# COMPARISON OF THE TWO-SOLITON COLLISION FOR SEVERAL NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT. As an application of the method of hereditary symmetries we show that the two-soliton solutions of several nonlinear partial differential equations are intimately related.

## INTRODUCTION

We consider the following equations for $u = u(x, t)$:

| | | |
|---|---|---|
| (1) | $u_{xt} = \frac{1}{2}\sin(2u)$ | sine-Gordon equation, |
| (2) | $u_t = u_{xxx} + 6\,u_x u$ | Korteweg-de Vries equation (KdV), |
| (3) | $u_t = u_{xxx} + 6\,u_x u^2$ | modified KdV. |

Then we have the following explicit formulas between the two-soliton solutions:

THEOREM 1. *Let v be a function in x and t, then the following are equivalent:*

(i) $u(x,t) = \int_{-\infty}^{x} v(\xi, t)\,\mathrm{d}\xi$ *is a two-soliton solution of the sine-Gordon equation with asymptotic speeds* $1/c_1$ *and* $1/c_2$.

(ii) $u(x,t) = v(x - (c_1 + c_2)t, -c_1c_2t)$ *is a two-soliton solution of the modified KdV with asymptotic speeds* $c_1$ *and* $c_2$.

(iii) $u(x,t) = v(x - (c_1 + c_2)t, -c_1c_2t)^2 + iv_x(x - (c_1 + c_2)t, -c_1c_2t)$ *is a complex two-soliton solution of the KdV with asymptotic speeds* $c_1, c_2$.

At first sight this appears as a remarkable coincidence. But, of course, there is a deeper reason behind it, and it is one of the purposes of this paper to point out that reason.

## SOLITONS AND COMPLEX SOLITONS

We deal in this paper with solutions in $S_-$, the space of $C_1^\infty$-functions in $x$, vanishing rapidly at $-\infty$.

It is well known that eqns. (1) – (3) admit traveling wave solutions of the form $u(x, t) =$



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$s_c(x - ct)$. These solutions are easily found by replacing $u_t$ by $-cu_x$, thus obtaining an ordinary differential equation. This equation then yields:

$$s_c(x) = c^{1/2} \{\cosh(c^{1/2}(x - x_0)\}^{-1}$$

for the modified KdV, and

$$s_c(x) = c^{-1/2} \{\cosh(c^{-1/2}(x - x_0))\}^{-1}$$

for the sine-Gordon equation.

But we like to remark that, in general, these are not the only traveling wave solutions, even if we require that the solutions vanish rapidly at $|x| = \infty$. Let us demonstrate this for the KdV. Insertion of $u_t = -c\, s_x$, $u_x = s_x$ in (2) gives:

$$s_{xxx} + 6\, ss_x = c\, s_x \ .$$

Integration (from $-\infty$ to $x$), multiplication with $2\, s_x$ and a second integration yield:

$$s_x^2 + 2\, s^3 = c\, s^2 \ .$$

One solution is

$$s_c(x) = c\{\cosh(c^{1/2}(x - x_0))\}^{-2} .$$

But one easily verifies that a second (complex) solution is given by:

(4) $$s_c(x) = c\{\cosh(c^{1/2}(x - x_0))\}^{-2} \{1 - i \sinh(c^{1/2}(x - x_0))\} .$$

These complex solutions are usually overlooked, because complex quantities are not relevant in the physical phenomena described by the KdV (shallow water waves, etc.). But Theorem 1 shows that complex solitons are important for a general understanding of the structure of soliton equations. They behave quite similar to real solitons, in particular it turns out that complex initial conditions yield solutions which decompose for large $t$ into different traveling wave solutions.

We recall that $u(x, t)$ constitutes an $N$-soliton solution if $u(x, t)$ decomposes asymptotically (i.e., $|t| \to \infty$) into $N$ traveling wave solutions with speeds $c_1, \ldots, c_N$ (called asymptotic speeds), such that the total energy is completely carried by these traveling waves [1]. To be more precise, we require that $u(x, t)$ is of the form

$$u(x,t) = \sum_{n=1}^{N} s_{c_n}(x - c_n t - \theta_n) + \Delta(x,t) ,$$

where the error term $\Delta(., t)$ converges for $|t| \to \infty$ to zero in an appropriate sense (usually $L^2$-norm which then, due to the special structure of the evolution equation, implies convergence in any other reasonable sense). A complete description of these $N$-soliton solutions is easily furnished by the theory of hereditary symmetries [2]. Explicit formulas for these solutions are found in [5], for the KdV, and in [3], for the sine-Gordon equation.

