iii) is an immediate consequence of (ii) and (i).

(iv) First, we extend δ to a superadditive $\delta' \in \bigwedge_{T(s_o)}(q,p)$ with $\delta'_{|T} \geq \delta$ by

(3a) $\delta'(u) = \sup\{\ \delta(t) + q(s) \mid t \in T,\ s \in T(s_o),\ t+s = u\ \}$.

The maximality of δ on T implies $\delta'_{|T} = \delta$. Unfortunately δ' is in general not exposed. But by (iii), any exposed $\tilde{\delta} \geq \delta'$ has to be maximal. So we construct a superadditive map attaining on $T(s_o)\setminus T$ the maximal value which is possible for elements in $\bigvee_{T(s_o)}(\delta',p)$. To be more precise, we define

(3b) $\tilde{\delta}(u) = \sup\{\ \delta'(t) + n\,[p,\delta'](s_o) \mid n \in \mathbb{N},\ t \in T(s_o), t + n s_o \leq u\}$

where $[p,\delta']$ is the subadditive map on $T(s_o)$ defined according to (2). For $u' \geq u$ the set over which the supremum in (3b) is taken increases, therefore $\tilde{\delta}$ must be monotone. $\tilde{\delta}$ is superadditive because of

$$\begin{aligned}
\tilde{\delta}(u_1+u_2) &= \sup\{\delta'(t) + n[p,\delta'](s_o) \mid n \in \mathbb{N}, t \in T(s_o),\ t+n s_o \leq u_1+u_2\} \\
&\geq \sup\{\delta'(t_1+t_2) + (n_1+n_2)[p,\delta'](s_o) \mid n_1,n_2 \in \mathbb{N},\ t_1,t_2 \in T(s_o), t_1+n_1 s_o \\
&\qquad\qquad \leq u_1,\ t_2\, n_2 + s_o \leq u_2\ \} \\
&\geq \sup\{\ \delta'(t_1) + \delta'(t_2) + n_1[p,\delta'](s_o) + n_2[p,\delta'](s_o) \mid \text{same set as above}\} \\
&= \tilde{\delta}(u_1) + \tilde{\delta}(u_2).
\end{aligned}$$

By definition we have $\tilde{\delta} \geq \delta'$. Thus by the maximality of $\delta = \delta'_{|T}$ we get $\delta = \tilde{\delta}_{|T}$. We claim that $\tilde{\delta} \leq p_{|T(s_o)}$. For this we have to prove that $\delta'(t) + n[p,\delta'](s_o) \leq p(u)$ if $t + n s_o \leq u$. We may assume $\delta'(t) \neq -\infty$ otherwise the inequality trivially holds. But from (2) we get $n\,[p,\delta'](s_o) \leq p(n\,s_o + t) - \delta'(t)$. Since p is monotone we have $p(n\,s_o + t) \leq p(u)$. Hence the desired inequality is proved and it remains to show that $\tilde{\delta}$ is exposed.

Let $q_{|T(s_o)} \leq q' \prec p' \leq p_{|T(s_o)}$. For $q'_{|T} \prec p'_{|T}$ (*) follows from the fact that $\delta = \tilde{\delta}_{|T}$ is exposed. Thus there remains the case where $q'_{|T} = p'_{|T}$ and both are additive on T. If $p'_{|T} \neq \delta$ then (*) follows from (ii). So the case remains where $p'_{|T} = q'_{|T} = \delta$. Now, $q' \neq p'$ implies $q'(s_o) < p'(s_o)$ and we obtain (*) from (2),(3) and $q' \preceq p'$ in the following way

$$\tilde{\delta}(s_o) \geq [p,\delta'](s_o) \geq [p,q'](s_o) \geq [p',q'](s_o) =$$

$$= \inf\{ \frac{1}{m} (p'(m\,s_o + t) - q'(t) \mid 0 \neq m \in \mathbb{N},\ t \in \tilde{T} \quad \text{with } q'(t) \neq -\infty \}$$

$$\geq p'(s_o) > q'(s_o).$$

The second inequality follows from $\delta' \leq q'$ which is an immediate consequence of $\delta = q'_{|T}$, $q_{|T(s_o)} \leq q'$ and

$$\delta(t) + q(s) \leq q'(t) + q'(s) \leq q'(t+s) \qquad \forall\, t \in T,\ s \in T(s_o). \blacksquare$$

<u>MAIN THEOREM</u>: <u>Let</u> q <u>be superadditive and</u> $p \geq q$ <u>be subadditive and monotone. Then there is a weakly exposed additive and monotone</u> $\delta \in \bigwedge(q,p)$.

