From Single Solitons to Auto-Bäcklund Transformations and Hereditary Symmetries

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Dedicated to Professor Dr.F.Beck on the occasion of his 60th birthday

ABSTRACT

A method is given to determine auto-Bäcklund transformations and hereditary recursion operators out of the hidden intrinsic algebraic structure of **single** soliton solutions. New auto-Bäcklund transformations are presented. Bianchi permutability and soliton phase shift are explained in terms of commutative groups.

I. Introduction

For completely integrable systems we claim that ABT (auto-Bäcklund transformations) can be computed from the single solitons. This does not mean that equations admitting solitary waves do have auto-Bäcklund transformations, only that solitary waves of completely integrable systems do have hidden symmetries which yield ABT's.

As an introductory example we perform the necessary computations for the KdV

$$u_t = u_{xxx} + 6uu_x \,. \tag{1.1}$$

For this equation the single soliton with asymptotic speed c is given by

$$s(x + ct) = \frac{c}{2}\cosh^{-2}\{\frac{\sqrt{c}}{2}(x - x_0 + ct)\}. \tag{1.2}$$

The KdV is translation invariant. We consider translations by $+\beta$ and $-\beta$, respectively for (1.2)

$$\frac{\sqrt{c}}{2}x \;\rightarrow\; \frac{\sqrt{c}}{2}x+\beta \;\text{ and }\; \frac{\sqrt{c}}{2}x \;\rightarrow\; \frac{\sqrt{c}}{2}x-\beta \tag{1.3}$$

and denote the corresponding solutions by s_β and $s_{-\beta}$. We decompose into odd and even parts

$$V_+ = \frac{1}{2}(s_\beta + s_{-\beta}), \quad V_- = \frac{1}{2}(s_\beta - s_{-\beta}), \tag{1.4}$$

and, in order to abbreviate notation, we introduce

$$\begin{aligned} k &= \frac{\sqrt{c}}{2} \\ \xi &= (x - x_0 + ct) \\ N(k\xi) &= \{\cosh^2(k\xi)\cosh^2(\beta) - \sinh^2(k\xi)\sinh^2(\beta)\}^{-1} \\ &= \{\cosh^2(k\xi) + \sinh^2(\beta)\}^{-1}. \end{aligned} \tag{1.5}$$

Now, we consider explicitly s_β and we apply the addition-theorem for the cosh-function. Splitting up s_β into odd and even parts yields

$$\begin{aligned} s_\beta &= 2k^2\cosh^{-2}(k\xi+\beta) \\ &= 2k^2\{\cosh(k\xi)\cosh(\beta) + \sinh(k\xi)\sinh(\beta)\}^{-2} \\ &= 2k^2N(k\xi)^2\{\cosh(k\xi)\cosh(\beta) - \sinh(k\xi)\sinh(\beta)\}^2 \\ &= V_+ + V_- \end{aligned} \tag{1.6}$$

where now the odd part V_- and the even part V_+ are given by

$$V_+ = 2k^2N(k\xi)^2\{\cosh^2(k\xi)\cosh^2(\beta) + \sinh^2(k\xi)\sinh^2(\beta)\} \tag{1.7}$$

$$V_- = -4k^2N(k\xi)^2\{\cosh(k\xi)\sinh(k\xi)\cosh(\beta)\sinh(\beta)\}. \tag{1.8}$$

Considered as functions in the variable ξ, the even part V_+ is a polynomial of second order in the common denominator $N(k\xi)$. Furthermore V_- is, apart from multiplication with a constant, the derivative of $N(k\xi)$. Thus we obtain a **relation** between V_+ and V_- . Let us write down this explicitly. Simple computations give

$$V_+ + \gamma N + \delta N^2 = 0 \tag{1.9}$$

where

$$\gamma = -2k^2\cosh(2\beta)$$

$$\delta = k^2\sinh^2(2\beta)$$

and

$$DN = (\ 2k\cosh(\beta)\sinh(\beta)\ )^{-1}V_-\,,\quad D = \frac{\partial}{\partial x}. \tag{1.10}$$

This gives

$$V_+ - \frac{2k\cosh(2\beta)}{\sinh(2\beta)}D^{-1}V_- + \{D^{-1}V_-\}^2 = 0. \tag{1.11}$$

Now, let β depend on k in such a way that

$$\lambda = -2k\coth(2\beta) \tag{1.12}$$

is independent of k . Then we have the following algebraic relation

$$B^{(\lambda)}(s_\beta\,,s_{-\beta}) := (s_\beta + s_{-\beta}) + \lambda D^{-1}(s_\beta - s_{-\beta}) + \frac{1}{2}\{D^{-1}(s_\beta - s_{-\beta})\}^2 = 0$$

where the coefficients are **independent** of $k=(\sqrt{c}\,)/2$. By translation invariance we then have

$$B^{(\lambda)}(s\,,s_{-2\beta}) = (s + s_{-2\beta}) + \lambda D^{-1}(s - s_{-2\beta}) + \frac{1}{2}\{D^{-1}(s - s_{-2\beta})\}^2 = 0 \tag{1.13}$$

This relation between s and $s_{-2\beta}$ we consider for general arguments

$$B^{(\lambda)}(u\,,\bar{u}) = u + \bar{u} + \lambda D^{-1}(u - \bar{u}) + \frac{1}{2}\{D^{-1}(u - \bar{u})\}^2 \tag{1.14}$$

u and $\bar{u}$ on the manifold S of C^∞-functions vanishing rapidly with all their derivatives at infinity. We consider u as fixed. The map

$$u \to \bar{u} = f_{B^{(\lambda)}}(u) \tag{1.15}$$

given implicitly by

$$B^{(\lambda)}(u\,,\,\bar{u}) = 0 \tag{1.16}$$

is injective around u if the implicit function theorem condition is fulfilled. This condition requires that for $B^{(\lambda)}$ the kernel of the variational derivative $B^{(\lambda)}{}_u$ with respect to u has to be trivial, i.e. for every v in the tangent space at u it must hold that whenever

$$B^{(\lambda)}{}_u[v] = 0 \;\text{ and }\; B^{(\lambda)}(u\,,\,\bar{u}) = 0 \tag{1.17}$$

then

$$v = 0\,. \tag{1.18}$$

Recall that the partial variational derivative $B^{(\lambda)}{}_u$ is defined to be

$$B^{(\lambda)}{}_u[v] = \frac{\partial}{\partial\varepsilon}\, B^{(\lambda)}(u + \varepsilon\, v\,,\,\bar{u})|_{\varepsilon=0}. \tag{1.19}$$

Those λ violating this condition for u we call the **spectral** points of u . The other λ are said to be **non-spectral** points.

Obviously (1.15), or (1.16), map the soliton solution s onto $s_{-2\beta}$ where

$$2\beta = \text{arcoth}\,(-\frac{\lambda}{\sqrt{c}}). \tag{1.20}$$

This relation is called the **phase shift relation**. In particular s and all its translations are mapped onto the zero function for $\lambda = \pm\sqrt{c}$. Hence $\pm 2k$ are spectral points for the one soliton having celerity $c = 4k^2$. All other λ's are nonspectral because for $B^{(\lambda)}(u\,,\,\bar{u}) = 0$ the operator $B^{(\lambda)}{}_u$ is a differential operator which is invertible on the vector space S.

