Progress of Theoretical Physics, Vol. 68, No. 4, October 1982

The Lie Algebra Structure of Degenerate Hamiltonian and Bi-Hamiltonian Systems

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(Received October 13, 1981)

A generalization of Noether's theorem is obtained via an extension of the well-known Poisson bracket formalism. It is shown that degenerate closed forms yield Lie algebra homomorphisms between vector fields and covector fields. A similar result holds for operators working in the opposite way. Application of these Lie algebra homomorphisms to a dynamical system having two (degenerate) Hamiltonian formulations yields a selfmap in the space of infinitesimal generators of one-parameter symmetry groups of this system. These Hamiltonian formulations are not assumed to constitute a Hamiltonian pair (in the sense of Gelfand-Dorfman). Thus infinite-dimensional symmetry groups for a wider class of equations can be constructed. Several new equations are shown to admit infinite dimensional symmetry groups.

§ 1. Introduction

Let us review the classical situation.

Consider a C^∞ manifold (eventually infinite dimensional) and its tangent space S and cotangent space S^*, respectively (i.e., the typical fibres of the tangent and cotangent bundles). Throughout this paper "continuous" means continuous with respect to the weak topology given by the duality (S, S^*). The application of $s^*\in S^*$ to $s\in S$ we denote by $\langle s^*, s\rangle$. By $T(p, q)$ we denote the space of $C^\infty(p, q)$-tensor fields (i.e., covariant of degree q and contravariant of degree p). As usual $[X, Y]$ denotes the Lie product in $T(1, 0)$ (i.e. the C^∞ vector fields) given by $[X, Y]=DY\cdot X-DX\cdot Y$ (D is the derivative). We recall[1] that a symplectic form ω is a closed two-form (i.e., $d\omega=0$, d exterior derivative) which is nondegenerate, where nondegenerate means that the operator $\Omega_\omega^*: S\to S^*$ given by $\langle\Omega_\omega^* s_1, s_2\rangle=\omega(s_1, s_2)$, $s_1, s_2\in S$ is bijective.

Now, consider a dynamical system consisting of a vector field $X\in T(1,0)$ with flow $F(X)_t=F_t$ (i.e., $(d/dt)F_t=X\cdot F_t$ for all t). This system is called *Hamiltonian system* if there is a symplectic form ω and a function $H\in T(0,0)$ such that $i_X\omega=dH$ where i_X denotes the interior product with X. Obviously this is equivalent to

$$X=\Omega_\omega^{*-1}dH \quad \text{or} \quad \frac{d}{dt}F_t=\Omega_\omega^{*-1}dH\cdot F_t\,.$$

It is well known[1] that for a Hamiltonian system the operator Ω_ω^{*-1} is then a Lie-algebra homomorphism from the Lie-algebra given by the gradients of the

Poisson brackets in $T(0,0)$ into the Lie-algebra $T(1,0)$. A consequence of this result is the classical Noether theorem about the correspondence between infinitesimal generators of one-parameter symmetry groups and gradients of conserved quantities.

In this paper we drop the nondegeneracy condition and we extend the Lie-algebra from $dT(0,0)$ to all covector fields such that we still have a canonical Lie algebra homomorphism. Then we obtain similar results for (degenerate) inverse-symplectic maps. We introduce bi-Hamiltonian systems and we show that, in general, they admit infinite-dimensional (eventually noncommutative) symmetry groups. This will be true even in the case that the two Hamiltonian formulations of the system do not constitute a Hamiltonian pair[2] or a symplectic Kähler manifold.[3]

In § 3 we compare the results of this paper with the work of Gelfand-Dorfman[2] and the work of Magri.[3]

In the second part we give various examples to illustrate our techniques. We give several new nonlinear evolution equations with infinite-dimensional symmetry groups. Among them there is a system describing the interaction of long waves for which Hirota and Satsuma[4] conjectured complete integrability.

1.1. *Lie algebras in the covector fields*

We define a *symplectic operator* to be a closed two-form $\omega \in T(0,2)$. And by *implectic operator* (short form for inverse-symplectic) we mean an operator which has the same Lie-algebraic properties as the inverse of a symplectic operator. To be precise: $\varphi \in T(2,0)$ is called *implectic* if it is skew-symmetric (i.e. $\varphi(s_1^*, s_2^*) = -\varphi(s_2^*, s_1^*)$ for all $s_1^*, s_2^* \in S$) and if the tensor field $D\varphi$ fulfills the following Jacobi identity:

$$(D\varphi)(s_1, s_2^*, s_3^*) + (D\varphi)(s_2, s_3^*, s_1^*) + (D\varphi)(s_3, s_1^*, s_2^*) = 0 \quad \forall s_1^*, s_2^*, s_3^* \in S^* . \tag{1}$$

where $s_i = \Omega_\varphi s_1^*$, $i=1,2,3$ and where Ω_φ: $S^* \to S$ is the operator given by $\langle \tilde{s}^*, \Omega_\varphi s^* \rangle = \varphi(s^*, \tilde{s}^*) \forall s^*, \tilde{s}^* \in S^*$. If Ω_φ is invertible, then (1) is equivalent[5] to the fact that the two-form ω given by

$$\omega(s, \tilde{s}) = \langle \Omega_\varphi^{-1} s, \tilde{s} \rangle$$

is closed. Let us denote by L_X the Lie-derivative with respect to the vector field X. Now, define for two covector fields the following bracket:

$$[X^*, Y^*]_\varphi = L_{\Omega_\varphi X^*} Y^* - i_{\Omega_\varphi Y^*} dX^* . \tag{2}$$

Because of

$$[X^*, Y^*]_\varphi = d(\varphi(X^*, Y^*)) + i_{\Omega_\varphi X^*} dY^* - i_{\Omega_\varphi Y^*} dX^*$$

$$=L_{\Omega_\varphi X^*}Y^*-L_{\Omega_\varphi Y^*}X^*-d(\varphi(X^*,\ Y^*)) \tag{3}$$

this bracket is for closed forms X^*, Y^* the gradient of the usual Poisson bracket.