PROOF: We replace p by p_q defined in remark 1.2. Then $q \leq q' \prec p' \leq p$ if and only if $q \leq q' \prec p' \leq p_q$ because p_q is the maximum of all the subadditive ω with $q \preceq \omega \leq p$. Therefore the exposed elements of $\bigwedge(q,p)$ and $\bigwedge(q,p_q)$ are the same. We consider the set of those (S_1,μ_1) such that $S_1 \supset \{0\}$ is a subsemigroup of S and μ_1 is exposed in $\bigwedge_{S_1}(q,p_q)$.

This set is not empty because it has $(\{0\},\ 0 \to 0)$ as element. We define an order relation by

$$(S_1, \mu_1) \le (S_2, \mu_2) \Leftrightarrow S_1 \subset S_2,\ \mu_{2|S_1} = \mu_1 .$$

Then the set is inductively ordered and the theorem follows via Zorn's Lemma from Lemma 1 (iv). ∎

<u>REMARK 2</u>: (i) If μ is additve on S and if T is a subgroup of S then μ does not attain the value $-\infty$ on T. This follows immediately from $0 = \mu(0) = \mu(t) + \mu(-t) \quad \forall\, t \in T$.
(ii) Let R have an order unit [6] and let us assume in addition that S is a preordered cone (i.e. an $\mathbb{R}_+$- module such that inequalities can be multiplied by elements of $\mathbb{R}_+$). If the subadditive p has the property that $\lim\limits_{\varepsilon \downarrow 0} p(\varepsilon\, s) = 0$ for all $s \in S$
(lim with respect to the order unit norm) then every additive $\mu \le p$ is linear ($\mathbb{R}_+$- homogeneous and additive). This is proved as usual by approximating the reals from below by rationals.
(iii) The main theorem remains true if one leaves out the notion "monotone". (One has to consider the special preorder given by the equality relation).

2. GEOMETRIC PROPERTIES

In the following we assume that R has an order unit I.
By the fundamental Stone-Kakutani theorem R is then isomorphic to the vector lattice $C(K)$ of real continuous functions on some compact K([6],p.104).Therefore we have in a natural way a multiplication in R because $C(K)$ is an algebra.
If $0 \le \lambda \le I$ then $\lambda a + (I - \lambda)b$ is called a <u>convex combination</u> of $a, b \in \mathrm{Add}(f,g)$. Note that $\mathrm{Add}(f,g)$ is convex since it contains all convex combinations of its elements. Denote with S^* the set of additive maps $S \to \bar{R}$. A subset $X \subset S^*$ is called <u>bounded</u> if $\sup\{ x(s) \mid x \in X \}$ exists for all $s \in S$. For bounded nonempty X the maps $s \to \bigvee X(s) = \sup\{ x(s) \mid x \in X \}$ and $s \to \bigwedge X(s) = \inf\{ x(s) \mid x \in X \}$ are subadditive and superadditive respectively. We define $\mathrm{Add}(\bigwedge X, \bigvee X)$ to be the <u>closed-convex hull</u>

of X. We shall denote the closed-convex hull of X by $\langle X \rangle$. Note that $\mathrm{Add}(f,g)$ is obviously always closed-convex.

LEMMA 2: *Let p be subadditive, $\mathbb{N}$-homogeneous and monotone and let q be superadditive with $q \leq p$. Then the following are equivalent:*

(i) $q \preceq p$

(ii) $p = \bigvee X$, *where* $X = \{\nu \mid \nu \in \mathrm{Add}(q,p),\ \nu \text{ monotone}\}$.

PROOF: (ii) $\Rightarrow$ (i): By definition we have

$$\nu(t) + q(s) \leq \nu(t) + \nu(s) = \nu(t+s) \leq p(t+s) \quad \text{for all } \nu \in X.$$

Now, one takes the supremum over $\nu \in X$ on the left side and obtains the desired result.

