ADVANCES IN COMPUTER METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS • V
R. VICHNEVETSKY and R.S. STEPLEMAN (Editors)
Publ. IMACS - 1984 ©

DISTRIBUTION ALGEBRAS AND ELEMENTARY SHOCK WAVE ANALYSIS

Benno Fuchssteiner
Gesamthochschule Paderborn
D 4790 Paderborn

We review the main features and the canonical structure of the algebra of almost-bounded distributions which was introduced in order to treat distribution solutions of nonlinear differential equations. It is shown that this algebra is the canonical extension of a well known construction in shock wave analysis. All results are discussed in context of the evolution equation of shallow water wave theory (lowest order). The impact of distribution solutions on the Hamiltonian structure and the existence of symmetry groups and conservation laws is discussed.

Introduction

Describing the evolution of a dynamical system from its infinitesimal viewpoint is certainly in many respects superior to any other description. For example, finding for a completely integrable nonlinear flow on some infinite dimensional manifold the symmetry group explicitely seems to be an impossible task, whereas finding the infinitesimal generators of one-parameter symmetry groups is a routine matter nowadays [8]. This would not be possible if we were not able to describe the flow by its infinitesimal behaviour. In fact, the strength and the beauty of areas like theoretical mechanics, with all the impact it had on the development of pure mathematics, is based on the infinitesimal aspects of the systems under consideration. To include also noncontinuous solutions into this framework is one of the reasons which led to the invention of distribution theory. But alas, distributions do not constitute an algebra and many of the relevant flows are nonlinear, at least when interaction is involved. So it seems as if noncontinuous solutions (for example shock waves) of nonlinear systems cannot be treated from the infinitesimal viewpoint thus forbidding the application of the heavy machinery of classical mechanics to these systems. To show that this is not necessarily so, is the content of this paper. Moreover, we demonstrate that the usual computational concept for shock waves imposes a canonical algebraic structure on a suitable subspace of distributions.

Let us first review the algebra of almost-bounded distributions (see [7] and [10]). We are interested in distribution algebras having the following properties:

(i) the product rule for the differentiation is valid

(ii) the product is associative

(iii) the algebra is an extension of the usual algebra for functions

(iv) the algebra is translation invariant.

It is well known [23] that there is no algebraic structure in $\mathcal{D}(\mathbb{R})$ (Schwartz's distributions) fulfilling (i) to (iv). So we restrict our considerations to a subspace, the space $B(\mathbb{R})$ of almost-bounded distributions. A distribution $\varphi \in \mathcal{D}(\mathbb{R})$ is said to be almost-bounded if, for every $n \in \mathbb{N}$, its n-th derivative is of the form

$$\varphi^{(n)}(x) = b(x) + \Delta(x) \qquad (1)$$

where b is a locally bounded function and where Δ has discrete support without accumulation point. Of course, the decomposition in (1) is unique. The map $\varphi \to \varphi^{(n)}$ we denote by D^n (usual differentiation operator) whereas the map $\varphi \to b$ is denoted by d^n. Observe that d is a derivation on the algebra of those almost-bounded distributions which are functions.

Theorem [7]: There are exactly two algebraic structures in $B(\mathbb{R})$ fulfilling (i) to (iv). These algebras are, for $\varphi, \Psi \in B(\mathbb{R})$, given by

$$\varphi(x)\Psi(x) \underset{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \varphi(x+\varepsilon)\Psi(x) \qquad (2.1)$$

and

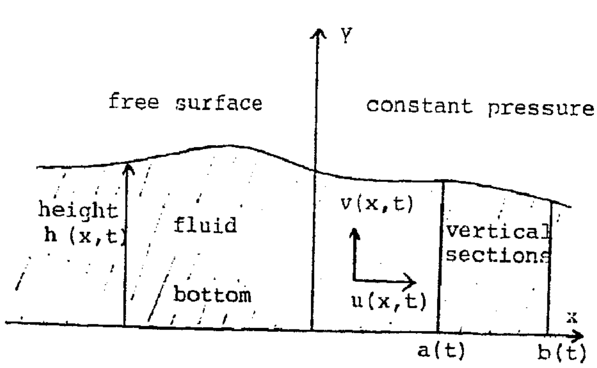
$$\varphi(x)\Psi(x) \underset{\text{def}}{=} \lim_{\varepsilon \uparrow 0} \varphi(x+\varepsilon)\Psi(x) \; . \qquad (2.2)$$

