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EXPOSED FIXPOINTS IN ORDER-STRUCTURES

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Dedicated to Professor Leopoldo Nachbin on the occasion of his sixtieth birthday

1. Introduction

In his beautiful book on topology and order [19], as well as in many other fundamental papers, Leopoldo Nachbin has given invaluable contributions waving the fibre of order-theory into the fabric of analysis, a concept which has proved its advantages more and more over the years. In fact, an order theoretic frame provides an effective setup for many concrete situations which, on the first glance, do not seem to be related at all to order-structures.

In contrast to that the applicability of order theoretic fixpoint theory seems not to be widely known. But indeed, many of the fundamental theorems in abstract analysis deal with objects which are maximal (or minimal) in some intuitive sense. For example: The minimal sublinear functionals which are automatically linear, or 'minimal' sets of points of a convex set where the linear maps attain their maximum (which lead to extreme points), and so on. This list could be continued for quite a while.

In the context of fixpoint theory this means that there is a demand for fixpoints being maximal in some sense. But alas, such fixpoints do exist in general only under additional assumptions on the maps which are involved. For this reason in this paper the notion of maximality is weakened and replaced by exposed fixpoints which are introduced in the next section. There are also given the basic definitions of this paper and an existence theorem for exposed fixpoints is proved for decreasing maps and families of such maps.

In Section 3 these results are applied to the lattice of superlinear vector-valued functionals on some abelian semigroup. There, the fixpoint theorem yields strong combinations of Hahn–Banach and Krein–Milman type theorems.

In Section 4 an iteration theorem for almost commuting families of decreasing maps is presented, generalizing the iteration theorem which was proved in [8] for the case of a single map. Already this special case had turned out to be a rather efficient tool in many areas of analysis (see [8], [1] and [15]).

A few parts of this paper follow the widely circulated but unpublished preprint [7]. But the relevant definitions have been made much more transparent, the corresponding theorems have been extended and the proofs have been cast into a short and digestible form.

2. Exposed Fixpoints

We consider a partially ordered set $(X, \leqslant)$. If $x \leqslant y$, then we say y is *above* x, or, x is *below* y. For $x \leqslant y$ and $x \neq y$ we write $x < y$. In the following we assume that X is *inductively ordered*, which means, that every chain (i.e. linearily ordered subset of X) has to have an infimum. Of course, $\inf \emptyset$ then is the supremum of X (and will be denoted by I).

Now, let us consider a *decreasing* map $\varphi : X \to X$, i.e. $\varphi(x) \leqslant x$ for all $x \in X$. A trivial consequence of Zorn's lemma is (see [1]):

Lemma (2.1). (*Zermelo's fixpoint lemma.*) *φ has a fixpoint.*

It should be noted, that Zorn's lemma is not at all necessary in order to prove this lemma. On the contrary, this lemma actually is the essential step in the proof of Zorn's lemma based on the axiom of choice. Since the average reader may not be interested in set-theoretic technicalities we resist the temptation to include an effective[1] proof of this lemma at this point, but for completeness we add such a proof in the appendix.

We say that z is *φ-above* x if $\varphi(z) \geqslant x$. This implies that z is above x, since φ is decreasing. We say that x *φ-exposes* y, if every z which is above y and φ-above x must be φ-above y. Finally, x is called *φ-exposed*, if x exposes only the elements below x. In other words: x is φ-exposed iff for every y not below x there is some $z \geqslant y$ with $\varphi(z) \geqslant x$ such that y is not below $\varphi(z)$.

In case that only one map φ is involved we say for short 'exposes' instead of 'φ-exposes'.

[1] Effective is called everything which can be proved in ZF (Zermelo–Fraenkel) without AA (axiom of choice).

The following are easy to verify:

Remark (2.2). (1). I is exposed.

(2). x exposes every element below x.

(3). Let $x_1 \geq x_2$ and assume that x_2 exposes y. Then x_1 exposes y.

(4). Let Y be a chain in $\mathrm{ex}(X)$ (the set of exposed points), then the X-infimum of Y is again exposed, hence $\mathrm{ex}(X)$ again is inductively ordered.

Proposition (2.3). *The set* $\mathrm{ex}(X)$ *of* φ*-exposed points is* φ*-invariant.*

Proof. Let x be exposed and assume that $\varphi(x)$ exposes y. We have to prove that $y \leq \varphi(x)$. We know that x exposes y (Remark (2.2) (3)). This implies $x \geq y$ since x is exposed. So, $z = x$ is above y and φ-above $\varphi(x)$. Therefore z must be φ-above y, since $\varphi(x)$ exposes y. This gives $\varphi(z) = \varphi(x) \geq y$. □

Theorem (2.4). φ *has an exposed fixpoint.*

Proof. Since $\mathrm{ex}(X)$ is φ-invariant and inductively ordered we can apply Zermelo's fixpoint lemma to $\mathrm{ex}(X)$ instead of X. □

In order to show that exposed fixpoints are, in a certain sense, maximal we give some examples.

