GAME THEORY AND MATHEMATICAL ECONOMICS
O. Moeschlin, D. Pallaschke (editors)

DISINTEGRATION METHODS IN MATHEMATICAL ECONOMICS

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It is shown that problems in Mathematical Economics which are connected with submeasures can be treated by means of suitable disintegration methods. A flow theorem for nonfinite networks and a production-distribution theorem are given to illustrate these disintegration techniques.

INTRODUCTION

Looking at some problems in mathematical economics from a general point of view one easily discovers that they are intimately connected with the notion of submeasure. There is some reason to expect that in the future submeasures will play an important role in a unified theory for several areas of mathematical economics. As a modest contribution in the direction of such a development we show in this paper that disintegration with respect to submeasures is an essential tool for handling infinite networks as well as infinite supply-demand problems.

A *submeasure* $p : \Sigma \to \mathbb{R}_+$ is a map from a σ- algebra Σ (on some set X) into the nonnegative real numbers which has the following properties:

(1) $p(A \cup B) \leq p(A) + p(B)$ for all $A, B \in \Sigma$

(2) if $A_n \in \Sigma$ is a decreasing sequence, i.e. $A_{n+1} \subset A_n$ for all n, then $\inf_{n \in \mathbb{N}} p(A_n) = p(\bigcap_{n \in \mathbb{N}} A_n)$.

In other words, a submeasure is a positive measure where additivity is replaced by subadditivity. An example for such a submeasure is easily given: Take a positive finite measure m on $X \times X$ (endowed with the product σ- algebra $\Sigma \otimes \Sigma$) and define $p(A) = m(A \times \complement A)$ for all $A \in \Sigma$, where $\complement A$ denotes the complement $X \setminus A$. As we will see later on, submeasures of this kind play an essential role in network theory and related areas (for example the theory of zero-one matrices [16]).

A similar kind of submeasure turns up in the study of supply-demand models. Let X now be a commodity set (i.e. the set of indices of the commodity space) and consider aggregate production and consumption. Assume that the producers do have alternatives in their production program. For example certain producers may be

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A similar kind of submeasure turns up in the study of supply-demand models. Let X now be a commodity set (i.e. the set of indices of the commodity space) and consider aggregate production and consumption. Assume that the producers do have alternatives in their production program. For example certain producers may be able to produce , within certain limits, a commodity b instead of a commodity $\tilde{b}$. Denote for $A \subset X$, by $\sigma^*(A)$ the aggregate *maximal production* of A. That is the maximal production (per time unit) of commodities of the species A, provided that the producers concentrate all their efforts on A, i.e. whenever a producer has the alternative to produce something out of A instead of something out of $\complement A$ he does so. The quantity σ^* is a submeasure and in most cases a submeasure of a very special kind.

Let a similar situation be given on the consumer side. Assume that some consumers allow, according to their specific tastes and needs, that certain commodities may be replaced by others. Then we define $\nu^*(A)$ to be the aggregate *minimal demand* for commodities out of A, i.e. the demand which is given by the provision that elements out of A are replaced by those out of $\complement A$ whenever a consumer allows this. For reasonable σ-algebras Σ is the quantity ν^* a *supermeasure*. That is a map $\nu^* : \Sigma \to \mathbb{R}_+$ with the properties:

(3) $\nu^*(A \cup B) \geq \nu^*(A) + \nu^*(B)$ for all $A,B \in \Sigma$ with $A \cap B = \phi$

(4) If $A_n \in \Sigma$ is an increasing sequence then

$$\sup_{n\in\mathbb{N}} \nu^*(A_n) = \nu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

In other words, a supermeasure is a measure where additivity is replaced by superadditivity.

In the study of supply-demand models the basic problem is whether or not the production can satisfy the demand. This is more or less equivalent to the question if there is a measure m with $\nu^* \leq m \leq \sigma^*$. Of course, a necessary condition for an affirmative answer is $\nu^* \leq \sigma^*$. But in contrast to the case where sub - and superlinear functionals are considered we do not have [2] a sandwich theorem [4] for sub- and supermeasures. So, from the beginning it is far from obvious that the condition $\nu^* \leq \sigma^*$ is sufficient. Nevertheless, a detailed analysis of the quantities σ^* and ν^* reveals that in this special case we have in fact a sandwich theorem, i.e. the production satisfies the demand if and only if maximal production dominates minimal demand. We get even stronger results, namely for the measure between ν^* and σ^* disintegrations with respect to the actual demand and the production capacity. These two disintegrations correspond to production and distribution plans.