The set of spectral points in general is finite because (1.17) is an ordinary differential equation for v having only for certain λ's solutions vanishing rapidly at infinity. We shall see this in detail later on.

Now we return to the special case where $u = s$ and $\bar{u} = s_{-2\beta}$. Recall that

$$cu_x = K(u) = u_{xxx} + 6uu_x \tag{1.21}$$

$$c\bar{u}_x = K(\bar{u}) = \bar{u}_{xxx} + 6\,\bar{u}\bar{u}_x \tag{1.22}$$

because both functions are one-solitons. Translation invariance of $B^{(\lambda)}(u\,,\,\bar{u})=0$ given by the free parameter x_0 yields

$$B^{(\lambda)}{}_u\,[\,u_x\,] + B^{(\lambda)}{}_{\bar{u}}[\bar{u}_x] = 0 \tag{1.23}$$

and insertion of (1.21) and (1.22) gives

$$B^{(\lambda)}{}_u\,[K(u)] + B^{(\lambda)}{}_{\bar{u}}[K(\bar{u})] = 0. \tag{1.24}$$

Observe that for nonspectral points λ, where $\bar{u}$ locally is uniquely given by

$$B^{(\lambda)}(u\,,\,\bar{u}\,) = 0 \tag{1.25}$$

the equation (1.23) is a differential equation for u (or rather the integral of u). And if this equation is nontrivial then on the manifold **S** under consideration this equation has the same number of integration constants as equation (1.21).

But there is an essential difference between (1.21) and (1.24) + (1.25), coming from the important fact that the relation $B^{(\lambda)}$ is independent of c or k :

The solutions for (1.21) are a one-parameter family (parameter x_0) whereas the solutions for (1.24)+(1.25) are a two-parameter family (parameters x_0 and c). Thus the system (1.24) + (1.25) has to many integration parameters. This can only be if the system is trivial, i.e. if (1.24) is identically fulfilled whenever (1.25) holds. And, since the nonspectral points are dense, this must also hold for those λ which are spectral.

Thus we have found for **arbitrary** arguments: Whenever

$$B^{(\lambda)}(u\,,\,\bar{u}\,) = 0$$

then

$$B^{(\lambda)}{}_u\,[K(u)] + B^{(\lambda)}{}_{\bar{u}}[K(\bar{u})] = 0. \tag{1.26}$$

This relation is equivalent [4] to $B^{(\lambda)}(u,\bar{u}\,)$=0 being an auto-Bäcklund transformation for the KdV (1.1), i.e. whenever $u(t)$ is a solution of the KdV and $\bar{u}(t)$ is such that $B^{(\lambda)}\,(u(t),\bar{u}(t))$=0 then $\bar{u}(t)$ is again a solution of the KdV. Of course, (1.26) can also be proved by explicit computation since we obtain easily

$$\{D^3 + 3(u + \bar{u}\,)D\}\; B(u\,,\,\bar{u}\,) = 0. \tag{1.27}$$

This auto-Bäcklund transformation is related to the ABT

$$(u + \bar{u}) + \delta + \frac{1}{2}\{D^{-1}(u - \bar{u})\}^2 = 0 \tag{1.28}$$

which is usually found in the literature [1]. By the formal substitution

$$D^{-1} \;\rightarrow\; D^{-1} + constant$$

coming out of the existence of integration constants, (1.28) goes over into (1.26).

However, there is a notable difference between (1.28) and (1.25) since (1.25) is compatible with the boundary condition on the manifold **S** , i.e. with the requirement that u and $\bar{u}$ vanish at infinity, whereas (1.28) is not compatible with this requirement. This point is essential in the following where we turn our interest to the spectral points of u.

Since $B^{(\lambda)}(u\,,\,\bar{u}\,) = 0$ is an auto-Bäcklund transformation for (1.1) the property of being a spectral point is obviously invariant against this flow. So we have arrived at some kind of spectral problem for which (1.1) constitutes an isospectral flow (see [9]). However, this is a **nonlinear**

Spectral problem :
Given a solution u of (1.1), find those λ's such that there is some nonzero vectorfield ω and some $\bar{u}$ on the manifold under consideration such that

$$B^{(\lambda)}{}_u(u,\bar{u})[\omega] = 0 \tag{1.29}$$

and

$$B^{(\lambda)}(u,\bar{u}) = 0\ . \tag{1.30}$$

We do not know of any criteria which give reasonable answers to the question under what circumstances such a nonlinear spectral problem is equivalent to a linear one. However, in case of the KdV this problem is easily linearized (see [9]):

Variational derivative of (1.14) with respect to u yields the operator:

$$B_u = I + (D^{-1}(u - \bar{u}))D^{-1} + \lambda D^{-1}\ . \tag{1.31}$$

And the spectral problem (1.29) reads as follows

$$0 = \omega + (D^{-1}(u - \bar{u}))D^{-1}\omega + \lambda D^{-1}\omega\ . \tag{1.32}$$

Abbreviation $D^{-1}\omega = v$ gives

$$D^{-1}(u - \bar{u}) = -(\frac{v_x}{v} + \lambda)\ . \tag{1.33}$$

Writing $u + \bar{u}$ as $2u - (u-\bar{u})$ and then replacing all terms $u-\bar{u}$ in (1.14) by (1.33) we obtain

$$2u + (\frac{v_x}{v} + \lambda)_x + \frac{1}{2}(\frac{v_x}{v} + \lambda)^2 - \lambda(\frac{v_x}{v} + \lambda) = 0$$

which is certainly a nonlinear eigenvalue equation. By multiplication with v^2 we obtain

$$2u\,v^2 + v_{xx}v - \frac{1}{2}v_xv_x = \frac{1}{2}\lambda^2v^2\ . \tag{1.34}$$

If this problem can be linearized there must be operators $A(v)$ and $\Psi(u)$ such that $A(v)v = Cv^2$ and $A(v)\Psi(u)v$ is equal to the left side of (1.34). Comparison of suitable terms yields:

$$D^{-1}vD\{v_{xx} + 2uv + 2\,D^{-1}\,(u\,v_x)\} = \lambda^2D^{-1}\,vDv\ . \tag{1.35}$$

Hence $A(v) = D^{-1}vD$ and $\Psi(u) = D^2 + 2u + 2D^{-1}uD$.

Going back to $\omega = v_x$ we see that ω is a solution of (1.29) if and only if ω is an eigenvector of

$$\Phi(u) = D\Psi(u)D^{-1} = D^2+2u+2DuD^{-1}. \tag{1.36}$$

And if λ is the spectral point given by (1.29) then λ^2 is the corresponding eigenvalue of $\Phi(u)$.

