The Dynamical Behavior of Interacting Solitons

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ABSTRACT

In this survey we demonstrate that interacting solitons for exactly solvable equations can be described by nonlinear evolution equations where the nonlinear terms are due to selfinteraction. This means that the quantity representing the soliton appears as the only field variable in the nonlinear evolution equation. The dynamics of the interacting soliton turns out to be again exactly solvable and inherits all its structure from the original system. Thus the process of soliton decomposition can be repeated, this leads to "virtual" solitons. It is shown that, for example, two soliton solutions can be understood as the exchange of one virtual soliton. Details are carried out for the KdV, where also the solutions for the interacting soliton equation are plotted.

I. INTRODUCTION

We are interested in solitons for

$$u_t=K(u)\,, \qquad u=u(x,t). \tag{1.1}$$

The solutions u are required to vanish rapidly at either $+\infty$ or $-\infty$. The evolution equation is assumed to be exactly solvable. To be precise, we either assume that there is a translation invariant recursion operator $\Phi(u)$ such that

$$K(u)=\Phi(u)u_x \tag{1.2}$$

or that we have a one parameter family of auto-Bäcklund transformations

$$B(u,\tilde{u},\lambda)=0 \tag{1.3}$$

with parameter λ.

Recall that being an auto-Bäcklund transformation [8] means that

$$B_u[K(u)]+B_{\tilde{u}}[K(\tilde{u})]=0 \qquad (1.4)$$

must hold whenever u and $\tilde{u}$ fulfill (1.3). Here B_u means the variational derivative with respect to u , i.e.

$$B_u[K(u)]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} B(u+\varepsilon K(u),\tilde{u},\lambda). \qquad (1.5)$$

Recall that $\Phi(u)$ is a recursion operator for (1.1) if its Lie derivative in direction of K vanishes [7], i.e. if

$$\Phi_u[K(u)]=K_u\Phi-\Phi K_u. \qquad (1.6)$$

If, for example $\Phi(u)$ is a hereditary operator ([7], [9]), then Φ automatically is a recursion operator for $K(u)=\Phi(u)u_x$. In this case the Lie algebra generated in the vector field Lie algebra by

$$\{\Phi(u)^n u_x \mid n\in N_0\} \qquad (1.7)$$

is abelian [9]. A solution $u(x,t)$ of (1.1) of the form

$$u(x,t)=s(x+ct) \qquad (1.8)$$

is said to be a traveling wave solution [16]. The quantity c is the speed of that wave. Obviously, s then must be a solution of

$$K(s)=cs_x. \qquad (1.9)$$

In case of the existence of a recursion operator this means that

$$\Phi(s)s_x=cs_x \qquad (1.10)$$

i.e. s_x then must be an eigenvector of $\Phi(s)$ with eigenvalue c . A solution u out of which there emerges asymptotically for $t\to\pm\infty$ a traveling wave is said to contain a soliton.

If a solution asymptotically decomposes completely into solitons we call it a multisoliton. Here by complete decomposition we mean that there is some suitable energy-norm such that all the energy is carried by the asymptotic solitons.

The problems addressed in this paper are:

I. Is there a way to describe the dynamics of solitons s in fields u of interacting solitons such that only selfinteraction occurs in the description of s? To be precise: Can we find a nonlinear evolution equation

$$s_t=G(s) \qquad (1.11)$$

for s such that u does not occur in this equation.

II. Can we choose this nonlinear evolution equation for the interacting soliton in such a way that all the structure (Hamiltonian formulation, complete integrability, [hereditary] recursion operator, angle variable, auto Bäcklund transformation, etc.) which may exist for the evolution of u carries over to the evolution of s.

The reasons why we are interested in these questions are (among others):

A. Analysis of the evolution of s may eventually lead to a simple qualitative description for interaction of solitons.

B. Furthermore, an analysis of evolution of s may give a simple way to define trajectories for the movement of solitons.

C.We may be able to find new completely integrable systems and to get new insight into complete integrability for flows on infinite dimensional manifolds.

The necessary steps to give a satisfactory solution to problems I and II are:

1. Find a way to identify for all finite t the soliton s which emerges asymptotically out of a field u.

2. Describe the dynamics of s in terms of an equation coupled to the external field u , i.e. find an evolution equation for s which is of the form

$$s_t=\Gamma(u,s). \tag{1.12}$$

A natural requirement for this evolution equation should be that it inherits the structural properties which do hold for the evolution (1.1) of the external field.

3. Decouple the evolution (1.12), i.e. find a way to eliminate the external field u such that the nonlinear terms in the evolution are only due to selfinteraction of s.

4. Check, whether or not the evolution equation for s provides a solution for problem II.

Surprisingly, to carry out steps 1 to 4 turns out to be extremely simple and straightforward; at least in the presence of a recursion operator. In the following we briefly indicate how that can be done. For more details the reader is referred to the existing literature (in case of step 1 and 2) and to the papers [12] and [13] for the remaining steps.

II. Coupled Dynamics of Interacting Solitons

For the Korteweg-de Vries equation

$$u_t=u_{xxx}+6uu_x \tag{2.1}$$

it was observed in 1974 [14] that a for a multisoliton solution u there is always a decomposition of u_x in terms of the squared eigenfunctions of the Schrödinger operator

$$u_x=\sum_{n=1}^{N}\Psi_{n_x}{}^2. \tag{2.2}$$

This was proved via inverse scattering theory. In fact, inverse scattering can be completely avoided for obtaining this decomposition [6]. Since the x-derivatives of the squared eigenfunctions are the eigenfunctions of the recursion operator of the KdV

$$\Phi(u)=D^3+2u+2DuD^{-1} \tag{2.3}$$

we find that a solution is a multisoliton if and only if u_x is the linear sum of eigenvectors of the recursion operator. This characterization holds for all equations with recursion operator ([7], [9]). It subsequently appeared many times in the literature (see for example [21] - [24], [17], [10]).