DISINTEGRATION TOOLS

Let E be a vector space and let p_1,p_2 be sublinear functionals on E. Consider further a linear functional $\mu : E \to \mathbb{R}$ with $\mu \leq p_1 + p_2$. The following result can be found in Choquet's 1968 Princeton lectures [1 , p.273] (see also H. König [13]):

Sum Theorem: *There are linear functionals* μ_1,μ_2 *with* $\mu_1 \leq p_1$ *and* $\mu_2 \leq p_2$ *such that* $\mu = \mu_1 + \mu_2$.

Although, from the point of view of duality theory of vector spaces, this result may not be surprising it seems to me of fundamental importance. Not only that this lemma is far more basic than all of duality theory,but in addition it puts the emphasis where it belongs, namely on the elementary facts of decomposition of linear functionals. Heinz König combined this result with other fundamental observations [13] and opened the road towards a complete theory of sublinear functionals.

The theorem mentioned above has been generalized in several ways. We are going to describe the different stages of these generalizations . Then we are going to give a theorem which comprises all the different aspects. Let us start by rephrasing the above situation in a rather fancy way.

Consider the trivial measure space $\Omega = \{1,2\}$, $\Sigma = P(\Omega)$ endowed with the counting measure m . The the pair p_1, p_2 defines a sublinear operator $P : E \to L^1(m)$ by putting

$$P(x)(\omega) = p_\omega(x), \quad x \in E, \ \omega \in \Omega = \{1,2\}.$$

The condition $\mu \le p_1 + p_2$ reads

$$\mu(x) \le \int_\Omega P(x)\, dm \quad \text{for all} \quad x \in E$$

and the lemma states that there is a linear operator $T : E \to L^1(m)$ with $T \le P$ such that

$$\mu(x) = \int_\Omega T(x) dm \quad \text{for all} \quad x \in E.$$

It is not hard to imagine that this result does not depend on the specific nature of our measure space Ω . In fact, the result has been carried over to rather arbitrary measure spaces, see H. König [13], M. Wolff [19], M. Valadier [18], M. Neumann [15]. The resulting theorem is very much related to the celebrated Strassen disintegration theorem [17], an observation which can also be found in [1].

The second line of generalization deals with convex cones or even with abelian semigroups, if one likes. This work [4] follows the usual routine of replacing $\mathbb{R}$ by $\mathbb{R} \cup \{-\infty\}$ and replacing the equality $\mu = \mu_1 + \mu_2$ be the inequality $\mu \le \mu_1 + \mu_2$.

But generalizing the Sum Theorem to convex cones a new problem arises. Let $\le_1$ and $\le_2$ be preorder relations (order relations without antisymmetry) which are *compatible* (with the algebraic structure of the cone or the vector space, i.e. inequalities can be handled in the usual way). Assume that p_i are monotone with respect to the $\le_i$, $i=1,2$, where monotone means that $x \le_i y$ always implies $p_i(x) \le p_i(y)$. Then in case of a vector space the linear $\mu_i \le p_i$ do inherit automatically the monotonicity from the p_i . This, because the order relations $\le_i$ are characterized by their positive cones $E_i^+ = \{x \in E \mid 0 \le_i x\}$. Not so in the case of convex cones. Here, some additional routine work [4] is needed to ensure that the μ_i and p_i do have the same monotonicity properties.

Now, one can combine all these generalizations to obtain an abstract disintegration theorem for the following:

Situation: Let (Ω,Σ,m) be a measure space, where the measure m is positive and σ- finite. Denote by $L^1_*(m)$ the cone of measurable real-valued functions f on Ω such that the positive part $f_+ = (f \vee 0)$ is in $L^1(m)$, i.e.

$$L^1_*(m) = \{f - g \mid 0 \le f,g \text{ are measurable and } f \in L^1(m)\} .$$

Let F be a convex cone and consider a family $\le_\omega, \omega \in \Omega$, of compatible preorder relations in F. Define a map $\Psi : F \to L^1_*(m)$ to be *Ω- monotone* if for all $\varphi,\tilde{\varphi} \in F$ we have that $\Psi(\varphi) \le \Psi(\tilde{\varphi})$ m- almost everywhere on the set $\{\omega \in \Omega \mid \varphi \le_\omega \tilde{\varphi}\}$.