THEOREM 1: *Let φ be implectic. Then $[\ ,\ ]_\varphi$ defines a Lie product in the covector fields and Ω_φ is a Lie-algebra homomorphism with respect to this Lie product.*

Proof

For the moment we abbreviate $X=\Omega_\varphi X^*$, etc. For arbitrary Z^* we get from (3) and the skew-symmetry of φ:

$$\begin{aligned}\langle Z^*, \Omega_\varphi[X^*,\ Y^*]_\varphi\rangle &= -\langle [X^*,\ Y^*]_\varphi, Z\rangle\\ &= -(D\varphi)(Z, X^*,\ Y^*)-\varphi(DX^*\cdot Z,\ Y^*)\\ &\quad + dX^*(Y, Z)+\varphi(DY^*\cdot Z, X^*)-dY^*(X, Z).\end{aligned}$$

Now, observe that

$$\varphi(DX^*\cdot Z,\ Y^*)-dX^*(Y, Z)=-\langle DX^*\cdot Y, Z\rangle\,,$$

$$(D\varphi)(X,\ Y^*, Z^*)-\langle DY^*\cdot X, Z\rangle=\langle Z^*, DY\cdot X\rangle\,.$$

Hence we get with (1)

$$\begin{aligned}\langle Z^*, \Omega_\varphi[X^*,\ Y^*]_\varphi\rangle &= -(D\varphi)(Z, X^*,\ Y^*)+\langle DX^*\cdot Y, Z\rangle-\langle DY^*\cdot X, Z\rangle\\ &=(D\varphi)(X,\ Y^*, Z^*)+(D\varphi)(Y, Z^*, X^*)\\ &\quad +\langle DX^*\cdot Y, Z\rangle-\langle DY^*\cdot X, Z\rangle\\ &=\langle Z^*, DY\cdot X\rangle-\langle Z^*, DX\cdot Y\rangle=\langle Z^*, [X,\ Y]\rangle\,.\end{aligned}$$

Since Z^* was arbitrarily chosen, we have

$$\Omega_\varphi[X^*,\ Y^*]_\varphi=[X,\ Y]. \tag{4}$$

The skew-symmetry $[X^*,\ Y^*]_\varphi=-[Y^*,\ X^*]_\varphi$ follows immediately from (3). To prove the Jacobi identity we have to observe that (1) is equivalent to:

$$i_Y L_X Z^*+(\text{cyclic})=0\,. \tag{5}$$

Now, recalling the usual calculation rules for the Lie-derivative we obtain with the help of (3) and (4):

$$\begin{aligned}[Z^*, [X^*,\ Y^*]_\varphi]_\varphi &= L_Z L_X Y^*-L_Z L_Y X^*-L_{\Omega_\varphi[X^*,\ Y^*]_\varphi}Z^*\\ &\quad +d(\varphi([X^*,\ Y^*]_\varphi, Z^*))-dL_Z\varphi(X^*,\ Y^*)\\ &=-di_Z([X^*,\ Y^*]_\varphi+d\varphi(X^*,\ Y^*))\end{aligned}$$

$$+L_Z L_X Y^* - L_Z L_Y X^* - L_{[X,Y]} Z^*$$
$$= di_Z(L_Y X^* - L_X Y^*) + L_Z L_X Y^*$$
$$-L_Z L_Y X^* - L_X L_Y Z^* + L_Y L_X Z^* .$$

Hence, (5) yields

$$[Z^*, [X^*, Y^*]_\varphi]_\varphi + (\text{cyclic}) = 0 .$$

Now, let ω be a symplectic operator, i.e., $d\omega = 0$. Then a somewhat similar identity holds. To see this we define a bracket $[\,,\,]_\omega$ mapping $T(1,0)\times T(1,0)$ into $T(0,1)$:

$$[X, Y]_\omega = L_X i_Y \omega - i_Y d i_X \omega . \tag{6}$$

Since ω is a form, we have in general

$$L_X i_Y \omega = i_Y L_X \omega + i_{[X,Y]} \omega .$$

Inserting $d\omega = 0$ and $L_X = di_X + i_X d$, we obtain

$$[X, Y]_\omega = i_{[X,Y]} \omega . \tag{7}$$

Remark: Let $\Omega_\omega{}^*$ be the operator given by $\langle \Omega_\omega{}^* s, \tilde{s}\rangle = \omega(s, \tilde{s})$. If the kernel of $\Omega_\omega{}^*$ (i.e., those vector fields X with $\Omega_\omega{}^* X = 0$) is an ideal in the Lie algebra of vector fields, then we may define a Lie-algebra in $\Omega_\omega \cdot T(1,0) \subset T(0,1)$ by

$$[\Omega_\omega{}^* X, \Omega_\omega{}^* Y]_{\omega^*} = [X, Y]_\omega . \tag{8}$$

The condition that the kernel is an ideal is necessary to ensure that this product is properly defined (i.e., that it vanishes for $\Omega_\omega{}^* X = 0$ or $\Omega_\omega{}^* Y = 0$, respectively). Then the meaning of (7) is that $\Omega_\omega{}^*$ is a Lie algebra homomorphism from the vector fields into $\Omega_\omega{}^* T(1,0)$. It is pretty obvious that, if the symplectic ω is nondegenerate (i.e., $\Omega_\omega{}^*$ is invertible), and if we define φ by $\Omega_\varphi = \Omega_\omega^{*-1}$ then φ is implectic and the Lie-algebras (2) and (8) coincide. The same holds if we start with a nondegenerate implectic φ.

1.2. *Generalized bi-Hamiltonian systems*

Now, let us define the basic notions. Consider dynamical systems consisting of the vector fields $Y, X \in T(1,0)$ and let us denote their flows by $F(Y)_t$ and $F(X)_t$ respectively. The vector field Y is said to be an *infinitesimal generator of a symmetry* of the flow $F(X)_t$ if $[X, Y] = 0$. This notion is selfexplanatory since $[X, Y] = 0$ means that the flow $F(Y)_t$ of Y commutes with $F(X)_t$.

If $L_X Z^* = 0$ then we call a C^∞ covector field Z^* a *conserved covariant* form (for the dynamical system given by X).

Now, let us in addition consider an implectic operator φ and some symplectic operator ω. We call (X, φ) a *generalized Hamiltonian system* if there is some $H \in T(0,0)$ such that $X = \Omega_\varphi dH$. And we call (ω, X) *inverse-Hamiltonian* if there is some $\tilde{H} \in T(0,0)$ such that $i_X\omega = d\tilde{H}$. The system (ω, X, φ) is said to be a *generalized bi-Hamiltonian system* if (X, φ) is generalized Hamiltonian and if (ω, X) is inverse-Hamiltonian. Of course, the bi-Hamiltonian formulation of the flow generated by X is not necessarily unique.

THEOREM 2

(i) *Let (X, φ) be generalized Hamiltonian. Then Ω_φ maps conserved covariant forms onto infinitesimal generators of symmetries.*
(ii) *Let (ω, X) be inverse-Hamiltonian then $\Omega_\omega{}^*$ maps infinitesimal generators of symmetries onto conserved covariant forms.*
(iii) *Let (ω, X, φ) be generalized bi-Hamiltonian then $\Omega_\varphi \cdot \Omega_\omega{}^*$ and $\Omega_\omega{}^* \cdot \Omega_\varphi$ are selfmaps in the space of infinitesimal generators of symmetries and in the space of conserved covariant forms, respectively.*

Proof

(iii) is an immediate consequence of (i) and (ii).
(i) Consider $Z^* \in T(1,0)$ such that $L_X Z^* = 0$ and let $H \in T(0,0)$ such that $X = \Omega_\varphi dH$. Then by theorem 1 we obtain:

$$[\Omega_\varphi Z^*, X] = \Omega_\varphi [Z^*, dH]_\varphi$$
$$= \Omega_\varphi \{d(\varphi(Z^*, dH)) + i_{\Omega_\varphi Z^*} ddH - i_X dZ^*\}$$
$$= -\Omega_\varphi \{d\langle Z^*, X\rangle + i_X dZ^*\} = -\Omega_\varphi L_X Z^* = 0\,.$$

(ii) Choose $Y \in T(1,0)$ such that $[X, Y] = 0$ and take $\tilde{H} \in T(0,0)$ with $i_X\omega = d\tilde{H}$. From (7) we obtain

$$0 = \Omega_\omega{}^*[X, Y] = i_{[X,Y]}\omega = L_X i_Y \omega - i_Y d i_X \omega$$
$$= L_X(\Omega_\omega{}^* Y) - i_Y dd\tilde{H} = L_X(\Omega_\omega{}^* Y).$$

Hence, $\Omega_\omega{}^* Y$ must be a conserved covariant form.