We like to know what the meaning of this operator $\Phi(u)$ is. Since the KdV (1.1) is an isospectral flow for this differential (or rather integro-differential) operator we look for the second component $\Lambda(u)$ of the corresponding Lax-pair

$$\frac{d}{dt}\Phi(u) = [\Lambda(u),\Phi(u)] \tag{1.37}$$

describing the time evolution of $\Phi(u)$ if u evolves according to the KdV. This operator $\Lambda(u)$ is easily found by using the fact that $B^{(\lambda)}(u\,,\bar{u})=0$ is an auto-Bäcklund transformation.

Take the time-derivative of (1.29) and replace the time derivatives of u and $\bar{u}$ by the right hand side of the KdV to obtain

$$B^{(\lambda)}{}_{uu}(u\,,\bar{u})[\omega\,,K(u)\,] + B^{(\lambda)}{}_{u\bar{u}}(u\,,\bar{u})[\omega\,,K(\bar{u})\,] + B^{(\lambda)}{}_{u}(u\,,\bar{u})[\omega_t] = 0. \tag{1.38}$$

Since (1.29) holds u can be changed in (1.30) infinitesimaly by ω without changing $\bar{u}$. Hence variational derivative of u by ω and $\bar{u}$ by zero in (1.24) yields that the first two terms in (1.38) are equal to

$$-B^{(\lambda)}{}_{u}[K\,'(u)[\omega]].$$

Hence, we can choose ω such that

$$\omega_t = K'(u)[\omega]. \tag{1.39}$$

Inserting this into the time derivative of the eigenvalue equation

$$\Phi(u)\omega = \lambda^2\omega, \tag{1.40}$$

where λ is time-independent we obtain

$$\frac{d}{dt}\Phi(u) = K'(u)\Phi(u) - \Phi(u)K'(u) \tag{1.41}$$

and*)

$$\Lambda(u) = K'(u) = D^3+6Du. \tag{1.42}$$

Hence $\Phi(u)$ must be a recursion operator ([12]) or a strong symmetry ([7]) for the KdV. In fact this is the well known hereditary Lenard operator [11] which generates all symmetry generators of the KdV recursively.

Also from our construction via single-solitons follows the meaning of the spectral points in terms of multi-solitons. Consider an N-soliton solution u of the KdV, i.e. a solution which for large t decomposes asymptotically into N single solitons such that the overlap of these single solitons becomes exponentially small. Then by simple asymptotic arguments the spectral points of u must be those points which are spectral for one of these asymptotically occurring single soliton solutions.

*)Actually on first sight this equation only holds when applied to an arbitrary eigenvector ω of $\Phi(u)$. But application of suitable local-global arguments then shows that this must be true in general (see the widely referenced paper [6]).

By (1.20) the spectral points of these single solitons are the $\lambda_n = \pm 2k_n = \sqrt{c_n}$ (c_n speed of the n-th soliton) which correspond to annihilations of these solitons by translating them by

$$\beta = \infty = \frac{1}{2}\operatorname{arcoth}(-\frac{\lambda}{2k}) \tag{1.43}$$

out of finite sight.

Thus two facts are shown, namely

i) that the eigenvalues of $\Phi(u)$ are the speeds of the asymptotic solitons,

ii) that whenever u is some N-soliton solution then for spectral λ_n the $\bar{u}$ appearing in $B^{(\lambda_n)}(u,\bar{u})=0$ is the $(N-1)$-soliton solution where the n-th soliton is missing.

REMARK:

For the explanation of **phase shift** consider a solution $u=u(x,t)$ of the KdV such that asymptotically for $t \to \pm\infty$ a soliton with speed $c_1 = 4k_1^2$ is emerging. Let $B^{(\lambda)}(u,\bar{u})=0$ be a Bäcklund transformation between this solution and another solution $\bar{u}$. Because of the local structure of $B^{(\lambda)}(\cdot,\cdot)$ and the asymptotic behavior of u the effect of this Bäcklund transformation on the emerging soliton is the same as if it were a single soliton. Hence the corresponding soliton emerging out of $\bar{u}$ is, compared to u , shifted by a translation of the amount

$$\frac{4\beta}{\sqrt{c_1}} = \frac{2}{\sqrt{c_1}}\operatorname{arcoth}(-\frac{\lambda}{\sqrt{c_1}}). \tag{1.44}$$

Since the square root is multivalued we have no information about the direction of this shift. So, comparing the asymptotic solitons emerging at $t=-\infty$ with those at $t=+\infty$ we see that they are either shifted in the same direction or in opposite directions. Therefore the total phase shift of this soliton, compared to the corresponding soliton of u, is either zero or

$$\pm\frac{4}{\sqrt{c_1}}\operatorname{arcoth}(-\frac{\lambda}{\sqrt{c_1}}) \tag{1.45}$$

In fact all cases are possible. If in addition another soliton with speed $c_2 = 4k_2^2$ is emerging one can show by simple arguments that if the parameter λ is in between the square roots of the two velocities c_1 and c_2 then the smaller soliton of $\bar{u}$ compared with that of u is forwarded and the faster one is retarded. This can be shown without looking at explicit solutions just by considering small perturbation of the Bäcklund transformation. Now consider $\lambda = \sqrt{c_2}$, i.e. the Bäcklund transformation which annihilates the second soliton with speed c_2 . Assume the case $c_2 > c_1$. Then the annihilation is done by shifting this second soliton for $t \to +\infty$ into plus infinity. The phase shift of the soliton with speed c_1 is then

$$+\frac{4}{\sqrt{c_1}}\operatorname{arcoth}(\sqrt{c_2/c_1}) \tag{1.46}$$

In case $c_2<c_1$ we have to change the sign of this phase shift. Now, take a pure multisoliton solution and annihilate successively all emerging solitons except the one with speed c_1 . Then the phase shifts have to be added. Because of Bianchi permutability (see section II) the order of annihilation does not play any role. And the phase shift of the soliton, emerging out of u with speed c_1 , compared to the corresponding single soliton, which is the result of all these annihilations, must be

$$\frac{4}{\sqrt{c_1}}\sum_i \varepsilon_i \operatorname{arcoth}(\sqrt{c_i/c_1}) \tag{1.47}$$

where

$$\varepsilon_i = \begin{cases} -1 & \text{if } c_i > c_1 \\ +1 & \text{if } c_i < c_1 \end{cases} .$$

A result well known from the literature [10].

So the result of all our considerations is that looking at the hidden algebraic structure of the single soliton of the KdV we find the N-soliton structure (via the auto-Bäcklund transformation), the abelian symmetry group (via the hereditary recursion operator). Furthermore, via the operator Φ , one can find the bi-Hamiltonian structure ([4]) and the angle-variables of the system (via the mastersymmetries [8]) constructed by the recursion operator, furthermore information about details of soliton interaction. To me this seems to be a most remarkable fact.

II. General principles

What was done in the last chapter, with respect to the translation group, for the KdV can be done in general for integrable systems with respect to arbitrary symmetry groups. Let us resume the general aspects.

Consider evolution equations given by differential equations and being of the form

$$u_t = K(u), \tag{2.1}$$

where $u = u(x,t)$ and where u_t denotes partial derivative with respect to t (time). Eq. (2.1) is regarded as a dynamical system, i.e. $K(u)$ is to be a C^∞-vector field on the manifold of all admissible functions in the variable x . Of course, u may be vector valued, i.e. $u=(u_1, \cdots ,u_n)$. Only for simplicity, the space variable x considered is is one dimensional.