(i) $\Rightarrow$ (ii): We fix $t_o \in S$ and define a superadditive $q \leq q' \leq p$ by $q'(t) = \sup\{n\, p(t_o) + q(s) \mid n\, t_o + s = t,\ n \in \mathbb{N}\}$. By the main theorem there is a monotone $\nu \in \mathrm{Add}(q',p)$. By definition we have $p(t_o) = q'(t_o)$ and $p(t_o) \geq \nu(t_o) \geq q'(t_o)$. This implies (ii) because t_o was chosen arbitrarily. ∎

COROLLARY 1: *Let $X = \mathrm{Add}(f,g) = \mathrm{Add}(f',g')$. Then $\delta \in X$ is weakly exposed with respect to f,g iff it is weakly exposed with respect to f',g'.*

PROOF: Let δ be exposed with respect to f',g' and take $q' \in \bigwedge(f,g)$, $p' \in \bigvee(f,g)$ with $q' \prec p'$. We show that (∗) can be obtained. Consider $Y = \{\nu \in X \mid q' \leq \nu \leq p'\}$. According to lemma 2 we have $p' = \bigvee Y$. For $\tilde{q} = \bigwedge Y \geq q'$ we have either $\tilde{q} \prec p'$ or $\tilde{q} = p'$. In the first case (∗) can be obtained from $f' \leq \tilde{q} \prec p' \leq g'$ and the fact that δ is (f',g')- exposed. In the second case we have either $\delta \neq \tilde{q} = p'$ or $\delta = \tilde{q} = p'$. For $\delta \neq \tilde{q} = p'$ (∗) follows from lemma 1(ii). If $\delta = p'$, (∗) is a consequence of $p' \gneq q'$. Hence δ is exposed with respect to f,g. ∎

REMARK 3: If $X \subset S^*$ is closed-convex then the property of $\delta \in X$ to be weakly exposed does not depend on the particular representation $X = \mathrm{Add}(f,g)$ but is a geometric property depending only on the closed-convex set and the semigroup S.

LEMMA 3: *Consider a monotone subadditive* p *and let* $T \supset \{0\}$ *be a subsemigroup of* S. *Then a weakly exposed monotone* $\delta \in \bigwedge_T(q,p)$ *can be extended to a weakly exposed monotone* $\mu \in \bigwedge_S(q,p)$ *if and only if*

$$\delta(t) + q(s) \leq p(s+t) \quad \forall\, t \in T,\ s \in S.$$

PROOF: Because of lemma 1(i) the condition is necessary. Now, consider an exposed monotone $\mu \in \bigwedge(\tilde{q},p)$, where
$\tilde{q}(\tilde{t}) = \sup\{\ \delta(t) + q(s) \mid t \in T,\ s \in S,\ \tilde{t} = t + s\} \leq p(\tilde{t})$.
By lemma 1(iii) μ is an extension of δ. For showing that μ is also exposed in $\bigwedge(q,p)$ we take arbitrary $q' \in \bigwedge(q,p)$ and $p' \in \bigvee(q,p)$ with $q' \prec p'$.
If $q'_{|T} \neq p'_{|T}$ then $(*)$ follows from the fact that δ is exposed in $\bigwedge_T(q,p)$. We assume therefore $q'_{|T} = p'_{|T}$. This implies that $q'_{|T}$ and $p'_{|T}$ are additive. If $\delta \neq q'_{|T} = p'_{|T}$ then $(*)$ follows from lemma 1(ii). For $q'_{|T} = \delta$ we obtain $q' \geq \tilde{q}$ from the superadditivity of q' and $(*)$ follows because μ is exposed in $\bigwedge(\tilde{q},p)$. ∎

DEFINITION 2: Let $X \subset S^*$ be bounded and closed-convex. Then $\delta \in X$ is called *extreme point* of X if $\delta \leq \lambda\, \nu_1 + (I-\lambda)\, \nu_2$ with $\nu_1, \nu_2 \in X$ and $0 \lneqq \lambda \lneqq I$ implies $\nu_1 = \nu_2 = \delta$. Further ∂X and $\exp(X)$ denote the set of extreme points of X and weakly exposed points of X, respectively.

THEOREM 2: *Let* $\phi \neq X \subset S^*$ *be bounded and closed-convex.*
(i) $\exp(X) \subset \partial X$. (*This implies* $\partial X \neq \phi$).
(ii) *every* $s_o \in S$ *attains its supremum on* X *at some element of* $\exp(X)$; *i.e. there is a* $\mu \in \exp(X)$ *such that* $\mu(s_o) = \bigvee X(s_o)$.