1.3. *Comparison with other work*

In this section the results of the preceding chapters are compared with the work done by Gelfand-Dorfman,[2] Magri[3] and by Fokas-Fuchssteiner.[5] Magri started his deep Lie-algebraic contribution to the field by a paper in 1978.[6] Later on,[3] he observed in addition the very important fact that the evolution equations under consideration provide an isospectral flow for the corresponding Nijenhuis operators[3] (a fact which, in another context, was already observed earlier[7]). But since Magri's Lie-algebraic contributions are identical with those of Gelfand-Dorfman we concentrate our explicit comparison on the papers of Gelfand-

Dorfman.

These authors consider two Lie products fulfilling a compatibility condition, and out of the combination of these products they obtain recursion formulas for conserved quantities (or symmetries via Noether's theorem). Their first Lie product, defined in a certain subalgebra $\mathcal{H}$ of zero-forms is given by some (eventually degenerate) closed two form ω. We extend this construction by our formulas (6)~(8). In fact, speaking in our terminology, the subalgebra $\mathcal{H}$ of Gelfand and Dorfman corresponds uniquely to the potentials of those exact one-forms which are of the form $\Omega_\omega{}^*X$, X some vector field. It is easy to see that formula (8) defines an honest Lie-algebra in the gradients of these potentials. This is then the gradient of the Poisson brackets considered by Gelfand-Dorfman. So, our arguments extend the Gelfand-Dorfman situation to the less restrictive situation, where the covector fields are not assumed to be closed. This will turn out to be important in applications.

Gelfand-Dorfman have to ensure that their successive construction of conservation laws does not lead outside of $\mathcal{H}$, and therefore they have to require a compatibility condition for the second Lie product. This second Lie product results out of an operator which they call Hamiltonian (implectic in our terminology) and which is defined for all covector fields given by exact forms. Again we extend this to a situation where the forms are not required to be exact. Our theorem 1 is a proper extension of their theorem 1.3. This generalization is necessary in order to get rid of the restriction that the Hamiltonians are required to be a Hamiltonian pair (which means the same as Magri's symplectic Kähler structure[3] or our notion of compatible symplectic operators.[5]

Let us explain in what way our theorem 2 extends Gelfand-Dorfman's theorem 3.4, and what the drawbacks of this generalization are. In the case of generalized bi-Hamiltonian systems an immediate consequence of theorem 2 is that $G_n=(\Omega_\omega{}^*\Omega_\varphi)^n dH$, $n\in N_0$, constitutes a sequence of conserved covariant forms for the flow given by $X=\Omega_\varphi dH$. Contrary to Gelfand-Dorfman's theorem 3.4 we do not assume that the Hamiltonians are a Hamiltonian pair (or equivalently, that $\Phi^+=\Omega_\omega{}^*\cdot\Omega_\varphi$ has to be regular). Later on we give nontrivial examples, where this generalization turns out to be essential. But apart from that generalization, and apart from the corresponding symmetry assertion, our theorem 2 contains even more information. For example, it is shown that if we start with a conserved covariant form $\tilde{G}$ which is *not* among the G_n then by $\tilde{G}_n=(\Phi^+)^n\tilde{G}$ another sequence of conserved covariant forms is given. Even for regular Φ^+ this result is not covered by Gelfand-Dorfman's theorem 3.4. In this case, regularity does not yield additional properties for the $\tilde{G}_n$ (see example 2.1.d). But, for the G_n regularity of Φ^+ yields important additional information. Namely, all the G_n are closed and they are in involution.[3),2),5] This, of course, is an important drawback of the generalization presented in theorem 2. But quite often, there are other, although less transparent, properties which ensure commutativity and exactness of the G_n.

Let us mention that regular operators are a special case of the hereditary operators introduced earlier.[7),8)] To be precise, regular operators are local hereditary operators possessing a symplectic-implectic factorization. Thus hereditary operators include regular operators of Gelfand-Dorfman. There are important examples where hereditary operators occur, which are either not local (Benjamin-Ono equation,[9)] Kadomtsev-Petviashvili equation[10)] or without symplectic-implectic factorization (Burgers equation[7),8)])).

We recall the abstract definition[8)] of a hereditary operator. A linear operator Φ in a Lie-algebra L is said to be *hereditary* if, for all $a, b \in L$, we have

$$\Phi^2[a, b]+[\Phi(a), \Phi(b)]=\Phi\{[a, \Phi(b)]+[\Phi(a), b]\}. \tag{9}$$

One might ask under what conditions on φ and ω the operator $\Phi=\Omega_\varphi\cdot\Omega_\omega{}^*$ (or equivalently $\Phi^+=\Omega_\omega{}^*\cdot\Omega_\varphi$) is hereditary. For invertible Ω_φ this question is answered in Refs. 2), 3) and 5). Following the spirits of these proofs, one finds that $\Omega_\varphi\cdot\Omega_\omega{}^*$ is hereditary if and only if φ and ω are *compatible* in the sense that the mixed Schouten bracket

$$\{Y_1, Y_2, Y_3\}=\langle\Omega_\omega{}^*Y_2, ((D\Omega_\varphi)\cdot Y_1)\Omega_\omega{}^*Y_3\rangle+\langle(((D\Omega_\omega{}^*)\cdot(\Omega_\varphi\Omega_\omega{}^*Y_1))Y_3, Y_2\rangle \tag{10}$$

has to fulfill, for all C^∞ vector fields, the Jacobi identity. This property then ensures that the G_n, $n\in N_0$, are exact and in involution. But one of the points of this paper is that even in the well-known cases it is almost impossible to check this property because of the horrible calculations which are necessary. Then it is much easier, to work with theorem 2 in its general version and to prove commutativity by other means (see example 2.1.b).