Consider a finite-dimensional abelian symmetry group **G** for (2.1) having infinitesimal generators $G_1, \ldots, G_n$. Recall that a vectorfield G is a symmetry group generator for (2.1) if $[K,G]=0$ in the vectorfield Lie algebra. The commutator $[K,G]=L_K G$ is given by the Lie-derivative, which in case the manifold is a vectorspace is defined by

$$L_K G = \frac{\partial}{\partial\varepsilon}_{|\varepsilon=0}\{G(u+\varepsilon K(u)) - K(u+\varepsilon G(u))\}. \tag{2.2}$$

In the example of section I the group **G** was the one-dimensional translation group and its generator was $G(u)=u_x$.

Now, we consider the manifold of **G**-soliton solutions of (2.1). That is the set of all special solutions $u(x,t)$ of (2.1) where the time-development is given by the action of some one-parameter subgroup of **G** , i.e. those $u(x,t)$ such that there is some $k=(k_1,k_2, \ldots ,k_n)$ with

$$k_1G_1(u)+k_2G_2(u) \cdots +k_nG_n(u)=K(u). \tag{2.3}$$

These solutions are easily determined: Take a solution $s(k,x)$ of (2.3) and the one-parameter subgroup $\{g(\tau),\tau\in \mathbb{R}\}$ of **G** having the infinitesimal generator

$$<k,G>=k_1G_1+k_2G_2+ \cdots +k_nG_n\,, \quad k=(k_1,\ldots,k_n)$$

Then

$$u(x,t)=g(t)(s(k,x)) \tag{2.4}$$

is the **G**-soliton with initial condition $u(x,t=0)=s(k,x)$. This solution we denote by $s(k)$ or $s(k,t)$, it plays the role of the function given in (1.2).

Observe that, because of invariance with respect to time translation, the solution $s(k,x)$ of (2.3) has at least one free integration parameter, i.e. whenever $\beta\in$ **G** then application of β to $s(k,x)$ yields

$$s_\beta(k,x) = \beta(s(k,x)) \tag{2.5}$$

which again is a solution of (2.3). For abbreviation we denote this solution by $s_\beta(k)$ and the one obtained by application of β^{-1} we denote by $s_{-\beta}(k)$.

Now, we proceed as before. We look for a relation $B^{(\lambda)}(\cdot,\cdot)$ such that

$$B^{(\lambda)}(s_\beta,s_{-\beta}) = 0 \tag{2.6}$$

holds independently of the special group parameter k we had chosen. Since we have to do with a several parameter group our $\lambda=(\lambda_1, \ldots ,\lambda_n)$ now also has several components.

The function $B^{(\lambda)}(u,\bar{u})$ is said to be **admissible** around u and $\bar{u}$ if i) and ii) are fulfilled:

i) $B^{(\lambda)}$ is invariant against **G**

ii) $B^{(\lambda)}$ defines a diffeomorphism around (2.6)

The relation $B^{(\lambda)}$ is said to be **G**-invariant if whenever $B^{(\lambda)}(u,\bar{u})=0$ then

$$B^{(\lambda)}(g(u)\ ,\ g(\bar{u}\,)) = 0 \tag{2.7}$$

must hold for every $g\in\mathbf{G}$. An equivalent infinitesimal condition is that

$$B^{(\lambda)}{}_u\,[G(u)] + B^{(\lambda)}{}_{\bar{u}}[G(\bar{u})] = 0 \tag{2.8}$$

holds for all infinitesimal generators G of one-parameter subgroups of **G** . And $B^{(\lambda)}$ is said to be a diffeomorphism around (2.6) if the implicit function theorem condition holds, i.e. if the operators

$$B^{(\lambda)}{}_u\,(s_\beta,s_{-\beta})[\cdot], \qquad B^{(\lambda)}{}_{\bar{u}}(s_\beta,s_{-\beta})[\cdot]$$

are invertible operators between the tangent spaces of **S** at s_β and $s_{-\beta}$, respectively.

PROPOSITION:
Let Λ *be the set of those* λ *where for every* $k=(k_1,\cdots,k_n)$ *there is some* $\beta=\beta(k)$ *with*

$$B^{(\lambda)}(s_\beta(k),\ s_{-\beta}(k)) = 0$$

such that $B^{(\lambda)}$ *is admissible around these points. Then for every element* λ *in the closure of* Λ

$$B^{(\lambda)}(u\ ,\ \bar{u}\,) = 0 \tag{2.9}$$

is an auto-Bäcklund transformation for (2.1).

The arguments leading to this proposition are the same as in section 1. For the moment we put $u=s_\beta$ and $\bar{u}=s_{-\beta}$. Then, since B is **G**-invariant, we have

$$B^{(\lambda)}{}_u\,[G_i(u)] + B^{(\lambda)}{}_{\bar{u}}[G_i(\bar{u})] = 0,\ \ i=1,\cdots,n.$$

Because of (2.3) this yields

$$B^{(\lambda)}{}_u\,[K(u)] + B^{(\lambda)}{}_{\bar{u}}[K(\bar{u})] = 0. \tag{2.10}$$

Since around the special u and $\bar{u}$ the quantity $B^{(\lambda)}$ defines a diffeomorphism this is an ordinary differential equation admitting as solution all solutions of (2.3) - and this for any arbitrary set $k_1,\ldots,k_n$. Now, since (2.10) has the same degree as (2.3) it has to many integration parameters (namely in addition to those of (2.3) the parameters $k_1,\ldots,k_n$). Hence (2.10) must be an identity which holds for **all** $u,\bar{u}$ connected by

$$B^{(\lambda)}(u\ ,\ \bar{u}\,) = 0. \tag{2.11}$$

Therefore it must be an auto-Bäcklund transformation. The isospectral problem given by the Bäcklund transformations obtained this way is the same as before:

Spectral problem :
Given a solution u *of (2.1), find those* λ*'s such that there is some nonzero*

vectorfield ω *and some* $\bar{u}$ *on the manifold under consideration such that*

$$B^{(\lambda)}{}_u(u,\bar{u})[\omega] = 0 \tag{2.12}$$

and

$$B^{(\lambda)}{}_u(u,\bar{u}) = 0 \, . \tag{2.13}$$

Those λ*'s are said to be spectral points of the Bäcklund transformation and the* ω *appearing in (2.12) are called eigenvectors. This spectral problem is invariant under the flow (2.1).*

Again, the time development of its eigenvector ω is given by

$$\omega_t = K'(u)[\omega]. \tag{2.14}$$

Hence, if the spectral problem can be linearized, then the corresponding linear operator $\Phi(u)$ is part of a Lax-pair

$$\frac{d}{dt}\Phi(u) = K'(u)\Phi(u) - \Phi(u)K'(u) \tag{2.15}$$

which shows in particular that its eigenvectors are solutions of the linearized equation (or symmetry equation) (see [7]). In case the general solution decomposes asymptotically into single soliton solutions the interpretation of spectral points and eigenvectors is the same as before.