We call φ *order-convex* if $x \leq y \leq z$ and $\varphi(x) \leq \varphi(z)$ always imply $\varphi(x) \leq \varphi(y) \leq \varphi(z)$.

Proposition (2.5). *Let* φ *be order-convex. Then for every exposed fixpoint* x *we have* $x \leq \varphi(y) < y$ *whenever* $x < y$. *In particular: Every exposed fixpoint is a maximal fixpoint.*

Proof. Because of $y > x$ we have that x does not expose y (since x is exposed). Hence, there must be some z with $z \geq y$ and $\varphi(z) \geq x$ but not $y \leq \varphi(z)$. Hence $\varphi(z) \geq \varphi(y) \geq \varphi(x) = x$ which means in particular $\varphi(y) \neq y$. □

The map φ is said to be *monotone* if $x \leq y$ always implies $\varphi(x) \leq \varphi(y)$.

Proposition (2.6). *In case that φ is monotone there is only one exposed fixpoint. This exposed fixpoint is the maximum of all fixpoints.*

Proof. Let x be an exposed fixpoint and y some fixpoint. Then, if z is above y we have $\varphi(z) \geq \varphi(y) = y$ (monotonicity). This means that x exposes y. Hence $y \leq x$ since x was exposed. So x must be the maximum of all fixpoints, in particular it is unique. □

For technical reasons we add:

Proposition (2.7). *There is always an exposed fixpoint $x \leq \varphi(I)$.*

Proof. Obviously, $X^\circ = \{x \in X \mid x \leq \varphi(I)\}$ is φ-invariant since φ was decreasing. So, apply Theorem (2.4) to X° instead of X to find an exposed fixpoint x of X°. We claim that x is exposed in X as well. Assume that x exposes $y \in X$. Since I is above y and φ-above x we must have $\varphi(I) \geq y$. Hence y is in X° and therefore below x since x was exposed in X°. □

It is simple to transfer the preceding results to families of decreasing maps. Let Φ be a non-empty family of decreasing maps $X \to X$. A point is said to be a *common fixpoint* if it is a fixpoint for all $\varphi \in \Phi$.

A point $x \in X$ is said to be *Φ-exposed* if for every $y \in X$, which is not below x there are $\varphi \in \Phi$ and $z \in X$ with $z \geq y$ and $\varphi(z) \geq x$ such that y is not below $\varphi(z)$. Theorem (2.4) together with the axiom of choice easily yields the existence of common exposed fixpoints.

Theorem (2.8). *There is an exposed common fixpoint for Φ.*

A little bit more refined:

Theorem (2.8′). *For any $\varphi_0 \in \Phi$ there is an exposed common fixpoint x for Φ such that $x \leq \varphi_0(I)$.*

Proof. For $y \in X$ define $M_y = \{\varphi \in \Phi \mid \varphi(y) \neq y\}$ and let γ be a choice function picking out of every nonempty M_y a single element and having the additional property that $\gamma(M_I) = \varphi_0$ if $\varphi_0(I) \neq I$. Define a decreasing $\varphi^\circ : X \to X$ by

$$\varphi^\circ(y) = \gamma(M_y)(y) \text{ if } M_y \neq \emptyset \quad \text{and} \quad \varphi^\circ(y) = y \text{ if } M_y = \emptyset$$

From Proposition (2.7) we obtain a φ°-fixpoint $x \leqslant \varphi^\circ(I) = \varphi_0(I)$, which by construction of φ° must be a common fixpoint for Φ because $\varphi^\circ(x) = x$ implies $M_x = \emptyset$. Now observe that φ°-exposed implies Φ-exposed. □

Remark (2.9). (1). Even if all the elements φ of Φ are order-convex then in general the map φ° occurring in the proof of Theorem (2.8′) is not order-convex. Nevertheless, the proof of Proposition (2.5) carries over. That means, if all the $\varphi \in \Phi$ are order-convex then for every Φ-exposed common fixpoint x with $x < y$ there is some $\varphi \in \Phi$ such that $x \leqslant \varphi(y) < y$. In particular x is a maximal fixpoint.

(2). Monotonicity does not carry over from the elements of Φ to the map needed in the proof of Theorem (2.8). In general, the result of Proposition (2.6) is false for families. However it carries over to the situation where all elements of Φ commute (an easy exercise or a consequence of Proposition (4.3)(5).

Remark (2.10). In case that Φ is a countable set, then the choice function in the proof of Theorem (2.8) exists without the help of AA. Hence, in this case Theorem (2.8) can be proved effectively. In general, already the existence of a common fixpoint is effectively equivalent to the axiom of choice as shown in the appendix.

At the end of this section we give a version of Theorem (2.8) which is technically more involved, but rather useful in applications.

Definition (2.11). A set $X' \subset X$ is said to be *completely Φ-invariant* if
(1). $\varphi X' \subset X'$ for all $\varphi \in \Phi$.
(2). The X-infimum of every chain in X' again belongs to X'.
(3). $I \in X'$.

