Progress of Theoretical Physics, Vol. 70, No. 6, December 1983

Mastersymmetries, Higher Order Time-Dependent Symmetries and Conserved Densities of Nonlinear Evolution Equations

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(Received April 11, 1983)

As examples, for the Lie algebras of mastersymmetries, all time-dependent symmetries and constants of motion of the Benjamin-Ono equation, the Kadomtsev-Petviashvili equation, and all their generalizations are explicitly constructed. It is shown that these quantities exist in any polynomial order of time, that they are not in involution and that they do not coincide for different members of the hierarchies. It turns out that the corresponding Lie algebras are finitely generated and that the crucial role in this generating-process is played by vector fields which are constant on the manifold under consideration. The general method for the construction of the relevant quantities is described in detail, so that it can be applied to other nonlinear evolution equations as well.

§ 1. Introduction

Symmetries and conservation laws of nonlinear evolution equations provide a major contribution to a better understanding of these equations. Apart from that, these quantities play an important role in different areas of application. Let alone the interest constants of motion can claim in their own right, these quantities (symmetries and constants of motion alike) allow reductions of infinite dimensional flows to finite dimensional invariant submanifolds; thus selecting special solutions which usually are of particular physical interest (soliton solutions, etc.).

With the new discovery of so many completely integrable evolution equations, there is a growing demand for simple, transparent and direct methods to obtain these quantities in an explicit form. In order to contribute to a partial satisfaction of this demand, we introduce in this paper the notion of mastersymmetry. For a specified equation a mastersymmetry (of degree n) is a derivation in the Lie algebra of vectorfields having the property that an n-fold application leaves the commutant of the flow under consideration invariant. Mastersymmetries are in a correspondence with symmetries depending explicitly on time. Mastersymmetries of degree one were first discovered for the Benjamin-Ono equation and the Kadomtsev-Petviashvili equation[1),2)] in fact they exist for all the popular completely integrable equations.

The method was extended by Chen, Lee and Lin [3)~6)] in order to describe time dependent symmetries of first order in t. By methods similar in nature (but different in conception) Broer and Eikfelder[7)] even could construct higher order time dependent symmetries for the Benjamin-Ono equation. But still, a lot of guesswork was involved in guessing the relevant vector fields in order to start the recursion procedure.

In this paper we study the Lie algebra background of all these methods on a systematic base and we show how to obtain the higher order mastersymmetries from simple constant fields. At first glance, it seems to be a tremendous set back that, in passing over to time dependent symmetries, one loses the commutativity of the symmetry group as well

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as the property that the symmetry groups for different members of the hierarchy coincide (in fact both properties only hold for the time independent quantities). But, actually, this apparent disadvantage turns out to be a major advantage. This, insofar, that now the Lie algebra of the symmetry group is *finitely generated*. For the BO and the KP (as well as many other equations) it turns out that the Lie algebra of the trivial symmetries together with a constant vector field is finite, and that insertion of *one* additional nontrivial symmetry generates the infinite algebra of *all* time dependent symmetries. So we come very close to proving the conjecture that a relevant nonlinear evolution equation has infinitely many symmetries as soon as it has one nontrivial.

In the first part of this paper we point out the systematic background of mastersymmetries and time dependent symmetries. Then the Benjamin-Ono equation is treated in great detail ; from the viewpoint of symmetries as well as from that of conserved densities. Furthermore, we show for this equation how the first nontrivial symmetry can be constructed out of the information given by the constant mastersymmetry x.

Then the same kind of analysis is carried out for the *KP* equation. It should be remarked that this analysis can be performed for other equations as well (see for example Ref. 8)).

§ 2. Time dependent symmetries

Let M be some C^∞-manifold and let u denote the variable running through M. The manifold may eventually be of infinite dimension. Let $\mathcal{L}^*$ be a Lie algebra of C^∞-vector fields and let $\mathcal{L}$ stand for some suitable subalgebra. Recall that, if $A(u)$, $B(u)$ are C^∞-vector fields then, in case that M itself is a vector space, the commutator [A, B] is given by

$$[A(u),B(u)]=\frac{\partial}{\partial\varepsilon}\{A(u+\varepsilon B(u))-B(u+\varepsilon A(u))\}\Big|_{\varepsilon=0}. \tag{1}$$

Consider some evolution equation

$$u_t=K(u), \qquad u=u(t)\in M \tag{2}$$

with $K\in\mathcal{L}$. Then $G\in\mathcal{L}^*$is said to be a *symmetry* (abbreviation for symmetry group generator) of (2) if $[K,G]=0$. This is the same as saying that the flow given by $u_t=G(u)$ provides a one-parameter symmetry group for (2), or, that the infinitesimal transformation

$$u(t)\to u(t)+\varepsilon G(u(t)), \qquad \varepsilon \text{ infinitesimal}$$

leaves (2) form-invariant.

Now, take a family $G(u,t)$, $t\in R$, of C^∞-vector fields depending in a C^∞-way on the parameter t. Then we call this a *time-dependent symmetry* for (2) if the infinitesimal transformation

$$u(t)\to u(t)+\varepsilon G(u(t),t), \qquad \varepsilon \text{ infinitesimal}$$

leaves (2) form-invariant. We easily see that this is equivalent to

$$G_t=[K,G], \tag{3}$$

where the right-hand side is the usual Lie bracket (1) and the left-hand side is the partial derivative with respect to the parameter t. Observe that the time dependent symmetries are a Lie algebra with respect to the Lie bracket given by (1). If one is willing to work within the framework of formal power series then time dependent symmetries are easily found.