Because of this observation we define s to be an interacting soliton in the field u - even in case that u is not a multisoliton - if and only if s_x is the eigenfunction of the recursion operator $\Phi(u)$.

This seems to be a reasonable definition since:

α. The flow (1.1) is always - because of (1.6) - an isospectral flow for the operator Φ . Hence an eigenfunction is present for all time t if it is present for one t_0.

β. If $\Phi(u)$ is reasonably localized and if asymptotically there is a soliton then, because of (1.10), asymptotically there is a corresponding eigenvector. Hence, because of (α.), there is always an eigenvector of $\Phi(u)$ which corresponds to this soliton.

γ. The dynamics of eigenvectors of the recursion operator is uniquely determined. These eigenvectors have the same dynamical behavior as infinitesimal generators of one-parameter symmetry groups ([7], [9]).

Let us make (γ.) a little bit more precise:

If $w(t_0)$ is an eigenvector of $\Phi(u(t_0))$ then we may choose $w(t)$ such that for all t we have

$$\Phi(u(t))w(t)=cw(t) \tag{2.4}$$

$$w_t=K_u[w]. \tag{2.5}$$

Combining this with the definition of the soliton

$$s_x = w \tag{2.6}$$

we find that equations (2.4) to (2.6) completely determine the dynamics of s. Since w is a solution of the linearization (perturbation equation) of (1.1), and because these linearizations inherit, as coupled systems, the structure from (1.1), we may expect that the evolution of s has the same structural properties as the evolution of u .

III. Dynamics of Selfinteraction

The idea for obtaining the dynamics of s in terms of s alone is extremely simple:

Consider equation (2.4) as a Bäcklund transformation between u and s . Use this to express u by s and insert then $u=F(s)$ in the evolution equation (2.5). Thus we obtain the evolution for the interacting soliton.

By using the fact that Bäcklund transformations preserve structure (Hamiltonian formulation, [hereditary] recursion operators, etc.) we then can transfer the structural properties from equation (1.1) to the evolution equation for the interacting soliton. To do that explicitly, we only need the transformation formulas from [5] or [8].

On first view there seem to be the following difficulties:

(1) The Bäcklund transformation is an eigenvector equation and solving eigenvector equations is difficult.

(2) The transformation formulas of [8] or [5] for Bäcklund transformations were only derived for diffeomorphisms between u and s . But certainly (2.4) not even defines a map from u to w since, obviously, the implicit function theorem for

$$B(u,w)=(\Phi(u)-c)w=0 \tag{3.1}$$

does not hold because w itself lies in the kernel of the variational derivative B_w.

But both difficulties are easily discarded for the following reasons:

(3) Of course, eigenvectors are difficult to find. But given an eigenvector then going the other way, namely to find the potential, often is extremely simple. And this is what only is required in our case.

(4) Although the implicit function theorem is violated we nevertheless can apply all the transformation formulas given in [8] or [5]. This because we know that the kernel of B_w consists of the function w and this function is a symmetry group generator [7]. Hence, for the equations of the interacting soliton, we can work in the algebra modulo an additional and obvious symmetry (see [12] for details).

So let us carry out the necessary computations in case of the KdV.

EXAMPLE 1:
For the KdV

$$u_t=u_{xxx}+6uu_x \tag{3.2}$$

equations (2.4) to (2.6) have the form

$$cs_x = s_{xxx} + 4us_x + 2u_xs \quad \textit{(eigenvector eq.)} \tag{3.3}$$

$$s_t = s_{xxx} + 6us_x \quad \textit{(coupled evolution eq)}, \tag{3.4}$$

here already w was replaced by s_x.

Now, solving (3.3) for u in terms of s we find

$$u = \pm\frac{1}{2}\frac{cs - s_{xx}}{\sqrt{cs^2 - s_x{}^2 + const.}}. \tag{3.5}$$

Inserting the boundary condition at infinity we find for the integration constant $const.{=}0$. Hence we finally have

$$u = \pm\frac{1}{2}\frac{cs - s_{xx}}{\sqrt{cs^2 - s_x{}^2}}. \tag{3.6}$$

Insertion of this into (3.4) yields

$$s^2 s_t = s^2 s_{xxx} - 3ss_x s_{xx} + \frac{3}{2}s_x{}^3 + \frac{3}{2}c\, s^2 s_x \tag{3.7}$$

which describes the evolution of interacting solitons for the KdV (no matter how many other solitons are present).

The same simple procedure can be applied to obtain the evolution of interacting solitons for:
Burgers equation:

$$ss_t = ss_{xx} - 2s_x s_x + 2c\, ss_x \tag{3.8}$$

mKdV-equation:

$$s_t = s_{xxx} + \frac{3(c\, s - s_{xx})^2}{2(c\, s^2 - s_x{}^2)} s_x \tag{3.9}$$

sine-Gordon equation:

$$s_{xt} = \frac{1}{2}s \cos\left[\int_{-\infty}^{x} \frac{c\, s(\xi) - s_{\xi\xi}(\xi)}{2\sqrt{cs(\xi)^2 - s_\xi(\xi)^2}} d\xi\right] \tag{3.10}$$

and the nonlinear Schrödinger equation:

$$|\psi|^2\psi_t = -i\psi_{xx}\,|\,\psi\,|^2 + i\psi\,|\,m\psi + \frac{i}{2}\psi_x\,|^2 - i(m\psi + i\psi_x)^2\overline{\psi}. \tag{3.11}$$

In the last equation ψ is the x-derivative of the interacting soliton.
See [12] for the details of the computation.