Observe that intersections of completely Φ-invariant subsets are again completely Φ-invariant. Remark that, in case that $\Phi = \{\varphi\}$ is a singleton, the set of Φ-exposed points is completely Φ-invariant (Proposition (2.3) together with obvious considerations). This fact does not remain true for arbitrary families. Nevertheless we obtain:

Theorem (2.12). *Let X' be completely Φ-invariant. Then for every $\varphi_0 \in \Phi$ there is an exposed common fixpoint x for Φ such that x belongs to X' and fulfills $x \leqslant \varphi_0(I)$.*

Proof. Take φ° as in the proof of Theorem (2.8′). Then X' is also completely $\{\varphi^\circ\}$-invariant. Since the set $\mathrm{ex}(X)$ of φ°-exposed points is completely $\{\varphi^\circ\}$-invariant the set $X^\circ = X' \cap \mathrm{ex}(X)$ must have this property. Take a φ°-fixpoint x in X° with $x \leqslant \varphi_0(I)$. Then x has the required properties. □

3. Subadditive Functionals

Let $S = (S, +, \leqslant)$ be some preordered abelian semigroup with neutral element 0. In a preordered semigroup the semigroup structure has to be compatible with the given preorder relation $\leqslant$ in the sense that inequalities can be added.

Furthermore, we consider a complete vector lattice $(R, \leqslant)$ (for example the real numbers). Suprema and infima in R are denoted by sup{ } and inf{ }. We write $\max\{A\}$ in case that $\sup\{A\} \in A$. For technical reasons an element $-\infty = \inf(R)$ is adjoint. To $\boldsymbol{R} = R \cup \{-\infty\}$ addition is extended in the obvious way. Furthermore we define $\lambda(-\infty) = -\infty$ for real $\lambda > 0$ and $0(-\infty) = 0$.

Let $\pi : S \to \boldsymbol{R}$ be a subadditive functional. *Subadditive* means that π is $\mathbb{N}$-homogeneous

$$\pi(ns) = n\pi(s) \qquad \text{for all } s \in S \text{ and } n = 0, 1, 2, \ldots$$

and

$$\pi(s_1 + s_2) \leqslant \pi(s_1) + \pi(s_2) \qquad \text{for all } s_1, s_2 \in S\,.$$

π is said to be *monotone* if

$$s_1 \leqslant s_2 \Rightarrow \pi(s_1) \leqslant \pi(s_2) \qquad \text{for all } s_1, s_2 \in S\,.$$

An $\mathbb{N}$-homogeneous functional $\omega : S \to \boldsymbol{R}$ is called *superadditive* if

$$\omega(s_1 + s_2) \geqslant \omega(s_1) + \omega(s_2) \qquad \text{for all } s_1, s_2 \in S\,.$$

Since $\boldsymbol{R}$ has an order relation, the functionals $S \to \boldsymbol{R}$ are canonically endowed with an order relation given by pointwise order on S. This order relation we denote also by $\leqslant$ since no confusion can arise. So $\omega \leqslant \pi$

means $\omega(s) \leq \pi(s)$ for all $s \in S$. In case that $\omega \leq \pi$ then $\Lambda(\omega, \pi)$ denotes the set of superadditive functionals q between ω and π, i.e. $\omega \leq q \leq \pi$. Observe that the pointwise infimum of two elements of $\Lambda(\omega, \pi)$ is again in $\Lambda(\omega, \pi)$ and that every ascending chain in $\Lambda(\omega, \pi)$ does have a supremum.

Of course, a functional which is sub- and superadditive is called *additive*. We say that a superadditive ω *exhausts* a subadditive π if

$$\pi(s_1) + \omega(s_2) \leq \pi(s_1 + s_2) \qquad \text{for all } s_1, s_2 \in S .$$

This means in particular that $\omega \leq \pi$. As we shall see later, the real meaning of this definition is that π has to be the pointwise supremum of all additive functionals between ω and π, i.e. the additive $\mu \geq \omega$ 'exhaust' π. Observe that if $\omega \leq \pi$ then there is always a unique maximal subadditive functional $p = [\pi, \omega]$ with $\omega \leq p \leq \pi$ such that ω exhausts p. This functional is given by

(3.1)

$$p(s) = [\pi, \omega](s) = \inf\left\{\frac{1}{m}(\pi(ms + t) - \omega(t)) \,\middle|\, 1 \leq m \in \mathbb{N},\, t \in S,\, \omega(t) \neq -\infty\right\}.$$

By construction $[\pi, \omega]$ is monotone whenever π is monotone. Furthermore one observes that $[\pi, \omega] \leq [\pi^\circ, \omega^\circ]$ whenever $\pi \leq \pi^\circ$ and $\omega \leq \omega^\circ$.

Definition (3.1). A superadditive $\mu \in \Lambda(\omega, \pi)$ is called *weakly exposed* (in $\Lambda(\omega, \pi)$) if for every superadditive $q \geq \omega$ and every subadditive $p \leq \pi$ with $p \neq q$ such that q exhausts p there is some $s_0 \in S$ with

(3.2) $$q(s_0) < \mu(s_0)$$

and

(3.3) $$\mu(s_0) \geq p(s_0) .$$

The main point of this definition is that we have both inequalities for the same s_0 and that $\mu(s_0)$ is strictly greater than $q(s_0)$.