Observe, that the products (2.1) and (2.2) do always exist since, by definition, the singularities of φ and Ψ are not too close together. In any case the product of two distributions with discrete support is equal to zero. If $\eta(x)$ denotes the jump function ($\eta(x) = -1$ for $x < 0$ and $+1$ for $x \geq 0$) and if $\delta(x)$ denotes the Dirac δ- distribution then (2.1) implies $\eta\,\delta = \delta$ and $\delta\,\eta = -\delta$. Hence, the algebra is non-commutative. Noncommutativity implies that there are <u>at least</u> two structures fulfilling (i) to (iv), if there is any, since interchanging the order of factors gives an algebra isomorphism. Therefore, because we do not have more than the minimal number of algebras, we are entitled to say that the almost-bounded distributions have a canonical algebraic structure.

Let me add a short remark in order to avoid misunderstandings. The existence of a "unique" algebra depends heavily on the restrictions we have imposed by (i) to (iv). If one is willing to drop a few of these requirements then, for example, also commutative approaches to distribution multiplication make sense. In fact a number of excellent approaches to distribution multiplication which differ from my approach can be found in the literature (for example [1]-[3], [5], [6], [12]-[16], and especially [20], [21]).

An elementary example

In this section we look at the description of bores (hydraulic jumps) by distribution solutions of the nonlinear partial differential equations derived from shallow water wave theory (lowest oder) (see [4], [24], [27], [10]). We consider the flow of an incompressible fluid along the horizontal x-axis



Here $u(x,y,t)$ denotes the velocity along the x-axis. We assume that the viscosity is zero, that there is neither surface tension nor rotation, and that the only exterior force is gravitation. The assumption of shallow water wave theory (of lowest order) is that $u(x,y,t) = u(x,t)$ has to be independent of y and that the velocity $v(x,y,t)$ in direction of the y-axis is zero. This is the same as assuming that, over the motion, vertical sections remain vertical sections and that the pressure $p(x,y,t)$ in the fluid is the same as the hydrostatic pressure, i.e.

$$\rho\{g(h(x,t) - y) - p(x,y,t)\} = \text{const.}$$
$$= \text{exterior pressure.}$$

where ρ, g are physical constants. In other words, we assume conservation of mass between the moving vertical sections $a(t)$ and $b(t)$ (velocities $u(a(t),t)$ and $u(b(t),t)$)

$$\int_{a(t)}^{b(t)} h(x,t)\,dx = \text{constant} \qquad (3.1)$$

and that the change in momentum is given by the difference of the pressure acting on these sections

$$\rho \frac{d}{dt} \int_{a(t)}^{b(t)} h(x,t)u(x,t)\,dx = \qquad (3.2)$$
$$= \int_0^{h(a(t))} p(a(t),y,t)\,dy - \int_0^{h(b(t))} p(b(t),y,t)\,dy$$

Making $b(t) - a(t)$ infinitesimal and performing some elementary manipulations we obtain the usual nonlinear equations of gas dynamics

$$(h\,u)_x + h_t = 0 \qquad (4.1)$$

$$u\,u_x + g\,h_x + u_t = 0 \; . \qquad (4.2)$$

Now, inserting the ansatz $h(x,t) = h(x-ct)$, $u(x,t) = u(x-ct)$ one observes that, even based on our distribution algebra, no shock wave solutions occur. Certainly, this is contrary to the physical facts. But we completely forgot that the algebra was non-commutative and that we used commutativity freely by going from (3.1), (3.2) to (4.1) and (4.2). Doing this derivation again without commutativity we obtain

$$(h\,u)_x + h_t = 0 \qquad (5.1)$$

$$h\,u u_x + \frac{1}{2}\,g(hh)_x + h\,u_t = 0 \qquad (5.2)$$

Now, a shock wave ansatz really makes sense, we obtain the usual shock solutions with the well known [24] jump conditions (for details see [10]). And, fortunately, none of the results depends on the choice of either of the two algebras (2.1) or (2.2).