## HEREDITARY SYMMETRIES

Let $\Phi(w)$ be a differentiable function in $w \in S_-$ attaining values in the space of linear operators on $S_-$. We assume that $\Phi$ is invariant with respect to translation of the $x$-axis. Such an operator-valued function is then called a *hereditary symmetry* if $[\Phi'(w), \Phi(w)]$ is for all $w \in S_-$ a symmetric bilinear operator on $S_- \times S_-$. Here, the commutator is an abbreviation for the following bilinear operator:

$$[\Phi'(w), \Phi(w)]\,(\sigma, \nu) = \frac{\partial}{\partial\epsilon}\left\{\Phi(w + \epsilon\Phi(w)\sigma) - \Phi(w + \epsilon\sigma)\right\}_{\epsilon = 0} \nu,$$

where $\sigma$ and $\nu$ are arbitrary elements of $S_-$.

Now, let $\Phi$ be hereditary and consider any of the following evolution equations

$$(5.n) \qquad u_t = (\Phi(u))^n u_x \doteqdot K_n(u)\,, \qquad n \in \mathbb{Z}\,.$$

Then I showed [2] that the operator function $\Phi(u)$ completely describes the symmetries, the conservation laws and the soliton solutions for the sequence (5). The evolution equations (5) are usually called the *generalizations* of

$$u_t = \Phi(u)\, u_x\,.$$

Let us concentrate on the soliton solutions. An important property of $\Phi$ is, that for any solution $u(t)$ of $(5.n)$ the eigenvalues of $\Phi(u(t))$ are time-independent. Moreover, $(5.n)$ has a particular structural stability which says, that whenever we have for a fixed $t_0$ that

$$u_x(t_0) = \sum_{k=1}^{N} w_k(t_0),$$

then for all $t$ it holds that

$$(6) \qquad u_x(t) = \sum_{k=1}^{N} w_k(t)\,,$$

where the $w_k$

$$\Phi(u(t)\, w_k(t) = \lambda_k w_k(t), \qquad \lambda_k \text{ time-independent} \tag{7}$$

are eigenvectors of $\Phi(u)$ and, in addition, solutions of the linear perturbation equation

$$w_t = \frac{\partial}{\partial \epsilon} \{K_n(u + \epsilon w)\}_{\epsilon = 0}. \tag{8}$$

Usually (for suitable $\Phi$) eqns. (6) and (7) constitute a system of ordinary differential equations. And an asymptotic study of the eigenvectors $w_k$ shows that (6) is the *soliton decomposition.* The $w_k$ are asymptotically equal to the derivatives of the traveling wave solutions. Their asymptotic speeds $c_1, \ldots, c_N$ are easily determined. Because of (5.$n$) we have

$$u_t = \Phi(u)^n u_x = \sum_{k=1}^{N} \Phi(u)^n w_k = \sum_{k=1}^{N} \lambda_k^n w_k . \tag{9}$$

Hence, we get

$$c_k = -\lambda_k^n . \tag{10}$$

## TIME-HISTORY OF THE TWO-SOLITON SOLUTIONS

It is remarkable that for the whole class of equations given by (5) the $N$-soliton solutions are described by the same system of *ordinary* differential equations, namely by (6) and (7). Of course, the time evolution of these solutions may be rather different (for $N = 1$ these evolutions are given by (10) !) In general – due to the boundary condition at infinity – the system (6)–(7) admits $N$ different integration parameters, and the time evolution of the solution is given by assigning to every $t$ values for these integration parameters. In the case of two interacting solitons this story is extremely simple since one easily obtains explicit formulas between the corresponding solutions:

THEOREM 2. *Let $v$ be a function in $x$ and $t$, then the following are equivalent:*

(i) $u(x,t) \doteqdot v(x,t)$ *is a two-soliton solution of* (5.$n$) *with asymptotic speeds* $-\lambda_1^n$ *and* $-\lambda_2^n$ .