Denote the adjoint map given for $B \in \mathcal{L}^*$ by $\hat{B}$, i. e.,

$$\hat{B}A = [B,A] \text{ for all } A \in \mathcal{L}^*.$$

Furthermore denote by $\exp(t\hat{K})$ the operator defined on $\mathcal{L}^*$ by the Taylor series of the exponential function

$$\exp(t\hat{K}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \hat{K}^k . \tag{4}$$

Then for any $T \in \mathcal{L}^*$ the quantity

$$G_T(t) = \exp(t\hat{K})T \tag{5}$$

satisfies (3) formally. Of course, in the absence of a topological structure for $\mathcal{L}^*$ (5) does not make sense except for those T for which the series (5) reduces to a finite sum. In order to single out these T we call $T \in \mathcal{L}^*$ a *K-generator* of degree n if

$$\hat{K}^{n+1} T = 0. \tag{6}$$

If (6) holds then the series (5) is reduced to the sum

$$G_T(t) = \sum_{k=0}^{n} \frac{t^k}{k!} \hat{K}^k T, \tag{7}$$

which is an element of $\mathcal{L}^*$, hence a time dependent symmetry for (2). Thus K-generators and time dependent symmetries which are polynomial in t, are in a one-to-one correspondence (one easily sees that any such time dependent symmetry must have the form of (7)). Some obvious remarks about K-generators seem to be appropriate.

Observation 1:

(i) If T is a K-generator of degree $n+1$ then $[K,T]$ is a K-generator of degree n.

(ii) If T_1, T_2 are K-generators of degree n_1 and n_2, respectively, then from Leibniz' formula we obtain that $[T_1, T_2]$ is a K-generator of degree n_1+n_2-1. Hence, the linear hull of the K-generators (all degrees) is a sub-Lie algebra of $\mathcal{L}^*$.

(iii) The K-generators of degree 0 are exactly the symmetries of (2).

Let us denote the K-generators of degree 0 by K^0. This is then the commutant of K with respect to $\mathcal{L}^*$. The use of (ii) and (iii) is based on the fact that we are able to construct out of one nontrivial K-generator T(of degree $n \geq 2$) as many different K-generators of degree $n-1$ as we are given elements of K^0. Commutation with T results in K-generators of degree $2n-2$, and commuting these again with T we find K-generators of degree $3n-3 \geq 3$. And so forth until we have eventually found infinitely many K-generators of any degree. Commuting these an appropriate number of times with elements of K^0, we then obtain infinitely many elements of K^0. Thus, via this construction, meaningful K-generators yield the construction of an infinite dimensional symmetry group for (2).

§ 3. Master symmetries

In this section we show that if the commutant of K is abelian then, under mild additional assumptions, K-generators have a very strong additional property. Lie algebra elements with this property will be called mastersymmetries and, in the case of abelian commutants, one of the main consequences of this property will be that K-generators are automatically G-generators whenever $[K,G]=0$. This will lead to an elementary procedure to calculate out of one suitable K-generator all time dependent and time independent symmetries not only for (2) but for a whole hierarchy belonging to (2). Again we study the situation of a Lie algebra $\mathcal{L}$ embedded in some larger Lie algebra $\mathcal{L}^*$. For $K\in\mathcal{L}$ we denote by $K^\perp$ its commutant in $\mathcal{L}$, i. e.,

$$K^\perp=\{A\in\mathcal{L}|[A,\ K]=0\}.$$

Observe that this may be different from the commutant in $\mathcal{L}^*$ which was denoted by K^0. We call $T\in\mathcal{L}^*$ a *K-mastersymmetry* of degree 1 if

$$[T,\ A]\in K^\perp \qquad \text{for all } A\in K^\perp,$$

i. e., if $\hat{T}$ maps $K^\perp$ into itself. Observe that this definition requires

$$[[T,\ A],\ K]=0 \qquad \text{for all } A\in K^\perp$$

and, *in addition*, that

$$[T,A]\in\mathcal{L} \qquad \text{for all } A\in K^\perp,$$

since $K^\perp$ is supposed to be the commutant with respect to $\mathcal{L}$ (not with respect to $\mathcal{L}^*$). By induction we define $T\in\mathcal{L}^*$ to be a K-mastersymmetry of degree $n+1$ if, for all $A\in K^\perp$, the commutator $[T,\ A]$ is a K-mastersymmetry of degree n. Observe that T is a K-mastersymmetry of degree n if and only if, for arbitrary $A_1,\cdots,A_n\in K^\perp$, we have

$$\hat{A}_1\cdot\hat{A}_2\cdots\hat{A}_nT\in K^\perp.$$

Again, by Leibniz' formula, the commutator $[T_1,T_2]$ of two K-mastersymmetries of degree n_1 and n_2, respectively, is a K-mastersymmetry of degree n_1+n_2-1. Hence the linear hull of all K-mastersymmetries constitutes a sub-Lie algebra of $\mathcal{L}^*$. Of course, all K-mastersymmetries are K-generators (of the same degree), but in general, being a K-generator imposes a very weak condition compared to being a K-mastersymmetry.