PROOF: (i) We consider $\mu \in \exp(X)$ with

(4) $\quad \mu \leq \lambda\, \nu_1 + (I-\lambda)\, \nu_2,\ \nu_1, \nu_2 \in X,\ 0 \lneqq \lambda \lneqq I$.

and define $p' = \bigvee\{\mu, \nu_1, \nu_2\}$, $q' = \bigwedge\{\mu, \nu_1, \nu_2\}$. We have to show $p' = q'$ because that implies $\mu = \nu_1 = \nu_2$. If $p' \neq q'$ then we obtain from definition 1 an element $s \in S$ with

$\inf\{\nu_1(s), \nu_2(s)\} < \mu(s) \geq \sup\{\nu_1(s), \nu_2(s)\}$ in contradiction to (4).
(ii) We fix $s_o \in S$ and consider $\alpha = p(s_o)$, where $p = \bigvee X$ $\delta = \bigwedge X$. Then $\alpha \geq \sup\{x(s) \mid x \in X\}$. Now, we set $T = \{n s_o \mid n \in \mathbb{N}\}$ and define an exposed $\tilde{\mu} \in \bigwedge_T(\delta, p)$ by $\tilde{\mu}(n s_o) = n\alpha$. Lemma 3 then gives the desired result. ▮

REMARK 4: Every compact convex subset of a locally convex Hausdorff vector space E may be considered as a bounded closed-convex set(in the above sense) of linear functionals on E'. This shows that Theorem 2(i) has the Krein-Milman theorem as special case. Theorem 2(ii) is a generalization of H. Bauer's maximum principle [1]. Another immediate consequence of Theorem 2 is that every dual ball of a real Banach space has an extreme point. Since this statement already implies the axiom of choice ([2],[5]) our main theorem must be effectively equivalent to the axiom of choice. The Hahn-Banach theorem and its applications can be obtained from the main theorem in the same way as in [3].The generalization of the above mentioned theorems does not only stem from the generalization to vector lattices (instead of the reals) but also from the fact that in general exp(X) and ∂X are not equal.
In order to investigate further geometrical properties of exp(X) we need a technical lemma. For this purpose we consider a proper subsemigroup $T \supset \{0\}$ of S. For $s_1 \in S \setminus T$ we denote by $T(s_1)$ the subsemigroup $\{t + n s_1 \mid t \in T, n \in \mathbb{N}\}$. If $X \subset S^*$ then $X_{|T}$ stands for $\{x_{|T} \mid x \in X\}$.

LEMMA 4: Let X be closed-convex and nonempty.
(i) Every $\nu \in \exp(\langle X_{|T} \rangle)$ has one and only one exposed extension $\bar{\nu} \in \exp(\langle X_{|T(s_o)} \rangle)$ to $T(s_o)$.
(ii) Let $\nu \in \exp(\langle X_{|T} \rangle)$ and $\bar{\nu} \in \langle X_{|T(s_o)} \rangle$ with $\bar{\nu}_{|T} = \nu$. Then $\bar{\nu} \in \exp(\langle X_{|T(s_o)} \rangle)$ if and only if

$$\bar{\nu}(s_o) = \max\{\mu(s_o) \mid \mu \in X,\ \nu \leq \mu_{|T}\} = \max\{\mu(s_o) \mid \mu \in X,\ \nu = \mu_{|T}\}.$$

(iii) Let $\mu \in \exp(X)$ then there is an $s_o \in S \setminus T$ such that

$$\mu(s_o) = \max\{\nu(s_o) \mid \nu \in X,\ \mu_{|T} \leq \nu_{|T}\}.$$

PROOF: (i) In lemma 1(iv) the existence of an exposed extension was already shown. Consider two exposed extensions ν_1, ν_2 and assume $\nu_1 \neq \nu_2$. Lemma 1(ii) provides us with some $\bar{s} \in S$ such that $\nu_1(\bar{s}) < \nu_2(\bar{s})$. This implies $\nu_1(s_o) < \nu_2(s_o)$ because of $\nu_{1|T} = \nu_{2|T}$. Hence $\nu_1 \lneq \nu_2$ in contradiction to lemma 1(iii).