The spectral λ's correspond to the speeds of the asymptotic solitons. This correspondence can be found by looking at the single soliton solution where we pick out those λ's which are limits of group actions having parameters going to infinity in such a way that the soliton is removed out of sight into infinity. Of course, these λ's are depending now on the parameter k, from (2.3), characterizing the soliton under consideration. The eigenvector itself is the soliton which is lost by going from u to $\bar{u}$. This explains why $B^{(\lambda)}(u\ ,\ \bar{u}\ ) = 0$ for spectral λ connects N-solitons with $(N-1)$-solitons.

Also **Bianchi-permutability** has a very simple explanation, it is a consequence of the abelian structure of $\mathbf{G}$. To see this consider nonspectral λ and denote for the moment those $\bar{u}$ coming from $B^{(\lambda)}(u,\bar{u})=0$ by

$$\bar{u}=U(\lambda,u). \tag{2.16}$$

Recall that if u is a single soliton then $\bar{u}$ simply is u moved by some action of an element in $\mathbf{G}$. Since $\mathbf{G}$ is commutative and since $B^{(\lambda)}(\cdot,\cdot)$ is $\mathbf{G}$-invariant we must have

$$U(\lambda_2,U(\lambda_1,u))=U(\lambda_1,U(\lambda_2,u)) \tag{2.17}$$

or

$$u=\phi(u), \tag{2.18}$$

where

$$\phi(u)=U(-\lambda_2,U(-\lambda_1,U(\lambda_2,U(\lambda_1,u)))). \qquad (2.19)$$

Certainly

$$\bar{u}-\phi(u)=0 \qquad (2.20)$$

is an auto-Bäcklund transformation for (2.1) which reduces to the identity mapping on the G-solitons. Hence, this must be the identity mapping everywhere and (2.17) must hold for general u. However, only for nonspectral λ's on first view. But this induces also for all limits of nonspectral λ by density.

Let me emphasize that the condition of admissibility is an essential one. Condition i) was necessary in order to arrive at (2.10) and the following example will serve to see that condition ii) is equally important.

EXAMPLE:

Consider the square root of the single soliton solution of the KdV

$$\sigma(x+ct)=\sqrt{\frac{c}{2}}\cosh^{-1}\{\frac{\sqrt{c}}{2}(x-x_0+ct)\}. \qquad (2.21)$$

We perform a translation on the x-line by $(2\Theta/\sqrt{c})$ and obtain for σ_Θ (resulting from this translation)

$$\sigma_\Theta(x)=\frac{\sqrt{c/2}\cosh(\Theta)\cosh(y)-\sqrt{c/2}\sinh(\Theta)\sinh(y)}{(\cosh^2(\Theta)\cosh^2(y)-\sinh^2(\Theta)\sinh^2(y))} \qquad (2.22)$$

where

$$y=\frac{1}{2}\sqrt{c}\,(x-x_0+ct).$$

Splitting up into odd- and even parts

$$\sigma_\Theta=V_++V_-,\quad V_+=\frac{1}{2}(\sigma_\Theta+\sigma_{-\Theta}),\quad V_-=\frac{1}{2}(\sigma_\Theta-\sigma_{-\Theta}) \qquad (2.23)$$

we find

$$2\{D^{-1}(V_+V_-)\}^2-\sinh^2(\Theta)V_+^2+\cosh^2(\Theta)V_-^2=0 \qquad (2.24)$$

or

$$\{D^{-1}(\sigma_\Theta^2-\sigma_{-\Theta}^2)\}^2+2\{\sigma_\Theta^2+\sigma_{-\Theta}^2-2\alpha\sigma_\Theta\sigma_{-\Theta}\}=0. \qquad (2.25)$$

Another relation between σ_Θ and $\sigma_{-\Theta}$ is found by choosing Θ such that

$$\cosh(\Theta)\sinh(\Theta)=\alpha\sqrt{c}.$$

Then

$$2D^{-1}(\sigma_\Theta^2-\sigma_{-\Theta}^2)-\alpha\sigma_\Theta\sigma_{-\Theta}=0. \qquad (2.26)$$

Recalling that σ_Θ^2 is a single soliton of the KdV we have found the following relation between a single soliton u and suitable translates

$$\{\frac{1}{2}[D^{-1}(u-\bar{u})]^2+u+\bar{u}\}^2-4\alpha^2 u\bar{u}=0 \tag{2.27}$$

$$4\{D^{-1}(u-\bar{u})\}^2-\alpha^2 u\bar{u}=0. \tag{2.28}$$

Both relations are auto-Bäcklund transformations for the KdV **only** for very special choices of the parameters α . The reason they are not Bäcklund transformations is obviously that condition ii) in the admissibility requirements is not fulfilled. This because the inverse of the variational derivative of B leads out of the manifold under consideration. One might try to repair this by enlarging the manifold under consideration in dropping the boundary conditions at infinity. But then the manifold of single solitons also enlarges (theta-function solutions), and for those additional single solitons the relations (2.27) and (2.28) do not hold.

Let me conclude this section by some additional comments.

REMARKS:

1) It should be noted that if there is any **G**-invariant auto-Bäcklund transformation

$$B^{(\lambda)}(u\,,\bar{u})=0 \tag{2.29}$$

for (2.1) with nonspectral λ then it must be found by our procedure. To see this take u to be some **G**-soliton, i.e. a solution of (2.3). Since $B^{(\lambda)}$ is invariant with rspect to **G** and the one parameter group given by $K(u)$ we find

$$B^{(\lambda)}{}_u\,[K(u)-\sum k_iG_i(u)]+B^{(\lambda)}{}_{\bar{u}}[K(\bar{u})-\sum k_iG_i(\bar{u})]=0. \tag{2.30}$$

But $K(u)-\sum k_iG_i(u)=0$ since u is a **G**-soliton. Since λ is assumed to be nonspectral this relation must also hold for $\bar{u}$ instead of u. So $\bar{u}$ again is a **G**-soliton (having the same $k_1,\ldots,k_n$ as u). This implies that $\bar{u}$ results out of u by some action of an element in **G** . Hence, $B^{(\lambda)}(u,\bar{u})=0$ can be found as a relation connecting **G**-solitons.

2) One might ask what happens in the KdV-case if **G** , which was the group of x-translations, is enlarged by adding those one-parameter groups having the well known symmetries as generators. Then, in this case, the **G**-solitons are just the multisolitons (up to some order depending on the dimension of **G**) and the Bäcklund transformations found by our procedure are simply the iterations of the well known Bäcklund transformation.

