When does the Riesz Representation theorem hold?

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We consider the problem of characterizing those cones $F(X)$ of real functions on an arbitrary set X which have the property that every order-preserving linear map $F(X) \to \mathbb{R}$ has an integral representation.

In view of Choquet's theorem a reasonable guess for a necessary and sufficient condition would be that X has to contain the extreme points of the state space of $F(X)$. In fact this is true if $F(X)$ consists of upper-semicontinuous functions on a σ-compact space X (cf. [5]). However the following example due to Glicksberg [6] shows that in general it is not necessary that X contains all extreme points of the state space. Let X be a pseudocompact space and $F(X) = C(X)$ (continuous functions on X). Then the extreme points of the state space of $C(X)$ are the point evaluations on the Stone-Czech compactification of X, however every sup-norm continuous linear functional on $C(X)$ can be represented as an integral on X. (In a certain sense the reason for this counter-example is the following characterization: A completely-regular space X is pseudocompact if and only if every F_σ-subset $\supset X$ of its Stone-Czech compactification is compact.)

Our Main theorem (chapter IV) provides a simple characterization in terms of order properties of $F(X)$ (Dini-property) for the situation when all order-preserving linear functionals on $F(X)$ have integral representations. This theorem generalizes (cf. [5]) Choquet's theorem and the Riesz representation theorem. The basic idea of the proof is not complicated, it consists of exploiting decomposition properties for linear functionals. The main tool is the (countable) decomposition theorem of chapter II.

I. Remarks on Cones. Throughout this paper we consider cones F consisting of upper-bounded $[-\infty, +\infty[$-valued functions on some set X. f is *upper-bounded* if $\sup_X(f) \doteq \sup_{x \in X} f(x) < \infty$. Cones are always convex. Foı emphasizing on which set the functions are defined we write sometimes $F = F(X)$. The cone of the restrictions of all $f \in F$ to a subset $Y \subset X$ is then denoted by $F(Y)$. F is called an *order-unit cone* if it contains $\mathbb{R}$ (constant functions). F is said to be *max-stable* if with $f, g \in F$ the pointwise maximum $f \vee g$ is again in F. The max-stable cone generated by F is denoted by $\vee F$, it consists of the maxima of finite subsets of F.

A linear (i.e. positive-homogeneous and additive) map $\mu : F \to [-\infty, +\infty[$ is a *state* if it is dominated by $\sup_X$, that means $\mu \leqq \sup_X$ pointwise on F. A state μ is called *order-preserving* if $f \geqq g \Rightarrow \mu(f) \geqq \mu(g)$. μ is called *maximal* if it is not dominated by any other state. Every maximal state is order-preserving. This is a consequence of a suitable sandwich theorem for ordered cones (for example [3, Theorem 1]). Every state is dominated by a maximal state. This follows from Zorn's Lemma since the pointwise supremum on F of a maximal chain of states is a maximal state. In general a state μ is not order-preserving (but dominated by an order-preserving state). However $f \leqq 0$, $f \in F$ implies $\mu(f) \leqq 0$, therefore states on vector spaces are always order-preserving.

The set $S(F, X)$ of all states of F is called the *state-space*. Every $x \in X$ defines a state via the *Gelfand map* $x \to [f \to f(x)]$. Any $f \in F$ defines a function on $S(F, X)$ by $\mu \to \mu(f)$. This function we denote also by f since no confusion can arise.

We always consider in $S(F, X)$ the coarsest topology (F-topology) such that all elements of F are upper-semicontinuous. Let $Y \subset X$ and $S(F, Y) = \{\mu \in S(F, X) \mid \mu \leqq \sup_Y\}$ then $S(F, Y)$ is a compact subset of the state space because every ultrafilter converges. In particular is the state space itself compact. The same is true for $S_{\leqq}(F, X)$ the set of order-preserving states. We use compact in the sense that the space is not necessarily separated.

From Dini's lemma we get:

Lemma 1. *Let (f_n) be a decreasing sequence in F ($(f_n) \downarrow$ in short) then there is an order-preserving state μ such that:*

$$\inf_{n \in \mathbf{N}} \mu(f_n) = \inf_{n \in \mathbf{N}} \sup_X (f_n).$$

Sometimes it shall be useful to extend states.