THEOREM I: *Let $K^\perp$ be abelian and T a K-generator of degree n such that, for arbitrary $A_1,\cdots,A_n\in K^\perp$, we have*

$$(\star)\hat{A}_1\cdots\hat{A}_nT\in\mathcal{L}\,.$$

Then T is a K-mastersymmetry of degree n.

Proof: We start with $n=1$. Then the assumption ($\star$) means that $\hat{K}T\in K^\perp$. Consider $A\in K^\perp$. Then from the Jacobi identity we find

$$0=[[K,A],T]=[[K,T],A]+[K,[A,T]]=[\hat{K}T,A]+[K,\hat{A}T].$$

Since $K^\perp$ is abelian we have $[\hat{K}T,A]=0$, thus $[K,\hat{A}T]=0$. Hence, since $\hat{A}T\in\mathcal{L}$, $\hat{A}T$

must be an element of $K^{\perp}$. So T is a mastersymmetry (of degree 1).

Now, assume the theorem holds true for all $n \leq m$. Assume further that $(\hat{K})^{m+1}T \in K^{\perp}$ and $\hat{A}_1 \cdots \hat{A}_{m+1} T \in \mathcal{L}$ for arbitrary $A_1, \cdots, A_{m+1} \in K^{\perp}$. All what remains to be proved is that T must be a K-mastersymmetry (of degree $m+1$). But this is quite obvious. Put $T_1 = \hat{K}T$. Then $(\hat{K})^m T_1 \in K^{\perp}$ and $\hat{A}_1 \cdots \hat{A}_m T_1 \in \mathcal{L}$ for $A_1, \cdots, A_m \in K^{\perp}$. Since the theorem is true for $n=m$, T_1 must be a mastersymmetry (of degree m). Now, take again arbitrary elements $A_2, \cdots, A_{m-1} \in K^{\perp}$ and put $T_2 = \hat{A}_2 \cdots \hat{A}_{m+1} T$. Since $K^{\perp}$ is abelian we have $\hat{K}T_2 = \hat{A}_2 \cdots \hat{A}_{m+1} T_1$. Hence $\hat{K}T_2 \in K^{\perp}$ (because T_1 was already proved to be a mastersymmetry). Now, applying the theorem for n=1 we obtain that T_2 is a K-master symmetry (of degree 1), i. e., $\hat{A}_1 T_2 \in K^{\perp}$ for every $A_1 \in K^{\perp}$. So, we have shown

$$\hat{A}_1 \cdots \hat{A}_{m+1} T \in K^{\perp} \text{ for all } A_1, \cdots, A_{m+1} \in K^{\perp}. \qquad \blacksquare$$

The use of this theorem lies in the fact that, for abelian $K^{\perp}$ and for $G \in K^{\perp}$, suitable K-generators of degree n yield G-mastersymmetries of degree n. Hence performing the same construction as described at the end of the last section we will be able to construct out of one meaningful K-generator and one nontrivial element of $K^{\perp}$ the algebra of all K-mastersymmetries which is then identical with the algebra of all G-mastersymmetries, for $G \in K^{\perp}$.

Of course a crucial property is that $K^{\perp}$ is required to be abelian. This is checked either by direct inspection after the construction is carried out (for example, in Refs. 1), 2) and 8)) or seen from structural properties of the Lie algebra under consideration. For example when the Lie algebra $\mathcal{L}$ is *beautiful*. Here we call a subset $\mathcal{A}$ of $\mathcal{L}$ beautiful if for any $A \in \mathcal{A}$ either A is trivial (i. e., commutes with every element in $\mathcal{L}$) or if $A^{\perp}$ is abelian. In fact, as we shall see later on, some Lie algebras of vector fields, which turn up in real life, are indeed beautiful. A systematic study of this property has been carried out.[9]

At the end of this section we want to remark that the requirements in Theorem I may be weakened in order to obtain a useful variant of Theorem I.

THEOREM II : *Let $K^{\perp}$ be abelian and T a K-generator (of degree n) and $G \in K^{\perp}$. Assume $(\hat{G})^n T \in \mathcal{L}$. Then T is a G-generator (of degree n).*

Proof : The same as for Theorem I only replace all the A_i by G. $\blacksquare$

Combining this theorem with the fact that mastersymmetries are generators, which correspond uniquely to time dependent symmetries being polynomial in t, we obtain:

THEOREM III : *Assume that $K^{\perp}$ is abelian. Let $H(u,t) = \sum_{m=0}^{n} (t^m/m!) T_m(u)$ be a time dependent symmetry of $u_t = K(u)$. Let $G \in K^{\perp}$ and put $A_0 = T_0$, $A_m = (\hat{G})^m A_0$. If $A_n \in \mathcal{L}$ then $R(u,t) = \sum_{m=0}^{n} (t^m/m!) A_m(u)$ must be a time dependent symmetry of $u_t = G(u)$.*

§ 4. Examples and applications

4.1. *The one-dimensional case*

The manifold under consideration shall be a vector space S of C^{∞}-functions on the real line. We assume that S is closed under taking derivatives and performing additional operations which will be specified later on. The variable in S is denoted by $u; u_x, u_{xx}, \cdots$, $u_{(n)}, \cdots$ denote its derivatives. The Lie algebra $\mathcal{L}$ is generated by the polynomials in u, u_x, etc. (including the constant polynomial 1) and is assumed to be closed under the additional operations.

