Hamiltonian Structure and Integrability

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1 Introduction

Whenever a quantity, or a set of quantities, evolves with time then we call this a **dynamical system**. The evolution of the universe certainly is a dynamical system, however a complicated one. The laws of evolution which govern such a system are called the **dynamical laws**.

To describe dynamical systems we usually make suitable approximations in the hope of finding valid descriptions of their characteristic quantities. But even after such approximations we mostly cannot write down explicitly how these quantities depend on time. usually such a dependence is much to complicated to be computed explicitly. Therefore we commonly write down dynamical systems in their **infinitesimal form**.

Considering a dynamical system in its infinitesimal form has many advantages. The principal one is that such an infinitesimal description is possible even in those cases where a **global** description is not feasible at all. Technically speaking, an infinitesimal description leads to a differential equation, which in many cases has nonlinear terms due to the interaction between different quantities. To find such a differential equation we only have to know a suitable set of dynamical laws. However, solving such a nonlinear differential equation for arbitrary starting points (initial conditions) is often a hopeless endeavor.

Fortunately, the infinitesimal description sometimes gives an insight into the essential structures for the dynamics of the system, or at least into those parts of the dynamics which can be described locally.

Speaking from an abstract viewpoint the main objects of our interest are equations of the form

$$u_t = K(u) \tag{1.1}$$

where $K(u)$ is a **vector field** on some manifold M and where u denotes the general point on this manifold. Since we do not restrict the size of the dimension of the manifold M this equation still comprises an abundance of possible dynamical systems. For example u. could be the collection of all relevant data of an economy, then equation (1.1) describes the evolution of that economy. With regard to size of the manifold, this would be a rather simple dynamical system since the manifold certainly has finite dimension whereas most systems we consider later on will describe systems on **infinite dimensional manifolds**. Most notions which we use in the study of equation (1.1) do have a very intuitive meaning. For example, we call equation (1.1) a **flow** on the underlying manifold. Thus we imagine that a point is flowing along its path on the manifold. Such a path is called an **orbit** of the system. Since $K(u)$ describes the change in the position of u for infinitesimal times.

$K(u)$ must be tangential to the orbit of u. A simplified picture for the situation under consideration is:
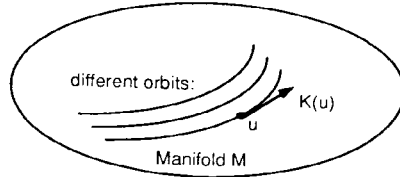

Fig. 1: Flow on a manifold M

Systems of particular importance are those describing the dynamics of particles in classical mechanics. For these systems the dynamical laws are determined by the total energy of the system. As an example, we consider the case when the energy $H = V + T$ is the sum of potential energy V and kinetic energy T, where $V = V(\vec{x}_1, ..., \vec{x}_n)$ only depends on the positions of the different particles and where the kinetic energy is

$$T = \frac{1}{2} \sum m_i \dot{x}_i^2. \quad (m_i = mass \; of \; particle \; i) .$$

In order to eliminate the masses m_i (which are irrelevant for the structure of the system) we introduce new coordinates in the space $\mathbb{R}^{2n}$ (phase space)

$$q_i = \vec{x}_i, \; p_i = m_i \dot{\vec{x}}_i, \; i = 1, ..., n. \tag{1.2}$$

Using Newton's law we find

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.3}$$

or, if we introduce the formal field variable $\vec{u}$ to be the transposed of the position-momentum vector $\vec{u} = (q_1, ..., q_n, p_1, ..., p_n)^T$, then the dynamics has the form

$$\vec{u}_t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla H \tag{1.4}$$

where I is the identity in n-space.