IV. Interacting Solitons in the Two-Soliton case

In the presence of a recursion operator Φ we find for the two-soliton ([10], [12])

$$u_x{=}w_1{+}w_2 \tag{4.1}$$

where

$$\Phi(u)w_i=c_iw_i\ , \quad i=1,2 \tag{4.2}$$

and where the c_i are the asymptotic speeds of the solitons. This yields

$$(\Phi(u)-c_1)(\Phi(w)-c_2)u_x=0. \tag{4.3}$$

Now, we apply the obvious identity

$$1=\frac{1}{c_2-c_1}(\Phi(u)-c_1)+\frac{1}{c_1-c_2}(\Phi(u)-c_2) \tag{4.4}$$

in order to obtain

$$u_x=\frac{1}{c_2-c_1}(\Phi(u)-c_1)u_x+\frac{1}{c_1-c_2}(\Phi(u)-c_2)u_x \tag{4.5}$$

or

$$u_x=\frac{1}{c_2-c_1}(K(u)-c_1u_x)+\frac{1}{c_1-c_2}(K(u)-c_2u_x).$$

Because of (4.3) $(\Phi-c_2)$ cancels the first term of the decomposition (4.5) and $(\Phi-c_1)$ does the same for the second, therefore we find

$$w_1=\frac{1}{c_1-c_2}(K(u)-c_2u_x) \tag{4.6}$$

and

$$w_2=\frac{1}{c_2-c_1}(K(u)-c_1u_x). \tag{4.7}$$

Here we made use of equation (1.2).

This shows that

$$s_1=\frac{1}{c_1-c_2}D^{-1}(K(u)-c_2u_x) \tag{4.8}$$

and

$$s_2=\frac{1}{c_2-c_1}D^{-1}(K(u)-c_1u_x) \tag{4.9}$$

are solutions of the nonlinear equations for interacting solitons.

In case, of multisolitons of higher order the same analysis goes through (see [12] or [10]). Only the $K(u)$ occurring in (4.8) and (4.9) then have to be replaced by suitable sums over higher order symmetries.

For the two-soliton of the KdV, which is plotted in Fig.1, the corresponding interacting solitons are plotted in Fig.2.A and Fig.2.B, respectively. (In order to have better plots in these pictures we have, compared to the variable x, enlarged the height of the solitons by a factor 7 and the time by a factor 4. The slices to be