3) The content of this section was that, in a certain sense, the auto Bäcklund relations are completely determined by the one-soliton-solutions. On first view this seems to be rather surprising, but in fact it is not if one thinks of exactly solvable systems in terms of linearizable flows. Because, according to what we know up to now, the variable transformation (for example the inverse- scattering transform) which transforms the given flow into a linear one $u_t=Au$, A linear, transforms the one-soliton solutions into eigenvectors of A and the multisoliton solutions into linear combinations of such eigenvectors. Taking further into account that the auto-Bäcklund transformations connect N-soliton-solutions and $(N+1)$-soliton-solutions,

one easily guesses that in terms of the new variable the auto-Bäcklund relation must be of the form $(A-\lambda)u_2=u_1$, where λ is the eigenvalue of that part of u_2 which corresponds to the disappearing soliton. But certainly, for such a linear relation we can check whether or not it is an auto-Bäcklund relation on the linear hull of the eigenvectors by checking if it has this property on all single eigenvectors. Now, going back to the original manifold, where the flow was nonlinear, this then corresponds to checking out the required properties on the one-soliton-solutions.

4) One computational advantage of our derivation of ABT's is that we know right away what happens to solitons occurring asymptotically in fields of interacting solitons. Since the group element $\beta=\beta(k)$ in the proposition was chosen in such a way that (2.9) holds, we now know from **G**-invariance of $B^{(\lambda)}$ that a single soliton $s(k)$ emerging asymptotically out of u is transformed by $B^{(\lambda)}(u,\bar{u})=0$ into a single soliton $s(k)_{-2\beta(k)}$ emerging out of $\bar{u}$. The information about the phase shift now can be build up in the same way as in case of the KdV, provided the Bäcklund transformation respects the local structure of emerging solitons.

EXAMPLES AND APPLICATIONS

III. The Benjamin-Ono equation

We are interested in the BO ([2] ,[13]) which is

$$u_t = K(u), \quad K(u) = (2uu_x + Hu_{xx}) \tag{3.1}$$

where H is the Hilbert transform

$$(Hf)(x) = \frac{1}{\pi}\int_{+\infty}^{-\infty} \frac{f(\xi)}{\xi-x}d\xi \quad \textit{(principal value integration)}$$

and where u is assumed to be an element of **S**, now **S** being the space of functions having the property that all derivatives are absolutely integrable on the real line. For convenience we have defined $H1=0$.

Observe that K commutes with $G(u)=u_x$, the infinitesimal generator of the translation group. Equation (3.1) has (see [1,p.204])

$$s(k,x) = \frac{ik}{k(x-x_0)+i} - \frac{ik}{k(x-x_0)-i}, \quad x_o \in \mathbf{R} \tag{3.2}$$

as one-soliton solutions, i.e. as solutions of $ku_x=K(u)$. This is easily seen from

$$iH(\frac{1}{x-\alpha}) = \begin{cases} \dfrac{1}{x-\alpha} & \text{if } \textit{imaginary part}(\alpha) < 0 \\ -\dfrac{1}{x-\alpha} & \text{if } \textit{imaginary part}(\alpha) > 0 \end{cases} \tag{3.3}$$

We restrict our attention to positive k. Replacing x_0 in (3.2) by $\beta+x_0$ we obtain

$$s_\beta(x) = \eta_\beta(x) + \eta^*_\beta(x) \tag{3.4}$$

where * denotes complex conjugation and where

$$\eta_\beta = \frac{ik}{k(x-\beta-x_0)+i} . \tag{3.5}$$

Decomposition of (3.4) into odd and even parts (with respect to β)

$$s_\beta = V_+ + V_-,\ \ V_+ = \frac{1}{2}(s_\beta + s_{-\beta}),\ \ V_- = \frac{1}{2}(s_\beta - s_{-\beta}) \tag{3.6}$$

gives

$$V_+ = \frac{1}{2}\{\eta_\beta + \eta^*_\beta + \eta_{-\beta} + \eta^*_{-\beta}\} \tag{3.7}$$

$$V_- = \frac{1}{2}\{\eta_\beta + \eta^*_\beta - \eta_{-\beta} - \eta^*_{-\beta}\}. \tag{3.8}$$

Observe that integration and exponentiation of a simple pole yields the same pole. Application of this together with a decomposition into partial fractions leads to:

$$\exp(2iD^{-1}V_-) = \frac{\eta_\beta \cdot \eta^*_{-\beta}}{\eta^*_\beta \eta_{-\beta}} = 1 + 2\frac{\beta}{(k\beta-i)}\{\eta_\beta + \eta^*_{-\beta}\}. \tag{3.9}$$

Now, observe that application of iH to V_- changes the sign of the second and the fourth term and leaves the others unchanged. Hence the second and third terms cancel in $V_+ + iHV_-$. This yields

$$V_+ + iHV_- = \eta_\beta + \eta^*_{-\beta}.$$

From this together with (3.9) we obtain

$$\exp(2iD^{-1}V_-) = 1 + 2\lambda(iHV_- + V_+), \tag{3.10}$$

where

$$\lambda = \beta/(k\beta-i). \tag{3.11}$$

Inserting

$$V_+ = \frac{1}{2}(u + \bar{u}),\ \ V_- = \frac{1}{2}(u - \bar{u}),\ \ u = s_\beta, \bar{u} = s_{-\beta}$$

we obtain

$$\exp(iD^{-1}(u-\bar{u})) = 1 + \lambda\{iH(u - \bar{u}) + (u + \bar{u})\}. \tag{3.12}$$

Since this relation is admissible it must be an auto-Bäcklund transformation for (3.1). The spectral point corresponding to a soliton with speed k one easily obtains by putting $\beta=\infty$ in (3.11). Hence, each soliton with speed k generates a spectral point $\lambda=1/k$.

In the literature [15] sometimes the more general Bäcklund transformation

$$\mu \exp(iD^{-1}(u-\bar{u})) = 1 + \lambda\{iH(u-\bar{u}) + (u+\bar{u})\} \tag{3.13}$$

is given. This relation is generated out of (3.11) by the formal substitution

$$D^{-1} \rightarrow D^{-1} + constant.$$

However, if and only if $\mu=1$ this auto-Bäcklund transformation stays on the required manifold of functions disappearing at infinity.

There are other auto-Bäcklund transformations for the Benjamin-Ono equation. Consider

$$iHV_+ = \frac{1}{2}\{\eta_\beta - \eta^*_\beta + \eta_{-\beta} - \eta^*_{-\beta}\} \tag{3.14}$$

$$iHV_- = \frac{1}{2}\{\eta_\beta - \eta^*_\beta - \eta_{-\beta} + \eta^*_{-\beta}\}. \tag{3.15}$$

Solving equations (3.7),(3.8),(3.14) and (3.15) for the η's we obtain

$$\eta_{\pm\beta} = \frac{1}{2}(1+iH)(V_+ \pm V_-) \tag{3.16}$$

and

$$\eta^*_{\pm\beta} = \frac{1}{2}(1-iH)(V_+ \pm V_-). \tag{3.17}$$

Integration and exponentiation yields the following two relations

$$\begin{aligned}\exp(iD^{-1}(1+iH)V_-) &= \eta_\beta/\eta_{-\beta} \\ &= 1 + 2i\beta\eta_\beta = 1 + i\beta(1+iH)(V_+ + V_-).\end{aligned} \tag{3.18}$$

$$\begin{aligned}\exp(iD^{-1}(1-iH)V_-)) &= \eta^*_{-\beta}/\eta^*_\beta \\ &= 1 - 2i\beta\eta^*_{-\beta} = 1 - i\beta(1-iH)(V_+ - V_-).\end{aligned} \tag{3.19}$$

Hence resubstitution by (3.11) yields a new auto-Bäcklund transformation consisting of the following pair of relations

$$\exp(\frac{i}{2}D^{-1}(1+iH)(u-\bar{u})) = 1 + i\beta(1+iH)u \tag{3.20}$$

$$\exp(\frac{i}{2}D^{-1}(1-iH)(u-\bar{u})) = 1 - i\beta(1-iH)\bar{u} \quad . \tag{3.21}$$

This transformation certainly is not equivalent to the one given by (3.11) since it has no spectral points. The transformation splits into two parts corresponding to the eigenspaces of the operator H . This decomposition is the usual decomposition into functions being analytic in the upper and lower half of the complex plane, respectively.