Proposition (3.2). *Every weakly exposed $\mu \in \Lambda(\omega, \pi)$ is additive.*

Proof. Put $q = \mu$ and $p = [\pi, \mu]$. Then q exhausts p. Since there can be no s_0 with $q(s_0) < \mu(s_0)$ we must have $q = p = \mu$. Hence μ is also subadditive since p is. □

In addition we have proved:

Proposition (3.3). *If $\mu \in \Lambda(\omega, \pi)$ is weakly exposed then $\mu = [\pi, \mu]$. In particular: If π is monotone then μ is monotone.*

Now, we define in $X = \Lambda(\omega, \pi)$ a preorder relation $<^*$ to be the inverse of $\leqslant$, i.e. $x_1 <^* x_2$ if and only if $x_1(s) \geqslant x_2(s)$ for all $s \in S$. Then $(X, <^*)$ is inductively ordered. For $x \in X$ and $t \in S$ we define $\varphi_t(x): S \to \boldsymbol{R}$
$\boldsymbol{R}$

$$\varphi_t(x)(s) = \sup\{x(\tau) + [\pi, x](nt) \mid \tau \in S,\, n \in \mathbb{N} \text{ with } s = \tau + nt\}.$$

The function $\varphi_t(x)$ is superadditive $\geqslant x$ and we have $\varphi_t(x) \leqslant \pi$ since x exhausts $[\pi, x]$. So $\varphi_t(\cdot)$ must be a decreasing map on $(X, <^*)$. By $\Phi = \{\varphi_t(\cdot) \mid t \in S\}$ we denote the set of all these maps.

Lemma (3.4). *If x is a common exposed fixpoint for Φ then it is weakly exposed.*

Proof. Observe that x is a Φ-fixpoint if and only if $x = [\pi, x]$. Now, take a superadditive $q \geqslant \omega$ and a subadditive $p \leqslant \pi$ with $p \neq q$ such that q exhausts p. If q would be $<^*$-below x then, since q exhausts p (i.e. $[p, q] = p$), the inequality

$$q \geqslant x = [\pi, x] \geqslant [p, q] = p,$$

would lead to the contradiction $p = q$. So, since x is Φ-exposed, there must be some $s_0 \in S$ and some $z \in X$ with (1) $z \leqslant q$ and (2) $\varphi_{s_0}(z) \leqslant x$ such that (3) $\varphi_{s_0}(z) \leqslant q$ does not hold. From (1) and (2) we obtain

$$x(s_0) \geqslant \varphi_{s_0}(z)(s_0) \geqslant [\pi, z](s_0) \geqslant [\pi, q](s_0) \geqslant [p, q](s_0) = p(s_0) \geqslant q(s_0).$$

If $q(s_0) < x(s_0)$ would not hold then $q(s_0) = x(s_0)$ and we get from $q(s_0) = x(s_0) = [\pi, z](s_0)$, using (1), a contradiction to (3): $z = \varphi_{s_0}(z) \leqslant q$. Hence, $q(s_0) < x(s_0)$ and x is weakly exposed. □

Combining this lemma with Theorem (2.8′) we obtain:

Theorem (3.5). *Let π be subadditive and monotone and let ω be superadditive with $\omega \leq \pi$. Then for every fixed $t_0 \in S$ there is a monotone, additive and weakly exposed $\mu \in \Lambda(\omega, \pi)$ (i.e. $\omega \leq \mu \leq \pi$) such that*

$$\mu(t_0) = \inf\left\{\frac{1}{m}(\pi(mt_0 + t) - \omega(t)) \middle| 1 \leq m \in \mathbb{N},\ t \in S,\ \omega(t) \neq -\infty\right\}.$$

Proof. Theorem (2.8′) and Lemma (3.4) give us a weakly exposed μ (which then must be additive and monotone, anyway) such that $\varphi_{t_0}(\omega) \leq \mu$, in particular $\mu(t_0) \geq [\pi, \omega](t_0)$. We have to prove $\mu(t_0) = [\pi, \omega](t_0)$. Because of $\mu \geq \omega$ we have $[\pi, \omega](t_0) \geq [\pi, \mu](t_0)$ and this is equal to $\mu(t_0)$ since μ is weakly exposed (Proposition (3.3)). □

Remark (3.6). In case that S has only countably many elements one easily observes that all inf's and sup's are going over countable sets. Hence, in this case it suffices that R is σ-complete and has the additional property that every bounded linearily ordered set has a supremum. So, in this case R can be replaced by any $C(S)$, S compact σ-Stonian.