We would like to remark that, at least in the absence of additional operations, this Lie algebra is in fact beautiful. The only trivial element is u_x, i. e., it commutes with all elements of the Lie algebra. All nontrivial elements have abelian commutants. This result is suggested by a surprising and ingenious result of Tu [10] who proved the beauty of all those vector fields where at least second derivatives occur. A thorough but lengthy analysis shows that this result can be extended to the whole of $\mathcal{L}$ (Ref. 9)).

Therefore we do not have to worry about the assumption that $K^{\perp}, K$ nontrivial in $\mathcal{L}$, has to be abelian. Nevertheless, I want to emphasize that we do not need this result in order to verify the subsequent results, since we can check, once the construction of $K^{\perp}$ has been done, whether or not $K^{\perp}$ is abelian, by straightforward computations (Ref. 1) or 2)). By $\mathcal{L}^*$ we denote the Lie algebra which is generated by $\mathcal{L}$ and which is closed against multiplication with the independent variable x. In the absence of additional operations this is the algebra of polynomials in $x, u, u_x, u_{xx}, \cdots$. This algebra is far from being beautiful. Special attention will be given to the constant vector fields $1, x, x^2, \cdots$. To me it was very surprising that a thorough analysis of, for example, the vector fields x yields a tremendous amount of information about very many nonlinear evolution equations. We would like to remark that in all examples the condition ($*$) of Theorem I is trivially fulfilled. So, no special attention will be given to this condition.

In the following we are going to analyse the equations

$$u_t = K(u), \qquad u = u(t) \in S, \tag{8}$$

where $K \in \mathcal{L}$. We concentrate our attention on the BO-equation; but the same kind of analysis can be carried out for some other equations.

4.1.1. *The Benjamin-Ono equation*

We consider the well-known[11)~14)] Benjamin-Ono equation

$$u_t = Hu_{xx} + 2uu_x, \qquad \text{(BO equation)} \tag{9}$$

where H stands for the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x} d\xi \qquad \text{(principal value)}.$$

Put $K(u) = Hu_{xx} + 2uu_x$ then a simple "leading term" analysis (Ref. 1)) shows that $K^{\perp} \subset \mathcal{L}$ is abelian.

Remark 1:

i) 1 is a K-generator of degree 1.

ii) x is a K-generator of degree 2.

iii) Since $K^{\perp}$ is abelian 1 and x are mastersymmetries for the Benjamin-Ono equation of degree 1 and 2, respectively.

The proof of that remark is completely trivial and consists of writing down the following obvious commutators:

$$[1,\ K]=-2u_x, \tag{10·a}$$

$$[u_x,\ K]=0, \tag{10·b}$$

$$\tau_0=[x,\ K]=-2xu_x-2u, \tag{10·c}$$

$$[\tau_0,\ K]=4K. \tag{10·d}$$

At first glance, one gets the impression that the only feature which distinguishes this result is its *overwhelming triviality* and that this result, certainly, cannot be of any practical use at all.

But far from being true!

Although the Lie algebra generated by $1, x, \tau_0, K$ is four-dimensional, it becomes right away infinite-dimensional as soon as one includes one additional symmetry of (9). In fact, by adding one nontrivial symmetry of (9) one generates the Lie algebra of *all* mastersymmetries of the Benjamin-Ono equation, including all time-independent symmetries and all mastersymmetries of any degree. Hence, the algebra then yields time dependent symmetries of *any* polynomial order in t. We leave out the proof that any additional symmetry (or K-generator) generates all of the Lie algebra of mastersymmetries, since this result is, at the moment, not of great practical importance (the interested reader may carry out the proof for himslf by adapting the methods proposed in Ref. 9)). Instead of doing this, we will present some of the relevant quantities in greater detail.

For simplicity we take one of the known symmetries of the BO. Later on, we show how to obtain even that systematically from the information provided in (10). The quantity (Refs. 13) and 1))

$$K_2(u)=(2u^3+3H(uu_x)+3uHu_x-2u_{xx})_x \tag{11}$$

is known to be a symmetry for BO. Thus

$$\tau_1=[x,\ K_2(u)]=-6xK(u)-6u^2-9Hu_x \tag{12}$$

must be a mastersymmetry of degree 1. To check this we compute explicitly

$$[\tau_1,\ K]=6K_2\,.$$

Hence, we obtain (as in Ref. 1)) an infinite sequence of commuting symmetries for BO by:

$$K_1=K,$$
$$\vdots$$
$$K_{n+1}=\frac{1}{6}[\tau_1,\ K_n]. \tag{13}$$

Infinitely many mastersymmetries (of degree 1) are obtained by

$$\tau_n=[x,\ K_{n+1}]. \tag{14}$$

Observe that commutation of τ_n with τ_m only yields again mastersymmetries of degree 1 (since $[K_n,\ K_m]=0$). This is the reason for the fact that the procedure in Ref. 4) only leads to time-dependent symmetries linear in t. From (7) we obtain that these symmetries are

$$G_{1,n}=\tau_n+t[K_1,\tau_n]=\tau_n+t[K_{n+1},\tau_0]. \tag{15·a}$$

Observe that these quantities do not constitute symmetries for other equations in the BO hierarchy. But, of course, the corresponding symmetries for the equations

$$u_t=K_m(u) \tag{9·m}$$

are given by :

$$G_{1,n}^{(m)}=\tau_n+t[K_m,\tau_n]. \tag{15·b}$$

Now, symmetries of higher order in t are constructed by use of higher order mastersymmetries. Observe that the quantities $\tau_{k,n}$ given by the recursion

$$\begin{aligned}&\tau_{2,n}=[x,\tau_n], \qquad n=1,2,\cdots\\ &\tau_{k+1,n}=[x,\tau_{k,n}], \qquad k=1,2,\cdots;\quad n=1,2,\cdots\end{aligned} \tag{16}$$

are mastersymmetries of degree k. Hence, we obtain from (7) time dependent symmetries for the BO

$$G_{k,n}=\sum_{l=0}^{k}\frac{t^l}{l!}(\hat{K})^l\tau_{k,n}=\exp(t\hat{K})\tau_{k,n}, \tag{17·a}$$