Lemma 2. *Let $F = F(X)$ and $G = G(X)$ be order-unit cones such that $F \subset G$. Then every order-preserving state μ of F can be extended* (i.e. $\tilde{\mu}(f) = \mu(f)\ \forall f \in F$) *to an order-preserving state $\tilde{\mu}$ of G.*

Proof. We have to show [3, Cor. 1.3] that $f_1, f_2 \in F$ and $g \in G$ with $f_1 \leqq f_2 + g$ implies $\mu(f_1) \leqq \mu(f_2) + \sup_X(g)$. We abbreviate $\alpha = \sup_X(g)$ then $F \ni f_1 - \alpha \leqq f_2$ and $\mu(\alpha) = \alpha$ since $F \supset \mathbb{R}$ and μ is linear on $\mathbb{R}$. From the assumption that μ is order-preserving we obtain:

$$\mu(f_1 - \alpha) = \mu(f_1) - \alpha \leqq \mu(f_2).$$

Which is the desired inequality. ∎

In chapter V we shall investigate weighted cones. Let $\omega \geqq 0$ be a weight function on X then a cone $\mathcal{F}$ of functions (this time not necessarily upper-bounded) together with the sublinear functional P_ω defined by $f \to \sup_X(\omega f)$ is called a *weighted cone* if $P_\omega(f) < \infty\ \forall f \in \mathcal{F}$.

A linear $\mu : \mathcal{F} \to [-\infty, +\infty[$ with $\mu \leqq P_\omega$ is called ω-state of $\mathcal{F}$.

Lemma 3. *Let (F, P_w) be a weighted cone and μ be a ω-state. Then there is a state $\tilde{\mu}$ of $\omega\mathscr{F} \doteq \{\omega f \mid f \in \mathscr{F}\}$ such that $\mu(f) \leqq \tilde{\mu}(\omega f)\ \forall f \in \mathscr{F}$.*

Proof. Define a superlinear $\delta \leqq \sup_X$ on $\omega\mathscr{F}$ by $f \to \sup\{\mu(g) \mid \omega f = \omega g, g \in \mathscr{F}\}$. Then the sandwich theorem [3, Theorem 1] gives us a linear $\tilde{\mu}$ on $\omega\mathscr{F}$ with $\delta \leqq \tilde{\mu} \leqq \sup_X$. ▮

Now, let $F = F(X)$ be again a cone of upper-bounded functions and Y a *sup-boundary* of X. That is a subset $Y \subset X$ such that $\sup_Y(f) = \sup_X(f)\ \forall f \in F$. Then by considering the characteristic function of Y as weight function we get:

Corollary. 1. *For every state μ of $F = F(X)$ there is a state $\tilde{\mu}$ of $F(Y)$ such that $\mu(f) \leqq \tilde{\mu}(f|_Y)\ \forall f \in F$.*

A major tool in proving our main theorem arises from the study of decomposition properties. Therefore we define these properties in a rather general context. Let $F = F(X)$ be a cone and $\mathscr{M}$ be a family of subsets of X. Then a state μ is said to have the *countable decomposition property* (CD in short) with respect to $\mathscr{M}$ if for every countable covering $\{M_n \mid n \in \mathbf{N}\} \subset \mathscr{M}$ of X there are order-preserving states $\mu_n \leqq \sup_{M_n}$ and $\lambda_n \geqq 0\ (n = 1, 2, \ldots)$ such that $\sum_{n \in \mathbf{N}} \lambda_n = 1$ and $\mu = \sum_{n \in \mathbf{N}} \lambda_n \mu_n$.

The state space is said to have the *dominated countable decomposition property* (abbreviated by DCD) if for every family $\{Y_n \mid n \in \mathbf{N}\}$ of subsets of X with $\bigcup\{Y_n \mid n \in \mathbf{N}\} = X$ and every state μ there are $\lambda_n \geqq 0$ such that $\sum_{n \in \mathbf{N}} \lambda_n = 1$ and $\mu \leqq \sum_{n \in \mathbf{N}} \lambda_n \sup_{Y_n}$.