§ 2. Examples and applications

Let us first explain the notation we are going to adopt. The manifold under consideration will be S, some real or complex locally convex Hausdorff vector space. By S^* we denote the dual of this vector space, and $\langle s^*, s\rangle$ denotes the application of $s^*\in S^*$ to $s\in S$. The zero-forms are C^∞ maps from S into the scalars. The vector fields $T(1,0)$ are $C^\infty(S, S)$ (C^∞ maps from S into S) or a suitable subspace. The covector fields $T(0,1)$ are either $C^\infty(S, S^*)$ or, again, a suitable subspace. For a C^∞ function F from S into a vector space the derivative DF is given by:

$$DF(u)[v]=\frac{\partial}{\partial\varepsilon}F(u+\varepsilon v)_{|\varepsilon=0}, \qquad u, v\in S.$$

The Lie algebra in $T(1,0)$ is given by

$$[X,\ Y](u)=DX(u)[Y(u)]-DY(u)[X(u)]$$
$$=\frac{\partial}{\partial\varepsilon}\{X(u+\varepsilon Y(u))-Y(u+\varepsilon X(u))\}_{|\varepsilon=0}\,,$$

where $u\in S$ is arbitrary. The Lie-algebra module under consideration is $T(0,0)$ and the multiplication of an element m of the module with an element X of the Lie algebra is defined to be

$$(Xm)(u)=Dm(u)[X(u)],\qquad u\in S\,.$$

Obviously, that gives a representation, since

$$XYm-YXm=[X,\ Y]m\,.$$

Two-tensors $\omega\in T(0,2)$, $\varphi\in T(2,0)$ are identified with operator valued functions $\Omega_\omega{}^*(u)$ and $\Omega_\varphi(u)$ by

$$\omega(u)(s_1,\ s_2)=\langle\Omega_\omega{}^*(u)s_1,\ s_2\rangle\,,\quad \varphi(u)(s_1{}^*,\ s_2{}^*)=\langle s_1{}^*,\ \Omega_\varphi(u)s_2{}^*\rangle\,.$$

The adjoint of an operator Ω is denoted by $\Omega^\dagger$. Then the exterior derivative for a one-form Z^* is given by

$$dZ^*(u)=DZ^*(u)-(DZ^*(u))^\dagger\,.$$

2.1. *Examples which are, more or less, known*

We start with some examples which are either trivial or generally known. The amount of new information revealed by them may not be overwhelming. Nevertheless, we believe that they may serve for the understanding of the proposed methods.

2.1.a *The KdV and other popular equations*

Fix $S=S(\boldsymbol{R})$ to be the space of C^∞ functions on the line which vanish rapidly at infinity. By ∂ and ∂^{-1} we denote the differential operator and its inverse

$$(\partial^{-1}s)(\xi)=\int_{-\infty}^{\xi}s(x)dx\,,\qquad s\in S\,.$$

The dual S^* of S is defined to be

$$S^*=\{\partial^{-1}s+r|s\in S(\boldsymbol{R}),\ r\in\boldsymbol{R}\},$$

where the evaluation on S shall be given by

$$\langle s^*,\ s\rangle=\int_{-\infty}^{+\infty}s^*(\xi)s(\xi)d\xi\,.$$

The topology, which is necessary in order to define what C^∞ vector fields are, is assumed to be the weak topology.

For $u\in S$ consider the operators $\theta(u)$: $S^*\to S$ and $J(u)$: $S\to S^*$ which are given by:

$$\theta(u)=\alpha\partial+\beta\partial^3+\gamma(u\partial+\partial u)+\delta\partial u\partial^{-1}u\partial\,,$$
$$J(u)=\partial^{-1}\,,$$

where $\alpha, \beta, \gamma, \delta$ are scalar coefficients and where u stands for the multiplication operator given by u.

Let $\varphi\in T(2,0)$ (contravariant of degree 2) and $\omega\in T(0,2)$ (covariant of degree two) be the tensor fields given by

$$u\in M\to\varphi(s_1^*, s_2^*)=\langle s_1^*, \theta(u)s_2^*\rangle\,, \qquad s_1^*, s_2^*\in S^*\,, \tag{11·1}$$
$$u\in M\to\omega(s_1, s_2)\ =\langle J(u)s_1, s_2\rangle\,, \qquad s_1, s_2\in S\,, \tag{11·2}$$

i.e., using the notation we adopted so far, we have

$$\Omega_\varphi(u)=\theta(u),$$
$$\Omega_\omega^*(u)=J(u),$$

and these operators are implectic and symplectic, respectively. Now, we study the flow determined by the vector field

$$X(u)=\Omega_\varphi(u)u=\alpha u_\xi+\beta u_{\xi\xi\xi}+3\gamma uu_\xi+\frac{3}{2}\delta u^2u_\xi\,. \tag{12}$$

Then we easily see that $X=\theta dp$, where d denotes the exterior derivative and where p is the zero-form

$$p(u)=\frac{1}{2}\int_{-\infty}^{+\infty}u^2(\xi)d\xi\,.$$

Furthermore,

$$(i_X\omega)(u)=\partial^{-1}X(u)=d\bar{p}(u)$$

with

$$\bar{p}(u)=\int_{-\infty}^{+\infty}\left\{\frac{\alpha}{2}u^2(\xi)-\frac{\beta}{2}u_\xi(\xi)^2+\frac{\gamma}{2}u(\xi)^3+\frac{\delta}{8}u(\xi)^4\right\}d\xi\,.$$

Hence, (ω, X, φ) is a bi-Hamiltonian system and

$$\Phi=\Omega_\varphi\Omega_\omega^* \tag{13}$$

must be a selfmap in the space of symmetry-generators of the evolution equation

$$u_t=X(u(t)). \tag{14}$$

Since the operator Φ is hereditary,[7),8)] all the vector fields:

$$\sigma_n(u)=\Phi^n(u)X(u)=\Phi^{n+1}(u)u_X\,,\qquad n=-1,0,1,2,\cdots \tag{15·1}$$

do commute and define symmetry-generators for (14). In fact, in view of Tu's[11)] result the commutativity is not surprising. Another consequence of the hereditariness of Φ is that all the covector fields

$$\gamma_n(u)=\Omega_\omega{}^*(u)\sigma_n(u)=\partial^{-1}\sigma_n(u) \tag{15·2}$$

are closed one-forms, i.e. gradients of conservation laws. And these conservation laws must be in involution (i.e., they commute with respect to the Poisson brackets).

Among the equations given by (14) there are (by an appropriate choice of the coefficients) the KdV, the modified KdV, the Gardner equation, etc. If the vector field $X(u)$ is replaced by $\Phi^{-1}(u)u_X$ then all the arguments go through and among the equations one finds the sine-Gordon equation[5),7)] and the "inverse KdV".[12)] Other equations can be treated in the same way.[5),7)] The tensor fields φ and ω are not the only bi-Hamiltonian formulations of the evolution equation under consideration, but they determine all other formulations insofar that one can only replace $J(u)$ and $\theta(u)$ by $J(u)\Phi(u)^n$ and $\Phi(u)^m\theta(u)$, respectively.