The corresponding quantities for higher order members of the hierarchy $(9\cdot m)$ are

$$G_{k,n}^{(m)}=\sum_{l=0}^{k}\frac{t^l}{l!}(\hat{K}_m)^l\tau_{k,n}=\exp(t\hat{K}_m)\tau_{k,n}\,. \tag{17·b}$$

4.1.2. *Time-dependent constants of motion for the BO*

In order to construct higher order time dependent constants of motion, we consider a suitable Lie algebra homomorphism from the appropriate Poisson brackets between densities into the algebra of vector fields. We only need to sketch the relevant constructions since they are more or less standard.

For the case of the BO hierarchy we regard two vector fields G_1, G_2 as equivalent :

$$G_1\equiv G_2$$

if there are vector fields (in the Lie algebra under consideration) G_3, G_4, G_5 such that

$$G_1-G_2=G_{3,x}+G_5HG_4+G_4HG_5\,.$$

The equivalence classes are called *densities*, and $\bar{G}$ denotes the class given by G. The notion makes sense insofar as, under suitable boundary conditions (for example, if $u(x)$ vanishes rapidly at $x=\pm\infty$), integration $\int_{-\infty}^{+\infty}\cdot\,dx$ cannot distinguish between equivalent fields. Now, define (G_1,G_2) to be the class given by $G_1\cdot G_2$, then this is a density-valued scalar product in the vector fields. The great advantage now is that the operators D and H, which generated our Lie algebra, have adjoints with respect to this scalar product ; in fact, both operators are skew-symmetric. Using this fact we discover that for each vector field G_1 (in the Lie algebra under consideration) there is a unique vector field ∇G_1 (called gradient of G_1) such that

$$G_1'[G_2]\equiv(\nabla G_1,G_2).$$

Observe that the gradient does not distinguish between different elements of density classes. To see some examples take

$$\mathcal{H}_0=\frac{u^2}{2},$$

$$\mathcal{H}_1=\frac{1}{2}uHu_x+\frac{u^3}{3},$$

$$\mathcal{H}_2=-uu_{xx}-\frac{3}{2}u_xHu^2+\frac{1}{2}u^4,$$

$$T_{-1}=-xu,$$

$$T_0=-xu^2,$$

$$T_{2,1}=\frac{1}{2}x^2u, \tag{18·a}$$

then the respective gradients are

$$\nabla\mathcal{H}_0=u,$$

$$\nabla\mathcal{H}_1=Hu_x+u^2,$$

$$\nabla\mathcal{H}_2=2u^3+3H(uu_x)+3uHu_x-2u_{xx},$$

$$\nabla T_{-1}=-x,$$

$$\nabla T_0=-2xu,$$

$$\nabla T_{2,1}=\frac{1}{2}x^2. \tag{18·b}$$

Now, we introduce poisson brackets among densities

$$\{\bar{G}_1,\bar{G}_2\}\underset{\text{def}}{=}(\nabla\bar{G}_1)\cdot(\nabla\bar{G}_2)_x. \tag{19}$$

Recall that densities are equivalence classes (which are denoted by a bar). Then the map

$$\bar{G}_1\to\Gamma\bar{G}_1\underset{\text{def}}{=}(\nabla G_1)_x \tag{20}$$

is a Lie algebra homomorphism into the vector fields, i. e.

$$\Gamma\{\bar{G}_1,\bar{G}_2\}=[\Gamma\bar{G}_1,\Gamma\bar{G}_2]. \tag{21}$$

Observe furthermore:

$$\Gamma\mathcal{H}_0=K_0(u)=u_x,$$

$$\Gamma\mathcal{H}_1=K_1(u),$$

$$\Gamma\mathcal{H}_2=K_2(u),$$

$$\Gamma T_{-1}=-1,$$

$$\Gamma T_0=-2(xu)_x=\tau_0,$$

$$\Gamma T_{2,1}=x. \tag{22}$$

We call a density $\overline{G(u)}$ a *conserved density* for the BO if

$$\frac{d}{dt}\bar{G}(u(t))\equiv 0$$

for all solutions $u(t)$ of (9). Taking any element G out of the density class of $\bar{G}$ we see that this means

$$0\equiv G_t+G'[K_1]\equiv G_t+(\nabla G,K_1)\equiv \bar{G}_t+\{\bar{G},\mathcal{H}_1\}.$$