Remark (3.7). A trivial observation is that Theorem (3.5) contains the classical Hahn–Banach theorem as well as the Hahn–Banach theorem for the case when R is a Dedekind complete vector lattice. (This latter theorem goes back to the fundamental paper [18] of L. Nachbin, see also [10] or [11].) In the case that S is a vector space and that $\pi(\lambda_n t) \to 0$ whenever $\lambda_n \to 0$ ($\lambda_n \in \mathbb{R}$) then the required $\mathbb{R}$-homogeneity follows rightway from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. Furthermore, in this case no additive μ can attain the value $-\infty$ because we always have $\mu(s) + \mu(-s) = \mu(0) = 0$.

Remark (3.8). If S is countable then Theorem (3.5) can be proved without AA by using the principle of dependent choice. The same is true in case that S is a vector space with a countable dense (with respect to the topology induced by π) subset. However, in general the theorem is effectively equivalent to the axiom of choice since a special consequence will be that every unit ball of the dual of a normed space has an extreme point and it is known that this statement is effectively equivalent to AA ([3] or [17]). This also shows that Theorem (3.5) is effectively independent of the Hahn–Banach theorem (which is a consequence of BPI, the Boolean prime ideal axiom) since the AA is effectively independent of BPI ([13] and [16]).

3.1. Simple Consequences

We adopt the same notation as before. Let Add(ω, π) be the set of additive maps in $\Lambda(\omega, \pi)$. Add(ω, π) is a convex set. A point $\mu \in$ Add(ω, π) is said to be an *extreme point* if $\mu \leqslant \lambda\nu_1 + (1-\lambda)\nu_2$, $0<\lambda<1$, $\nu_1, \nu_2 \in$ Add(ω, π) always implies $\nu_1 = \nu_2$.

Lemma (3.9). *Every weakly exposed μ is an extreme point of* Add(ω, π).

Proof. As above let (*) $\mu \leqslant \lambda\nu_1 + (1-\lambda)\nu_2$. Put $p = \max\{\mu, \nu_1, \nu_2\}$ and $q = \min\{\mu, \nu_1, \nu_2\}$. We have to show $p = q$. If $p \neq q$ then, since q trivially exhausts p, there would be some $s_0 \in S$ with

$$\min\{\nu_1(s_0), \nu_2(s_0)\} < \mu(s_0) \geqslant \max\{\nu_1(s_0), \nu_2(s_0)\}\,.$$

Clearly a contradiction to (*). □

Corollary (3.10). Add(ω, π) *has an extreme point.*

Corollary (3.11). (*Maximum principle.*) *Every $s_0 \in S$ attains its supremum on* Add(ω, π) *already on some weakly exposed element. The supremum is equal to* $[\pi, \omega](s_0)$.

Proof. Fix $s_0 \in S$. Since every $\nu \in$ Add(ω, π) is between ω and π it is easily seen from the additivity of ν that $\nu(s_0) \leqslant [\pi, \omega](s_0)$. Now, take the weakly exposed μ with $\mu(s_0) = [\pi, \omega](s_0)$ given in Theorem (3.5). □

Thus, via the biopolar theorem, the Krein–Milman theorem and Bauer's maximum principle are special cases of Theorem (3.5). Another consequence is that $[\pi, \omega]$ is the supremum of all additive elements between ω and π. Hence, q exhausts p if and only if p is the supremum of all additive elements between q and p.

3.2. Vectorielly Exposed Points

In the case that $R \neq \mathbb{R}$ we have additional interesting consequences.

Assume that ω and π do not attain the value $-\infty$ and that R has an order unit I. Then by the fundamental Kakutani–Krein–Stone–Yoshida theorem [11, p. 95] R must be isomorphic to the vector lattice $C(K)$ of real continuous functions on some extremally disconnected compact set K. So, let us put $R = C(K)$ which gives us a multiplication in R, since $C(K)$ is an algebra. By δ_k, $k \in K$, we denote the point evaluations on $C(K)$. By $\Lambda_k(\omega, \pi) = \Lambda(\delta_k \circ \omega, \delta_k \circ \pi)$ we denote the real-valued superadditive functionals between $\delta_k \circ \omega$ and $\delta_k \circ \pi$. In the same way we define $\mathrm{Add}_k(\omega, \pi)$.

A superadditive $\mu \in \Lambda(\omega, \pi)$ is called *vectorielly exposed* if there is a meagre subset $M \subset K$ of K such that for all $k \in K \setminus M$, the functionals $\delta_k \circ \mu$ are weakly exposed in $\Lambda_k(\omega, \pi)$. As an exercise for the reader we leave that vectorielly exposed implies weakly exposed.

In analogy, a $\mu \in \mathrm{Add}(\omega, \pi)$ is said to be *vectorielly extreme* if, except on a meagre subset M of K, the real-valued functionals $\delta_k \circ \mu$ are extreme points of $\mathrm{Add}_k(\omega, \pi)$ $(k \in K \setminus M)$.