Disintegration theorem: *Let $\mu : F \to \mathbb{R}$ be linear and let $P : F \to L^1_*(m)$ be a sublinear Ω- monotone operator with*

$$(5) \quad \mu(\varphi) \le \int_\Omega P(\varphi)\, dm \quad \textit{for all } \varphi \in F.$$

Then there is an Ω- monotone linear operator $T : F \to L^1_*(m)$ with

$$(6) \quad T \le P$$

such that

$$(7) \quad \mu(\varphi) \le \int_\Omega T(\varphi)\, dm \quad \textit{for all } \varphi \in F.$$

The proof of this theorem can be found in [7].

THE FLOW THEOREM

As an application of the disintegration theorem we prove an abstract flow theorem.

We start with a measure space (X,Σ,μ) where μ is a signed measure. Then we consider the cone F consisting of all nonnegative simple measurable functions on X. These are functions of the following kind:

$$\varphi = \sum_{n=1}^{m} \lambda_n 1_{A_n} ,$$

where $m \in \mathbb{N}$, $\lambda_n \ge 0$, $A_n \in \Sigma$ and where 1_A denotes the characteristic function of A. By $\Omega = X \times X$ we denote the cartesian product and we endow this set with the product σ- algebra $\Sigma \otimes \Sigma$. Now, take $\varphi \in F$ and define $\hat{\varphi} : \Omega \to \mathbb{R}_+$ by

$$\hat{\varphi}(x_1,x_2) = \max(\varphi(x_1) - \varphi(x_2),0) \qquad \forall\, x_1,x_2 \in X.$$

Clearly, the operator $\varphi \to P(\varphi) \underset{\text{def}}{=} \hat{\varphi}$ is sublinear.

If, in addition, a finite positive measure τ on Ω is given, then we can make the:

Observation: *The following are equivalent:*

i) $\int_X \varphi \, d\mu \le \int_\Omega \hat{\varphi} \, d\tau$ *for all* $\varphi \in F$

ii) $\mu(A) \le \tau(A \times \complement A)$ *for all* $A \in \Sigma$.

Proof:

(i) ⇒ (ii): Take $\varphi = 1_A$ and observe that $\hat{\varphi} = 1_{A \times \complement A}$. Now, apply (i) in order to obtain (ii).

(ii) ⇒ (i): Every $\varphi \in F$ can be written in the form

$$(*) \quad \varphi = \sum_{n=1}^{m} \lambda_n 1_{B_n} ,$$

with $B_1 \supset B_2 \supset B_3 \supset \ldots \supset B_m$ and $\lambda_1, \ldots, \lambda_n \ge 0$. One easily observes that $P(\varphi_1 + \varphi_2) = P(\varphi_1) + P(\varphi_2)$ for those $\varphi_1, \varphi_2 \in F$ with

$$\varphi_1(x) = \sup_{\xi \in X} \varphi_1(\xi) \quad \text{whenever} \quad \varphi_2(x) > 0.$$

Combining this partial linearity with (*) one obtains the inequality (i). □

We introduce the following family $\{\le_\omega | \omega \in \Omega = X \times X\}$ of preorders in F:

$$\varphi_1 \le_\omega \varphi_2 \;\underset{\text{def}}{\Leftrightarrow}\; \varphi_1(x) \le \varphi_2(x) \text{ and } \varphi_1(y) \ge \varphi_2(y) \text{ , where } \omega = (x,y).$$

Then the operator $P : F \to L^1_*(\tau)$ is clearly Ω- monotone and we are arrived at exactly the same situation leading to the disintegration theorem. Recall that a bimeasure ([11] or [12]) ν on $X \times X$ is a map $\nu : \Sigma \times \Sigma \to \mathbb{R}$ such that ν is a signed measure in each variable separately, i.e. $A \to \nu(A,B)$ and $A \to \nu(B,A)$ are both, for fixed $B \in \Sigma$, signed measures on X . Now, rephrasing the disintegration theorem one obtains:

Flow theorem [17]: *The following are equivalent:*

i) $\mu(A) \le \tau(A \times \complement A)$ *for all* $A \in \Sigma$

ii) *There is a bimeasure* ν *on* $X \times X$ *with the following properties*

(8) $\mu(A) \le \nu(A,X)$ *for all* $A \in \Sigma$

(9) $\nu(A,B) \le \tau(A \times (B \cap \complement A))$ *for all* $A,B \in \Sigma$

(10) $\nu(A,B) \ge 0$ *for all* $A,B \in \Sigma$ *with* $A \cap B = \emptyset$.