An interesting discovery [10] is made if we look at the conservation law given by the energy-conservation of the system. Between the moving sections $a(t)$ and $b(t)$ the change in energy (kinetic and potential) and the power given by pressure is

$$\frac{\partial E}{\partial t} = \rho\,\frac{d}{dt}\int_{a(t)}^{b(t)} \frac{1}{2}\,\{u^2\,h + g\,h^2\}dx +$$

$$+ u(b(t),t)\int_0^{h(b(t))} p(b,y,t)dy - \qquad (6.1)$$

$$- u(a(t),t)\int_0^{h(a(t))} p(a,y,t)dy \quad .$$

Insertion of our formula for the pressure yields

$$\frac{1}{\rho}\,\frac{\partial E}{\partial t} = \{\frac{1}{2}\,u^3 h + g\,uh^2\}_{a(t)}^{b(t)} + \qquad (6.2)$$

$$+ \frac{1}{2}\int_{a(t)}^{b(t)} (u^2 h + g\,h^2)_t\,dx \; .$$

From this we obtain the density

$$\frac{1}{\rho}\,\frac{\partial^2 E}{\partial t\partial x} = (\,\frac{1}{2}\,u^3 h + g\,uh^2)_x + \frac{1}{2}(u^2 h + g\,h^2)_t \qquad (7)$$

and by use of (5) from the above we infer

$$\frac{1}{\rho}\,\frac{\partial^2 E}{\partial t\partial x} = \frac{1}{2}\,g\,\{[u_x,h]h + \frac{1}{2}\,[u,(h^2)_x] \quad (8)$$

where [,] denotes the usual commutator, i.e. $[A,B] = AB - BA$. Looking at (8) we see that energy is conserved when the quantities under consideration are commuting. In case of shock waves, i.e. noncommuting quantities, energy is not conserved, and, in fact, the amount of violation of energy conservation is exactly what it shall be (see [24]). For extensive investigations see [18],[26].

Several questions and doubts have been raised in context with this elementary example. Questions like:

(a) Can we really be sure that the results of similar computations are independent of the choice of the algebra (2.1) or (2.2)?
(b) Are (5.1) and (5.2) really the relevant equations? Is it always necessary to go back to the physical problem in order to find the real equations? In particular: Are there other ways of finding the differential equations from the physical problem which lead - even noncommutativity is observed - to equations different from (5.1) and (5.2) ?

(c) What kind of conservation laws continue to hold for the case that noncommuting physical quantities are considered? Why is it, that energy conservation is violated whereas the corresponding symmetry of time translation still holds for discontinuous solutions?

(d) Having described discontinuous solutions by infinitesimal methods can we now consider linearizations of the equations describing small perturbations? In particular: Can we now describe - even for discontinuous solutions - infinitesimal generators of one-parameter symmetry groups as solutions of the linearized equations? Or in general: What becomes out of the methods of classical mechanics in case of distribution algebras.

And so on!

Indeed all these questions have satisfactory answers. Some of them are given in the sequel.

Shocks for systems of conservation laws

As in [17] we consider evolution equations given by systems of conservation laws, i.e. equations for $u_1(x,t)$, $u_2(x,t)$ of the form

$$u_{1_t} + f_x = 0, \quad u_{2_t} + g_x = 0 \qquad (9)$$

where f and g are nonlinear functions in u_1 and u_2. For simplicity we assume in this paper that they are polynomials. Here the quantities u_1 and u_2 are called conservation laws. The reason for this is evident, because if one considers - for example - the u_1,u_2 as elements in some space S of C^∞-functions in x which vanish sufficiently rapidly at infinity, then the scalar quantities $\int_{\mathbb{R}} u_i(x)dx$, $i=1,2$, are independent of time, i.e. they are conserved. Quite often the systems under consideration are Hamiltonian, for example if we have for the partial variational derivatives with respect to u_i, that

$$g_{u_2} = f_{u_1} \qquad (10.1)$$

i.e. if there is a scalar function $H(u_1,u_2)$ such that

$$g = H_{u_1} \;,\quad f = H_{u_2} \qquad (10.2).$$

In order to explain why this is so we introduce some notation. S^* denotes the space of those functions u^* with $u^*_x \in S$. The spaces $S^2 = S \oplus S$ and $S^{*2} = S^* \oplus S^*$ we call tangent space and cotangent space, respectively. They are the typical fibres of the tangent and cotangent bundles of the manifold S^2 on which a flow is given by (9). Instead of $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ or $\begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}$ we write $\vec{u}$ and $\vec{u}^*$.

Between tangent and cotangent space we introduce a duality

$$< \vec{u}^*, \vec{u} > = \int_{\mathbb{R}} (u_1^* u_1 + u_2^* u_2)dx \qquad (11)$$

thus variational derivatives of scalar quantities become covector fields.