IV Burger's equation

It is obvious that the BO contains Burgers equation as a special case. For example if a solution u of BO is analytic in the upper half of the complex plane then $Hu=iu$ and u therefore must be a solution of

$$u_t = (2uu_x + iu_{xx}). \tag{4.1}$$

The same holds true for the lower half plane, we only have to replace i by $-i$. This means that the flow of the BO restricted to certain invariant manifolds reduces to versions of Burgers equation.

However this does not mean that auto-Bäcklund transformations for BO automatically yield ABT's for Burgers equation because such an ABT is not necessarily compatible with the restriction to invariant manifolds. For this reason (3.12) cannot be used to obtain an ABT for Burgers equation. In contrast to that (3.20) + (3.21) give right away an ABT for Burgers equation since these are compatible with the restriction to functions analytic either in the upper or lower half of the complex plane. Let us carry out the details for the case of the upper half of the complex plane. In this case only (3.21) is meaningful. Replacing H by i we obtain

$$\exp(iD^{-1}(u-\bar{u})) = 1 + 2i\beta\bar{u} \tag{4.2}$$

as an ABT for (4.1). Performing the substitution $t\to it/\delta$, $x\to ix/\delta$ we obtain for arbitrary values of α that

$$\exp(\delta^{-1}D^{-1}(\bar{u}-u)) = \alpha\bar{u} + 1 \tag{4.3}$$

must be an ABT for

$$u_t = 2uu_x + \delta u_{xx} \tag{4.4}$$

which is the general form of Burgers equation. Those readers who do not feel at ease with the methods we proposed in this paper should take the opportunity to check that statement by a simple direct computation.

This example actually shows that our method constitutes a helpful tool. It enabled us to compute out of the symmetry structure of a special family of functions (3.2) an ABT for an evolution equation for which these functions were not even solutions.

In this case the recursion operator is easily obtained by the nonlinear spectral problem given by this auto-Bäcklund transformation (see [9]) although the boundary conditions at infinity have to be changed in the formulation of the spectral problem.

One might ask if in case of BO the nonlinear spectral problem also leads to a recursion operator. Indeed this is the case, however the procedure is very complicated and the recursion operator obtained is of a form involving u and other functions only determined by u via an implicit function.

V. A Nonlinear System

We investigate a nonlinear system of differential equations for $w_i = w_i(x,t)$, $i=1,2$, which is intimately connected to the isospectral flow of the KdV.

We introduce the differential operators

$$L(u) = D^2 + u \tag{5.1}$$

$$T(\phi) = 2\phi D + 6\phi_x \tag{5.2}$$

and we use the notation

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{5.3}$$

$$|w|^2 = w_1{}^2 + w_2{}^2 \tag{5.4}$$

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5.5}$$

$$\tau(w) = \begin{bmatrix} T(w_1) & 0 \\ 0 & T(w_2) \end{bmatrix} . \tag{5.6}$$

The system we are interested in is the following

$$\tau(w)Mw_t = \tau(w)L(\,|\,w\,|^2)w. \tag{5.7}$$

or in case that $\tau(w)$ is invertible

$$w_t = ML(\,|\,w\,|^2)w.$$

This equation has two obvious symmetry groups, namely the translation group with infinitesimal generator w_x and a gauge group (rotation in $\boldsymbol{R}_2$ or $\mathbb{C}_2$ leaving $|w|^2$ invariant). The infinitesimal generator of the gauge group is Mw .

Up to now, we have only used one-soliton manifolds defined by the infinitesimal generator of x-translation. In our present example we take the generator of the gauge group instead, i.e. we define for the moment the one-soliton manifold by

$$\kappa\tau(w)w = \tau(w)L(\,|\,w\,|^2)w \tag{5.8}$$

where the speed parameter is now called κ .

In order to compute the one-soliton manifold we observe that the single-soliton of the KdV is the well-known Bargmann potential [10] which is the same as saying that for the single-soliton of the KdV the square of the corresponding eigenvector of the Schrödinger operator is equal to u (see [5]). The reason for this fact is the following identity which holds for all $\phi = \phi(x)$ and $u = u(x)$

$$\Theta(u)\phi^2 = T(\phi)L(\,u\,)\phi \tag{5.9}$$

where

$$\Theta(u) = D^3 + 2Du + 2uD.$$

Because of the special form of $T(\phi)$ this yields that the relation

$$\Theta(u)\phi^2 = 4\alpha(\phi^2)_x \tag{5.10}$$

holds if and only if

$$T(\phi)(L(u)-\alpha)\phi = 0. \tag{5.11}$$

The function

$$s(c) = s(c,x) = \frac{c}{2}\cosh^{-2}\{\frac{\sqrt{c}}{2}(x - x_0 + ct)\} \tag{5.12}$$

given in (1.2) fulfills by definition

$$cs_x = \Theta(s)s \tag{5.13}$$

because the KdV is of the form $u_t=\Theta(u)u_x$. So, in case that $4\alpha=c$ the function $\phi^2=s(c)$ must be a solution of (5.10). Hence, $\phi = \sqrt{s(c)}$ solves (5.11) and we have

$$T(\phi)L(\phi^2)\phi = \frac{1}{4}cT(\phi)\phi\ ,\quad \phi = \sqrt{s(c)}. \tag{5.14}$$

Comparing this with (5.8) we have found the one-soliton-manifold for (5.7). In order to see this we observe that the time evolution on the one-soliton manifold, now given by the gauge-group, must be of the form

$$w(x,t) = \exp(t\kappa M)\psi(x)\ ,\quad \psi = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{5.15}$$

where the components of ψ must satisfy (5.8). If we require $\psi_2 = 0$ and consider $\kappa = c/4$ then because of (5.14) $\psi_1 = \phi$ is a solution of this equation. Looking at the kernel of $T(\phi)$ and knowing that the eigenvalues of the Schrödinger operator have multiplicity 1 we easily observe that the general solution of (5.8) must come out of this solution by an application of the gauge group. Hence the general solution of (5.8) must be

$$w = \begin{bmatrix}\cos(\rho)\sqrt{s(c,x)}\\ \sin(\rho)\sqrt{s(c,x)}\end{bmatrix},\quad \kappa = c/4. \tag{5.16}$$

