Signed Representing Measures

By

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The problem of representing all sup-norm continuous linear functionals on a vector space of functions on some non-compact set X by integrals on X appeared in the examination of the strict topology and in the study of weighted spaces $CV_0(X)$ (continuous functions f on X such that vf vanishes at infinity for all v in a given family of upper-semicontinuous weight functions). In both cases such a representation is always possible (cf. [5], [6], [7], [9], [10]).

However, very little is known for more general situations, for example spaces where the weighted functions do not vanish at infinity. Nevertheless one can construct a lot of subspaces of $C(X)$ where all sup-norm continuous functionals can be represented as integrals on X. One example is the space of all continuous functions on a pseudocompact X.

In this note we prove results of a more general type. We characterize (in terms of order-properties) all cones $F(X)$ of real-valued functions on an arbitrary set X such that every linear $\mu\colon F(X) \to \mathbb{R}$ with $\mu(f) \leqq \sup_{x \in X} |f(x)|\ \forall f \in F(X)$ can be represented by integration with respect to a signed measure on X with total variation $\leqq 1$. This yields analogous results for weighted spaces and weighted cones. The main tool is a generalization of the Riesz Representation Theorem obtained in [4].

I. The simple case: Vector Lattices. Let us first introduce some notation. Troughout this paper F denotes a cone of bounded real-valued functions on an arbitrary set X. For emphasizing on which set the functions are defined we sometimes write $F = F(X)$ and denote the restrictions to a subset $Y \subset X$ by $F(Y)$. A linear (i.e. additive and positive-homogeneous) functional $\mu\colon F \to \mathbb{R}$ is called a *state* of F if

$$\mu(f) \leqq \sup_X(f) \doteq \sup_{x \in X} f(x) \quad \forall f \in F.$$

μ is called normed if $\mu(f) \leqq \sup_X |f| \quad \forall f \in F.$

By Σ_F we mean the smallest σ-algebra in X such that all $f \in F$ are Σ_F-measurable. A Σ_F-measure τ on X with total variation $|\tau| \leqq 1$ is said to be a *signed representing measure* for the linear μ if

$$(*) \qquad \mu(f) \leqq \int_X f\, d\tau \quad \forall f \in F.$$

τ is called a *strict* signed representing measure if we have equality in (*) for all $f \in F$. Of course, every signed representing measure is strict if F is a vector space. A (strict) signed representing measure τ which is a probability measure (i.e. positive and $\tau(X) = 1$) is called a (strict) *representing measure*. For a maximal (pointwise order or F) state every representing measure is strict because the right-hand side in (*) defines a state.

The cone $F(X)$ is called a *Dini-cone* if $F(X)$ contains all constant functions and if it has the Dini property, i.e. if for every decreasing sequence (f_n) in $F(X)$ the following holds:

$$\inf_{n \in \mathbb{N}} \sup_X (f_n) = \sup_X (\inf_{n \in \mathbb{N}} f_n).$$

One example of a Dini cone is $\mathbb{R} + UC_\infty^+(X)$, where $UC_\infty^+(X)$ are the non-negative upper-semicontinuous functions f on the topological space X vanishing at infinity (i.e. $\forall \varepsilon > 0$, $\{x \mid |f(x)| \geqq \varepsilon\}$ quasicompact). In [4, Main Theorem] we proved:

Let $F(X)$ contain the constants. Then every state of $F(X)$ has a representing measure if and only if $F(X)$ is a Dini cone. If in addition X is a topological space and $F(X)$ contains only upper-semicontinuous functions then every state has a representing measure which can be extended to a σ-algebra containing all closed quasicompact sets.

Unfortunately this theorem does not immediately yield results about signed representing measures since the following simple example shows that there are vector spaces such that all states do have signed representing measures but some do not have representing measures.

Example 1. Define v: $[0, 1] \to \mathbb{R}$ by $v(x) \doteqdot \{1 - x$ for $0 \leqq x \leqq 1/4$, x for $1/4 < x \leqq 1\}$ and consider $vC[0, 1] \doteqdot \{vf \mid f: [0, 1] \to \mathbb{R}$ continuous$\}$. Then the state $\mu: f \to \lim_{\varepsilon \downarrow 0} f(1/4 + \varepsilon)$ has no representing measure, but $1/3\, \delta_{1/4}$ ($\delta_{1/4}$ Dirac measure at the point $1/4$) is certainly a signed representing measure for μ.