Since our manifold under consideration is a vector space the notions "closed" and "exact" coincide, i.e. the result given by Darboux's theorem not only holds locally but also globally. To be precise, an element

$$\vec{\varphi} = \begin{pmatrix} \varphi_1(u_1,u_2) \\ \varphi_2(u_1,u_2) \end{pmatrix} \in S^{*2}$$

is said to be closed if $(\varphi_1)_{u_2} = (\varphi_2)_{u_1}$,

i.e. if there is a scalar quantity $\Phi : S^2 \to \mathbb{R}$

$$\vec{\varphi} = d\Phi \underset{\text{def}}{=} \begin{pmatrix} \Phi_{u_1} \\ \Phi_{u_2} \end{pmatrix} \qquad (12) .$$

In this case $\vec{\varphi}$ is said to be the gradient of Φ. An evolution equation on S^2

$$\vec{u}_t = K(\vec{u}) \qquad (13)$$

is said to be Hamiltonian if there is a closed two-form $\omega((.,.)$ such that $\omega(K(\vec{u}),\cdot)$ is a gradient. Introduce the operators

$D : S^* \to S$ (derivative with respect to x), $D^{-1} : S \to S^*$ (integration from $-\infty$ to x) and $\vec{D} : S^{*2} \to S^2$, $\vec{D}^{-1} : S^2 \to S^{*2}$

$$\vec{D} = \begin{vmatrix} 0 , D \\ D , 0 \end{vmatrix} \quad , \quad \vec{D} = \begin{vmatrix} 0 & D^{-1} \\ D^{-1} & 0 \end{vmatrix} .$$

Then the two-form $\omega : S^2 \quad \mathbb{R}$ given by

$$\vec{\omega}(\vec{u},\vec{v}) = < \vec{D}^{-1} \vec{u}, \vec{v} > \qquad (14)$$

is closed. Provided that (10) holds the system (9) is then Hamiltonian because it can be written as

$$\vec{u}_t = -\vec{D}\, dH, \quad \text{i.e.} \quad K(\vec{u}) = -\vec{D}\, dH$$

which yields $\omega(K(\vec{u}),\cdot) = -dH$.

One advantage of having Hamiltonian formulations for flows - which are by definition infinitesimal formulations - is that if the closed two-form (symplectic form) under consideration is like (14) then $\vec{D} = (\vec{D}^{-1})^{-1}$ is an operator with the property that it maps gradients of conservation laws onto infinitesimal generators of one-parameter symmetry groups of the system (Noether's theorem). This observation has striking consequences for the construction of the symmetry group of the system [9]. Of course, a conservation law is a map going from the manifold into the real (or complex) numbers such that it is invariant under the flow. And infinitesimal generators of symmetry groups are given by vector fields $G(\vec{u})$ which are solutions of the linearization of the flow, i.e. in case of (13) solutions of

$$G(\vec{u})_t = \frac{\partial}{\partial\varepsilon} K(\vec{u} + \varepsilon\, G(\vec{u}))\,_{|\varepsilon=0} \qquad (15).$$

It should be remarked that (10) is not necessary for the system to be Hamiltonian since the symplectic form responsible for the Hamiltonian structure of the system may differ from that given in (14). In these cases one sometimes also has to change the duality structure under consideration (according to the boundary conditions at infinity). The flow given by (4) is Hamiltonian, because for $\vec{u} = \binom{u}{h}$ it can be written

$$\vec{u}_t = -\vec{D}\, dH(u) \qquad (16.1)$$

where

$$H(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}} (u^2 h + g\, h^2)\, dx \qquad (16.2)$$

is the energy of the system (which is conserved for continuous solutions). Other conservation laws are

$$\int_{\mathbb{R}} h\, u\, dx \quad \text{(conservation of momentum)} \tag{17.1}$$

$$\int h\, dx \quad \text{(conservation of mass)} \tag{17.2}$$

$$\int u\, dx \quad \text{(conservation of mean signal velocity)}. \tag{17.3}$$

The gradient of (17.1) is mapped by $\vec{D}$ onto the infinitesimal generator of the translation group, whereas the gradients of (17.2) and (17.3) are annihilated by $\vec{D}$.