(ii) By lemma 3 the set $Y = \{\mu \in X | \nu \le \mu_{|T}\}$ is not empty. And because of lemma 1(iii), Y is equal to $\{\mu \in X | \nu = \mu_{|T}\}$. From theorem 2(ii) we know that s_o attains its maximum on Y, say at $\bar{\mu}$. Let $\bar{\nu} \in \exp(\langle X_{|T(s_o)} \rangle)$ be the unique exposed extension of ν to $T(s_o)$. According to lemma 3, $\bar{\nu}$ can be extended to some $\hat{\nu} \in \exp(X)$. Obviously, we have $\hat{\nu} \in Y$ and $\hat{\nu}(s_o) \le \bar{\mu}(s_o)$. Now, we have $\bar{\nu} = \hat{\nu}_{|T(s_o)} \le \bar{\mu}_{|T(s_o)}$ because of $\hat{\nu}_{|T} = \bar{\mu}_{|T}$ and $\hat{\nu}(s_o) \le \bar{\mu}(s_o)$. Finally, with the help of lemma 1(iii) we conclude $\bar{\nu} = \bar{\mu}_{|T(s_o)}$ because $\bar{\nu}$ is exposed.

(iii) Consider $Y = \{x \in X | \mu_{|T} \le x_{|T}\}$ and define $q' = \bigwedge Y$, $p' = \bigvee Y$. If $q' = p'$ then Y contains only one element and this must be μ. In this case we can take for s_o any element of $S \setminus T$. If $q' \neq p'$ then $q' \prec p'$ and there is an s_o with $q'(s_o) < \mu(s_o) \ge p'(s_o)$. Now, $q'(s_o) < \mu(s_o)$ clearly implies $s_o \notin T$. ∎

Let $X, Y \subset S^*$, then $X+Y$ stands for $\{x+y | x \in X, y \in Y\}$. We are interested in the relations between the exposed points of $X+Y$ and those of X and Y respectively. Even for closed-convex X and Y, unfortunately, $X+Y$ is not always closed convex. But the following theorem shows that in certain cases $X+Y$ is indeed closed-convex.

<u>THEOREM 3</u>: <u>Let p_1, p_2 be subadditive and μ additive with $\mu \le p_1 + p_2$. Then there are additive $\mu_1 \le p_1$ and $\mu_2 \le p_2$ such that $\mu \le \mu_1 + \mu_2$.</u>

PROOF: On the semigroup $S^2 = \{(s_1,s_2) | s_1, s_2 \in S\}$ we define a subadditive π and a superadditive ρ by:

$$\pi((s_1,s_2)) = p_1(s_1) + p_2(s_2)$$

$$\rho((s_1,s_2)) = \begin{cases} \mu(s) & \text{if } s = s_1 = s_2 \\ -\infty & \text{otherwise} \end{cases}$$

By the main theorem there is an additive $\bar{\mu}$ on S^2 with $\rho \le \bar{\mu} \le \pi$. Now, $\mu_1(s) \underset{\text{def}}{=} \bar{\mu}((s,0))$ and $\mu_2(s) \underset{\text{def}}{=} \bar{\mu}((0,s))$ do the job. ∎

COROLLARY 3: *Let* S *be a group and let* $X \subset S^*$ *be closed-convex.*

(i) $\mu \in S^*$ *is in* X *if and only if* $\mu \le \bigvee X$.

(ii) *If* $Y \subset S^*$ *is closed-convex then* $X + Y$ *is closed-convex.*

PROOF: (i) The necessity of $\mu \le \bigvee X$ is trivial. Now, let $\mu \le \bigvee X$. We have to prove $\mu \ge \bigwedge X$. But this follows from $\mu(s) = -\mu(-s) \ge -(\bigvee X)(-s) = (\bigwedge X)(s)$.
(ii) Let $\nu \in \langle X+Y \rangle$, then $\nu \le \bigvee X + \bigvee Y$. According to theorem 3 there are additive $\mu_1 \le \bigvee X$, $\mu_2 \le \bigvee Y$ with $\nu \le \mu_1 + \mu_2$. We have just proved that $\mu_1 \in X$, $\mu_2 \in Y$. It remains to show that $\nu \ge \mu_1 + \mu_2$. This follows via $\nu \le \mu_1 + \mu_2$ from $\mu(s) = -\mu(-s) \ge -\mu_1(-s) - \mu_2(-s) = \mu_1(s) + \mu_2(s)$. ∎

Now, we fix two subadditive maps p_1, p_2 and we consider the closed-convex sets $X_{p_i} = \{x \in S^* \mid x \le p_i\}$ $(i = 1,2)$.