In fact all states do have signed representing measures (cf. [5, Theorem 2] or [6, Theorème 4.9]).

However $vC[0, 1]$ is a vector lattice and for vector lattices the Main Theorem gives results about signed representing measures via a Jordan decomposition for linear functionals.

Lemma 1. *Let $E = E(X)$ be a vector lattice of bounded functions on X. Then every normed linear μ: $E \to \mathbb{R}$ is of the form $\mu = \lambda\mu_1 - (1 - \lambda)\mu_2$ where $0 \leqq \lambda \leqq 1$ and μ_1, μ_2 are normed linear such that $\mu_{1|E_+}$ and $\mu_{2|E_+}$ are maximal states of*

$$E_+ \doteqdot \{f \in E \mid f \geqq 0\}.$$

If in addition μ is a state then we may choose μ_1, μ_2 such that $\mu_1 = \mu_2$.

Proof. The lemma is a consequence of the fact that $E^* \doteqdot (E, \text{sup-norm})^*$ is an abstract L-space. For completeness we sketch a simple proof. Define

$$p(v) \doteqdot \sup\{v(f) \mid f \in E_+,\ \sup_X(f) \leqq 1\} \quad \text{for} \quad v \in E^*.$$

Then $p(\ )$ is linear on the cone of order-preserving functionals because

$$\{f \in E_+ \mid \sup_X(f) \leqq 1\}$$

is upwards directed. For normed ν we have $p(\nu) \leqq 1$ and we claim that $\nu_{|E_+}$ is then a maximal state of E_+ if and only if $p(\nu) = 1$. The Jordan decomposition (cf. [8, Summensatz] or [2, p. 10]) gives for normed ν (i.e. $\nu \leqq \max(\sup_X, -\inf_X)$ pointwise on E) states ν_+, ν_- of E and $0 \leqq \lambda \leqq 1$ such that $\nu = \lambda\nu_+ - (1-\lambda)\nu_-$.

If $p(\nu) = 1$ then $\lambda = 1$ and for a state $\tilde{\nu}_{|E_+} \geqq \nu_{+|E_+}$ of E_+ (its unique extension to E denoted by $\tilde{\nu}$) the difference $\tilde{\nu} - \nu$ and $\tilde{\nu}$ itself must be order-preserving. Using the partial additivity of $p(\ )$ we get

$$1 \geqq p(\tilde{\nu}) = p(\nu) + p(\tilde{\nu} - \nu) = 1 + p(\tilde{\nu} - \nu) \Rightarrow p(\tilde{\nu} - \nu) = 0 \quad \Rightarrow \quad \nu = \tilde{\nu},$$

which implies that $\nu_{|E_+}$ has to be maximal. Now, let $\nu_{|E_+}$ be maximal then $\varrho = \nu/p(\nu)$ is normed and $\varrho_{|E_+}$ is a state of $E_+ \geqq \nu_{|E_+}$. This has $p(\nu) = 1$ as consequence. From this characterization and the partial additivity of p it follows that the E_+-maximal states are a convex set. By virtue of the Jordan decomposition it is therefore sufficient to prove the lemma for states. Assume therefore $\mu = \mu_+$ then we have:

$$\mu = \mu_+ = \left(\frac{1 + p(\mu)}{2}\right)\nu - \left(\frac{1 - p(\mu)}{2}\right)\nu, \quad \nu := \frac{\mu}{p(\mu)}$$

and $p(\nu) = 1$. For showing that ν is maximal it remains to prove that it is normed but this is a consequence of $\mu = \mu_+$. ∎

The following theorem is for truncated $E(X)$ well known (cf. [1, p. 199]).