Hence, $\bar{G}$ is a conserved density if and only if

$$0=\bar{G}_t+\{\bar{G},\mathcal{H}_1\}, \tag{23}$$

where G_t denotes the partial derivative with respect to time. Since the Γ-transform of that equation is (3), and because (22) tells us that the crucial operators in (10) and (12) are just the Γ-images of the operators considered in (18) we see that all the recursion formulas for the time dependent symmetries have their corresponding analogues in the density space. Writing down these quantities explicitly we get, starting with $\bar{T}_{1,1}$ and $\mathcal{H}_0$ or $\mathcal{H}_1$, respectively, the densities:

$$\mathcal{H}_{n+1}=\frac{1}{6}\{\bar{T}_{1,1},\mathcal{H}_n\}, \qquad \text{where } \bar{T}_{1,1}=\{\bar{T}_{2,1},\mathcal{H}_2\}. \tag{24·a}$$

This is the Γ-preimage of (13). The Γ-preimage of (14) we write in the way

$$\bar{T}_{1,n}=\{\bar{T}_{2,1},\mathcal{H}_{n+1}\}. \tag{24·b}$$

Further recursion yields

$$\bar{T}_{2,n}=\{\bar{T}_{2,1},\ \bar{T}_{1,n}\}, \qquad n=1,2,\cdots \tag{24·c}$$

and

$$\bar{T}_{k+1,n}=\{\bar{T}_{2,1},\bar{T}_{k,n}\}, \qquad n=1,2,\cdots. \tag{24·d}$$

Now, all the $\mathcal{H}_n$ are conserved densities for any member of the hierarchy

$$u_t=K_m(u)=(\nabla\mathcal{H}_m(u))_x \tag{9·m}$$

and conserved densities (in any polynomial order of t) for (9·m) are given by

$$\mathcal{H}^{(m)}_{k,n}=\sum_{l=0}^{k}\frac{t^l}{l!}(\hat{\mathcal{H}}_m)^l\bar{T}_{k,n}=\exp(t\hat{\mathcal{H}}_m)\bar{T}_{k,n}, \tag{25}$$

where $\hat{\mathcal{H}}$ is the adjoint map coming from $\mathcal{H}$, i. e.,

$$\hat{\mathcal{H}}\bar{T}\underset{\text{def}}{=}\{\mathcal{H},\bar{T}\}.$$

The corresponding constants of motion are given by integration.

4.1.3. *The modified BO*

We did not yet mention how one can find systematically the first nontrivial symmetry K_2 of the BO. Instead of solving this problem we shall do something slightly more general.

We replace the Hilbert transform operator H in (9) by an arbitrary operator L. We assume that L commutes with x and the differential operator. We would like to find out under what conditions on L the equation

$$u_t = Lu_{xx} + 2uu_x = K_1(u) \tag{26}$$

has a nontrivial symmetry.

Obviously, all the relations (10) remain true for this case. Therefore all the formulas given for the BO remain true if L replaces H, provided, we can find at least one higher order symmetry. So, we have shown that (26) has infinitely many symmetries (and conservation laws) if and only if it has one nontrivial symmetry. So, all has been reduced to the computation of one additional polynomial symmetry for (26). We briefly sketch the analysis necessary for this computation (we implicitly use some ideas from Ref. 9)).

Assume that $K_2(u)$ is a nontrivial symmetry of (26). We look in $K_2(u)$ at the term of highest order in u, which must commute with uu_x, the highest order term of (26). The only vector fields commuting with that, are of the form $u^n u_x$ (simple calculation, Ref. 9)). If $n=1$, then $K_2(u)$ must be a scalar multiple of $K_1(u)$ (the right-hand side of (26)). So, this is ruled out. If $n>2$ then we commute $K_1(u)$ $(n-2)$-times with 1. Since 1 is a K-generator (of degree 1) we obtain by that a symmetry with highest order term $u^2 u_x$. Now, we are going to work in the equivalence classes modulo arbitrary interchanges of L and polynomials in u, u_x, $u_{xx}\cdots$, etc. That means we treat L as a number. Hence, (26) is now equivalent to Burger's equation, and $K_2(u)$ must be equivalent to a symmetry of Burger's equation (Ref. 15) for the construction of these symmetries), i. e., $K_2(u)$ is of the form

$$K_2(u) \sim (L^*)^2 u_{xxx} + 3L^*(uu_x)_x + 3u^2 u_x. \tag{27}$$

Here, the star at the L shall be a remainder that eventually we have to interchange L with polynomials in u, $u_x, \cdots$. Now, everything is reduced to a straightforward computation. Since x is a K-generator (of degree 2)

$$3\tau = [K_2, x] = 3x\tilde{K}(u) + B(u),$$

(where $\tilde{K}(u) \sim L^* u_{xx} + 2uu_x$, $B(u) \sim 6L^* u_x + 3u^2$) must be a K-generator (of degree 1). Commuting this with the symmetry u_x we find $\tilde{K} = K_1$. Also, the commutator $[K_1, \tau]$ must be a symmetry. Since its highest order term is twice that of K_2 we obtain

$$[K_1, \tau] = 2K_2. \tag{28}$$

This condition determines the position of the L's in K_2, as well as in τ, uniquely. Let us show this (via explicit calculations, for simplicity): The term $\sim 6L^* u_x$ in $B(u)$ must be of the form $(6-3\alpha)Lu_x + 3\alpha u_x L1$. Since u_x commutes with K_1 the term $3\alpha u_x L1$ does not contribute in (28). So, we skip this term, and it remains to determine α. We find

$$[K_1, \tau] = 2\left(L^2 u_{xxx} + \left(\frac{1+\alpha}{2}\right)L(u^2)_{xx} + (2-\alpha)(uLu_x)_x + (u^3)_x\right).$$