LEMMA 5: *Let* $\mu \in \exp(\langle X_{p_1} + X_{p_2} \rangle)$, *then there are* $\mu_1 \in X_{p_1}$, $\mu_2 \in X_{p_2}$ *such that* $\mu = \mu_1 + \mu_2$. *Furthermore,* μ_1 *and* μ_2 *are unique.*

PROOF: By theorem 3 there are $\mu_1 \in X_{p_1}$, $\mu_2 \in X_{p_2}$ with $\mu \le \mu_1 + \mu_2$ and the maximality of μ (lemma 1(iii)) implies equality. Now, let $\bar{\mu}_i \in X_{p_i}$ $(i = 1,2)$ such that $\mu = \bar{\mu}_1 + \bar{\mu}_2$.

Then we consider $q' = \bigwedge\{\mu_1, \bar{\mu}_1\} + \bigwedge\{\mu_2, \bar{\mu}_2\}$, $p' = \bigvee\{\mu_1, \bar{\mu}_1\} + \bigvee\{\mu_2, \bar{\mu}_2\}$. For $q' \neq p'$ there is an $\hat{s} \in S$ with $q'(\hat{s}) < \mu(\hat{s}) \geq p'(\hat{s})$, because μ is exposed. This inequality is in contradiction to $\mu(\hat{s}) = \mu_1(\hat{s}) + \mu_2(\hat{s}) = \bar{\mu}_1(\hat{s}) + \bar{\mu}_2(\hat{s})$. Therefore we must have $q' = p'$. Hence $\mu_1 = \bar{\mu}_1$ and $\mu_2 = \bar{\mu}_2$. ▮

The second part of the following theorem shows that the μ_i occuring in lemma 5 have to be exposed.

<u>THEOREM 4</u>: (i) <u>For every</u> $\nu \in \exp(X_{p_1})$ <u>there is a</u> $\delta \in \exp(X_{p_2})$ <u>such that</u> $(\nu+\delta) \in \exp(\langle X_{p_1} + X_{p_2} \rangle)$.

(ii) <u>Let</u> $\nu \in X_{p_1}$, $\delta \in X_{p_2}$ <u>such that</u> $\nu + \delta \in \exp(\langle X_{p_1} + X_{p_2} \rangle)$, <u>then</u> $\nu \in \exp(X_{p_1})$ <u>and</u> $\delta \in \exp(X_{p_2})$.

PROOF: (i) We consider the set of (T,μ) where $T \supset \{0\}$ is a subsemigroup of S and $\mu \in \exp(\langle X_{p_2|T} \rangle)$ such that $\nu_{|T} + \mu \in \exp(\langle X_{p_1|T} + X_{p_2|T} \rangle)$. This set is not empty because it contains $(\{0\}, 0 \to 0)$ and it is inductively ordered with respect to the order relation considered in the proof of the main theorem. Now, let (T,μ) be a maximal element. We prove the statement by showing $T = S$. By way of contradiction we assume $T \neq S$. Lemma 4(iii) then provides us with some $s_o \in S\setminus T$ such that $\nu(s_o) = \max\{x(s_o) \mid x \in X_{p_1}, \nu_{|T} \leq x_{|T}\}$. From this and lemma 4(ii) it follows that $\nu_{|T(s_o)} \in \exp(\langle X_{p_2|T(s_o)} \rangle)$.

Let $\bar{\varphi} \in \exp(\langle X_{p_1|T(s_o)} + X_{p_2|T(s_o)} \rangle)$ and $\bar{\mu} \in \exp(\langle X_{p_2|T(s_o)} \rangle)$ be the unique (lemma 4(i)) extensions of $\varphi = \mu + \nu_{|T}$ and μ to $T(s_o)$. We claim $\bar{\varphi} = \bar{\mu} + \nu_{|T(s_o)}$. This will be in contradiction to the maximality of (T,μ) and thus finishes the proof.