Notable is that in this equation only ∇H the *gradient* of the energy (taken in the phase space) enters. These equations, either in the form (1.3) or (1.4), are prototypes of *Hamiltonian systems*. [1] Systems of this form have remarkable properties. Not only do they seem to be solely determined by their energy, but also there is the surprising property that whenever the energy functional H is invariant under either translation or rotation then we have conservation of momentum or angular momentum, respectively. So there must

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[1] Sir William Rowan Hamilton (August 4, 1805 - September 2, 1865) studied at Trinity College in Dublin where he obtained the Chair of Astronomy in the age of 22. Apart from his work in mechanics (principle of least action) he gave fundamental contributions to optics and mathematics. For example, he introduced complex numbers and the quaternions.

be some kind of relation between the conservation laws of the system and its symmetry structure. Indeed such a relation was revealed for classical Hamiltonian systems by Emmy Noether[2] in her habilitation [18] [3] and this relation is nowadays generalized to symmetries of so-called Lie-Bäcklund type for systems on infinite dimensional manifolds. The special form of (1.3) or (1.4) is due to the special coordinate system which was chosen, whereas the fundamental relation between symmetries and conserved quantities certainly must go beyond a structure which is the consequence of a special coordinate system. So, in studying Hamiltonian equations they must be analyzed in a differential geometric invariant setup such that their structure becomes independent of special charts which were chosen to parametrize the underlying manifold. We shall do that in the following sections 4 and 5.

The best known examples of Hamiltonian systems probably are the Harmonic Oscillator and the nonlinear pendulum, described below:

Example 1.1: Harmonic Oscillator

The evolution equations

$$x_t = y , y_t = -x ,$$

where $x(t), y(t) \in \mathbb{R}$, describe the time dependence of the harmonic oscillator. In matrix form this can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{1.5}$$

which certainly has the form (1.4) since

$$\begin{pmatrix} x \\ y \end{pmatrix} = \nabla H , \quad H = \frac{1}{2}(x^2 + y^2) .$$

The manifold under consideration is $M = \mathbb{R}^2$. Introducing the abbreviations

$$K(u) = Au, u = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the evolution equation (1.5) is clearly an example for (1.1). Looking at this system we detect many characteristic features which carry over to some nonlinear systems. The evolution of this flow is of the form

$$\exp(tA) : u(0) \longrightarrow u(t) \tag{1.6}$$

which shows that the map from the initial condition $u(0)$ to $u(t)$ defines a set of diffeomorphisms on the manifold $M = \mathbb{R}^2$. Because of the exponential function these diffeomorphisms

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[2] Emmy Noether (March 3, 1882 - April 4, 1935) was the daughter of the renowned mathematician Max Noether. With her contributions to the theory of ideals and the field of non-commutative algebra she influenced the development of modern algebra to a great extent. After she lost in 1933 her venia legendi (right to teach) in Nazi Germany she emigrated to Princeton.

[3] A work for which Emmy Noether herself had no high opinion (see [1]). Later on she refused to take any more notice of this work of fundamental importance and she even claimed that it had been lost ("verschollen"). Hermann Weyl had a completely different opinion and acknowledged this in his memorial address delivered in Bryn Mawr College on April, 26, 1935: "For two of the most significant sides of general relativity theory she gave at that time the genuine and universal mathematical formulation: First, the reduction of the problem of differential invariants to a purely algebraic one by use of "normal coordinates"; second, the identities between the left sides of Euler's equations of a problem of variation which occur when the (multiple) integral is invariant with respect to a group of transformations..."

form a representation of the additive group $(\mathbb{R}, +)$. Furthermore, we observe the advantage of introducing polar coordinates $r = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(y/x)$. Then in this new coordinate space the system becomes a flow with constant velocity along the coordinate lines $r = constant$. Thus in this case we are able to split up the coordinates into two sets, one set (*action variables*) which remains constant under the flow, and another set (*angle variables*) which grows on the orbits linear with time. If such special coordinates having these properties can be introduced then we call such a system *completely integrable*. Looking back at our example we see that this notion of complete integrability must be related to the existence of one-parameter symmetry groups. This is because changing one of the coordinates and leaving the others unchanged moves orbits into orbits. So this movement along coordinate lines constitutes a *symmetry group*. □