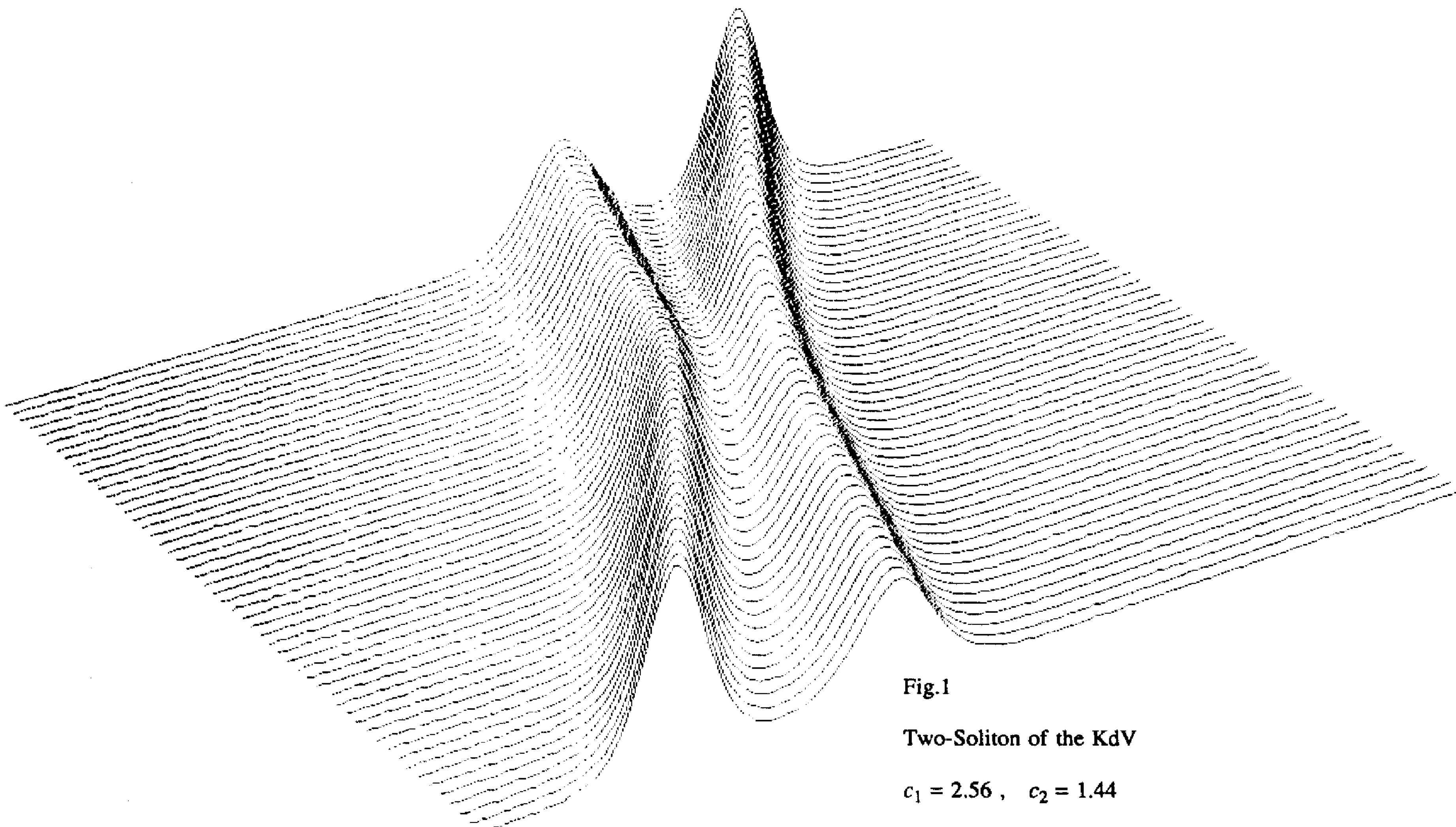

Fig.1

Two-Soliton of the KdV

$c_1 = 2.56$, $c_2 = 1.44$

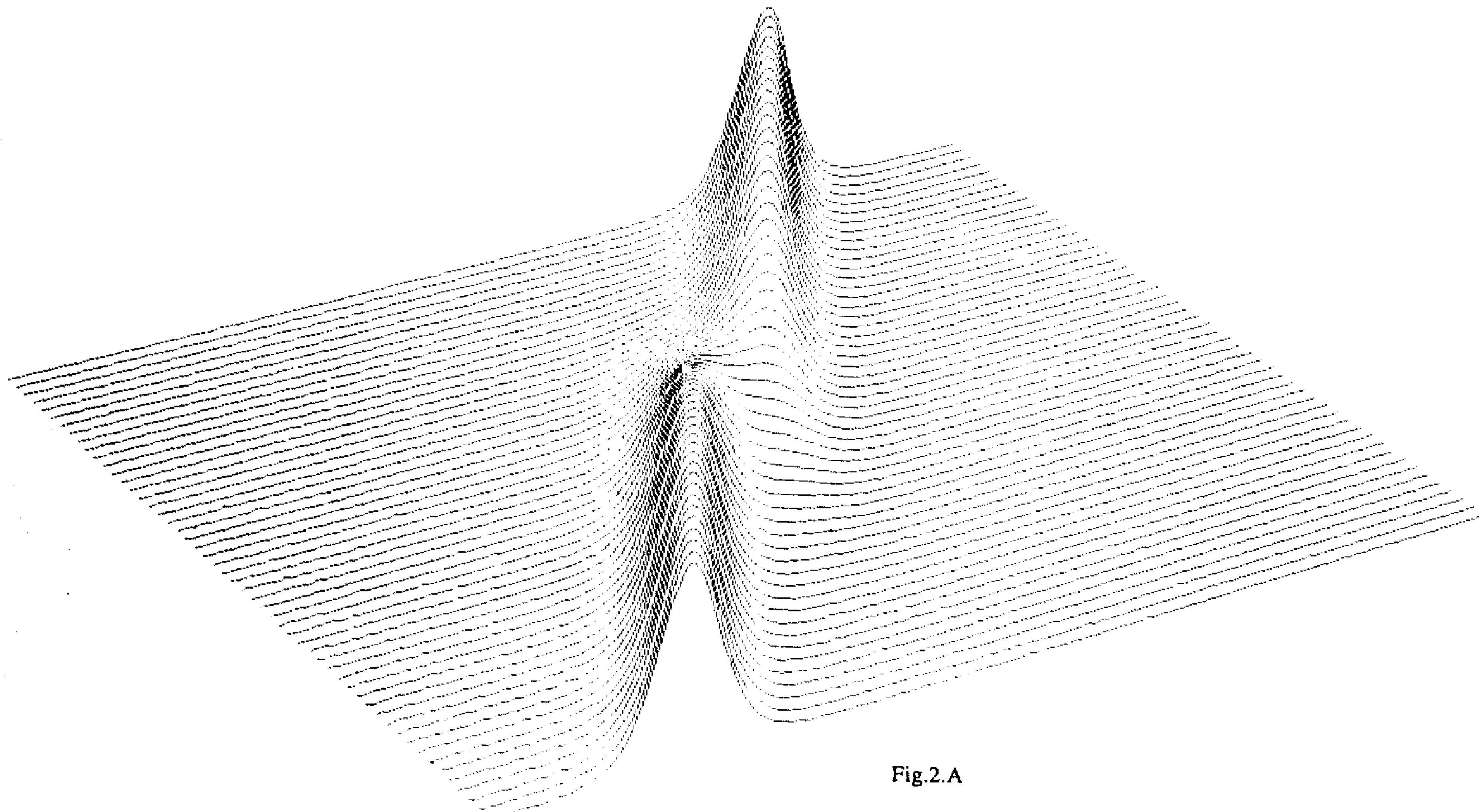

Fig.2.A

Larger Soliton $c_1 = 2.56$

interacting with $c_2 = 1.44$

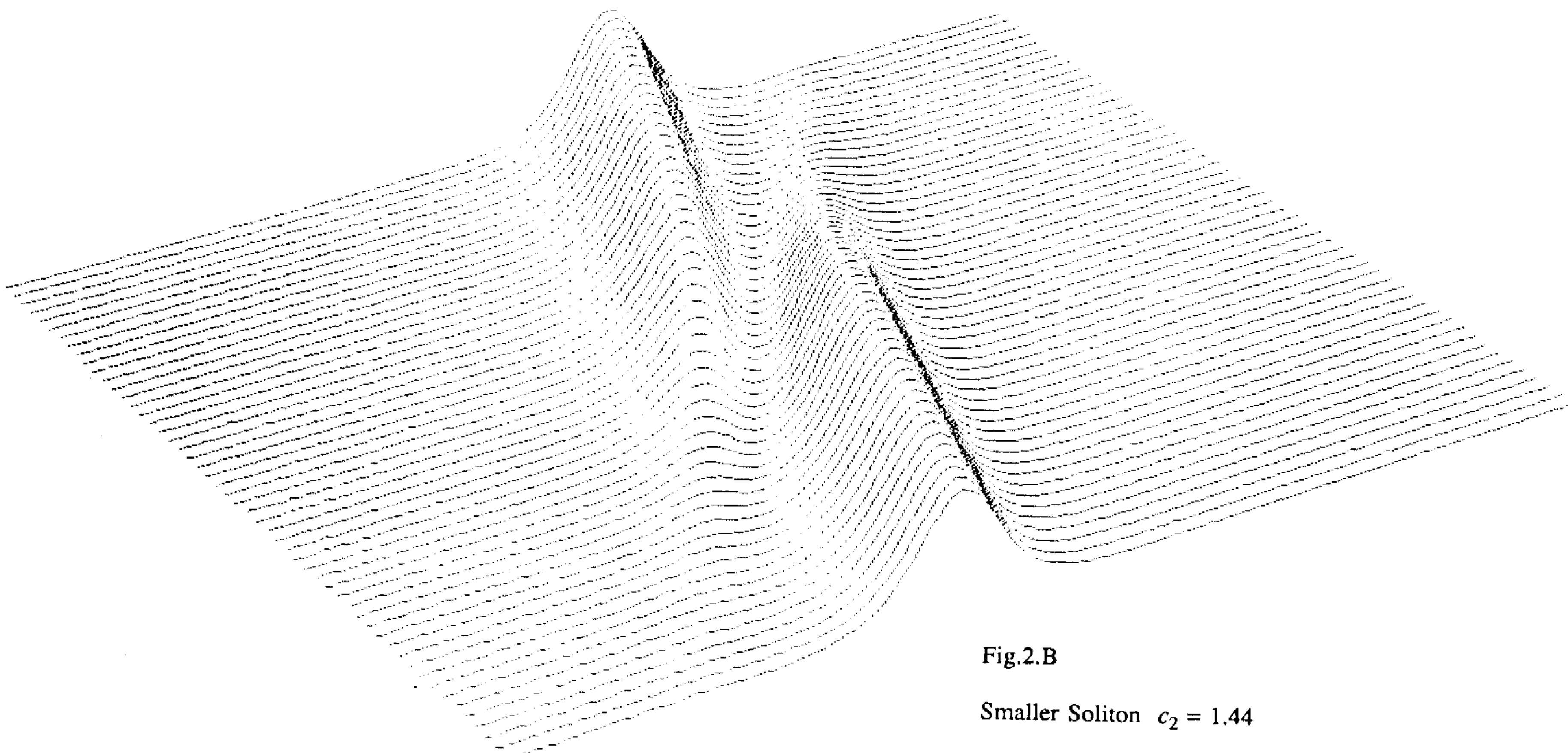

Fig.2.B

Smaller Soliton $c_2 = 1.44$

interacting with $c_1 = 2.56$

seen are the functions for constant time t.)

In a fundamental paper [16] a detailed discussion of different types of interaction can be found. However, this beautiful discussion only applies to the superposition of solitons (i.e. solutions like the one on Fig.1) and not to the interacting soliton itself. Looking at different interacting solitons one finds that there really is only one type of interaction (see [12] for details). Thus the qualitative description of soliton interaction can be considerably simplified by considering the interacting solitons instead of their superposition fields.

V. Algebraic Structure and Virtual Solitons

By application of transformation formulas for Bäcklund transformations ([8],[5]) we find that if $\Phi(u)$ is the recursion operator for (1.1) then by, a straightforward computation (see [12]),

$$\Psi(s) = D^{-1}\Phi'(u)[s_x]F'(s) + D^{-1}\Phi(u)D \quad where\ u = F(s) \tag{5.1}$$

must be the recursion operator for the evolution of the interacting soliton. Here $u=F(s)$ is the solution of the eigenvector equation (2.4)

$$\Phi(u)w=cw\ , \quad s_x=w. \tag{5.2}$$

It turns out that $\Psi(s)$ is hereditary whenever $\Phi(u)$ has that property. Thus an abelian hierarchy of nonlinear equations is found

$$s_t = \Psi(s)^n s_x, \quad n = 0,1,2,... \tag{5.3}$$

of which the first nontrivial member is the equation for the interacting soliton.

The reason why we are so interested in this hierarchy is the following:

To this hierarchy we can apply the same considerations which were applied before to the hierarchy of equation (1.1). This is obvious because the recursion operator was the only technical tool we needed.

Since the spectral properties of $\Phi(u)$ are the same as those for $\Psi(s)$ we find that the interacting soliton coming out of an N-soliton solution u must be an N-soliton solution of the interacting soliton equation (i.e. a decomposition into eigenvectors like the one given in (2.2) for u must hold for s).