Whenever μ is vectorielly extreme such that $\mu \leq \lambda \nu_1 + (1-\lambda)\nu_2$, with $\lambda \in C(K)$, $\nu_1, \nu_2 \in \mathrm{Add}(\omega, \pi)$ and $0 \leq \lambda(k) \leq 1$ (for all $k \in C(K)$) then $\nu_1|_H = \nu_2|_H = \mu|_H$, where $H = \{k \in K \mid 1 > \lambda(k) > 0\}$. Hence, 'vectorielly extreme' is a stronger property than 'extreme'. An application of Lemma (3.9), for the case $R = \mathbb{R}$, yields that vectorielly exposed points are vectorielly extreme.

Now, it is interesting to observe that the Bauer maximum principle also applies to this situation:

Corollary (3.12). *(Vector valued maximum principle.) Every $s_0 \in S$ attains its supremum on* $\mathrm{Add}(\omega, \pi)$ *already on some vectorielly exposed element.*

Let us sketch the proof. Take $X' \subset \Lambda(\omega, \pi)$ the set of those superlinear q such that $\delta_k \circ q$ is for all $k \in K \setminus M$ (M some meagre subset) the maximum of weakly exposed elements in $\Lambda_k(\omega, \pi)$. In fact, this set is completely Φ-invariant. Hence, the μ in Theorem (3.5) can be assumed to be in X', which implies that μ is vectorielly exposed. If the reader wants to check in detail that X' is completely Φ-invariant, he should keep in mind that for lower bounded subsets B of $C(K)$ the functions

$$f = \inf B \qquad \text{(infimum taken in } C(K)\text{)},$$
$$f^\circ(k) = \inf\{b(k) \mid b \in B\} \qquad \text{(pointwise infimum)},$$

coincide except on a meagre subset of K ([22] or [11, p. 99]).

At this point we like to remark that not only Bauer's maximum principle carries over to this situation but also Choquet's theorem [10].

In general, even in the situation $R = \mathbb{R}$ the set of weakly exposed points is strictly smaller than the set of extreme points (see [6] or [9] for situations which may serve as counterexamples).

4. Other Applications

4.1. Boundaries for Compact Sets

Let K be some compact set and Φ a point-separating family of upper semicontinuous real valued functions on K. Max$\{K\}$ denotes the set of those $k \in K$ such that for every compact $K \supset K^\circ \supset \{k\}$ with $K \neq \{k\}$ there is some $\varphi \in \Phi$ with $\varphi(k) = \max \varphi(K^\circ) > \inf \varphi(K^\circ)$.

Theorem (4.1). *Every $\varphi_0 \in \Phi$ attains its K-maximum already on* Max$\{K\}$.

Proof. Let X be the set of nonempty compact subsets of K ordered by inclusion. For every $\varphi \in \Phi$ denote by φ' the decreasing map $\varphi'\colon X \to X$ given by

$$\varphi'(K^\circ) = \{k \in K^\circ \mid \varphi(k) = \max \varphi(K^\circ)\} \qquad K^\circ \in X.$$

The set $\Phi' = \{\varphi' \mid \varphi \in \Phi\}$ consists of order convex maps, hence (by Remark (2.9)(1)) there is some fixpoint $K_0 \in X$ such that:

(4.1) $$\varphi_0'(K_0) \subset \varphi_0'(K).$$

(4.2) For every $K_0 \subset K^\circ \in X$ with $K_0 \neq K^\circ$ there is some $\varphi' \in \Phi'$ with $K_0 \subseteq \varphi'(K^\circ) \subset K^\circ$.

Since Φ is point separating any Φ'-fixpoint must be a singleton $K_0 = \{k_0\}$. (4.1) means that φ_0 attains on k_0 its K-maximum and (4.2) yields that k_0 is in Max(K). □

The interest in this Max-boundary stems from the fact that it is always a subset of the Choquet boundary ([6] or [11]). But in fact it may happen that Max(K) is strictly smaller than the Choquet boundary of K.

4.2. Iterations

Consider a partially ordered set $(Z, \leqslant)$ and a family Φ of decreasing maps on Z such that:

(4.3) Φ is *composition-closed*, i.e. if $\varphi_1, \varphi_2 \in \Phi$ then $\varphi_1 \circ \varphi_2 \in \Phi$.

(4.4) Φ is *almost commuting*, i.e. if $\varphi_1, \varphi_2 \in \Phi$ then there are $\varphi_3, \varphi_4 \in \Phi$ with $\varphi_3 \circ \varphi_1 = \varphi_4 \circ \varphi_2$.

Standard examples for a composition-closed almost commuting set are given by the composition-closed families of maps which are generated by families of commuting maps. A subset $Y \subset Z$ is said to be *Φ-directed* if for all $y_1, y_2 \in Y$ and $\varphi \in \Phi$ there is some $y_0 \in Y$ with $y_0 \leqslant \varphi(y_1)$ and $y_0 \leqslant \varphi(y_2)$. We assume about Z:

(4.5) Every nonempty Φ-directed subset of Z has an infimum.