The cone F has the *semi-interpolation property* (in short SIP, cf. [3, p. 7]) if for $f, g, h_1, h_2 \in F$ with $f \leqq g + h_1$ and $f \leqq g + h_2$ there is always an $h_3 \in F$ such that $h_3 \leqq \min(h_1, h_2)$ and $f \leqq g + h_3$. In particular has a min-stable cone the SIP.

Lemma 4. *Let the state space $S(F, X)$ have the* DCD. *Then for every state μ and every covering $\{Y_n \mid n \in \mathbf{N}\} \subset \mathscr{P}(X)$ of X there are $\lambda_n \geqq 0$ with $\sum_{n \in \mathbf{N}} \lambda_n = 1$ and states $\mu_n \in S(F, Y_n)$ such that $\mu \leqq \sum_{n \in \mathbf{N}} \lambda_n \mu_n$. If F has the* SIP *and μ is order-preserving then we can choose the λ_n, μ_n such that we have equality: $\mu = \sum_{n \in \mathbf{N}} \lambda_n \mu_n$.*

Corollary 2. *Let $S(F, X)$ have the DCD then every maximal state has the CD with respect to $\mathscr{P}(X)$. If in addition F has the SIP then every order-preserving state has the CD with respect to $\mathscr{P}(X)$.*

Proof. We may confine ourself to proving the lemma for order-preserving states because they are dominating all states. Let $S(F, X)$ have the DCD and fix a covering $\{Y_n \mid n \in \mathbf{N}\} \subset \mathscr{P}(X)$ of X. It is sufficient to prove that for every order-preserving state μ we have $\lambda_n \geqq 0$, $\mu_n \in S(F, Y_n)$ and $\lambda \geqq 0$, $\mu^* \in S(F, X)$ such that $\mu \leqq (=$ if F has SIP) $\lambda\mu^* + \sum_{n \in \mathbf{N}} \lambda_n \mu_n$ with $\lambda \leqq 1/2 \leqq \sum_{n \in N} \lambda_n = 1 - \lambda$. Because then applying our statement to μ^* and so forth and adding up all those sums gives the desired decomposition. In [3, Theorem 3] it was shown that for order-preserving linear

$\nu \leqq p_1 + p_2$ (where p_1, p_2 are order-preserving and sublinear) we can find linear $\nu_1 \leqq p_1$, $\nu_2 \leqq p_2$ such that $\nu \leqq (=\text{if } F \text{ has SIP})\ \nu_1 + \nu_2$. By induction this decomposition property also holds for $\mu \leqq p_1 + p_2 + \cdots + p_{m+1}$. Using the DCD of the state space we obtain $\mu \leqq \sum_{n\in N} \lambda_n \sup_{Y_n}$. Now, we fix an $m \in \mathbb{N}$ such that $\sum_{n\leqq m} \lambda_n \geqq 1/2$ and define $p_k = \lambda_k \sup_{Y_k}$ for $k \leqq m$ and $p_{m+1} = \lambda \sup_X$ where $\lambda = \sum_{k>m} \lambda_k$. Then by our previous statement we find $\mu_n \in S(F, Y_n)$ and $\mu^* \in S(F, X)$ such that $\mu \leqq (=\text{if } F \text{ has SIP})\ \lambda\mu^* + \sum_{n\in\mathbb{N}} \lambda_n \mu_n$. ▮

II. Dini Cones. A cone of functions $F = F(X)$ is said to have the *Dini property* if for every pointwise decreasing sequence $(f_n)\downarrow$ in F we have:

$$\sup_X \left(\inf_{n\in\mathbb{N}} f_n\right) = \inf_{n\in\mathbb{N}} (\sup_X f_n).$$

For max-stable order-unit cones this is equivalent to saying that every decreasing sequence pointwise converging to 0 converges uniformly to 0.

An order-unit cone with Dini property is called a *Dini cone*. A simple example for a Dini cone are the upper-semicontinuous functions on a compact space (Dini's Lemma).

Another example is $\mathbb{R} + UC_\infty^+(X)$ (cf. Lemma 6) where X is a topological space and $UC_\infty^+(X)$ are the nonnegative upper-semicontinuous functions on X vanishing at infinity. Here, we say that $f \geqq 0$ is vanishing at infinity if for every $\varepsilon > 0$ $\{x \in X \mid f(x) \geqq \varepsilon\}$ is compact. More examples of Dini-cones can be found in [5].