But there is one notable difference: Asymptotically only one soliton can be seen emerging out of the interacting soliton even if that comes out of an N-soliton solution for u. Hence, for the equation of the interacting soliton, there are solitons which cannot be seen asymptotically. Those we call **virtual** solitons.

The virtual solitons in the two-soliton case can be computed in the same manner as before. For the decomposition of s_i, i=1,2 given in (4.8) and (4.9) we obtain:

$$s_i=\sigma_{i,1}+\sigma_{i,2}\ ,\quad i=1,2 \tag{5.4}$$

where

$$\sigma_{21}=\sigma_{12} \tag{5.5}$$

is the virtual soliton. As before, we can easily express these quantities in terms of s and s can be expressed in terms of u. This gives

$$\sigma_{2,1}=\sigma_{1,2}=-(c_1-c_2)^{-}2\{4D^{-1}uK(u)-2uD^{-1}K(u)\} \tag{5.6}$$

(see [12] for details). The σ_{11} and σ_{22} are then easily computed by (5.4) and (5.6).

For the two-soliton of the KdV (given by Fig.1) these parts of the "second soliton decomposition" are plotted in Fig.3.a, Fig.3.b and Fig.3v. Looking at the virtual soliton (Fig.3.v) one easily sees that this quantity only pops up during the time of interaction. Hence the shapes and sizes of the virtual solitons can be considered as a measure for the interaction.

VI. Absence of a Recursion Operator

All the results which were presented so far depended heavily on the existence of the recursion operator. Furthermore, it was essential that this recursion operator was of local structure. Hence the problem remains what to do in case of absence of a local recursion operator (like Benjamin-Ono equation, KP and the like).

Actually, in this case the ABT (auto-Bäcklund transformation) can replace the recursion operator. In [11] we have demonstrated that a one-parameter auto-Bäcklund transformation yields a nonlinear spectral problem which - in case a local recursion operator exists - is equivalent to the spectral problem given by the recursion operator.

Let us indicate briefly how to use this equivalence for the derivation of evolution equations of interacting solitons. This method then carries over to the case where a local recursion operator does not exist (see [13]).