Example 1.2: Pendulum
The time development in this case is

$$\varphi_{tt} + \sin(\varphi) = 0. \tag{1.7}$$

Introducing

$$q = \varphi, \ p = \varphi_t \ and \ u = \begin{pmatrix} q \\ p \end{pmatrix}$$

we see that (1.7) is of the form (1.1)

$$u_t = \begin{pmatrix} q \\ p \end{pmatrix}_t = \begin{pmatrix} p \\ -\sin(q) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin(q) \\ p \end{pmatrix}. \tag{1.8}$$

The manifold under consideration again is $M = \mathbb{R}^2$. In contrast to (1.5) this equation constitutes a nonlinear flow. Again the dynamics has the form (1.4) since

$$\begin{pmatrix} \sin(q) \\ p \end{pmatrix} = \nabla H, \quad H = 1 - \cos(q) + \frac{1}{2}p^2 .$$

Although this is a nonlinear flow it can be linearized locally by introducing a suitable coordinate system. But this coordinate system is no longer given by polar coordinates. Obviously the part of the coordinate lines which we called action variables should now be given by the lines $H = constant$ and the remaining part should be chosen in a suitable way. How to do this will be described later on. □

The scope of this article is to rephrase these simple observations which we made for the harmonic oscillator in a general framework so that they can be carried over to other more complicated systems. Furthermore, we want to formulate the corresponding notions and relations in such a way that they are independent of the coordinate systems which we choose.

We organize the article in the following way: In the next section, we introduce some basic notions which lead in Section 3 to a description of the connection between symmetries and conserved quantities. At that point we shall not yet choose the most abstract setup for the description. Instead of formulating everything in a differential geometric invariant way we still will work with coordinate systems. This we do in order to keep the level of abstraction at the beginning as low as possible. Results in these sections are mostly

presented without proofs because later on proofs will be given in a short and concise way by using a higher level of abstraction. In Section 4 we introduce Lie algebra modules, Lie derivatives and tensors in order to have a notation which allows one to see which notions are geometrically invariant. In Section 5 we introduce the notion of bi-hamiltonian fields on a general level. Then, in the following section, we introduce compatibility, especially for hamiltonian pairs, and illustrate the power of this notion by a set of suitable examples (Section 7). In the final part, Section 8, we discuss complete integrability in the finite dimensional case and we show how that notion is connected to the situation considered before. In addition the action/angle structure of the multisoliton manifolds is given

2 Basic Notions in Chart Representation

I hope that most readers are acquainted with notions like manifolds, vector fields, tangent space, differentiability and so on. However, I do not believe that a knowledge of the theoretical background in manifold analysis is really necessary for understanding the concepts described in this article. For the most part a more intuitive grasp of infinitesimal calculus and a heuristic idea of manifolds as being something like smooth surfaces seems sufficient.

For the sake of completeness however, we include some remarks on this subject since notation will differ somewhat from the conventional notation, insofar as we avoid the calculus of exterior forms.

For infinite dimensional manifolds we will use the notion of Hadamard differentiability [26.] [27.] This is a fairly weak notion which nevertheless ensures the validity of the chain rule. A function $F : E_1 \to E_2$ between two linear spaces [4] is said to be *Hadamard-differentiable* at $u \in E_1$ if there is a continuous linear map $L : E_1 \to E_2$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ F(u + \epsilon v) - F(u) - \epsilon L[v] \} = 0 \qquad (2.1)$$

uniformly in v on each compact subset of E_1. The linear operator L, and its application $L[v]$ to v are then denoted by $F'(u)$ and $F'(u)[v]$, respectively. Of course, $F'(u)[v]$ is most easily computed from the *directional derivative* of F

$$F'[v] = F'(u)[v] = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} F(u + \epsilon v). \qquad (2.2)$$

If not otherwise mentioned functions are usually assumed to be C^∞-functions, i.e. infinitely often differentiable.