Proof:

Let us briefly indicate how to obtain this result form the Disintegration theorem. One considers the operator $P : F \to L^1_*(\tau)$ given by $\varphi \to \hat{\varphi}$ and the linear functional given by $\varphi \to \int_X \varphi \, d\mu$. Then i) of the Flow theorem is equivalent to the fact that the linear functional under consideration is dominated by $\int_\Omega P(\varphi)d\tau$. Hence, we apply the Disintegration theorem to obtain a suitable linear operator $T : F \to L^1_+(\tau)$ and we define

$$\nu(A,B) = \int_{X\times B} T(1_A)d\tau \ .$$

Then (8) is a consequence of (7), (9) follows from (6) and (10) is an immediate consequence of the Ω- monotony. Since T is linear ν must by definition be finitely additive in both variables. The σ- additivity in the second variable follows from the σ- additivity of the integral. The critical point is the proof of the σ- additivity in the first variable. For that purpose we consider $A_n \in \Sigma$ with $A_1 \supset A_2 \supset A_3 \dots \supset A_n \supset \dots$. Then using the fact that ν is finitely additive in both variables we write:

$$\nu(A_n,B) = \nu(A_n,B \cap \complement A_n) + \nu(A_n \cap \complement B,B \cap A_n) + \nu(A_n \cap B,X) - \nu(A_n \cap B, \complement(B \cap A_n)).$$

In the first, second and fourth term of the right side the intersection of both variables is empty, hence these terms converge to zero because of (9). Using (8) and (9) one gets for the third term the following inequality

$$\mu(A_n \cap B) \le \nu(A_n \cap B,X) \le \tau((A_n \cap B) \times X).$$

Since the upper bound as well as the lower bound converge to zero $\nu(A_n \cap B,X)$ also converges to zero. □

INFINITE NETWORKS.

Let us give one interpretation of the flow theorem in the context of networks. First we explain the situation considered in Gale's Flow theorem (see [3] or [10 , p. 38]). There is given a finite set X of - let us say - oil consumers, the consumption of $i \in X$ is measured by μ_i . Negative consumption $\mu_i < 0$ means production in the amount of $|\mu_i|$. Furthermore, there are pipelines between the consumers. τ_{ik} measures the capacity of the pipeline running from k to i. Note, that in general $\tau_{ik} \neq \tau_{ki}$ (e.g. the pipeline is going up or down a hill). So it is best, to imagine that the pipelines are one-way streets. Of course, if there is no pipeline from k to i then we assume τ_{ik} to be zero.

Now, the problem is to give conditions for a positive and possible flow which satisfies the consumption. To be more precise, we assume that a flow is represented by numbers ν_{ik} (flow from k to i). The flow is called *positive* if $\nu_{ik} \ge 0$ for $i \neq k$, and it is said to be *possible* if $\nu_{ik} \le \tau_{ik}$ for all $i,k \in X$. The flow *satisfies the consumption* if, for every $i \in X$, the total amount flowing to i minus the total amount flowing from i dominates the consumption at i:

$$\sum_{k\in X} (\nu_{ik} - \nu_{ki}) \ge \mu_i \ .$$

We abbreviate for $A,B \subset X$:

$$\mu(A) = \sum_{i \in A} \mu_i$$

and

$$\tau(A \times B) = \sum_{\substack{i \in A \\ k \in B}} \tau_{ik} \; ; \; \nu(A,B) = \sum_{\substack{i \in A \\ k \in B}} \nu_{ik} \; .$$

Then the last condition is trivially equivalent to:

$$\mu(A) \leq \nu(A,X) - \nu(X,A) \quad \text{for all} \quad A \subset X.$$

For $A \subset X$ we define the *import capacity* to be $\tau(A \times \complement A)$, that is the capacity of all pipelines coming from the outside into A.