So, K_2 must be of the form:

$$K_2=L^2u_{xxx}+(\frac{3}{2}+\lambda)L(uu_x)_x+(\frac{3}{2}-\lambda)(uLu_x)_x+3u^2u_x\,. \tag{29}$$

And a sufficient and necessary condition for (26) to have one (and therefore infinitely many) symmetry is that this K_2 commutes with K_1. Again explicit calculation yields that this is equivalent to $\lambda=0$ and

$$2uL^2D^2u_x-2L^2D^2uu_x+6LDu_xLu_x-3D(Lu_x)^2=0 \tag{30}$$

for all u. Now, $L=H$ (Hilbert transform) certainly fulfills this equation, since

$$2H(uHu)=(Hu)^2-u^2\,.$$

There are other operators fulfilling this relation. Right now, they do not seem of practical interest. But certainly, the method we adopted, will be of some importance in more complicated situations (several dimensions, systems of equations).

4.2. *The KP-equation*

We consider the two-dimensional KdV (or Kadomtsev-Petviashvili equation)

$$u_{tx}=(6uu_x-u_{xxx})_x-3u_{yy}$$

which we formally write as

$$u_t=6uu_x-u_{xxx}-3D^{-1}u_{yy}, \tag{31}$$

where $D^{-1}=\int_{-\infty}^{x}\cdot\,dx$ is the inverse of the differential operator. Applying to a constant $\hat{C}$ we define the Lie algebraic meaning of D^{-1} by

$$D^{-1}C=xC.$$

The Lie algebra $\mathcal{L}^*$ under consideration is now replaced by all polynomials in $u(x,y)$ and $D^\alpha\partial^\beta u(x,y)$, with $\alpha+\beta\geq 0$ and $\beta\geq 0$. Here ∂ and D stand for

$$\partial=\frac{\partial}{\partial y}\,,\qquad D=\frac{\partial}{\partial x}.$$

Equation (31) has the following obvious symmetries:

$$K_0(u)=u_x,$$

$$K_1(u)=-2u_y,$$

$$K_2(u)=6uu_x-u_{xxx}-3D^{-1}u_{yy}\,.$$

An infinite sequence of commuting symmetries was constructed in Ref. 2). In Ref. 5) the method was extended and an infinite sequence of time dependent (first order in t) symmetries was constructed.

We construct all time dependent symmetries (any order in t) and constants of motion by the following observation:

Remark 2: $G_{3,1}=y^2$ is a mastersymmetry of degree 3 for the KP.

Proof: Explicit computation yields

$$G_{2,2}=[G_{3,1},K_2]=6u_xy^2-6x,$$

$$G_{1,1}=[G_{2,2},K_2]=6y^2DK_2(u)-(6Du-D^3-3D^{-1}\partial^2)(6u_xy^2-6x)$$
$$=6y^2(-u_{xxxx}+6(uu_x)_x-3u_{yy})$$
$$+6y^2u_{xxxx}-36y^2(uu_x)_x+36u+2\cdot 36yu_y+18y^2u_{yy}+36(xu_x)_x$$
$$=36(xu_x+2u+2yu_y).$$

And finally, we obtain

$$[G_{1,1},K_2]=-72K_2$$

which indeed proves that y^2 is a mastersymmetry of degree 3. ■

Now, the procedure is straightforward. We only need one nontrivial symmetry for the KP in order to generate all time dependent symmetries (and constants of motion) for the KP and all members of its hierarchy. We could proceed as in § 1. 3, but in order not to bore the reader we skip this part.

From Ref. 2) we take the following symmetry of the KP

$$K_3(u)=12(u_{xxy}-4uu_y-2u_xD^{-1}u_y+D^{-2}u_{yyy}). \tag{32}$$

Then, via the mastersymmetry (of degree 2)

$$\phi_+=-\frac{1}{12}G_{2,2}=-\frac{1}{2}(u_xy^2-x)$$

we obtain a suitable mastersymmetry

$$\tau_+=[K_3,\phi_+]=yK_2-2xu_y-4D^{-1}u_y \tag{33}$$

of degree 1. This mastersymmetry was already given in Ref. 2). Now, starting with K_0(or K_1,K_2,K_3) we obtain from

$$K_{n+1}=[K_n,\tau_+] \tag{34}$$

a sequence of commuting time independent symmetries of the KP. And sequences of mastersymmetries are obtained from:

$$\tau_{1,n}=[K_n,\phi_+],$$
$$\tau_{2,n}=[\tau_{1,n},\phi_+]$$
$$\vdots$$
$$\tau_{k+1,n}=[\tau_{k,n},\phi_+].$$

Remark, that $\tau_{k,n}$ is a mastersymmetry (of degree k) for every member of the KP-hierarchy. Hence, the general time dependent symmetry for

$$u_t=K_m(u) \tag{31$\cdot m$}$$

is of the form

$$G_{k,n}^{(m)}=\sum_{l=0}^{k}\frac{t^l}{l!}(\hat{K}_m)^l\tau_{k,n}=\exp(t\hat{K}_m)\tau_{k,n}\,. \tag{35}$$