Lemma 5. *An order-unit cone F is a Dini cone if and only if $\vee F$ is a Dini cone.*

Proof. Obviously the Dini property for $\vee F$ implies the same for F. For the other implication we consider a sequence $(g_n)\downarrow$ in $\vee F$ and define $\beta \doteq \sup_X \left(\inf_{n\in\mathbb{N}} g_n\right)$, $\alpha = \inf_{n\in\mathbb{N}}(\sup_X(g_n))$. Since $\alpha \geqq \beta$ it remains to prove $\alpha \leqq \beta$ when $\alpha > -\infty$. Without loss of generality we may assume $\sup_X(g_n) \leqq \alpha + 1/n$ (otherwise we go over to a subsequence). Every g_n is of the form: $g_n = f_n^1 \vee f_n^2 \vee \cdots \vee f_n^{k_n}$ with $f_n^1, \ldots, f_n^{k_n} \in F$. Now, take an ultrafilter Φ of X containing the following decreasing sequence of nonempty sets: $X_n = \{x \in X \mid g_n(x) \geqq \alpha - 1/n\}$. As a consequence of the maximality of Φ we may find for every n a number $\varrho_n \leqq k_n$ such that

$$Y_n = \{x \in X \mid f_n^{\varrho_n}(x) \geqq \alpha - 1/n\} \in \Phi.$$

Let $h_m = \sum_{n\leqq m} 1/n\,(f_n^{\varrho_n} - \alpha - 1/n)$ then $h_m \in F$ and $(h_m)\downarrow$ because $f_n^{\varrho_n} \leqq g_n \leqq \alpha + 1/n$. Choose $y_m \in \bigcap\{Y_i \mid i \leqq m\}$ (which is nonempty) then the Dini property for F gives us an $x_0 \in X$ such that:

$$\sum_{n\in\mathbb{N}} 1/n\,(f_n^{\varrho_n}(x_0) - \alpha - 1/n) \geqq \inf_{m\in\mathbb{N}} \sup_X h_m - 1 \geqq$$

$$\geqq \inf_{m\in\mathbb{N}}(h_m(y_m)) - 1 \geqq -\left\{\sum_{n\in\mathbb{N}} \frac{2}{n^2}\right\} - 1 \geqq -\frac{2\pi^2}{6} - 1 > -\infty.$$

Combining this inequality with $\inf_{n\in\mathbf{N}} f_n^{\varrho_n}(x_0) \leqq \inf_{n\in\mathbf{N}} g_n(x_0) \leqq \beta$ we obtain

$$\sum_{n\in\mathbf{N}} 1/n(\beta+\varepsilon-\alpha) > -\infty \quad \forall \varepsilon > 0.$$

Since $\sum_{n\in\mathbf{N}} 1/n$ diverges we get finally $\beta \geqq \alpha$. ▌

Lemma 6. *Let $F = F(X)$ be a Dini cone consisting of upper-semicontinuous functions on the topological space X. Then $F + U C_\infty^+(X)$ is a Dini cone.*

Proof. By virtue of lemma 5 we can assume that F is max-stable. Let $(h_n)\downarrow$ with $h_n = f_n + \varphi_n$, $f_n \in F$, $\varphi_n \in U C_\infty^+(X)$ and (*) $\inf_{n\in\mathbf{N}} \sup_X(h_n) \doteqdot \beta > \alpha > \sup_X \inf_{n\in\mathbf{N}}(h_n)$. Assume that we have found integers $1 = k_1 < k_2 < \cdots < k_m$ such that the functions $g_n \doteqdot (f_{k_n} \vee \alpha) + 1/n$ are decreasing with n for $n \leqq m$.

Obviously there must be a $k_{m+1} > k_m$ such that

$$h_{k_{m+1}}(y) \leqq \alpha \quad \forall y \in Y \doteqdot \{x \in X \mid \varphi_{k_m}(x) \geqq 1/m - 1/(m+1)\}.$$

Otherwise Dini's Lemma would provide us with a contradiction to (*) by considering the restrictions of h_n to the compact set Y. This inequality together with $f_{k_{m+1}} \leqq h_{k_{m+1}} \leqq \leqq h_{k_m}$ implies $f_{k_{m+1}} + 1/(m+1) \leqq (f_{k_m} \vee \alpha) + 1/m$. Hence, by induction we obtain a decreasing sequence $g_n = (f_{k_n} \vee \alpha) + 1/n$ in F. It remains to show $\sup_X(g_n) \geqq \beta \, \forall n \in \mathbf{N}$.