2.1.b *The Caudrey-Dodd-Gibbon-Sawada-Kotera equation*

Replace in the last example the operators $\theta(u)$ and $J(u)$ by

$$\theta(u)=\partial^3+(u\partial+\partial u)=\Omega_\varphi(u), \tag{16·1}$$

$$J(u)=2\partial^3+(\partial^2u\partial^{-1}+\partial^{-1}u\partial^2)+\frac{1}{4}(u^2\partial^{-1}+\partial^{-1}u^2)=\Omega_\omega{}^*(u) \tag{16·2}$$

and let ω and φ be the corresponding tensor fields given by (11). Then ω and φ are again symplectic and implectic, respectively. If $X(u)$ is the following vector field

$$X(u)=u_{\xi\xi\xi\xi\xi}+\frac{5}{2}uu_{\xi\xi\xi}+\frac{5}{2}u_\xi u_{\xi\xi}+\frac{5}{4}u^2u_\xi\,,$$

then (14) is the well-known Caudrey-Dodd-Gibbon-Sawada-Kotera[13)] equation. This equation is Hamiltonian since we can write:

$$X(u)=\theta(u)dp(u),$$

where p is the following zero-form

$$p(u)=\int_{-\infty}^{+\infty}\left\{\frac{1}{12}u(\xi)^3-\frac{1}{2}u_\xi(\xi)^2\right\}d\xi\,.$$

Furthermore, one can show [14] that in fact $i_X\omega$ is a closed one-form. Hence (ω, X, φ) is a bi-Hamiltonian system and

$$\Phi(u)=\theta(u)J(u)=\Omega_\varphi(u)\Omega_\omega{}^*(u)$$

must be a selfmap in the space of symmetry generators of the CDGSK-equation. Again, this operator seems to be hereditary (horrible calculation). Hence, the Lie-algebra given by (15·1) is commutative and the covector fields (15·2) are closed and the corresponding conservation laws are in involution. The details of the calculation, together with an analysis of similar equations, are published separately.[14]

Let us emphasize what the point of the last example is. In fact, it is highly probable that the operator Φ occurring for the CDGSK is hereditary and that results are covered by the theory of Gelfand-Dorfman. But a proof of the hereditariness of Φ which is digestible is not known (several hundred terms of a differential operator have to be checked). Hence, the CDGSK is not accessible through results given by the beautiful theory of Hamiltonian pairs, whereas the calculations we have sketched here are easily manageable.

In this connection it is desirable to mention that recently there has been considerable progress in the study of the CDGSK (and other nonlinear evolution equations) through deep contributions of Date, Jimbo, Kashiwara and Miwa.[15] Their method seems to differ considerably from ours. Of course, there are relations, only these relations are not yet completely clear to the author in this paper.

2.1.c *Translation invariance—a noncommutative example*

We keep the notation of the last two sections. Consider the trivial—but nevertheless instructive—example of the translation group. Let X

$$X(u)=u_\xi$$

be the infinitesimal generator of that group. Then (14) has an infinite amount of essentially different bi-Hamiltonian formulations. For example, we can write

$$X(u)=\Omega_\varphi(u)dp_1(u),$$

where

$$p_1(u)=\int_{-\infty}^{+\infty}u(\xi)d\xi$$

and where

$$\Omega_\varphi(u)=\partial u+u\partial$$

is implectic. For the symplectic operator given by

$$\Omega_\omega^*(u)=\partial^{2n-1}$$

we have that $i_X\omega$ is closed:

$$i_X\omega=\Omega_\omega^* X=dp\,,$$

$$p(u)=\frac{1}{2}\int_{-\infty}^{+\infty}\left\{u(\xi)\frac{\partial^{2n}}{\partial\xi^{2n}}u(\xi)\right\}d\xi\,.$$

Hence, (ω, X, φ) must be bi-Hamiltonian. But the operator $\Phi=\Omega_\varphi\Omega_\omega^*$ is not hereditary, so we cannot expect that the symmetry group given by the infinitesimal generators (15·1) is commutative and that all the γ_n given by (15·2) are closed. Clearly, this is not the case. Of course, it is not at all difficult to determine in this case the Lie algebra of the symmetry group: It is given by all translation invariant vector fields $\tilde{X}$.

2.1.d *The three-dimensional harmonic oscillator*

We consider the movement of a particle in a three-dimensional rotation-invariant potential V. This example will show us:

i) Degenerate bi-Hamiltonian formulations are important and give rise to the construction of symmetry groups.

ii) The non-uniqueness of bi-Hamiltonian formulations can be used—even in the case that the operators Φ are hereditary—to construct noncommutative symmetry groups which are not accessible through the usual theory of Hamiltonian pairs.

First we fix a (conjugate-variable) notation which is consistent with the first part of this paper. The manifold under consideration is $M=\boldsymbol{R}^6$, the manifold variable is denoted by u. We split u up in the following way:

$$u=\begin{pmatrix} v\\ w\end{pmatrix},\qquad v=\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix},\qquad w=\begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix}.$$

Operators in the tangent space $S=M=\boldsymbol{R}^6$ are then given by two-by-two-matrices whose entries are operators in three-space, i.e., three-by-three-matrices. The infinitesimal generator of the movement of a particle in a rotation-invariant potential V is then given by

$$X(u)=\Omega_\varphi dH(u) \tag{17}$$

with

$$\Omega_\varphi=\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}, \tag{17·1}$$

$$H(u)=\frac{1}{2}w^2+V\left(\frac{1}{2}v^2\right). \tag{17·2}$$

This is clearly a Hamiltonian system since Ω_φ is implectic. Let us see if there is an essentially different inverse Hamiltonian formulation of that system. We have to find a symplectic operator $\Omega_\omega{}^*$ which maps X onto a gradient of a conserved quantity $p(u)$ of the flow given by (17). The available conserved quantities are: energy $H(u)$ (which was already used for (17)) and the three components of angular momentum:

$$j_1(u)=v_2w_3-v_3w_2\,,$$

$$j_2(u)=v_3w_1-v_1w_3\,,$$

$$j_3(u)=v_1w_2-v_2w_1\,.$$

Let us try $p(u)=j_i(u)$, $i=1, 2$ or 3. The only skew-symmetric operator mapping $X(u)$ onto $p(u)$ is

$$\Omega^*_{\omega_i}(u)=\begin{pmatrix}\theta_i & 0\\ 0 & \beta^{-1}\theta_i\end{pmatrix},\qquad \beta^{-1}=V'\left(\frac{1}{2}v^2\right),$$

where

$$\theta_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\qquad \theta_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\qquad \theta_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

But $\Omega^*_{\omega_i}$ is symplectic if and only if β is constant. Hence, only the harmonic oscillator has a meaningful bi-Hamiltonian formulation. In fact, not only one but three different degenerate ones, namely for $i=1, 2, 3$. Without loss of generality we assume $\beta=\pm 1$, and we treat only the stable case $\beta=+1$. The unstable case $\beta=-1$ is, from the mathematical point of view, in complete analogy.