Theorem 1. *Let $E = E(X)$ be a vector lattice of bounded functions on X and E_+ its positive cone. Then every normed linear $\mu\colon E \to R$ has a signed representing measure if and only if $E_+ + \mathbb{R}$ is a Dini cone. In this case there is for every state μ a positive measure representing μ. If in addition E_+ consists of upper-semicontinuous functions on the topological space X then every μ has a signed representing measure which can be extended to a σ-algebra containing all closed quasicompact subsets of X.*

Proof. A consequence of the quoted Main Theorem and Lemma 1 together with the observation that signed representing measures for maximal states on E_+ are strict and probability measures. ∎

Corollary 1. *Let E be a vector lattice of functions on X and $v\colon X \to \mathbb{R}$ be a weight function such that*

$$p_v(f) = \sup_X |vf| < \infty \quad \forall f \in E.$$

And let $\mathbb{R} + vE_+ = \mathbb{R} + \{vf \mid f \in E_+\}$ be a Dini cone. Then for every linear

$$\mu\colon E \to \mathbb{R} \quad \textit{with} \quad \mu(f) \leqq p_v(f) \quad \forall f \in F$$

there is a signed measure τ_μ (with respect to the smallest σ-algebra in X such that vE consists of measurable functions) such that

$$\mu(f) = \int_X vf\, d\tau_\mu \quad \forall f \in E \quad \textit{and} \quad |\tau_\mu| \leqq 1.$$

If μ is positive τ_μ can be chosen to be positive. If vE_+ consists of upper-semicontinuous functions on the topological space X then there is always a τ_μ which can be extended to a σ-algebra containing all closed quasicompact sets.

Proof. Follows from Theorem 1 and the observation that if $\tilde{\mu}: vE \to \mathbb{R}$ is normed linear then $\mu: f \to \tilde{\mu}(vf)$ is linear $\leqq p_v$ and if $\mu: E \to \mathbb{R}$ is linear $\leqq p_v$ then $\tilde{\mu}: vf \to \mu(f)$ is normed linear on vE. ∎

Example 2 (cf. [5], [6], [7], [9], [10]). Let $v \geqq 0$ be an upper-semicontinuous weight function on the Hausdorff space X and $C_v(X) \doteq \{f \mid f: X \to \mathbb{R}$ is continuous and vf vanishes at infinity$\}$. Then $vC_v(X)_+ + \mathbb{R}$ is a Dini cone because it is a subset of $\mathbb{R} + UC_\infty^\pm(X)$.

By the Corollary the dual unit ball of $C_v(X)$ is given by the functionals

$$f \to \int_X v f \, d\tau$$

where τ is a measure on a σ-algebra containing the compact sets. In these integrals τ can be replaced by a suitable tight measure $\tilde{\tau}$ without changing the value of the integrals. For example choose a sequence K_n of compact sets such that

$$|\tau|\left(\bigcup\{K_n \mid n \in \mathbb{N}\}\right) = \sup\{|\tau|(K) \mid K \text{ compact} \subset X\}$$

and define $\tilde{\tau}$ to be the restriction of τ to $\bigcup\{K_n \mid n \in \mathbb{N}\}$.

II. The general case. For the general situation we shall use the technique of antisymmetric functions. However, first one definition. Let $F = F(X)$ be a cone of bounded functions on X. F is said to have the *weak Dini-property* if for every sequence $(\alpha_n, f_n) \in \mathbb{R} \times F$, $n = 1, 2, \ldots$, such that $\alpha_n + f_n$ and $\alpha_n - f_n$ are both decreasing sequences we do have

$$\sup_X \left(\inf_{n \in \mathbb{N}} (\alpha_n + |f_n|)\right) = \inf_{n \in \mathbb{N}} \sup_X (\alpha_n + |f_n|).$$

Now, consider a set Z, a subset $Y \subset Z$ and an involutory (i.e. $j \circ j = \mathrm{id}|Z$) map $j: Z \to Z$ and assume that Y is j-generating. Here, j-generating means $Y \cup j(Y) = Z$. Furthermore we consider a cone $\Phi = \Phi(Z)$ of bounded real-valued antisymmetric functions on Z, where φ is called antisymmetric if $\varphi(z) = -\varphi(j(z))$ $\forall z \in Z$. Let $\hat{\varphi}$ be the restriction of φ to Y and $\hat{\Phi} = \{\hat{\varphi} \mid \varphi \in \Phi\}$. Then $\wedge: \Phi \to \hat{\Phi}$ is bijective because Y is j-generating and Φ consists only of antisymmetric functions.