Consider the one-parameter family of ABT's:

$$B(u,\tilde{u},\lambda)=0. \tag{6.1}$$

Then the nonlinear spectral problem is:

SPECTRAL PROBLEM:
Given u, and $\tilde{u}$ by (6.1), pick those λ's such that there is a nonzero w with

$$B_u[w]=0. \tag{6.2}$$

Then the w plays the role of the eigenvector of the recursion operator. Thus we find the following set of equations for the evolution of the soliton s

$$B(u,\tilde{u},\lambda)=0 \tag{6.3}$$

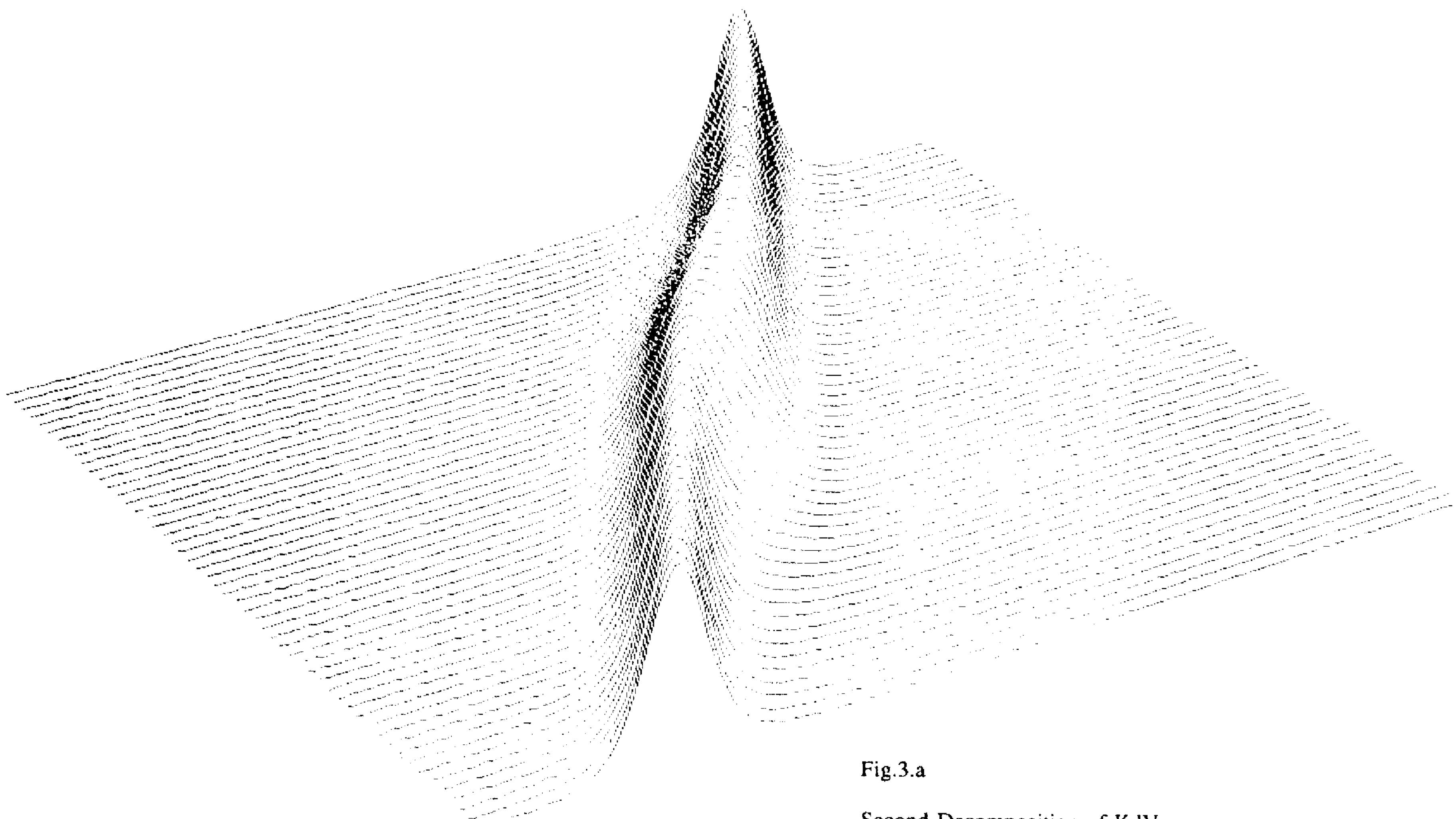

Fig.3.a

Second Decomposition of KdV

Larger Soliton