Now, combining the two Hamiltonian formulations, we obtain that

$$\Phi_i=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix}\theta_i & 0\\ 0 & \theta_i\end{pmatrix}=\begin{pmatrix}0 & -\theta_i\\ \theta_i & 0\end{pmatrix}$$

maps symmetry-generators onto symmetry-generators. This operator is hereditary since it does not depend on u. Thus the

$$\sigma_{i,n}(u)=\Phi^n{}_iX(u),\qquad n\in \boldsymbol{N}_0$$

are generators of a commuting group. And the

$$\gamma_{i,n}=\begin{pmatrix}0, & -I\\ I, & 0\end{pmatrix}\sigma_{i,n}, \qquad n\in N_0$$

do define gradients of conserved quantities which are in involution. The conserved quantities obtained this way are, $H(u)$, $j_i(u)$ and

$$p(u)=v^2+w^2-v_i^2-w_i^2 .$$

But the hereditariness of Φ_i does not mean that we cannot use this operator to construct non-commuting symmetry groups. In order to do that we only have to replace X by a suitable symmetry-generator. For example, take

$$\tilde{X}(u)=\begin{pmatrix}0, & -I\\ I, & 0\end{pmatrix}dj_2(u),$$

then, since angular-momentum is conserved, all $\tilde{\sigma}_n=\Phi_1{}^n\tilde{X}$ must be infinitesimal generators of one-parameter symmetry groups. Explicit calculation yields

$$\tilde{\sigma}_0(u)=\tilde{X}(u), \quad \tilde{\sigma}_1(u)=\begin{pmatrix}0\\ -w_1\\ 0\\ 0\\ v_1\\ 0\end{pmatrix}, \quad \tilde{\sigma}_2(u)=\begin{pmatrix}0\\ 0\\ v_1\\ 0\\ 0\\ w_1\end{pmatrix}, \quad \tilde{\sigma}_3(u)=\tilde{\sigma}_1(u).$$

The one-parameter symmetry group given by $\tilde{\sigma}_1$ is

$$u\to u(\tau)=\begin{pmatrix}v_1\\ v_2+\tau v_1\\ v_3\\ w_1\\ w_2+\tau w_1\\ w_3\end{pmatrix},$$

which clearly does not preserve angular-momentum. Hence, $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ do not commute. Among these generators only $\tilde{\sigma}_0$ corresponds to a conserved quantity whereas $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ do not.

2.2. *Coupled systems which describe long wave interactions*

2.2.a Introduction and principal results

By $S(\boldsymbol{R})$ we denote the space which was already considered in 2.1.a. The manifold under consideration will be $S=S(\boldsymbol{R})\oplus S(\boldsymbol{R})$. Its dual will be the direct sum of the duals considered before. Vectors in S are denoted by $\binom{u}{v}$, where $u, \varphi \in S(\mathbf{R})$. Operators in S are two-by-two matrices whose entries are operators in $S(\mathbf{R})$. The same holds for operator-valued functions, etc. An evolution equation in S is the same as a coupled system of two evolution equations in $S(\mathbf{R})$.

We consider dynamical systems of the following form:

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t = K(u, \varphi), \tag{18}$$

where $K(u, \varphi)$ is a two-component vector

$$K(u, \varphi) = \begin{pmatrix} K_1(u, \varphi) \\ K_2(u, \varphi) \end{pmatrix}.$$

Especially we are interested in the following systems:

a) *Hirota-Satsuma equation*[4]

$$u_t = \frac{1}{2} u_{xxx} + 3uu_x - 3\varphi\varphi_x, \tag{19·1}$$

$$\varphi_t = -\varphi_{xxx} - 3u\varphi_x. \tag{19·2}$$

b) *The symmetrically-coupled KdV*

$$u_t = u_{xxx} + \varphi_{xxx} + 6uu_x + 4u\varphi_x + 2u_x\varphi, \tag{20·1}$$

$$\varphi_t = u_{xxx} + \varphi_{xxx} + 6\varphi\varphi_x + 4\varphi u_x + 2\varphi_x u. \tag{20·2}$$

c) *The complexly-coupled KdV*

$$u_t = u_{xxx} + 6uu_x + 6\varphi\varphi_x, \tag{21·1}$$

$$\varphi_t = \varphi_{xxx} + 6u\varphi_x + 6u_x\varphi. \tag{21·2}$$

d) *Another system*

$$u_t = \varphi_{xxx} + 6\varphi_x u + 6\varphi u_x, \tag{22·1}$$

$$\varphi_t = u_{xxx} + 6uu_x + 6\varphi\varphi_x. \tag{22·2}$$

All these equations admit soliton solutions. For a) complete integrability was conjectured by Hirota and Satsuma.[4] We claim that all these equations do have bi-Hamiltonian formulations, but that only c) and d) are described by Hamiltonian pairs in the sense of Gelfand-Dorfman. All the equations do have countably many conserved covariants and symmetry generators (for which we give explicit formulas). But—roughly speaking—c), as well as d), have twice as many conserved covariants as the other equations. Hence, only they should be called completely integrable (I admit that this is more of a philosophical statement than a mathematical statement since complete integrability is not well defined for infinite dimensional manifolds). In all cases the conserved covariants do have potentials and the symmetry groups are commutative. For c) and d) this is a consequence of the hereditariness of the equations. For b) this can be seen by careful inspection, and for a) a careful and lenghty analysis has to be carried out (since the analysis is based on completely different methods, it will be published separately).

2.2.b Systematic coupling

Let us make some remarks about how to couple symplectic structures in order to obtain new composite symplectic structures. This is one aspect of the important problem: How should completely integrable systems interact without losing complete integrability? A systematic study of that problem will be published separately, here we only concentrate on some elementary results which are needed in the analysis of Eqs. a)~d).

Let $\theta_1(u)$, $\theta_2(u)$ be implectic operators with respect to the manifold $S(\boldsymbol{R})$ (for example, the operators considered in § 2.1). Then obviously

$$\theta(u, \varphi)=\begin{pmatrix}\theta_1(u), & 0 \\ 0, & \theta_2(\varphi)\end{pmatrix} \tag{23}$$

is an implectic operator with respect to the manifold $S=S(\boldsymbol{R})\oplus S(\boldsymbol{R})$. Another implectic operator for that manifold is given, for arbitrary λ, by

$$\theta(u, \varphi)=\begin{pmatrix}\theta_1(u+\lambda\varphi)+\theta_2(u-\lambda\varphi), & \frac{1}{\lambda}\{\theta_1(u+\lambda\varphi)-\theta_2(u-\lambda\varphi)\} \\ \frac{1}{\lambda}\{\theta_1(u+\lambda\varphi)-\theta_2(u-\lambda\varphi)\}, & \frac{1}{\lambda^2}\{\theta_1(u+\lambda\varphi)+\theta_2(u-\lambda\varphi)\}\end{pmatrix}. \tag{24}$$

This fact can be proved by direct calculation. A more elegant method to prove this is to consider a linear Bäcklund-transformation of the form

$$\begin{pmatrix}\tilde{u} \\ \tilde{\varphi}\end{pmatrix}-\begin{pmatrix}u+\lambda\varphi \\ u-\lambda\varphi\end{pmatrix}=0\,.$$

Then, the transformation formulas for implectic operators[5] yield immediately that (24) is the Bäcklund transformation of (23), and therefore again implectic. Now, let us have a look on symplectic composite operators. Let $J_1(u)$, $J_2(u)$ be symplectic with respect to $S(\boldsymbol{R})$ then obviously

$$J(u, \varphi)=\begin{pmatrix}J_1(u), & 0 \\ 0, & J_2(\varphi)\end{pmatrix} \tag{25}$$

must be symplectic. Furthermore, if S is some arbitrary constant skew-symmetric operator in $S(\boldsymbol{R})$ then

$$J(u,\varphi)=\begin{pmatrix} 0, & S\varphi \\ \varphi S, & 0 \end{pmatrix} \tag{26}$$

must be symplectic (direct calculation). Since symplectic operators are constituting a vector space any linear combination of (25) and (26) is again symplectic. This will be used in the following examples.