Again, the same analysis can be performed on the density side of the KP. Let us briefly sketch the relevant constructions. We regard vector fields G_1,G_2 as equivalent

$$G_1 \equiv G_2$$

if there are vector fields $\tilde{G}_r$ such that

$$G_1 - G_2 = \sum_r D^{\alpha_r} \partial^{\beta_r} \tilde{G}_r$$

with $\beta_r \geq 1$ and $\alpha_r + \beta_r \geq 1$. Densities are the equivalence classes and again $(\bar{G}_1, \bar{G}_2)$ is the class given by $G_1 G_2$. Now, the operators D and ∂ are skewsymmetric with respect to this density-valued scalar product. As before, the gradient $\nabla \bar{G}_1$ of $\bar{G}_1$ is defined by

$$(\nabla \bar{G}_1, \bar{G}_2) \equiv G_1'[G_2] \qquad \text{for all } G_2$$

and as Poisson brackets we take

$$\{\bar{G}_1, \bar{G}_2\} = (\nabla \bar{G}_1) \cdot (\nabla \bar{G}_2)_x .$$

The map $\Gamma : \bar{G}_1 \to (\nabla \bar{G}_1)_x$ is then a Lie algebra homomorphism. Observe that the Poisson brackets and the Lie algebra homomorphism are depending very much on the special equation under consideration. They are given by the requirement that the equation is Hamiltonian with respect to these quantities (see Ref. 16) for details). The fact that Poisson brackets and Lie algebra homomorphism are the same for KP and BO is pure coincidence. It is easily seen that the $K_0, \cdots, K_3$ have Hamiltonians:

$$\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2} u^2, \\
\mathcal{H}_1 &= - u D^{-1} u_y, \\
\mathcal{H}_2 &= \frac{1}{2} (u_x)^2 + u^3 - \frac{3}{2} u D^{-2} u_{yy}, \\
\mathcal{H}_3 &= 6 \left\{ (u_{xy})^2 - 2 u^2 D^{-1} u_y + \frac{1}{2} u D^{-3} \partial^3 u \right\},
\end{aligned} \tag{36}$$

i. e., we have

$$\Gamma \mathcal{H}_i = K_i, \qquad i = 0,1,2,3.$$

Now, take the mastersymmetry

$$\bar{\Phi}_+ = -\frac{1}{4} (y^2 u^2 - x^2 u). \tag{37}$$

Check $\Gamma \bar{\Phi}_+ = \phi_+$ and proceed with the usual routine.

$$\begin{aligned}
\bar{T}_+ &= \{\mathcal{H}_3, \bar{\Phi}_+\}, \\
\mathcal{H}_{n+1} &= \{\mathcal{H}_n, \bar{T}_+\}, \\
\bar{T}_{1,n} &= \{\mathcal{H}_n, \bar{\Phi}_+\} \\
&\vdots \\
\bar{T}_{k-1,n} &= \{\bar{T}_{k,n}, \bar{\Phi}_+\}
\end{aligned}$$

in order to obtain the general time dependent conserved density for

$$u_t = K_m(u) \tag{31·m}$$

in the form:

$$\mathscr{K}_{k,n}^{(m)}=\sum_{l=0}^{k}\frac{t^l}{l!}(\mathscr{K}_m)^l\bar{T}_{k,n}=\exp(t\mathscr{K}_m)\bar{T}_{k,n}\,. \tag{38}$$

Some lower order members of this hierarchy have been calculated in Ref. 17). In connection with the KP, we would like to draw the readers attention to the significant results obtained by E. Date, M. Jimbo, M. Kashiwara and T. Miwa.[18] A comparison of their methods and the methods proposed in this paper will be carried out in a subsequent paper.

I am greatly indebted to the referee whose constructive comments helped to improve this paper considerably.

References

1) A. S. Fokas and B. Fuchssteiner, Phys. Lett. **86A** (1981), 341.
2) W. Oevel and B. Fuchssteiner, Phys. Lett. **88A** (1982), 323.
3) H. H. Chen and Jeng-Eng Lin, Phys. Lett. **89A** (1982), 163.
4) H. H. Chen and Jeng-Eng Lin, Phys. Lett. **91A** (1982), 381.
5) H. H. Chen, Y. C. Lee and Jeng-Eng Lin, preprint, 1982.
6) H. H. Chen, Y. C. Lee and Jeng-Eng Lin, preprint, 1982.
7) L. J. F. Broer and H. M. M. Ten Eikfelder, Phys. Lett. **92A** (1982), 56.
8) B. Fuchssteiner, preprint, 1983.
9) B. Fuchssteiner, preprint, 1983.
10) Gui-Zhang Tu, Commun. Math. Phys. **77** (1980), 289.
11) T. B. Benjamin, J. Fluid Mech. **29** (1967), 559.
12) H. Ono, J. Phys. Soc. Jpn. **39** (1975), 1082.
13) T. L. Bock and M. D. Kruskal, Phys. Lett. **74A** (1979), 173.
14) A. S. Fokas and M. J. Ablowitz, Studies Appl. Math. **68** (1983), 1.
15) B. Fuchssteiner, Nonlinear Anal. TMA 3 (1979) 849; Prog. Theor. Phys. **65** (1981), 861.
16) B. Fuchssteiner, Prog. Theor. Phys. **68** (1982), 1082.
17) E. Infeld, Phys. Lett. **86A** (1981), 205.
18) E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Preprints of RIMS No. 3356-361, Kyoto (1981).