If the manifold is a vector space $M = E$, then vector fields are the continuous maps $K : E \to E$ assigning to each $u \in M$ some vector $K(u) \in E$. Again, we assume vector fields

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[4] All linear spaces E are assumed to be locally convex Hausdorff topological vector spaces. Usually we do not describe explicitly the topology on E. We rather introduce a vector space E^* of linear functionals on E, which separate points, and we assume that E is endowed with the weakest locally convex topology such that the elements of E^* are continuous (i.e. the weak topology with respect to E^*). Spaces $L(E_1, E_2)$ of linear maps $E_1 \to E_2$ are then endowed with the weakest topology given by the dual pairs E_1, E_1^* and E_2, E_2^*, i.e. the weakest convex topology such that all linear functionals μ on $L(E_1, E_2)$ given by $L \to \mu(L) = \rho(L(u))$ with $\rho \in E_2^*$, $u \in E_1$ are continuous.

to be C^∞. Thus they constitute a Lie algebra with respect to the **commutator** defined by

$$
\begin{aligned}
[K,G](u) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \{G(u + \epsilon K(u)) - K(u + \epsilon G(u))\} \\
&= G'(u)[K(u)] - K'(u)[G(u)].
\end{aligned}
\tag{2.3}
$$

This Lie algebra is referred to as the **vector field Lie algebra** .

Recall that the definition of a Lie-algebra implies that the map $(K,G) \rightarrow [K,G]$ is bilinear, antisymmetric ($[K,G] = -[G,K]$) and such that for all K, G, L the **Jacobi identity**

$$
[[K,G],L] + [[L,K],G] + [[G,L],K] = 0
\tag{2.4}
$$

holds. In fact this identity is easily verified by using the chain rule of differentiation.

If the manifolds M which we consider are not linear spaces, then derivatives are defined in the usual way by parametrizing, or modeling, manifolds by linear spaces. Although in most of our examples the underlying manifold is a vector space we briefly illustrate that procedure for the sake of completeness. Those readers who do not care for technicalities should skip the following paragraphs up to the introduction of conserved quantities.

Let M be some Hausdorff topological space and E some linear space, then we call M a C^∞-manifold if there are given an open covering $\{U_\alpha | \alpha \in some\ index\ set\}$ of M and homeomorphisms

$$
p_\alpha : U_\alpha \rightarrow V_\alpha, \; V_\alpha \text{ open susets of } E
$$

such that for all α and β the overlap map $p_\alpha \circ p_\beta^{-1}$ is a C^∞-map $V_\beta \cap p_\beta(U_\alpha \cap U_\beta) - E$. These p_α can be considered to be local coordinates for the corresponding U_α. The collection of these (U_α, p_α) is defined to be an *atlas*. Such an atlas allows transfer of all aspects of the differential structure from E to M. For example, a map φ on M is defined to be C^∞ if all the $\varphi \circ p_\beta^{-1}$ are C^∞. Consider $u \in M$, then a *chart* around u is a homeomorphism p from an open neighborhood U of u into the model space E such that for all α the map $p \circ p_\alpha^{-1}$ defined on $p_\alpha(U) \cap V_\alpha$ is C^∞. Now, the notion of tangent space is easily introduced. The *tangent space* $T_u M$ at the point u is represented by the model space together with a chart. The formal definition has to be such that it does not depend on the special chart which is chosen, hence it must be given by equivalence classes with respect to different charts. So, for fixed u, we consider pairs (p, v) consisting of a chart around u and a vector v in the model space. In these pairs we introduce an equivalence relation $(p_1, v_1) \equiv (p_2, v_2)$ defined by $(p_2 \circ p_1^{-1})'(p_1(u))[v_1] = v_2$. Then $T_u M$ is defined to be the set of equivalence classes endowed with the obvious topology inherited from the model space, and TM denotes the collection of all these tangent spaces and is called the *tangent bundle*. However, working with these equivalence classes by fixing some chart p around u and representing the tangent spaces by fixing some chart p around u and representing the tangent spaces of the points around u jointly by (p, E). Then a map $K : M \rightarrow TM$ which assigns to each $u \in M$ some element of $T_u M$ is said to be a C^∞-vector field if it is locally C^∞ with respect to such a common representation. Similarly, we define locally the *co-vector space* $T_u^* M$ by (p, E^*) instead, and *co-vector fields* to be suitable C^∞-maps from M into the *cotangent bundle* (collection of all $T_u^* M$.) Of course, strictly speaking, elements of $T_u^* M$ are again equivalence classes as before (only in the definition above, one has to replace the derivatives

by suitable adjoints in order to leave the application of a co-vector to a tangent vector invariant under coordinate changes). It should be observed that the choice of a common representation around some point $u \in M$ is the same as choosing a particular chart in the manifold given by the tangent bundle.