2.2.c The Hirota-Satsuma system

From § 2.1.a we know that

$$\theta_1(u)=\left(\frac{1}{2}\partial^3+\partial u+u\partial\right)$$

is implectic. Putting $\theta_2(u)=\theta_1(u)$ and $\lambda=(1/\sqrt{2})$ (24) tells us that

$$\theta(u,\varphi)=\begin{pmatrix} \frac{1}{2}\partial^3+\partial u+u\partial, & \partial\varphi+\varphi\partial \\ \partial\varphi+\varphi\partial, & \partial^3+2\partial u+2u\partial \end{pmatrix} \tag{27}$$

is implectic. Now, the exterior derivative of

$$p_1(u,\varphi)=\frac{1}{2}\int_{-\infty}^{+\infty}(u^2-\varphi^2)dx \tag{28·1}$$

is equal to

$$dp_1(u,\varphi)=\gamma_1(u,\varphi)=\begin{pmatrix} u \\ -\varphi \end{pmatrix}. \tag{28·2}$$

A simple calculation shows that

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t=\theta(u,\varphi)dp_1(u,\varphi) \tag{29}$$

is the Hirota-Satsuma equation (19). Hence, the operator $\theta(u,\varphi)$ provides a Hamiltonian formulation for this equation.

To find a suitable inverse-Hamiltonian formulation for (19) we first restrict our tangent space. This is done for technical reasons. Instead of putting the tangent space equal to S itself we take $\hat{S}=(\partial S(\boldsymbol{R}))\oplus S(\boldsymbol{R})$, i.e., the space of those $\binom{v}{\cdot}\in S$ with

$$\int_{-\infty}^{+\infty} v\,dx=0\,.$$

Now, clearly the operator valued functions ∂^{-1} and $(\partial^{-1}u+u\partial^{-1})$ are skew-symmetric with respect to $\partial S(\boldsymbol{R})$. Furthermore they are symplectic. Taking linear combinations of symplectic operators which are of the forms (25) and (26), we find that

$$J(u, \varphi)=\begin{pmatrix} \partial+2\partial^{-1}u+2u\partial^{-1}, & -2\partial^{-1}\varphi \\ -2\varphi\partial^{-1}, & -2\partial \end{pmatrix} \tag{30·1}$$

is again symplectic. Application of $J(u, \varphi)$ to the right-hand-side of (19)

$$K(u, \varphi)=\begin{pmatrix} \frac{1}{2}u_{xxx}+3uu_x-3\varphi\varphi_x \\ -\varphi_{xxx}-3u\varphi_x \end{pmatrix} \tag{30·2}$$

results in the closed one-form

$$J(u, \varphi)K(u, \varphi)=dp_3(u, \varphi), \tag{30·3}$$

where

$$p_3(u, \varphi)=\int_{-\infty}^{+\infty}\Big\{\frac{1}{4}uu_{xxxx}+\frac{5}{4}u^2u_{xx}+\frac{5}{4}u^4 - \frac{3}{2}\varphi^2u^2-u\varphi\varphi_{xx}-4u\varphi_x\varphi_x+\frac{3}{4}\varphi^4+\varphi\varphi_{xxxx}\Big\}dx \tag{30·4}$$

(apart from a misprint of one coefficient, this corresponds to the density I_4 which was found by Hirota and Satsuma[4]). Hence, (30·1) provides an inverse-Hamiltonian formulation for (19).

Now, taking the operators

$$\Phi(u, \varphi)=\theta(u, \varphi)J(u, \varphi), \tag{31·1}$$

$$\Phi^\dagger(u, \varphi)=J(u, \varphi)\theta(u, \varphi), \tag{31·2}$$

we obtain, by virtue of theorem 2, the following symmetry generators σ_n for the Hirota-Satsuma equation:

$$\sigma_{n+2}(u, \varphi)=\Phi(u, \varphi)\sigma_n(u, \varphi) \tag{32·1}$$

with

$$\sigma_1(u, \varphi)=\begin{pmatrix} u_x \\ \varphi_x \end{pmatrix} \tag{32·2}$$

and

$$\sigma_2(u, \varphi)=K(u, \varphi). \tag{32·3}$$

The corresponding conserved covariants are given by

$$\gamma_{n+2}(u, \varphi)=\Phi^\dagger(u, \varphi)\gamma_n(u, \varphi), \tag{33·1}$$

$$\gamma_0(u, \varphi)=\begin{pmatrix}1\\0\end{pmatrix}, \tag{33·2}$$

$$\gamma_1(u, \varphi)=\begin{pmatrix}u\\-\varphi\end{pmatrix}. \tag{33·3}$$

An explicit calculation shows that Φ (or equivalently $\Phi^\dagger$) is not hereditary. Hence, the bi-Hamiltonian formulation of the Hirota-Satsuma equation does not result out of a Hamiltonian pair.

As mentioned before, neither the commutativity of the σ_n nor the fact that the γ_n do have potentials does come out of theorem 2. To prove these facts we have to apply ideas which are similar to those presented in Refs. 9) and 14). For completeness we give here the explicit form of the potentials. One has to write down the two components of σ_{n+1}

$$\sigma_{n+1}(u, \varphi)=\begin{pmatrix}\sigma_{n+1}^{(1)}(u, \varphi)\\\sigma_{n+1}^{(2)}(u, \varphi)\end{pmatrix}.$$

Then the following integration over the first component

$$p_n(u, \varphi)=\int_{-\infty}^{+\infty} x\sigma_{n+1}^{(1)}(u, \varphi)dx$$

yields the conserved quantities of (19). (To obtain the densities found by Hirota and Satsuma one has to eliminate x by partial integration).