Furthermore we obtain from the antisymmetry:

(**) $$\sup_Z \varphi = \sup_Y |\hat{\varphi}| \quad \forall \varphi \in \Phi.$$

This immediately implies:

Lemma 2. *$\Phi + \mathbb{R}$ is a Dini cone if and only if $\hat{\Phi}$ has the weak Dini property.*

Since $\wedge$ is bijective and because of (**) we get:

Lemma 3. *Let μ be a state of Φ then $\hat{\mu}: \hat{\Phi} \to \mathbb{R}$ defined by $\hat{\varphi} \to \mu(\varphi)$ is normed and linear. If $\hat{\nu}: \hat{\Phi} \to \mathbb{R}$ is normed and linear then ν defined by $\varphi \to \hat{\nu}(\hat{\varphi})$ is a state.*

The next lemma is a little bit more difficult. Let Σ be the σ-algebra in Z generated by Φ and $\hat{\Sigma}$ the σ-algebra in Y generated by $\hat{\Phi}$. One difficulty in the lemma stems from the fact that Y is not always Σ-measurable. Fortunately we have

$$\hat{\Sigma} = \{B \cap Y \mid B \in \Sigma\}.$$

Lemma 4. *For every signed $\hat{\Sigma}$-measure $\hat{\tau}$ with $|\hat{\tau}| \leqq 1$ there is a Σ-probability measure τ such that*

$$(***) \qquad \int_Z \varphi \, d\tau = \int_Y \hat{\varphi} \, d\hat{\tau} \quad \forall \varphi \in \Phi.$$

And for every Σ-probability measure τ there is a signed $\hat{\Sigma}$-measure $\hat{\tau}$ with $|\hat{\tau}| \leqq 1$ such that (***) holds.

Proof (first part of the lemma). Let $\hat{\tau}$ be a signed $\hat{\Sigma}$-measure with $0 < |\hat{\tau}| \leqq 1$. Consider the Jordan decomposition $\hat{\tau} = \hat{\tau}_+ - \hat{\tau}_-$ of $\hat{\tau}$ and define for $B \in \Sigma$:

$$\tau(B) = (1 + \varepsilon)\hat{\tau}_+(B \cap Y) + \varepsilon \hat{\tau}_-(B \cap Y) + \\ + (1 + \varepsilon)\hat{\tau}_-(j(B) \cap Y) + \varepsilon \hat{\tau}_+(j(B) \cap Y)$$

where $\varepsilon = (1/(2\hat{\tau}) - \frac{1}{2})$. Then τ is a positive Σ-measure with

$$\tau(Z) = (1 + 2\varepsilon)\hat{\tau}_+(Y) + (1 + 2\varepsilon)\hat{\tau}_-(Y) = |\hat{\tau}|/|\hat{\tau}| = 1,$$

and the antisymmetry of $\varphi \in \Phi$ implies (***).

(Second part of the lemma). Let τ be a Σ-probability measure and take $Y \supset B_n \in \Sigma$ such that

$$\tau(B_n) \geqq (1 - 1/n) \sup\{\tau(B) \mid Y \supset B \in \Sigma\}.$$

Now, consider $\Omega = \bigcup\{B_n \mid n \in \mathbb{N}\}$; then $\Omega \subset Y$ and

(i) $$\tau(\Omega) = \sup\{\tau(B) \mid Y \supset B \in \Sigma\}.$$

We define $\tau^*(B) \doteq \tau(B \cap \Omega) - \tau(j(B)\backslash\Omega)$ and we claim:

(ii) $$\tau^*(B) = \tau^*(\tilde{B}) \quad \text{for} \quad B, \tilde{B} \in \Sigma \quad \text{with} \quad B \cap Y = \tilde{B} \cap Y.$$