$c_1 = 2.56$, $c_2 = 1.44$

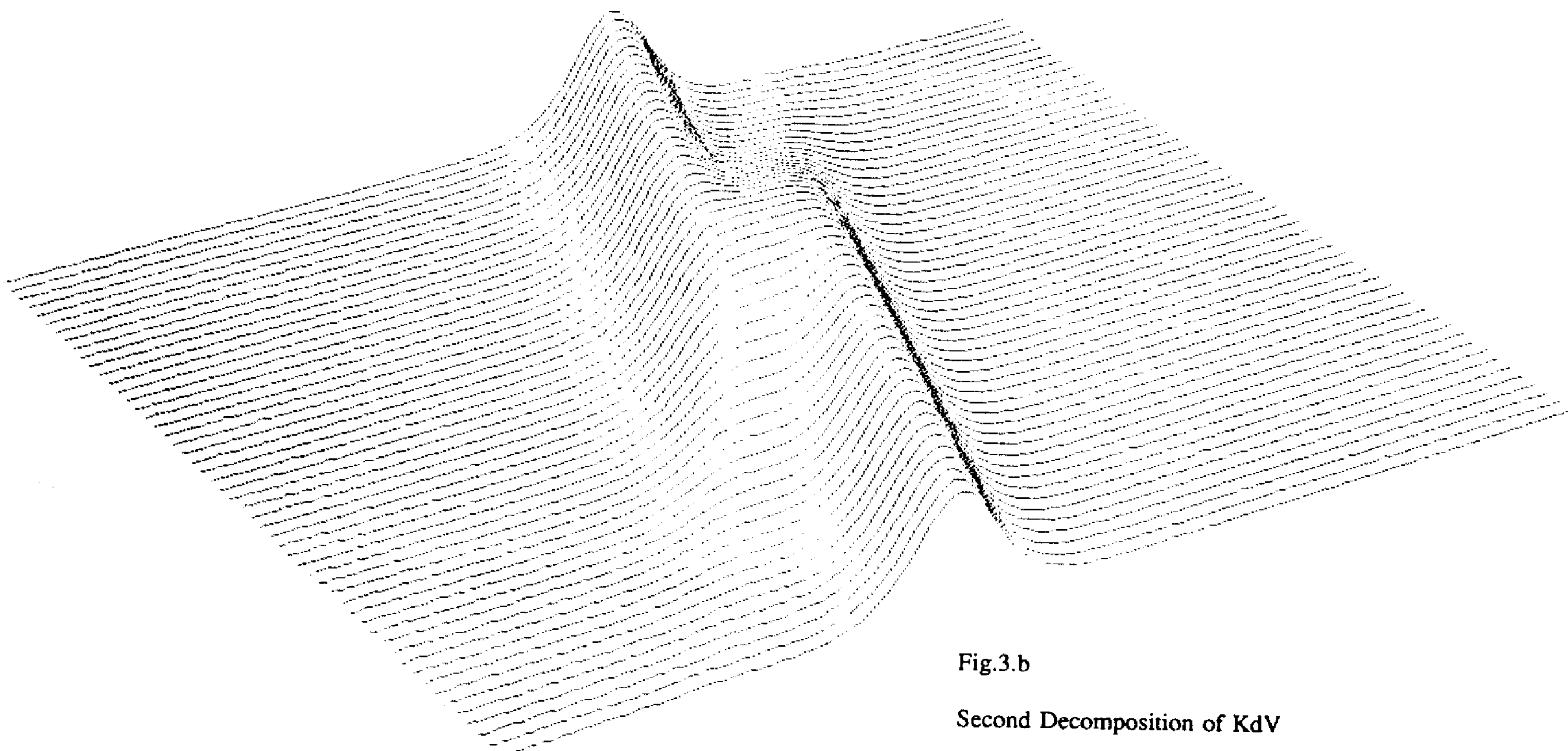

Fig.3.b

Second Decomposition of KdV

Smaller Soliton

$c_1 = 2.56$, $c_2 = 1.44$

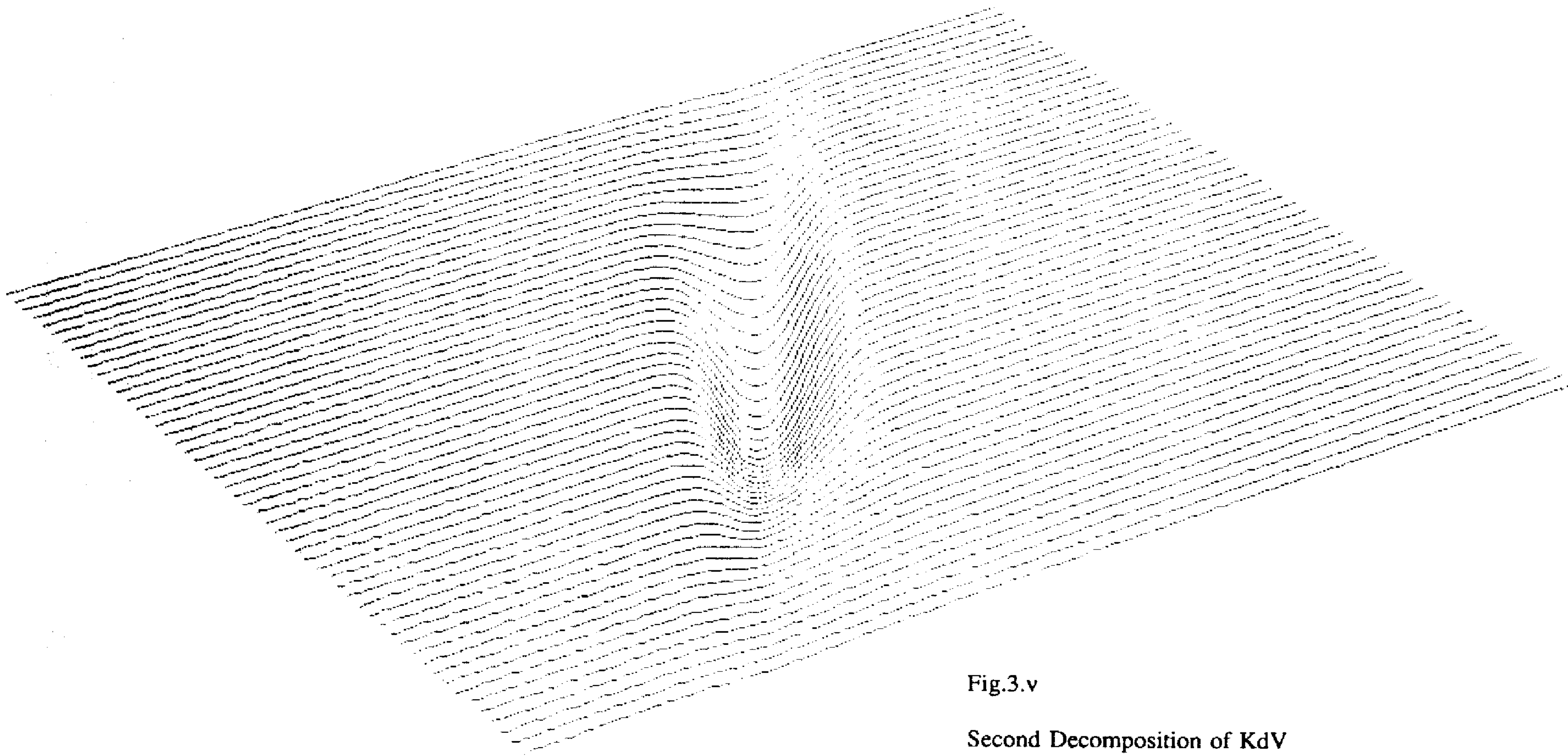

Fig.3.v

Second Decomposition of KdV

Virtual Soliton

$c_1 = 2.56$, $c_2 = 1.44$

$$B_u(u,\tilde{u},\lambda)[w]=0 \tag{6.4}$$

$$w_t=K'(u)[w] \tag{6.5}$$

$$s_x = w. \tag{6.6}$$

Now, solve (6.3) to (6.4) for u and $\tilde{u}$ and insert these in the dynamics given by (6.5). This nonlinear equation then gives the evolution of the interacting soliton.