If the manifold under consideration is a linear space, then we do not really need all these constructions because we then model the manifold by itself and for simplicity we choose the canonical chart given by the identity function on the model space. The validity of the requirements above then follows from the usual transformation formulas of differential calculus. In this case the tangent spaces $T_u M$ and cotangent spaces $T_u^* M$ can be identified with E and E^*, respectively, and we are back in the situation $M = E$ which we studied at the beginning.

We have chosen this formal approach to manifolds in order to indicate that differential calculus on abstract manifolds is indeed an easy task and that nevertheless for practical computations it mostly is sufficient to do analysis on linear spaces.

To proceed, we consider again

$$u_t = K(u), \quad u \in M, \quad M \text{ some manifold}. \tag{1.1}$$

C^∞-maps from the manifold M into the scalars (either $\mathbb{R}$ or $\mathbb{C}$) are called *scalar fields*. A scalar field $I(u)$ is said to be a **conserved quantity** for (1.1) if

$$I'(u)[K(u)] = 0 \tag{2.5}$$

for all $u \in M$. The reason why this name has been chosen is obvious: Take an orbit $u(t)$ of (1.1), then by the chain rule we find

$$\frac{d}{dt} I(u(t)) = I'(u(t))[K(u(t))] = 0. \tag{2.6}$$

Hence, (2.5) guarantees that I is constant along the orbits of (1.1).

Observe that, for every $u \in M$, the quantity $I'(u)$ is a continuous linear functional on the tangent space $T_u M$, i.e. $I'(u)$ must be a cotangent vector. Derivatives of scalar fields are called **gradients** and f. Therefore we use for scalar quantities I the notation $\nabla I(u)$ instead of $I'(u)$. If we write $<,>$ for the duality between tangent and cotangent vectors. then (2.5) is written as

$$< \nabla I, K > = 0. \tag{2.7}$$

Sometimes there is some advantage in looking at conserved quantities which depend explicitly on time. A family $F(u,t)$ of scalar fields depending in a C^∞-way on the parameter t is said to be a **time dependent conserved quantity** if

$$F_t(u,t) + < \nabla F(u,t), K(u) > = 0. \tag{2.8}$$

Here sub-t denotes partial derivative with respect to t and $\nabla F = F'$ is taken by ignoring the parameter t. The notion makes sense because it gives

$$\frac{d}{dt} F(u(t), t) = 0 \tag{2.9}$$

which implies that $F(u(t), t)$ is constant along the orbits of (1.1). From the physical point of view such a quantity does not seem to be very significant, since it is not invariant with respect to a translation of time. Nevertheless it turns out that it is a rather interesting quantity from the computational point of view.

Of special interest are those conserved quantities which are linear in t. Let

$$F(u, t) = f_0(u) + f_1(u)t \tag{2.10}$$

be such a quantity. Inserting F into (2.8) we then obtain by comparison of coefficients

$$f_1 + <\nabla f_0, K> = 0 . \tag{2.11}$$

Hence, $F(u, t)$ is uniquely determined by its absolute term $f_0(u)$. Furthermore, the term $f_1(u)$ must be a conserved quantity which is time independent.