Because this means that $(g_n)\downarrow$ is in contradiction to the Dini property of F a sequence in F with

$$\inf_{n\in\mathbf{N}} \sup_X(g_n) \geqq \beta > \alpha = \sup_X\left(\inf_{n\in\mathbf{N}}(g_n)\right).$$

Assume therefore $\sup_X(g_{n_0}) = \delta < \beta$ for some n_0. Then the left side of (*) implies that all $h_m\,(m \geqq k_{n_0})$ attain their sup on the compact set $\{x \in X \mid \varphi_{n_0}(x) \geqq \beta - \delta\}$ and (*) contradicts Dini's Lemma. ▌

Lemma 6 suggests to conjecture that the sum of two Dini cones is always a Dini cone. But this is easily disproved by a simple counterexample.

An important characterization is the following theorem, Implicitly it is already contained in [5].

Decomposition Theorem. *Let $F = F(X)$ be an order-unit cone. F is a Dini cone if and only if its state space $S(F, X)$ has the dominated countable decomposition property.*

Corollary 3. *Let $F = F(X)$ be a Dini cone then every maximal state has the countable decomposition property with respect to $\mathscr{P}(X)$. If in addition F has the SIP then every order-preserving state has the countable decomposition property with respect to $\mathscr{P}(X)$.*

Proof. First, let $(g_n)\downarrow$ be in F with $\sup_X\left(\inf_{n\in\mathbf{N}} g_n\right) = \beta < \alpha = \inf_{n\in\mathbf{N}} \sup_X(g_n)$ and $\{Y_n \mid n \in \mathbf{N}\}$ be the covering of X defined by $Y_n \doteqdot \left\{x \in X \mid g_n(x) \leqq \frac{\alpha+\beta}{2}\right\}$. Then

Lemma 1 provides a state μ such that $\inf_{n\in\mathbb{N}} \mu(g_n) = \alpha > \frac{\alpha+\beta}{2}$. Clearly, μ is not dominated by a countable convex-combination of $\{\sup_{Y_n} \mid n\in\mathbb{N}\}$ and $S(F, X)$ does not have the DCD. Now, let F be a Dini cone and consider a covering $\{Z_n \mid n\in\mathbb{N}\}$ of X. Define a σ-compact subset of the state space by $Z = \bigcup\{\bar{Z}_n \mid n\in\mathbb{N}\}$ where $\bar{Z}_n = S(F, Z_n)$. Then $F(Z)$ is a Dini cone since Z contains the Gelfand image of X and a state μ of F can be considered as a state of $F(Z)$.

Now, we proceed exactly as in the proof of [5, Satz 1]. We show that Z contains all extreme points of $S(F, X)$, then the required decomposition follows from [4, theorem 3]. Assume therefore that there is an extreme point μ_0 of $S(F, X)$ not contained in Z. By the extreme point criterion of [5, p. 187] there are $f_k\in F$, $k = 1, 2, \ldots$, such that $f_k \leqq 0$, $\mu_0(f_k) \geqq -1/k^2$ and $-3 \geqq \max\{\nu(f_k) \mid \nu\in S(F, Z_k)\}$. Then $h_m = \sum_{k\leqq m} f_k$ is a decreasing sequence in F with

$$\inf_m \sup_X (h_m) \geqq \inf_m \mu_0(h_m) \geqq -\frac{\pi^2}{6} > -3 \geqq \sup_X \inf_m (h_m).$$

Hence h_m is in contradiction to the Dini property of F. The corollary is a consequence of Cor. 2. ▮

A different approach to this theorem was recently given by M. Neumann [7].