$\tau(B \cap \Omega) = \tau(\tilde{B} \cap \Omega)$ is trivial. $B \triangle \tilde{B} \subset Z\backslash Y$ gives $j(B) \triangle j(\tilde{B}) = j(B \triangle \tilde{B}) \subset Y$ because j is involutory and Y is j-generating. From this and (i) easily follows:

$$\tau(j(B)\backslash\Omega) = \tau(j(\tilde{B})\backslash\Omega).$$

Now, (ii) means that $\hat{\tau}$ defined by $\hat{\tau}(Y \cap B) \doteq \tau^*(B)$ is a signed measure with $|\hat{\tau}| \leqq 1$. For $\varphi \in \Phi$ we obtain finally:

$$\int_Y \hat{\varphi} \, d\hat{\tau} = \int_\Omega \varphi \, d\tau - \int_{Z/\Omega} \varphi \cdot j \, d\tau = \int_Z \varphi \, d\tau. \quad \blacksquare$$

Theorem 2. *Let $F = F(X)$ be a cone of bounded real-valued functions on X. Then for every normed linear $\mu: F \to \mathbb{R}$ there is a measure τ_μ (with respect to the smallest*

σ-algebra in X generated by F) of total variation $|\tau_\mu| \leqq 1$ such that

$$\mu(f) \leqq \int_X f \, d\tau_\mu \quad \forall f \in F$$

if and only if F has the weak Dini property.

Proof. Consider $Z \doteq (X \times \{1\}) \cup (X \times \{-1\})$, $Y = X \times \{1\}$ and $j: Z \to Z$ defined by $j(x \times 1) = x \times (-1)$, $j(x \times (-1)) = x \times 1$ for $x \in X$. For $f \in F$ let φ_f be the function

$$\varphi_f(x \times 1) = f(x), \quad \varphi_f(x \times (-1)) = -f(x) \quad \forall x \in X$$

and $\Phi = \Phi(Z) = \{\varphi_f \mid f \in F\}$. Obviously every normed linear μ on F has a signed representing measure iff there is such a measure for every normed linear μ on $\Phi(Y)$. This is equivalent (Lemma 3, 4) to the assertion that every state on $\Phi(Z)$ has a representing measure. That is equivalent (Main Theorem) to $\Phi(Z) + \mathbb{R}$ being a Dini cone and by Lemma 2 to $\Phi(Y)$ having the weak Dini property. Finally $\Phi(Y)$ has the weak Dini property if and only if $F(X)$ has this property. ∎

Corollary 2. *Let $\mathscr{F} = \mathscr{F}(X)$ be a cone of real-valued (not necessarily bounded) functions on X and v a (not necessarily positive) function $X \to \mathbb{R}$ such that vf is bounded for every $f \in F$. Then for every linear $\mu: \mathscr{F} \to \mathbb{R}$ with $\mu(f) \leqq \sup_X |vf|$ $\forall f \in \mathscr{F}$ there is a measure τ_μ with respect to the smallest σ-algebra in X generated by*

$$v\mathscr{F} \doteq \{vf \mid f \in \mathscr{F}\}$$

of total variation $|\tau_\mu| \leqq 1$ such that

$$\mu(f) \leqq \int_X f v \, d\tau_\mu \quad \forall f \in \mathscr{F}$$

if and only if $v\mathscr{F}$ has the weak Dini property.

Proof. We observe that every normed linear δ on $v\mathscr{F}$ defines a linear $\mu: \mathscr{F} \to \mathbb{R}$ with (*) $\mu(f) \leqq \sup_X |vf|$ $\forall f \in \mathscr{F}$ by $f \to \delta(vf)$ and for every μ with (*) there is a normed linear $\tilde{\delta}$ on $v\mathscr{F}$ such that $\mu(f) \leqq \delta(vf)$ $\forall f \in \mathscr{F}$ (consequence of the Sandwich Theorem [2, p. 2]). Now, everything follows from Theorem 2. ∎

At the end of this paper we like to mention some problems for which we do not know satisfactory answers.

Problem 1. Under what kind of additional assumption on F can we extend the signed representing measure given by Theorem 2 to a σ-algebra containing all compact subsets of the topological space X?

Problem 2. Under which additional condition on F is τ_μ for every μ unique?

Problem 3. Assume that F does not have the weak Dini property. Give a simple characterization (in terms of order properties for F) for those μ which have signed representing measures.

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