Proof of the claim: We have to show $\bar{\varphi}(s_o) = \bar{\mu}(s_o) + \nu(s_o)$. Because we know already $\bar{\varphi}_{|T} = \bar{\mu}_{|T} + \nu_{|T}$. Lemma 4(ii) applied to $\bar{\varphi}$ tells us that $\bar{\varphi}(s_o) \geq \bar{\mu}(s_o) + \nu(s_o)$. So it remains to prove $\bar{\varphi}(s_o) \leq \nu(s_o) + \bar{\mu}(s_o)$. We can extend $\bar{\varphi}$ to an $\hat{\varphi} \in \exp(\langle X_{p_1} + X_{p_2} \rangle)$ (lemma 3). According to lemma 5 we have $\hat{\varphi} = x_1 + x_2$ with $x_i \in X_{p_i}$. A second application of lemma 5 to the restrictions of $\hat{\varphi}_{|T} = \nu_{|T} + \mu$, $x_{1|T}$, $x_{2|T}$ gives $x_{1|T} = \nu_{|T}$ and $x_{2|T} = \mu = \bar{\mu}_{|T}$. Lemma 4(ii) applied to the exposed $\nu_{|T(s_o)}$ and $\bar{\mu}$ allows us to conclude $\nu(s_o) \geq x_1(s_o)$ and $\bar{\mu}(s_o) \geq x_2(s_o)$. Therefore we have $\bar{\varphi}(s_o) = \hat{\varphi}(s_o) = x_1(s_o) + x_2(s_o) \leq \nu(s_o) + \bar{\mu}(s_o)$ and the claim is proved.

(ii) is proved in essentially the same way as (i). Let $T \supset \{0\}$ be a maximal subsemigroup of S such that $\nu_{|T} \in \exp(\langle X_{p_1|T} \rangle)$, $\delta_{|T} \in \exp(\langle X_{p_2|T} \rangle)$ and $(\nu + \delta)_{|T} \in \exp(\langle X_{p_1|T} + X_{p_2|T} \rangle)$.

We have to prove $T = S$. We assume $T \neq S$ and we take an $s_o \in S \setminus T$ such that

$$(5)\quad (\nu+\delta)(s_o) = \max\{x_1(s_o) + x_2(s_o) \mid x_i \in X_{p_i},\ (\delta+\nu)_{|T} \leq (x_1 + x_2)_{|T}\}$$

$$= \max\{x_1(s_o) + x_2(s_o) \mid x_i \in X_{p_i},\ \nu_{|T} = x_{1|T},\ \delta_{|T} = x_{2|T}\}$$

The last equality follows from the maximality of $(\delta+\nu)_{|T}$ and from lemma 5. Now, we consider the extensions $\bar{\nu} \in \exp(\langle X_{p_1|T(s_o)} \rangle)$ and $\bar{\delta} \in \exp(\langle X_{p_2|T(s_o)} \rangle)$ of $\nu_{|T}$ and $\delta_{|T}$ respectively.

Due to (5) and lemma 4 (ii) we have $(\nu+\delta)_{|T(s_o)} \leq \bar{\nu} + \bar{\delta}$. From (5) and lemma 4(ii) we know that $(\nu+\delta)_{|T(s_o)}$ is exposed, therefore $(\nu+\delta)_{|T(s_o)} = \bar{\nu} + \bar{\delta}$. Finally, application of lemma 5 to the exposed $(\nu+\delta)_{|T(s_o)}$ gives $\nu_{|T(s_o)} = \bar{\nu}$, $\delta_{|T(s_o)} = \bar{\delta}$.

This is in contradiction to the maximality of T. ∎

As a consequence of corollary 3 and lemma 5 we obtain:

COROLLARY 4: *Let* S *be a group and let* $X, Y \subset S^*$ *be nonempty and closed-convex.*

(i) *For every* $\nu \in \exp(X)$ *there is a* $\delta \in \exp(Y)$ *such that* $\nu + \delta \in \exp(X + Y)$

(ii) *Let* $z \in \exp(X + Y)$ *then there are unique* $\nu \in \exp(X)$ *and* $\delta \in \exp(Y)$ *such that* $z = \nu + \delta$.

REMARK 5: It should be noted that the results of lemma 4 through corollary 4 are also valid when R does not contain an order unit.

DEFINITION 3: Let $X \subset S^*$ be bounded and closed-convex. Then $\mu \in X$ is called a *strong extreme point* of X if for every bounded closed convex $Y \neq \phi$ there is a $\tilde{\mu} \in \partial Y$ such that $\mu + \tilde{\mu} \in \partial\,(X+Y)$ The set of strong extreme points of X is denoted by $\partial_s X$. (For the source of this definition see [4]).

As a consequence of theorem 2(i) and corollary 4(i) we have

COROLLARY 5: *Let* S *be a group and* $X \subset S^*$ *closed-convex. Then* $\partial_s X \supset \exp(X)$.