Since (1.14) gives an ABT for the functions $s(c,x)$ we are now able to rephrase this in order to obtain an ABT for the solution w of (5.7). We apply the translation (1.3) to the solution (5.12) and decompose into odd and even parts. Using the notation of section I and looking at formula (1.6) we find

$$V_+ = \sqrt{2}\ kN(k\xi)\{\cosh(k\xi)\cdot\cosh(\beta)\} \tag{5.17}$$

$$V_- = -\sqrt{2}\ kN(k\xi)\{\sinh(k\xi)\cdot\sinh(\beta)\}. \tag{5.18}$$

A straightforward computation yields

$$(V_+)_x = V_-\ k\coth(\beta)\ \{2\cosh^2(k\xi)\ N(k\xi)-1\} \tag{5.19}$$

$$= V_- \, k \coth(\beta) \, \frac{\cosh^2(k\xi) - \sinh^2(\beta)}{\cosh^2(k\xi) + \sinh^2(\beta)}$$

$$= V_- \sqrt{k^2 \coth^2(\beta) - \frac{4k^2 \cosh^2(k\xi) \cosh^2(\beta)}{(\cosh^2(k\xi) + \sinh^2(\beta))^2}}$$

$$= V_- \sqrt{\lambda^2 - 2V_+^2}, \quad \lambda = k \coth(\beta).$$

So in case of $\rho=0$ we have found for the special solution (5.16) the following relation between w and its translation $\bar{w}$

$$(w_1 + \bar{w}_1)_x = (w_1 - \bar{w}_1)\sqrt{\lambda^2 - \frac{1}{2}(w_1 + \bar{w}_1)^2}. \tag{5.20}$$

Since this relation is not gauge-invariant it cannot be the ABT we are looking for. But, certainly the ABT must be the gauge-invariant relation coming out of this

$$(w_i + \bar{w}_i)_x = (w_i - \bar{w}_i)\sqrt{\lambda^2 - \frac{1}{2}|\, w + \bar{w}\,|^2}, \quad i = 1,2. \tag{5.21}$$

However this relation is still unsatisfactory because we have not yet used the complete single soliton manifold given by the two one-parameter groups taken into account (translation and gauge). Considering this extended manifold we should get an auto-Bäcklund transformation with two free parameters instead of one.

For obtaining the second parameter we let the effect of the application of the gauge group depend on x since this can be compensated by adding to the equation under consideration a linear combination of infinitesimal generators of the translation group. To be precise: Replacing (5.16) by

$$w = \begin{bmatrix} \cos(\gamma x + \rho)\sqrt{s(c,x)} \\ \sin(\gamma x + \rho)\sqrt{s(c,x)} \end{bmatrix} \tag{5.22}$$

we see that this must be a solution to

$$\kappa w + 2\gamma M w_x + \gamma^2 w = L(|\, w\,|^2) w, \tag{5.23}$$

hence a solution on the soliton manifold given by gauge and translation. Insertion of (5.22) into (5.21) shows that the contributions coming from the x-derivatives of $\cos(\gamma x)$ and $\sin(\gamma x)$ can be compensated by replacing w_{1x} and w_{2x} by $w_{1x} + \gamma w_2$ and $w_{2x} - \gamma w_1$, respectively. The same has to be done for the components of $\bar{w}$. This yields

$$(w + \bar{w})_x = \gamma M(w + \bar{w}) + (w - \bar{w})\sqrt{\lambda^2 - \frac{1}{2}|\, w + \bar{w}\,|^2} \tag{5.24}$$

as an auto-Bäcklund transformation for (5.7).

VI. The nonlinear Schrödinger equation (NLS)

Since the NLS is the restriction of (5.7) to a special invariant submanifold we easily find an ABT for the NLS.

Consider solutions of (5.7) where both components vanish rapidly at $\pm\infty$. Then on this manifold the kernel of $\tau(w)$ consists only of the zero-element. So in this case (5.7) has the form

$$w_{1t} = -w_{2xx} - (w_1^2 + w_2^2)w_2 \qquad (6.1)$$

$$w_{2t} = +w_{1xx} + (w_1^2 + w_2^2)w_1 .$$

Consider real valued solutions of this equations and look at the corresponding evolution equation for $\psi = w_1 + iw_2$ instead of (6.1).

This equation is easily found to be

$$-i\psi_t = (D^2 + \psi\psi^*)\psi \qquad (6.2)$$

since M corresponds to multiplication with i . Hence, all those auto-Bäcklund transformations (5.23) which connect real valued solutions vanishing rapidly at $\pm\infty$ are ABT's for the NLS.

Rewriting (5.23) in complex form we obtain

$$(\psi + \overline{\psi})_x = i\gamma(\psi + \overline{\psi}) + (\psi - \overline{\psi})\sqrt{\lambda^2 - \frac{1}{2}|\psi + \overline{\psi}|^2} \qquad (6.3)$$

as ABT for (6.1). This is exactly the ABT known from the literature ([3] , [14]). In other words: The ABT (5.23) is the analytic extension of the well-known ABT for the NLS.

VII. The inverse problem

We can also run in the opposite direction through our recipes. For example, if we have a one-parameter family of functions admitting a relation (independent of the parameter) between these functions and corresponding translated functions, then, in the admissible case, this relation is an ABT for the evolution equation(s) having the translations of our one-parameter family of functions as one-soliton manifold.

Let us illustrate this for the most simple example. Consider a translation-invariant evolution equation having

$$s_k(x,t) = \cos(\frac{1}{k}(x-kt)) \qquad (7.1)$$

as one-soliton solutions. Such an equation is easily found

$$u_t^2 + u^2 = 1. \qquad (7.2)$$

Then looking at the special form of (7.1) one immediately sees that there should be a one-parameter family of ABT for (7.2), since translations of (7.1) can be expressed in cosine-functions.

Let us carry out the details. Consider

$$s(k,x) = \cos(x/k) \tag{7.3}$$

$$s_\beta(k,x) = \cos((x+\beta)/k) = \cos(\beta/k)\cos(x/k) - \sin(\beta/k)\sin(x/k). \tag{7.4}$$

This yields the relation

$$(s_\beta - s\cos(\beta/k))^2 = \sin^2(\beta/k)(1 - s^2). \tag{7.5}$$

Since this relation is independent of k it must be an ABT for the equation (7.2). Thus, replacing $\cos(\beta/k)$ by λ we have found an ABT for

$$(u - \lambda\bar{u})^2 = (1 - \lambda^2)(1 - \bar{u}^2) \tag{7.6}$$

or

$$u^2 + \bar{u}^2 - 2\lambda u\bar{u} = 1 - \lambda^2. \tag{7.7}$$

Actually, there is one dubious point: Equation (7.2) is not exactly of the form (2.1) which was considered in the second section. Since the square root is a multivalued function it cannot be transformed into (2.1). That means the notions we have given so far cannot be applied to this situation. But an intuitive feeling for what goes on tells us that certainly our notions and methods carry over to evolution equations which are given in (eventually multivalued) implicit form.

To phrase the result of this section vaguely: Two-parameter families of functions having strong symmetry properties can be considered as one-soliton solutions of evolution equations admitting auto-Bäcklund relations.

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