Now, after this detour into Hamiltonian mechanins let us come back to the shock solutions for (9). These are fairly simple to find: Just take the derivatives occuring in (9) in the distributional sense. For piecewise continuous solutions this leads to the usual jump conditions [17]

$$\frac{dx}{dt} = \frac{[f]_x}{[u_1]_x} = \frac{[g]_x}{[u_2]_x} \tag{18}$$

where $x = x(t)$ is the curve of discontinuity and where the symbol $[\ \ ]_x$

$$[f]_x \underset{\text{def}}{=} \lim_{\varepsilon \downarrow 0} (f(x+\varepsilon) - f(x-\varepsilon)) \tag{19}$$

denotes the difference between right and left-hand limits. This is a very consistent approach, but unfortunately it prevents us from speaking about the infinitesimal structure of the system. Thus impeding us from deciding whether or not the system is Hamiltonian at jumps, or from finding out if the connection between symmetry-generators and gradients of conservation laws given by Noethers theorem is still valid, or from looking at small pertubations of the system described by its linearization. To emphasize this point even further: The reason why our shock wave solutions are not consistent with the infinitesimal structure of the system lies in the fact that, for shock wave solutions, we are not allowed to go from (9), by application of the chain rule, to the evolution equations

$$u_{1_t} + f_{u_1} u_{1_x} + f_{u_2} u_{2_x} = 0$$
$$u_{2_t} + g_{u_1} u_{1_x} + g_{u_2} u_{2_x} = 0 \tag{9.1}$$

because in this case products like $f_{u_1} u_{1_x}$ may not be properly defined.

Further we should remark that there is an additional point of arbitraryness in (18) which comes out of the fact that there may be another pair of conservation laws for (19) yielding differential equations which are equivalent to (9) or (9.1)(in the sense that they have the same continuous solutions) but are leading to different jump conditions. Usually this arbitraryness is avoided by additional requirements (for example asking that the conservation laws have to be hyperbolic [17]).

The shock algebra

It is not uninteresting to look at the algebraic structure of the symbol given by (19). Consider, for example, the algebra A of almost-bounded functions, i.e. those elements of $B(\mathbb{R})$ which are everywhere defined, piecewise continuous and honest functions. Then for $f \in A$ we have

$$[f] = E^+ f - E^- f \tag{20.1}$$

where $E^\pm$ are the idempotent algebra-homomorphisms on A given by

$$(E^\pm f)(x) = \lim_{\varepsilon \downarrow 0} f(x \pm \varepsilon) . \tag{20.2}$$

Looking at the behaviour of $[\ \ ]$ for products we find

$$[fg] = [f]E^+(g) + E^-(f)[g] . \tag{21}$$

Observing that the operators $E^\pm$ are right-absorbing, i.e.

$$E^+E^- = E^+ , \; E^-E^+ = E^- \tag{22}$$

and introducing A_s , the set of singular elements in A , given by those elements of A which have singular support, we see that $d_1 = [\ \ ]$ is a derivation (product rule) from A into A_s with respect to the associative algebra [10] (shock algebra) defined in the following way:

$$f * g \underset{\text{def}}{=} (E^+f)\cdot g + f \cdot(E^-g) - (E^+f)\cdot(E^-g) \tag{23}.$$

This algebra coincides modulo singular elements with the usual algebra $(A,\cdot)$ of pointwise multiplication. Of course, the notion "modulo singular elements" makes perfect sense since A_s is a two-sided ideal in $(A,*)$ (as well as in $(A,\cdot)$. The quotients

$$\mathrm{A} = {(A,*)}/{A_s} = {(A,\cdot)}/{A_s}$$

are equal. These quotients are important since from the distributional point of view functions which coincide but on a discrete set are equal. Hence the distributions given by A are really the space A . Because $d_1 = [\ \ ]$ annihilates the elements of A_s, we can also interprete d_1 as a derivation from $A = {}^{(A,*)}/_{A_s}$ going into A_s, which can be considered as an A-bimodule since

$$A_s * A_s = 0 .$$

Now there is a canonical way of making out of $(A \times A_s)$ an algebra. One has to take the trivial extension [11] of A by A_s given via $A_s \cdot A_s = 0$. In order to make out of $d_1 : A \to A_s$ a derivation on some algebra containing the trivial extension $(A \times A_s)$ one again has to perform a canonical construction given by embedding $(A \times A_s)$ into the sequence structure $A \times A_s^\infty$, where A_s^∞ denotes the space of sequences in A_s . This extension of d_1 is given by d_1^* defined by

$$d_{1|_A}^* = d_1$$

$$d_1^* (s_o, s_1, s_2 \ldots) = (0, s_o, s_1 \ldots) \qquad (24)$$
$$\text{for } (s_o, s_1 \ldots) \in A_s^* .$$

The map d_1^* is a derivation on the algebra $A \times A_s^\infty$ (products given by $A_s^\infty \cdot A_s^\infty = 0$) . All these extensions are minimal, the details have been carried out in [10].