Obviously, a necessary condition for the existence of such a flow is that the import capacity is *sufficient* in the following sense:

$$\mu(A) \leq \tau(A \times \complement A) \quad \text{for all} \quad A \subset X.$$

Note, that this condition also requires that overall production dominates overall consumption, this because of

$$\mu(X) \leq \tau(X \times \phi) = 0 \; .$$

We are now going to generalize this problem to infinite consumer sets. On the first view this does not seem to be a relevant problem. But this generalization is absolutely necessary if one is - for example - interested in the dynamical behaviour of such a system. Let us for example consider the above problem for infinitely many different points T on the time scale, and with the provision that every consumer i has the possibility to store oil from t_1 to t_2 up to the amount $\sigma^i_{t_1,t_2}$. Then the mathematical consumer set is certainly $X \times T$ and $\sigma^i_{t_1,t_2}$ represents the pipeline capacity from (i,t_i) to (i,t_2). Certainly an infinite network system!

For the infinite system it seems appropriate to replace the quantities μ,τ and ν by suitable measures.

So, let (X,Σ) be a measurable space, where X is called the consumer set and where Σ is a σ- algebra on X. We consider a signed *consumption measure* μ on (X,Σ) measuring the consumption and the production, respectively. Furthermore, we consider a positive and finite measure τ on $\Omega = X \times X$ (with respect to $\Sigma \otimes \Sigma$) and we assume that $\tau(A \times B)$ measures the capacity of the pipelines going from B to A. Therefore τ is called the *capacity measure*. A bimeasure ν on $\Omega = X \times X$ is called a *positive flow* if

(11) $\nu(A,B) \geq 0$ for all disjoint $A,B \in \Sigma$. It is said to *satisfy the consumption* if

(12) $\mu(A) \leq \nu(A, \complement A) - \nu(\complement A,A)$ for all $A \in \Sigma$.

The flow ν is called *possible* (with respect to the capacity τ) if

(13) $\nu(A,B) \le \tau(A \times B)$ for all $A,B \in \Sigma$.

Of course, one should imagine $\nu(A,B)$ as the flow going from B to A.

Finally, we say that we have *sufficient import capacity* if

(14) $\mu(A) \le \tau(A \times \complement A)$ for all $A \in \Sigma$.

As an immediate application of our abstract flow theorem we get the following generalization to infinite networks of Gale's theorem ([3] or [10]).

Theorem: *There is a positive and possible flow which satisfies the consumption if and only if we have sufficient import capactiy.*

Proof: The necessity of the sufficient import condition is quite trivial, because from (11) to (13) we get:

$$\mu(A) \le \nu(A, \complement A) - \nu(\complement A,A) \le \nu(A, \complement A) \le \tau(A \times \complement A).$$

For the other implication, we observe that the sufficient import condition is nothing else than condition (i) in the Flow theorem. So, let us take the bimeasure given by the Flow theorem. From (10) we know that this is a positive flow, and (9) tells us that the flow is possible. A further consequence of (9) is that $\nu(X,B) \le 0$ for all $B \in \Sigma$. Hence we get from (8):

$$\mu(A) \le \nu(A,X) \le \nu(A,X) - \nu(X,A)$$

and because of the additivity of ν the last term is equal to $\nu(A, \complement A) - \nu(\complement A,A)$ since the $\nu(A,A)$ cancel. □

Other theorems of a similar nature can be obtained and generalized as well. For example the Ford-Fulkerson theorem and some of H. Ryser's results [16] about zero-one matrices. For the latter case it is useful to combine the Flow theorem with an extreme point argument or with a reasonable algorithm for the construction of the flow. We are not going into the details of these matters, the interested reader will find in [9] some material in that direction.

A SUPPLY - DEMAND PROBLEM

Certain quantities play a basic role in our model. They are essentially of a measure theoretic nature but can be understood best in the context of finite commodity sets. (On a less sophisticated level the model was already treated in [6]; it originates from the Lecture notes [14] of Heinz König; details about the present model can be found in [8].)