Corollary 5 also shows that $\exp(X)$ and ∂X, are in general not equal because in [4] there is an example for $\partial_s X \subsetneqq \partial X$.

The next theorem gives in a special case a characterization of $\exp(X)$ which does not involve S.

LEMMA 6: *Let* $X \subset S^*$ *be bounded and closed-convex. The following are equivalent.*

(i) $x_o \in X \backslash \exp(X)$

(ii) *there is a closed-convex* Y *with* $X \supset Y \supsetneqq \{x_o\}$ *such that* $y(s) = y'(s)$ *for all* $y, y' \in Y$ *whenever* $x_o(s) \geq \sup\{y(s) | y \in Y\}$.

PROOF: (ii) $\Rightarrow$ (i): (*) cannot hold for $p' = \bigvee Y$, $q' = \bigwedge Y$.
(i) $\Rightarrow$ (ii): Choose q', p' with $\bigwedge X \leq q' \prec p' \leq \bigvee X$ such that (*) never holds with respect to x_o.
We replace q', p' by $\tilde{q} = \bigwedge \tilde{Y}$, $\tilde{p} = \bigvee \tilde{Y}$, where $\tilde{Y} = \{x \in X | q' \leqq x \leqq p'\}$. Then (*) never holds for $\tilde{q}$, $\tilde{p}$ because $\tilde{q} \geq q'$ and $\tilde{p} = p'$ (Lemma 1). But $p' = x_o$ is not possible, otherwise (*) would hold for some s and p', q' because $p' \gneqq q'$.
Now, we take for Y the closed-convex hull of $\tilde{Y} \cup \{x_o\}$. ∎

THEOREM 5: Let S be a real vector space and let E be a space of linear functionals on S equipped with a topology of the dual pair (E,S) such that E is a Frechet space. Furthermore, let $X \subset E$ be a compact convex set. Then for every $x_o \in \partial X \setminus \exp(X)$ there is a compact convex set $Z \subset E$ such that $x_o + z \notin \partial (X+Z)$ for all $z \in \partial Z$.

COROLLARY 6: $\partial_s X = \exp(X)$

PROOF: Let d denote an invariant metric of E. Without loss of generality we can assume $x_o = 0$. According to lemma 6 there is a compact convex set Y with $X \supset Y \supsetneqq \{0\}$ such that 0 is not a support point of Y. But $0 \in \partial Y$ because of $Y \subset X$. We define $F_+ = \bigcup \{nY | n \in \mathbb{N}\}$, $F_- = \bigcup \{-nY | n \in \mathbb{N}\}$. Since $0 \in \partial Y$ we have $F_+ \cap F_- = \{0\}$. The closure of F_+ has to contain $F_+ \cup F_-$ otherwise the Hahn-Banach-separation theorem would provide a support functional of 0 with respect to Y. Now, we fix $0 \neq y_o \in (-Y) \subset F_-$ and choose inductively

$z_n \in F_+$ with $d(z_n, \frac{3}{2} y_{n-1}) \leqq \frac{1}{n}$ where $y_n = 3 y_{n-1} - 2 z_n$,

$n = 1,2,3\ldots.$ From $d(y_n, 0) \leq \frac{2}{n}$ we obtain that $\hat{Z} = \{y_n | n \in \mathbb{N}\} \cup \{z_n \mid n \in \mathbb{N}\} \cup \{0\}$ is compact. By Z we denote the compact convex hull of $\hat{Z}$. We know $\partial Z \subset \hat{Z}$ (compactness of $\hat{Z}$) and $\frac{1}{3} y_n + \frac{2}{3} z_n = y_{n-1}$ implies in addition $\partial Z \subset \{z_n | n \in \mathbb{N}\} \cup \{0\}$.

Now, $0 + 0 = 0 \notin \partial(Z+Y)$ because of $\frac{1}{2} y_o$, $-\frac{1}{2} y_o \in Z + Y$, and

$0 + z_n \notin \partial(Z+Y)$ because of $z_n \in F_+$. So Z has the required property. ∎

The corollary follows from Corollary 5.

PROBLEMS: (i) Give a complete characterization of those X with $\partial_s X = \exp(X)$.

(ii) Give a complete characterization of those X with $\exp(X) = \partial X$.

Partial answers for (ii) are known. This is the case for finite dimensional compact convex sets [4] or for Choquet simplexes (consequence of [1]Cor. II. 5.20).

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