Recall that A were distributions. Now, what is the distributional meaning of, say $(0,\ldots,0,s,0,\ldots)$ (n-th place) ? That is easy! Recall that s is unequal to zero only on a discrete set, say $s = a_k$ on the x_k, $k = 1,\ldots$. Then identify $(0,\ldots,s,\ldots)$ with $\sum_k a_k\, \delta^{(n)}(x-x_k)$ and there is found a one-to-one linear map from $A \times A_s^\infty$ onto $B(\mathbb{R})$ the space of almost-bounded distributions. Take on A the derivation d introduced at the beginning of this paper, define $d_{|A_s^\infty} = 0$, then $D = d + d_1^*$ is again a derivation on $A \times A_s^\infty$. And the most important point is that D is carried over, by the isomorphism between $A \times A_s^\infty$ and $B(\mathbb{R})$, into the usual differential operator.

Since $A \times A_s^\infty$ is an algebra we have found an algebra on $B(\mathbb{R})$ having the required properties.

The second algebra is found by interchange of products in (23). This algebra has also the property that (21) defines a derivation.

Now, what is the real meaning of all this algebraic humbug ?
The real meaning is

i) that the algebra of almost-bounded distributions is the canonical extension of the shock algebra given by (20.1) and (21) to the space of almost-bounded distributions

ii) that (relying on the isomorphism we have constructed) we can be sure to obtain, by use of the algebra of almost-bounded distributions, the same shock solutions as from (18), and, that our results do not depend on the choice of either of the two algebras (2.1) or (2.2).

One additional remark in defense of the algebraic considerations in this section: These considerations give the guideline for generalizations of the algebra to different situations since none of the results really depends of the meaning of $E^\pm$ but only on the algebraic properties of these operators.

Our example revisited

First, we observe that the equations (4.1) and (4.2) are of the form required in (9) since they are given by the conservation laws (17.2) (mass) and (17.3) (mean signal velocity). Therefore (18), applied to these conservation laws, should describe the shockfront.

At this point one really suffers a mild shock since (18) does not give the same solutions as (5.1) and (5.2) (which were the physical solutions as we claimed). This looks as if there is a contradiction to the algebraic considerations of the last section. Fortunately, this is not true. We only used the wrong conservation laws! We should have used (17.1) (momentum) and (17.2) (mass) instead. Then their evolution equations really correspond to the ones given by (5.1) and (5.2). Sure, they are equivalent to (4) in case of continuous solutions, but not so in case of discontinuous solutions.

This observation answers several of the questions which we posed in the second section: We are able to find the relevant equations for the noncommutative case of shock waves without going back to the physical problem by choosing the correct conservation laws. But the decision, which of the conservation laws are correct, depends either on our insight into the physical problem or on additional mathematical requirements (hyperboliticity etc.). Our algebra does not lead us in this choice.

In fact, from the algebraic point of view either two of the four conservation laws of the system lead to mathematical meaningful equations. All these equations coincide for continuous solutions but not for discontinuous ones.

Only those conservation laws which we used for the extension of our equation to the noncommutative case remain valid in this case, all others are - in accordance with the physical observation - destroyed by this extension.

This does not mean that all the corresponding symmetry-generators will be destroyed. In case of (5) for example, time translation invariance is still preserved whereas the corresponding energy conservation law is disturbed. The reason for this is the fact that the Hamiltonian structure is destroyed by shock solutions, i.e. in the non-commutative case. This is not surprising since for a one-form, say of polynomial type in the field variables, being à gradient is a rather restrictive condition, usually depending on the algebraic properties based on the commutativity of the field variables. These delicate properties are heavily disturbed by the noncommutative case.