Let $X = \{1,2,\dots,n\}$ be a finite *commodity* set. The *aggregate production capacity* and *consumption desire* are measured by functions $\bar{\alpha}$ and $\bar{\nu}$. These are nonnegative functions on $P_0(X) = \{Y \subset X \mid Y \neq \phi\}$ since the consumers allow, according to their tastes and needs, that certain goods may be replaced by others, and the producers have at their disposal certain production capacities which they use according to the maket situation.

To make things precise we assume that the producer consist of subunits $U(Y)$, $Y \in P_0(X)$, being the collection of all factories where the assembly lines can be switched arbitrarily to production of any item in Y but where no item outside of Y can be produced. The *production capacity* function $\bar{\alpha} : P_0(X) \to \mathbb{R}_+$ is given by the numbers $\bar{\alpha}(Y)$ measuring the maximal output (in pieces) of $U(Y)$ if no limitations (raw material shortage, government regulations etc.) are given. Furthermore, we assume that for the production of $i \in Y$ a specific raw-material is needed which is available to $U(Y)$ up to a certain amount, thus limiting the production of $i \in Y$ to a $\bar{\rho}(Y,i)$ pieces. Of course, if $i \notin Y$ then we define $\rho(Y,i) = 0$. The function $\bar{\rho}$ is called the *raw-material bound*.

The *aggregate demand* is given by $\bar{\nu} : P_0(X) \to \mathbb{R}_+$, where $\bar{\nu}(Y)$ measures that fraction of the total demand which can be satisfied by allocation of any arbitrary combination (of total amount $\bar{\nu}(Y)$) of commodities of the species Y. But it may happen that the consumer becomes tired by obtaining to many pieces of the same commodity instead of a well mixed variety. Therefore we give a *saturation bound* $\bar{\sigma}(Y,i)$, stating that to satisfy the demand $\bar{\nu}(Y)$ only those allocations are permitted which contain not more than $\bar{\sigma}(Y,i)$ pieces of the commodity i. Again, we put $\bar{\sigma}(Y,i) = 0$ for $i \notin Y$.

Now, we look for reasonable production and distribution plans $\bar{p}, \bar{v} : P_0(X) \times X \to \mathbb{R}_+$. The quantity $\bar{p}(Y,i)$ measures the number of pieces of i being actually produced by $U(Y)$ and $\bar{v}(Y,i)$ measures how many pieces of i are allocated to satisfy $\bar{\nu}(Y)$. The plan $\bar{p}$ is said to be *possible* if it observes the limitations given by $\bar{\alpha}$ and $\bar{\rho}$, i.e. if

$$(15) \quad \sum_{k \in X} \bar{p}(Y,k) \leq \bar{\alpha}(Y)$$

$$(16) \quad \bar{p}(Y,i) \leq \bar{\rho}(Y,i)$$

for all $Y \in P_0(X)$ and $i \in X$. The plan $\bar{v}$ is said to be *satisfactory* if it satisfies the demand and observes the limits given by $\bar{\sigma}$, i.e.

$$(17) \quad \bar{\nu}(Y) \leq \sum_{k \in X} \bar{v}(Y,k)$$

$$(18) \quad \bar{v}(Y,i) \leq \sigma(Y,i)$$

for all $Y \in P(X)$ and $i \in X$. The plans $\bar{p}$ and $\bar{v}$ are called *compatible* if, of each commodity, we do not distribute more than we produce, i.e.

$$(19) \quad \sum_{Y \in P_0(X)} \bar{v}(Y,i) \leq \sum_{Y \in P_0(X)} \bar{p}(Y,i) \quad \text{for all } i \in X.$$

Note, that in (15) and (17) the sum only goes over the $k \in Y$ since all other terms are equal to zero, in (20) the sum goes only over the Y containing i (for the same reason).

A map $\bar{\tau} : A \to \mathbb{R}$ on a finite set A can always be considered as a measure τ on A. Just define $\tau(B) = \sum_{a \in B} \tau(a)$ for all $B \subset A$. Now, all the properties of the functions $\bar{\alpha},\bar{\nu},\bar{\rho},\bar{\sigma},\bar{p},\bar{v}$ can easily be expressed as properties of the corresponding measures. But then it makes sense to extend all the notions and properties to arbitrary measure spaces, this enables us to treat infinite commodity sets as well. Hence we have to deal with the following:

Situation:

We are given a commodity set X. We endow X and $P_0(X) = \{Y \mid \phi \neq Y \subset X\}$ with σ- algebras Σ_X and Σ_{P_0}, respectively. Furthermore, we are given positive finite measures α (*production capacity*) and ν (*demand measure*) on $P_0(X)$ and positive finite measures ρ (*raw-material bound*) and σ on $P_0(X) \times X$ with

(20) $\rho(\Omega \times D) = \sigma(\Omega \times D) = 0$

for all $\Omega \in \Sigma_{P_0}$ and $D \in \Sigma_X$ such that $A \cap D = \phi$ for all $A \in \Omega$.