(ii) $\tilde{u}(x,t) \doteqdot v(x - \alpha t, \beta t)$ *is a two-soliton solution of* (5.$m$) with asymptotic speeds $-\lambda_1^m$ *and* $-\lambda_2^m$, where

$$\alpha = \frac{\lambda_2^n \lambda_1^m - \lambda_1^n \lambda_2^m}{\lambda_1^n - \lambda_2^n} , \qquad \beta = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1^n - \lambda_2^n} . \tag{11}$$

*Proof.* Let us consider two-soliton solutions $u, \tilde{u}$ of (5.$n$) and (5.$m$) with asymptotic speeds $-\lambda_1^n, -\lambda_2^n$ and $-\lambda_1^m, -\lambda_2^m$ respectively. We know then, that these solutions are of the form:

(12) $u_x = w_1 + w_2$, $\tilde{u}_x = \tilde{w}_1 + \tilde{w}_2$

where

(13) $\Phi(u)\, w_i = \lambda_i w_i$, $\Phi(\tilde{u})\tilde{w}_i = \lambda_i \tilde{w}_i$; $i = 1, 2$.

This system has a two-parameter family as solutions (in the solution space under consideration!) since the two-soliton solutions do form a two-parameter family. But, due to translation invariance, $u(x - x_0, t)$ as well as $\tilde{u}(x - x_0, t)$ are constituting such two-parameter families of solutions for (12)–(13) (the parameters are $x_0, t$). Hence we expect:

(14) $\tilde{u}(x, t) = u(x - a(t), b(t))$.

It remains to determine the functions $a(t)$ and $b(t)$. Put for the moment $\tilde{x} = x - a(t)$, $\tilde{t} = b(t)$ and take the time derivative of (14):

$$\tilde{u}_t(x, t) = \dot{b}(t) u_{\tilde{t}}(\tilde{x}, \tilde{t}) - \dot{a}(t) u_{\tilde{x}}(\tilde{x}, \tilde{t}) .$$

Expressing the time-derivatives by $(5.n)$ and $(5.m)$ this yields:

$$\Phi^m(\tilde{u}(x, t))\, \tilde{u}_x(x, t) = \dot{b}(t)\, \Phi^n(u(\tilde{x}, \tilde{t}))\, u_{\tilde{x}}(\tilde{x}, \tilde{t}) - \dot{a}(t)\, u_{\tilde{x}}(\tilde{x}, \tilde{t}) .$$

Now, express $\tilde{u}$ (with the help of (14)) in terms of $u$:

$$\{\Phi^m(u(\tilde{x}, \tilde{t})) - \dot{b}(t)\, \Phi^n(u(\tilde{x}, \tilde{t}))\}\, u_{\tilde{x}}(\tilde{x}, \tilde{t}) + \dot{a}(t)\, u_{\tilde{x}}(\tilde{x}, \tilde{t}) = 0.$$

Introducing the eigenvector decomposition given by (13) we obtain:

$$(\lambda_1^m - \lambda_1^n \dot{b}(t) + \dot{a}(t))\, w_1 + (\lambda_2^m - \lambda_2^n \dot{b}(t) + \dot{a}(t))\, w_2 = 0 .$$

Hence

$$\lambda_1^m - \lambda_1^n \dot{b}(t) + \dot{a}(t) = \lambda_2^m - \lambda_2^n \dot{b}(t) + \dot{a}(t) = 0 .$$

So, $\dot{a}$ and $\dot{b}$ must be the following time-dependent constants:

$$\beta = \dot{b}(t) = \frac{\lambda_1^m - \lambda_2^m}{\lambda_1^n - \lambda_2^n}, \quad \alpha = \dot{a}(t) = \frac{\lambda_2^n \lambda_1^m - \lambda_2^m \lambda_1^n}{\lambda_1^n - \lambda_2^n} .$$

So we have proved (i) $\rightarrow$ (ii) and the other implication follows by interchanging the role of $u$ and $\tilde{u}$. □

The reader should observe that this proof used ideas first expressed in [4].