III. Remarks on measures. Let $F = F(X)$ be a cone of $[-\infty, +\infty[$-valued functions. Throughout the following chapters Σ_F denotes the smallest σ-algebra in X such that all $f\in F$ are Σ_F-measurable. In particular all $g\in \bigvee F$ and all pointwise limits of sequences in $\bigvee F$ are Σ_F-measurable. A positive Σ_F-measure τ with $\tau(X) = 1$ is called a Σ_F-probability measure.

Theorem 1. *Let $E = E(X)$ be an order-unit vector lattice (i.e. a max-stable vector space of bounded functions on X containing the constants) and let $\mu\in S(E, X)$ be a state of E. Then there is a Σ_E-probability measure τ with*

$$\mu(f) = \int_X f\, d\tau \quad \forall f\in E$$

if and only if μ has the countable decomposition property with respect to Σ_E.

Proof. The only if part is trivial, it is proved by considering the restrictions of τ to the elements of Σ_E. The other direction is an application of the Daniell-Stone theorem [2, p. 160]. It is sufficient to show $0 = \inf_n \mu(f_n)$ for every decreasing sequence f_n in E with $\inf_n (f_n) = 0$. For this we consider the covering $X_n = \{x\in X \mid f_n(x) \leqq \delta\}$ where $\delta > 0$. By the decomposition property there are states $\mu_n\in S(E, X_n)$ and $\lambda_n \geqq 0$ with $\sum_{n\in\mathbf{N}} \lambda_n = 1$ such that $\mu = \sum_{n\in\mathbf{N}} \lambda_n\mu_n$. This implies $\delta \geqq \inf_n \mu(f_n)$ and $\inf_n \mu(f_n) = 0$ because δ was arbitrary. ▮

IV. Representing measures. Let $F = F(X)$ be a cone and μ be a state of F, then a probability measure τ with respect to the smallest σ-algebra Σ_F such that F consists of measurable functions is called a *representing measure* (on X) for μ if

(1) $$\mu(f) \leqq \int_X f\,d\tau \quad \forall f \in F.$$

If we have equality in (1) for all *bounded* $f \in F$ then τ is called a *strict representing measure*.

Main Theorem. *Let $F = F(X)$ be an order-unit cone. Then every state of F has a representing measure on X if and only if F is a Dini cone. If in addition F has the SIP and is max-stable then every order-preserving state does have a strict representing measure.*

Proof. First, let every state μ of F have a representing measure τ_μ. And consider $(f_n) \downarrow$ in F. Lemma 1 provides a state μ such that

$$\inf_{n\in\mathbb{N}} \sup_X (f_n) = \inf_{n\in\mathbb{N}} \mu(f_n) \leqq \inf_{n\in\mathbb{N}} \left(\int_X f_n\,d\tau_\mu \right).$$

By virtue of Lebesgue's dominated convergence theorem is the right hand side $\leqq \sup_X \left(\inf_{n\in\mathbb{N}} f_n \right)$ and F is a Dini cone.

Now, let F be a Dini cone then $\Phi \doteqdot \{f \in \vee F \mid f \text{ bounded}\}$ is also a Dini cone (lemma 5). Every maximal state μ^* of Φ has the CD with respect to $\mathscr{P}(X)$ (decomposition theorem + Cor. 2). If F is max-stable (i.e. $F = \vee F$) and has the SIP then Φ is max-stable and does have the SIP since $\Phi \subset F$. So, under this assumption does every order-preserving state of Φ have the CD (Cor. 2). Every order-preserving state μ^* of Φ can be extended to the vector lattice $E \doteqdot \Phi + (-\Phi)$ (lemma 2) and this extension must be *unique* since E is linearily generated by Φ. Now, let $\mu^* = \sum_{n\in\mathbb{N}} \lambda_n \mu_n$ with $\mu_n \leqq \sup_{M_n}$ where $\{M_n \mid n \in \mathbb{N}\} \subset \mathscr{P}(X)$ then we can extend (lemma 2) every μ_n to a $\tilde{\mu}_n \leqq \sup_{M_n}$ on E. Then $\sum_{n\in\mathbb{N}} \lambda_n \tilde{\mu}_n$ is an extension of μ^* and therefore the unique extension. This implies that the extension of μ^* has the CD if and only if μ^* has the CD. Hence, theorem 1 provides strict representing measures τ_{μ^*} for every maximal state of Φ (for every oder-preserving state if F has the SIP and is max-stable). We know (lemma 2) that every order-preserving state μ of F can be extended to an order-preserving state $\tilde{\mu}$ of $\vee F$ and that $\tilde{\mu}$ is dominated by a maximal state ν of $\vee F$. Obviously is its restriction $\mu^* \doteqdot \nu|_\Phi$ maximal on Φ otherwise would $\vee F \in f \to \inf\{\tilde{\nu}(\alpha \vee f) \mid \alpha \in \mathbb{R}\}$ be a state $\geqq \nu$ for any order-preserving state $\tilde{\nu} \geqq \mu^*$. Finally the observation that τ_{μ^*} is a representing measure for μ (and that $\tau_{\tilde{\mu}|\Phi}$ is a strict representing measure for μ if F has the SIP and is max-stable) finishes the proof. ▮