As usual, a set $Y \subset Z$ is called *Φ-invariant* if $\varphi(Y) \subset Y$ for all $\varphi \in \Phi$. A map $F : Z \to Z$ is said to be *Φ-absorbing* if

(4.6) $$F \circ \varphi = F \qquad \text{for all } \varphi \in \Phi .$$

and

(4.7) whenever F is constant on a Φ-invariant and Φ-directed set Y then F is constant on $Y \cup \{\inf Y\}$.

Finally, a decreasing map $\mathrm{It} : Z \to Z$ is called an *iteration* of Φ if it is Φ-absorbing and fulfils:

(4.8) $$F \circ \mathrm{It} = F \qquad \text{for every } \Phi\text{-absorbing map } F .$$

There can be only one iteration. For if F is any decreasing Φ-absorbing map then $F \circ \mathrm{It} = F$ implies $F(z) \leqslant \mathrm{It}(z)$ for all $z \in Z$. Hence, the iteration must be the maximum of all decreasing Φ-absorbing maps, therefore it is unique.

Theorem (4.2). *There is an iteration for Φ.*

Proof. Fix some arbitrary x in Z. Let X be the family of all Φ-invariant Φ-directed subsets J of $Z_x = \{z \in Z \mid z \leq x\}$ having the property:

(4.9) Whenever J° is a Φ-invariant Φ-directed subset of J with $\inf\{J^\circ\} > \inf\{J\}$ then $\inf\{J^\circ\}$ belongs to J.

Observe that X is not empty since $\Phi[x] = \{\varphi(x) \mid \varphi \in \Phi\}$ belongs to X. This follows from the fact that Φ is a composition-closed almost commuting set of decreasing functions. For $J_1, J_2 \in X$ write $J_1 \leq J_2$ if

$$J_2 \subset J_1 \tag{4.10}$$

and

$$\text{if } y_1 \in J_1 \backslash J_2 \text{ and } y_2 \in J_2 \text{ then } y_1 \leq y_2 . \tag{4.11}$$

Observe, that X is inductively ordered. Define a monotone (for that we need (4.9)) and decreasing map $\Phi^\circ : X \to X$ by

$$\Phi^\circ(J) = J \cup \{\varphi(\inf J) \mid \varphi \in \Phi \text{ or } \varphi = \mathrm{Id}_Z\} .$$

Now, take the maximal (or exposed) fixpoint J_x of Φ° (Proposition (2.6)). Observe that J_x is the largest, i.e. smallest with respect to $\subset$, element of X containing x and its infimum. We define

$$\mathrm{It}(x) = \inf J_x .$$

Then, by construction, $\mathrm{It}(\cdot)$ has the required properties: $\mathrm{It}(\cdot)$ is Φ-absorbing, because every non-empty

$$\mathrm{It}^{-1}(x_0) = \{z \in Z \mid \mathrm{It}(z) = x_0\}$$

is obviously Φ-invariant and Φ-directed and contains its infimum. Furthermore, if F is Φ-absorbing then it must be constant on J_x and if $\mathrm{It}(x') = \mathrm{It}(x)$ then the inf's of J_x and $J_{x'}$ are the same. Hence, F must be constant on $J_x \cup J_{x'}$. Therefore F is constant on any $\mathrm{It}^{-1}(x_0)$, which implies $F \circ \mathrm{It} = F$. □

Proposition (4.3). (1). *If x_0 is some Φ-fixpoint then* $\mathrm{It}(x_0) = x_0$.
(2). $\mathrm{It}(\cdot)$ *is idempotent, i.e.* $(\mathrm{It})^2 = \mathrm{It}$.

(3). It *maps* Z *onto the fixpoints of* Z.

(4). *Let* F *be* Φ*-absorbing such that* $F(z) \leq \varphi(z)$ *for all* $\varphi \in \Phi$ *and all* z *with* $F(z) \leq z$. *Then* $F(z) \leq \mathrm{It}(z)$ *whenever* $F(z) \leq z$.

(5). *If all* $\varphi \in \Phi$ *are monotone then* $\mathrm{It}(\cdot)$ *is monotone and* $\mathrm{It}(x)$ *is the maximum of all* Φ*-fixpoints* $\leq x$.

Proof. (1). If x_0 is a fixpoint, then obviously $J_{x_0} = \{x_0\}$ ($\subset$-minimality of J_{x_0}) hence $\mathrm{It}(x_0) = x_0$. (2) is an immediate consequence of (4.8) and the fact that $\mathrm{It}(\cdot)$ is itself Φ-absorbing. (3). Since $\mathrm{It}(\cdot)$ is Φ-absorbing and decreasing we obtain (with (2)) for every $\varphi \in \Phi$

$$\mathrm{It}(z) = (\mathrm{It})^2(z) = (\mathrm{It} \circ \varphi)\mathrm{It}(z) \leq \varphi \circ \mathrm{It}(z) \leq \mathrm{It}(z).$$