Since we have now for the flow an infinitesimal description which also includes shock waves we are able to consider small perturbations $v = \delta u$ and $\eta = \delta h$ in the usual way. They have to fulfill the linearization of (5.1) given by its variational derivatives:

$$(\eta u)_x + (hv)_x + \eta_t = 0$$

$$\eta u\, u_x + h\, v u_x + h\, u v_x + \frac{1}{2} g(\eta\, h + h\eta)_x + \\ + \eta u_t + h\, v_t = 0 \qquad (25)$$

where u and h have to be solutions of (5). Particular solutions of this system are given by the infinitesimal generators of time and x-translation for the flow (5) (easy exercise in noncommutative distribution multiplication). In this case these quantities v, η have δ-singularities if shock solutions for (5) are considered. This also shows that there are real physical quantities making evolution equations necessary which admit higher singularities (like δ-functions). For example, the equation for the accelleration of the particle and the velocity of the change of the height of our fluid. Equations like this can now be treated from an infinitesimal viewpoint in a consistent way.

References

[1] L. Berg, Construction of distribution algebras, Math.Nachr. 82 (1978), 255-262.

[2] G. Braunss, R. Liese, Canonical products of distributions and causal solutions of nonlinear wave equations, J. Differential Equations 16 (1974), 399-412.

[3] H.J. Bremermann, Some remarks on analytic representations and products of distributions, SIAM J. Appl.Math.15 (1969), 929-943.

[4] R. Courant, K.O. Friedrichs, Supersonic Flow and Shock Waves, New York-Heidelberg-Berlin 1976.

[5] L. Ehrenpreis, Solutions of some problems of division. Amer.J.Math. 76 (1954), 883-903.

[6] B. Fisher, Singular products of distrbutions, Math.Ann. 203 (1973), 103-116.

[7] B. Fuchssteiner, Eine assoziative Algebra über einem Unterraum der Distributionen, ibid.178 (1968), 302-314.

[8] - Application of hereditary symmetries to nonlinear evolution equations, Nonlinear Analysis TMA 3, 849-862 (1979).

- The Lie algebra structure of nonlinear evolution equations admitting infinite dimensional abelian symmetry groups, Progress of Theoretical Physics 65, 861-876 (1981).

[9] - The Lie algebra structure of degenerate Hamiltonian and Bi-Hamiltonian systems, Progress of Theoretical Physics 68, 1082-1104 (1982).

[10] - Algebraic foundation of some distribution algebras, Studia Math. 77, 419-432 (1983).

[11] A. Grothendieck, Éléments de géométrie algébrique (IV), Institut des hautes études, Publications mathématiques No. 20, Bures sur Yvette 1964.

[12] M. Itano, On the theory of the multiplicative product of distributions, Hiroshima Math.J. 6 (1976), 365-375.

[13] V.K. Ivanov, The algebra of a class of generalized functions, Dokl.Akad.Nauk SSR 237, 4 (1977).

[14] K. Keller, Analytic regularizations, finite part prescriptions and products of distributions, Math.Ann. 236 (1978), 48-84.

[15] - Irregular operations in quantum field theory I. Multiplication of distributions, Rep. Mathematiacal Phys. 14 (1978), 885.

[16] H. König, Multiplikation von Distributionen, Math.Ann. 128 (1954), 420-452.

[17] P.D. Lax, J. Glimm, Decay of solutions of systems of nonlinear hyperbolic conservation laws, Memoir of the Amer. Math.Soc. 101 (1970)

[18] R. Liese, Eine Distributionsalgebra und ihre Anwendungen in der nichtlinearen Gasdynamik, Diplomarbeit am Lehrstuhl für Mechanik, Darmstadt 1968.

[19] J.Lighthill, Waves in Fluids, Cambrigde University Press, Cambrigde 1978.

[20] E.E. Rosinger, Distributions and non-linear partial differential equations, Lecture Notes in Math. 684, Springer-Verlag, Berlin-Heidelberg-New York 1978.

[21] - Nonlinear partial differential equations, Mathematics studies 44, North Holland Publ, Amsterdam-New York-Oxford 1980.

[22] L. Schwartz, Théorie des distributions, Hermann, Paris 1951.

[23] - Sur l'impossibilité de la multiplication des distributions. C.R. Acad. Sci. Paris 239 (1954) 847.

[24] J.J. Stoker, Water Waves, Interscience Publishers, New York 1957.

[25] H.F. Walter, Über die Multiplikation von Distributionen in einem Folgenmodell, Math.Ann. 189 (1970), 211-221.

[26] J. Wellmann, Wirbelproduktion, Entropieproduktion und Zirkulationserhaltung an gekrümmten Stossfronten, Habilitationsschrift, Darmstadt 1980.

[27] G.B.Whitham , Linear and Nonlinear Waves, New York-London-Sidney-Toronto 1974.