Corollary 4. *Let $F = F(X)$ be a Dini cone consisting of upper-semicontinuous functions on the topological space X. Then every state μ of F has a representing measure τ_μ which can be extended to a σ-algebra containing all closed compact subsets of X.*

Proof. We replace F by the Dini cone $G \doteqdot F + UC_\infty^{\pm}(X)$ (Lemma 6) and apply the main theorem. Now, Σ_G contains all closed compact sets since $UC_\infty^{\pm}(X)$ contains the characteristic functions of those sets. ▮

Corollary 5. *Let F be a max-stable Dini cone with SIP of bounded functions on X. Then $F-F$ is a Dini cone, in particular is $-F$ a Dini cone.*

Proof. Consider any state μ of $F-F$. The main theorem provides a strict representing measure τ for $\mu|_F$. Obviously integration with respect to τ must be μ and τ is a representing measure for μ. Since μ was arbitrary $F-F$ must be a Dini cone by virtue of the main theorem. ▮

Corollary 6. *Let F be a max-stable Dini cone consisting of bounded continuous functions on the topological space X. If F has the SIP (in particular if F is min-stable) then every order-preserving state has a strict representing measure which can be extended to a σ-algebra containing all closed compact subsets of X.*

Proof. $F+(-F)$ and $\Phi \doteqdot F+(-F)+UC_\infty^+(X)$ are Dini cones (Cor. 5 and lemma 6). Every order-preserving state μ of F can be extended to an order-preserving state μ^* of Φ (lemma 2). μ^* has a representing measure (defined on Σ_Φ) which must be strict on the subspace $F+(-F)$. ▮

V. Weighted cones. In this chapter we shall reformulate the main theorem for weighted cones. Let $\omega \geqq 0$ be a weight function on X and $(\mathscr{F}, P_w)$ a weighted cone. That means $\mathscr{F} = \mathscr{F}(X)$ is a cone such that

$$P_w(f) = \sup_{x\in X} \omega(x) f(x) < \infty \quad \forall f \in \mathscr{F}.$$

A linear $\mu: \mathscr{F} \to [-\infty, +\infty[$ is a ω-state if $\mu \leqq P_w$ and a positive measure τ on the σ-algebra Σ_{wF} (where $\omega\mathscr{F} \doteqdot \{\omega f \mid f \in \mathscr{F}\}$) is called a ω-representing measure for μ if

$$\mu(f) \leqq \int_X f\omega\, d\tau \quad \forall f \in \mathscr{F}.$$

Theorem 2. *Every ω-state of $\mathscr{F}$ has a representing measure if and only if $\omega\mathscr{F}+\mathbb{R}$ is a Dini cone.*

Proof. $\mathscr{F} \ni f \to \mu(\omega f)$ is a ω-state if μ is a state of $\omega\mathscr{F}+\mathbb{R}$ and for every ω-state $\tilde{\mu}$ there is a state μ of $\omega\mathscr{F}+\mathbb{R}$ such that $\tilde{\mu}(f) \leqq \mu(\omega f)$ $\forall f \in \mathscr{F}$ (lemma 3 + sandwich theorem). Hence, every ω-state having a ω-representing measure is equivalent to every state of $\omega\mathscr{F}+\mathbb{R}$ having a representing measure which is equivalent to $\omega\mathscr{F}+\mathbb{R}$ being a Dini cone. ▮

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