2.2.d The symmetrically coupled KdV

Let us keep the notation of the last section. Replace (27) and (30·1) by

$$\theta(u, \varphi)=\begin{pmatrix}\theta_1(u), & 0\\0, & \theta_1(\varphi)\end{pmatrix}, \quad \text{where } \theta_1(u)=\partial^3+2(\partial u+u\partial), \tag{34}$$

$$J(u, \varphi)=\begin{pmatrix}\partial^{-1}, & \partial^{-1}\\\partial^{-1}, & \partial^{-1}\end{pmatrix}. \tag{35·1}$$

Then these operators are implectic and symplectic, respectively. Start with

$$\gamma_1(u, \varphi)=\begin{pmatrix}u+\varphi\\u+\varphi\end{pmatrix}=dp_1(u, \varphi), \tag{36·1}$$

where

$$p_1(u, \varphi)=\frac{1}{2}\int_{-\infty}^{+\infty}(u+\varphi)^2 dx\ . \tag{36·2}$$

Then

$$\theta(u, \varphi)\gamma_1(u, \varphi)=K(u, \varphi) \tag{37}$$

is the right-hand-side of (20). Hence, (34) yields a Hamiltonian formulation of this equation. The one-form

$$\gamma_3(u, \varphi)=JK(u, \varphi)$$

is closed since it is the gradient of

$$p_2(u, \varphi)=\int_{-\infty}^{+\infty}\left\{(u+\varphi)^3-\frac{1}{2}(u_x+\varphi_x)^2\right\}dx .$$

Hence we have found an inverse-Hamiltonian formulation. The operators θ and J are not compatible (i.e., they do not correspond to a Hamiltonian pair). This is the same as saying that neither

$$\Phi(u, \varphi)=\theta(u, \varphi)J \tag{38·1}$$

nor

$$\Phi^{\dagger}(u, \varphi)=J\theta(u, \varphi) \tag{38·2}$$

are hereditary. Nevertheless (theorem 2) the sequences

$$\gamma_{n+2}=\Phi^{\dagger}\gamma_n \tag{39·1}$$

with

$$\gamma_0=\begin{pmatrix}1\\1\end{pmatrix}, \qquad \gamma_1(u, \varphi)=\begin{pmatrix}u+\varphi\\u+\varphi\end{pmatrix} \tag{39·2}$$

and

$$\sigma_{n+2}=\Phi\sigma_n , \tag{40·1}$$

$$\sigma_1(u, \varphi)=\begin{pmatrix}u_x\\\varphi_x\end{pmatrix}, \qquad \sigma_2(u, \varphi)=K(u, \varphi) \tag{40·2}$$

yield conserved covariants and symmetry generators, respectively. It is easy to see that all the γ_n do have potentials p_n. These potentials are

$$p_n(u, \varphi)=Q_n(u+\psi),$$

where Q_n is the n-th conservation law of the KdV. This is obvious, because if $\binom{u}{\varphi}$ is a solution of (20) then $\tilde{u}=(u+\varphi)$ must be a solution of the KdV. Of course, the explicit form of the symmetry-generators does not follow out of this observation.

2.2.e The complexly-coupled KdV

Usually, one derives from the occurrence of infinitely many conservation laws the justification to call a dynamical system completely integrable. In this sense (19) and (20) are completely integrable. In order to show that one has to be careful with this notion we have included the systems (21) and (22). Both systems are of the same order as (19) or (20) and they have conservation laws of exactly the same order as these equations But, in addition, they have another sequence of conservation laws. To see this, replace (27) and (30·1) by

$$\theta(u,\varphi)=\begin{pmatrix}\theta_1(u), & 2\partial\varphi+2\varphi\partial\\ 2\partial\varphi+2\varphi\partial, & \theta_1(u)\end{pmatrix}, \tag{41·1}$$

where

$$\theta_1(u)=\partial^3+2\partial u+2u\partial, \tag{41·2}$$

and by

$$J=\begin{pmatrix}\partial^{-1}, & 0\\ 0, & \partial^{-1}\end{pmatrix}, \tag{42}$$

respectively.

Then these operators are implectic and symplectic respectively. Furthermore, they constitute a Hamiltonian pair in the sense of (10). This is equivalent to the fact that

$$\Phi(u,\varphi)=\theta(u,\varphi)J \tag{43·1}$$

and

$$\Phi(u,\varphi)^{*}=J\theta(u,\varphi) \tag{43·2}$$

are hereditary. The operator $\Phi(u,\varphi)$ commutes[8] with the vector field

$$\sigma_1(u,\varphi)=\begin{pmatrix}u_x\\ \varphi_x\end{pmatrix}. \tag{44·1}$$

Hence, the vector fields

$$\sigma_{n+1}(u,\varphi)=\Phi(u,\varphi)^n\sigma_1(u,\varphi) \tag{44·2}$$

all must commute. In addition $\Phi(u,\varphi)$ commutes with

$$\tilde{\sigma}_1(u,\varphi)=\begin{pmatrix}\varphi_x\\ u_x\end{pmatrix}. \tag{45·1}$$

Hence, all the vector fields

$$\tilde{\sigma}_{n+1}(u,\varphi)=\Phi(u,\varphi)^n\tilde{\sigma}_1(u,\varphi) \tag{45·2}$$

again do commute. Since σ_1 and $\tilde{\sigma}_1$ commute, the family of vector fields given by the σ_n *and* the $\tilde{\sigma}_n$ is commutative (elementary consequence of (9), the same proof as in Ref. 8)).

Therefore, any of the evolution equations given by either

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t = \sigma_m(u, \varphi) \tag{46·1}$$

or

$$\begin{pmatrix} u \\ \varphi \end{pmatrix}_t = \tilde{\sigma}_m(u, \varphi) \tag{46·2}$$

has all the other σ_n and $\tilde{\sigma}_n$ as symmetry-generators. Special cases of these equations are (21) (equal to (46.1) for $m=2$) and (22) (equal to (46·2) for $m=2$). Since $\gamma_1 = J\sigma_1$ as well as $\tilde{\gamma}_1 = J\tilde{\sigma}_1$ are closed one-forms, the compatibility of θ and J yields that the conserved covariants

$$\gamma_n = J\sigma_n \tag{47·1}$$

and

$$\tilde{\gamma}_n = J\tilde{\sigma}_n \tag{47·2}$$

are closed. The potentials p_n and $\tilde{p}_n$ of these conserved covariants are easily calculated, they are closely related to the KdV. If Q_n denotes the n-th conserved quantity of the KdV then the conserved quantities of (21) and (22) are given by

$$p_n(u, \varphi) = \frac{1}{2}\{Q_n(u+\varphi) + Q_n(u-\varphi)\} \tag{48·1}$$

and

$$\tilde{p}_n(u, \varphi) = \frac{1}{2}\{Q_n(u+\varphi) - Q_n(u-\varphi)\}. \tag{48·2}$$

Taking the sum of these conserved quantities one sees that all the conservation laws of (20) are conservation laws of (21), and (22) as well. But in addition to these, (21) and (22), although they are of the same order as (20), do have a second series of conservation laws. This demonstrates that (20) (and similarly (19)) should not be called completely integrable. Of course, the structure of Eqs. (21) and (22) is the same as the structure of the KdV since these equations go over to systems of uncoupled KdV equations by a simple variable transformation ($p = u+\varphi$, $g = u-\varphi$).

Acknowldgements

The author is indebted to the referees whose constructive comments helped to improve this paper.

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