The reason that the whole procedure works is founded on the Bianchi identity which is the equivalent of the hereditaryness for the recursion operator.

VII Concluding Remarks

Ever since the discovery of solitons there was constant work on decomposition of a field into its soliton components. This work started with the fundamental paper [14], where for multisolitons the decomposition into squared eigenfunctions is given, and goes until a recent series of interesting papers on this subject [22] - [24], and [17] and others.

At first glance, it looks as if all those decompositions were the same. And in fact they are, and they are the same as the one given in this paper. Nevertheless there is a fundamental difference between the present paper and the results of others.

A rather unessential difference lies in the methods applied in order to obtain the desired decomposition. Mostly, inverse scattering transform methods are used for the decomposition, as it was already the case in [14] . In fact this is not necessary as it was shown already in [6]. This change in method has two consequences. First, one discovers that the method must work also in cases where inverse scattering is unapplicable or only more difficult (e.g. other boundary conditions at infinity). Secondly, this observation opens the road for the discovery, that solitons, as dynamical systems being coupled to the superposition field, also make sense in cases where apart from solitons (discrete parts of the spectrum of the recursion operator or of the scattering method) also continuous parts of the spectrum contribute.

But even this discovery is rather ancient. For example in [7] [9] it was stated that the eigenvector of the recursion operator can be considered as soliton in interaction and that furthermore this soliton in interaction has a well defined dynamical behavior, namely that of a gradient of a conserved quantity.

But the present method, we believe, contains a new aspect. Namely, that in principle it is possible to find the dynamical behavior of the interacting system in such a way that no external field and no superposition with other solitons enters in the description of this dynamical behavior. Furthermore, that the dynamical system found this way also makes sense in cases which are not pure soliton solutions. So, the coupled systems which are given by other authors are decoupled, only self-interaction plays a role. To be precise: The dynamics given by other authors for interacting solitons is the one expressed by equation (2.5) which then in the multi-soliton case leads via (4.1), or its multisoliton analogue, to a coupled system. In contrast to that the dynamics expressed by (3.7) even holds in absence of the decomposition (4.1).

This decoupling is an essential prerequisite for finding a dynamical description of interacting solitons which is independent of the number of solitons present in the field.

Contrary to that, in the soliton-decompositions which can be found in the literature one will discover that the coupled equations change with the number of solitons present.

This decoupling, which mathematically turns out to be a triviality, then allows to study the structure of the interacting solitons (i.e. show their complete integrability in the general case, and find their recursion operators). Another important consequence seems to me that very many new systems, which are completely integrable, can be constructed this way, and, if one likes it, many new Lax pairs can be found.

Without finding the dynamical behavior in a selfinteracting way the second decomposition, which to my opinion offers a better qualitative understanding of soliton interaction, would have been impossible. With the knowledge of the dynamics in terms of uncoupled equations the second decomposition, as well as the third, the fourth etc. reduces to an observation which is more or less trivial.

There is another consequence of decoupling interacting solitons which seems even more important to me. One of the ultimate aims of decomposition into solitons seems to be to find simple dynamical descriptions for the "trajectories" of solitons. That means one tries to replace the dynamics (given by some nonlinear partial differential equation) by a system of ordinary differential equations describing the positions or "barycenters" of the different solitons. Thus a flow on some infinite dimensional manifold is described by a flow on a finite dimensional manifold. Of course, such a method, if it is known on a systematic basis would provide a better understanding and a simpler description of soliton interaction. There are very many interesting contributions towards such a method (see [20], [15], [4], [1], [2], [3] and the most interesting recent papers [18] [19]). But to my knowledge a systematic and foolproof method how to find particle systems imitating the soliton collision is still missing.

In principle, such a method is a consequence of our present results (although the technical details may be cumbersome). Let me describe this briefly, say for the case of the KdV:

Choose a set of trajectories $y_k(t)$, $k=1,...,N$ (to be specified later on) and consider a multi-soliton solution of the KdV, say some N-soliton. Define quantities $p_{i,k}^{(r)}$ ($r=0,1,2$; $i,k=1,...,N$) to be the values which the r-th derivative of the soliton s_i attains at y_k. Then by the eigenvector equation (2.4), and the decomposition given by (4.1) or its multisoliton analogue, the values at the y_k of higher derivatives than r of the s_i can be expressed by the $p_{i,k}^{(r)}$. Now, using the dynamics which is explicitly given for the s_i we can express the time evolution of the $p_{i,k}^{(r)}$ also by these quantities and the time derivatives of the y_k. This suggests that reasonable trajectories are those, where the time derivatives can be expressed also in terms of the $p_{i,k}^{(r)}$. Then for those we have a complete description of the dynamics of the $p_{i,k}^{(r)}$. But such trajectories are easily found, for example, take y_k to be the zero of s_{kx}. Then

$$\frac{d}{dt} s_k(y_k(t),t)_x = 0$$

together with (3.7) easily gives the desired relation between y_{kt} and the $p_{i,k}^{(r)}$. Here, again the explicit knowledge of the dynamics of the interacting soliton plays an essential role.

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