We denote by $D$ the operator of $x$-differentiation and by $D^{-1}$ the operator $w(x) \to \int_{-\infty}^{x} w(\xi)\, d\xi$. In [2] I proved that

$$\Phi_1(u) = D^2 + 4u + 2u_x D^{-1} \quad , \tag{15}$$

and

$$\Phi_2(u) = D^2 + 4u_x D^{-1} u + 4u^2 \, , \tag{16}$$

are hereditary. Let us first study the second operator. In particular, we are interested in the following evolution equations

$$u_t = \Phi_2(u) u_x \quad , \qquad \widetilde{u}_t = \Phi_2(\widetilde{u})^{-1}\, \widetilde{u}_x \, .$$

Evaluation yields that the explicit form of these equations is:

$$u_t = u_{xxx} + 6u_x u^2 \tag{17}$$

$$\widetilde{u}_t = \tfrac{1}{2} \sin\left( 2 \int_{-\infty}^{x} \widetilde{u}(\xi)\, d\xi \right). \tag{18}$$

Equation (17) is equal to the modified KdV and the substitution $\widetilde{u} = u_x$ makes out of (18) the sine-Gordon equation. So Theorem 2 applied to this special case gives (i) $\longleftrightarrow$ (ii) of Theorem 1.

Now, the reader can easily check the following operator identity for arbitrary $v \in S_{-}$:

$$\Phi_1(v^2 + iv_x)\,(2v + iD) = (2v + iD)\, \Phi_2(v). \tag{19}$$

This has as immediate

CONSEQUENCE 3. *$v(x, t)$ is a solution of $v_t = \Phi_2(v)\, v_x$ if and only if $u(x, t) = v(x, t)^2 + iv_x(x, t)$ is a solution of $u_t = \Phi_1(u)\, u_x$.*

This, of course, provides the missing link between the complex two-solitons of the KdV and the two-solitons of the modified KdV ( (ii) $\longleftrightarrow$ (iii) of Theorem 1).

Let us give another example. Consider

$$u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t - u_{xx} = 0 \, . \tag{20}$$

Then (20) can be written as $\Phi_1(u_x) u_{xt} = u_{xx}$ or

$$\widetilde{u}_t = \Phi_1(\widetilde{u})^{-1}\, \widetilde{u}_x \, , \qquad \widetilde{u} = u_x \, . \tag{21}$$

Hence, an immediate application of Theorem 2 yields:

OBSERVATION 4. *The following are equivalent:*

(i) $u(x, t)$ *is a two-soliton solution of the KdV with asymptotic speeds* $c_1, c_2$.

(ii) $v(x, t) \int_{-\infty}^{x} u(\xi - \alpha t, \beta t)\, d\xi$ *is a two-soliton solution of* (20). *(Inverse KdV) with asymptotic speeds* $1/c_1$, $1/c_2$, *where:*

$$\alpha = -\frac{c_1 + c_2}{c_2 c_1} \quad \text{and} \quad \beta = -\frac{1}{c_2 c_1}\,.$$

## FINAL REMARKS

(1) One should observe that Theorem 2 relates the two-soliton solutions of every soliton equation and its generalizations, like the KdV and its generalizations, and the Sakharov–Shabat equation and its generalizations.

(2) One might ask if the method applied in Theorem 2 also works for $N$-soliton solutions. In principle, yes, since we know for the eqns. ($5.n$) an infinite series of symmetries. But everything becomes more complicated because these symmetries are not linear in $u$. In fact, the only linear ones are $u_x$ and $u_t$, and these were used in the proof.

(3) I am quite sure that the same method works for those solutions which are not required to vanish at $-\infty$ (Novikov's theta function solutions). Although I did not carry out the details for this case.

(4) In the first moment the identity (19) looks surprising, but in fact it is not. It is well known [5] (Gardner transformation) that there is a Bäcklund transformation between the soliton-free KdV and the KdV. Making a complex variable substitution one obtains from there a Bäcklund transformation between the KdV and its modified version. And this complex Bäcklund transformation immediately yields (19).

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