Hence, $\mathrm{It}(z)$ must be a φ-fixpoint, and a Φ-fixpoint because φ was arbitrarily chosen. (4). The condition implies that F is constant on J_x and attains a value less than or equal to $\inf\{J_x\}$. (5). Let x_0 be some arbitrarily chosen fixpoint $\leq x$. Define $F(z) = x_0$ for all $z \in Z$. Since all $\varphi \in \Phi$ are monotone we have $\varphi(x_0) = x_0 \leq \varphi(y)$ whenever $x_0 \leq y$. Hence, F fulfils the condition required in (4). So, $x_0 = F(x) \leq \mathrm{It}(x)$. In particular, $x_0 \leq \mathrm{It}(x)$ which proves that $\mathrm{It}(\cdot)$ is monotone (for $z_1 \leq z_2$ put $x_0 = \mathrm{It}(z_1)$). □

For the case that Φ is generated by a single function Theorem (4.2) was first given in [8]. In this paper very many applications were given: Tarski's fixpoint theorem [23], Banach's fixpoint theorem for contractions and generalized contractions [4], Edelstein's fixpoint theorem for condensing maps [5], Sadovski's theorem for limit-compact maps [21], and the Kirk and the Belluce–Kirk theorems for normal structure [14] and [2]. The corresponding generalizations of these theorems to almost commuting families we leave for the fun of the reader of this paper.

Furthermore, we would like to call to attention the survey paper [1] of Amann, where similar ideas (based on [8]) have been applied to boundary value problems.

An extensive study of the iteration theorem has been carried out in [15] by Th. Landes.

5. Appendix

Here we gather those details which were off the mainstream of the paper and therefore left out.

5.1. Proof of Zermelo's Fixpoint Lemma

We adapt the proof of [20, p. 391] following an idea of Halmos [12, p. 64]. A family Γ of linearily ordered subsets of X is said to be *admissible* if

(5.1)
(i). $\{I\} \subset \Gamma$.
(ii). $I \in \gamma$ for all $\gamma \in \Gamma$.
(iii). $\gamma \in \Gamma$ then $\gamma \cup \{\varphi(\inf \gamma)\} \in \Gamma$.
(iv). If $G \subset \Gamma$ is upwards directed with respect to inclusion then $\bigcup\{\gamma \mid \gamma \in G\} \in \Gamma$.

The family of all linearily ordered subsets $\supset \{I\}$ is admissible. Obviously, the intersection Γ^* of all admissible families is again admissible. We claim that $\gamma^* = \bigcup\{\gamma \mid \gamma \in \Gamma^*\}$ is an element of Γ^*. If that is true, then $\inf\{\gamma^*\}$ must be a fixpoint since φ is decreasing. So it remains to prove the claim. Take

$$\Gamma^\circ = \{\gamma \in \Gamma^* \mid \text{for all } \beta \in \Gamma^* \text{ we have } \gamma \subset \beta \text{ or } \beta \subset \gamma\}.$$

Fix some arbitrary $\delta \in \Gamma^\circ$, consider

$$\Gamma(\delta) = \{\gamma \in \Gamma^* \mid \gamma \subset \delta \text{ or } \delta \cup \{\varphi(\inf \delta)\} \subset \gamma\}.$$

Both Γ° and $\Gamma(\delta)$ fulfill (i), (ii) and (iv) of (5.1). And the property which δ has from being an element of Γ° implies that $\Gamma(\delta)$ also fulfils (5.1)(iii). Hence $\Gamma(\delta) = \Gamma^*$. But then from the definition of $\Gamma(\delta)$ we obtain $\delta \cup \{\varphi(\inf \delta)\} \in \Gamma^\circ$. Hence, since $\delta \in \Gamma^\circ$ was arbitrary, Γ° must fulfill (5.1)(iii). So we have $\Gamma^\circ = \Gamma^*$ and this set must be linearily ordered. Then (5.1)(iv) implies $\gamma^* \in \Gamma^*$. □

5.2. Proof that the Existence of a Fixpoint is Effectively Equivalent to the Axiom of Choice

Let M be a set and denote by $\mathscr{X}$ the set of all pairs (N, f), where $N \subset \mathscr{P}(M)$ (power set) and where f is a choice function for N (if $\emptyset \in N$ then we define formally $f(\emptyset) = \emptyset$). We define an order relation on $\mathscr{X}$ by

$$(N_1, f_1) \leq (N_2, f_2) \quad \Leftrightarrow \quad N_1 \supset N_2 \text{ and } f_1|_{N_2} = f_2 .$$

Then $(\mathscr{X}, \leq)$ is inductively ordered and $(\{\emptyset\}, \emptyset \to \emptyset)$ is its supremum. For $y \in Y \subset M$ we define a map $\varphi_{y,Y}$ by

$$\varphi_{y,Y}(N,f)=\begin{cases}(N,f) & \text{if } Y\in N,\\ (N\cup\{Y\},f^\circ) & \text{where } \begin{cases}f^\circ|_N=f\\ f^\circ(Y)=y\end{cases}, \text{ otherwise}.\end{cases}$$

The maps $\varphi_{y,Y}$ are decreasing and the common fixpoint must be of the form $(\mathscr{P}(M),f^*)$. Hence there is a choice function f^* on all of M. □

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