The problem is: are there positive measures p (*production plan*) and v (*distribution plan*) on $P_0(X) \times X$ with:

(21) $p \leq \rho$ and $p(\cdot \times X) \leq \alpha$ (p is *possible*)

(22) $v \leq \sigma$ and $\nu \leq v(\cdot \times X)$ (v is *satisfactory*)

(23) $v(P_0 \times \cdot) \leq p(P_0 \times \cdot)$ (p and v are *compatible*).

Here, of course, $p(\cdot \times X)$ and $v(P_0 \times \cdot)$ denote the measures $\Omega \in \Sigma_{P_0} \to p(\Omega \times X)$ and $A \in \Sigma_X \to v(P_0 \times A)$, and so on.

By $m_1 \wedge m_2$ we denote the greatest lower bound of two measures m_1 and m_2. By m_1^+ we denote the positive part of m_1. Now, let us introduce the following quantities

(24) $\alpha_{max}(Y) = (\alpha \wedge \rho(\cdot \times Y))\,(P_0(X))$

(25) $\nu_{min}(Y) = (\nu - \sigma(\cdot \times \complement Y))^+(P_0(X))$.

α_{max} is a submeasure on X and ν_{min} is a supermeasure. If one goes back to finite commodity sets one easily finds the interpretation for these quantities:

- $\alpha_{max}(Y)$ denotes the *maximal production* of items out of Y, i.e. the number of pieces of elements of Y which is produced if in all subunits the assembly lines are switched to the production of commodities $\in Y$, whenever that is possible and compatible with the raw-material constraint ρ.

- $\nu_{min}(Y)$ is the *minimal demand* for commodities $\in Y$, i.e. whenever a good outside of Y can replace a good in Y without violating the limitation given by the saturation bound then the element in Y is replaced.

Supply - Demand theorem [8]: *There are possible, satisfactory and compatible production and distribution plans if and only if* $\nu_{min} \leq \alpha_{max}$.

Let us briefly sketch the lines of the proof. The necessity of $\nu_{min} \leq \alpha_{max}$ is clear. Now, take two disjoint copies of $P_0(X)$, say P_0 and P_0^*. Then in $\Gamma = P_0 \cup X \cup P_0^*$ one considers the largest σ- algebra such that the embeddings of P_0, P_0^* and X are measurable. Endow P_0 with the measure $-\alpha$ and P_0^* with ν. Denote by $-\hat{\alpha}$ and $\hat{\nu}$ the image measures in Γ with respect to the embeddings of P_0 and P_0^* and let $\mu = -\hat{\alpha} + \hat{\nu}$. Likewise, we endow $X \times P_0$ and $P_0^* \times X$ with the measures ρ and σ, denote by $\hat{\rho}$ and $\hat{\sigma}$ the image measures in $\Gamma \times \Gamma$ with respect to the embeddings. Define $\tau = \hat{\rho} + \hat{\sigma}$. Then it turns out that the condition $\nu_{min} \leq \alpha_{max}$ is equivalent to condition (i) (for μ and τ) in the Flow theorem. Hence, for $\nu_{min} \leq \alpha_{max}$, there is a bimeasure δ with the properties stated in the Flow theorem. This bimeasure we use to define positive bimeasures p and v on $P_0(X) \times X$ by:

$$(26) \quad p(\Omega \times A) = \delta(A,\Omega) \quad , A \in \Sigma_X, \ \Omega \in \Sigma_{P_0}$$

$$(27) \quad v(\Omega \times A) = \delta(\Omega,A) \quad , A \in \Sigma_X, \ \Omega \in \Sigma_{P_0^*} .$$

Since p and v are dominated by ρ and σ they are in fact honest measures on the product space. A straight forward calculation yields that they have the required properties. □

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