Related to conserved quantities are one-parameter groups of C^∞-diffeomorphisms on the manifold M. Recall that these are defined to be one-to-one C^∞-maps such that the inverse is again differentiable. A **one-parameter group** of diffeomorphisms is a map $(u, \tau) \to R(\tau)(u)$ which is differentiable on the product $M \times \mathbb{R} = \{(u, \tau) | u \in M, \tau \in \mathbb{R}\}$ and assigns to every $\tau \in \mathbb{R}$ some diffeomorphism $R(\tau) : M \to M$ such that

$$R(\tau_1 + \tau_2) = R(\tau_1) \circ R(\tau_2) \ and \ R(0) = I \tag{2.12}$$

for all $\tau_1, \tau_2 \in \mathbb{R}$. This implies that all the $R(\tau)$ do commute and that $R(-\tau)$ must be the inverse of $R(\tau)$. With other words: $R(\tau)$ defines a group representation of the additive group $(\mathbb{R}, +)$. One-parameter groups are completely determined by their τ-derivative at $\tau = 0$. To see this let $R(\tau)$ be a one-parameter group then

$$G = \frac{d}{d\tau} R(\tau)|_{\tau = 0} \tag{2.13}$$

is said to be its *infinitesimal generator*. Equation (2.13) is an abbreviation for $G(u) = (d/d\tau)\{R(\tau)(u)\}|_{\tau=0}$. Since $R(\tau)$ assigns to each point of the manifold another point, G must assign to each manifold point u a tangent vector at u . Hence G is a vector field. Because of the functional equation (2.12) the τ-derivative of $R(\tau)$ at arbitrary τ is easily expressed by G:

$$\frac{d}{d\tau} R(\tau) = G \circ R(\tau). \tag{2.14}$$

Hence $R(\tau)$ is uniquely determined by the vector field $G(u)$.

If the $R(\tau)$ are linear then G again is linear. Then the solution of the linear differential equation (2.14) can formally be written as $R(\tau) = \exp(\tau G)$. In general however, diffeomorphism groups are far from being groups of linear transformations. Nevertheless their structure is more or less given by the exponential function since by use of pull-backs equation (2.14) can be transformed into a linear differential equation (on some abstract manifold with rather high dimension, however).

To see this, consider $\mathcal{F} = C^\infty(M, \mathbb{R})$ or $C^\infty(M, \mathbb{C})$, respectively, the vector space of scalar fields. Let $R : M \to M$ be a C^∞-map then

$$(R^* f)(u) := f(R(u)) , \qquad f \in F \tag{2.15}$$

defines a map $R^* : \mathcal{F} \to \mathcal{F}$ which is linear on $\mathcal{F}$. R^* is said to be the **pull-back** given by R. Similarly, if K is a vector field we define a map $L_K : \mathcal{F} \to \mathcal{F}$

$$(L_K f)(u) = < \nabla f, K(u) >, \quad f \in \mathcal{F} \tag{2.16}$$

by assigning to each $f \in \mathcal{F}$ its derivative in the direction K. L_K is said to be the **Lie-derivative** given by K. Again, this is a linear map on $\mathcal{F}$.

The space of all Lie-derivatives is a vector space. The usual commutator of linear maps

$$[L_K, L_G] = L_K \circ L_G - L_G \circ L_K \tag{2.17}$$

endows this vector space in a natural way with a Lie-algebra structure.

Observation 2.1: *The map $K \to L_K$ is a Lie-algebra isomorphism from the Lie algebra of vector fields onto the Lie-derivatives, i.e. we have*

$$L_{[K,G]} = [L_K, L_G] \tag{2.18}$$

for all vector fields K and G.

The proof is simple, since by differentiation we see that the commutator bracket is a representation of the vector field bracket. Moreover, the required fact that $L_K \neq L_G$ whenever $K \neq G$ follows from the observation that for any two different tangent vectors in $T_u M$ we can find a scalar field (by application of a suitable co-vector) having different derivatives in the direction of these tangent vectors. By similar arguments we find $R_1^* \neq R_2^*$ whenever $R_1 \neq R_2$. Hence, it suffices to study the pull-backs and the Lie-derivatives instead of the original objects. For these new quantities equation (2.14) translates into

$$\frac{d}{d\tau} R^*(\tau) = L_K R^*(\tau). \tag{2.19}$$

which is clearly a linear differential equation since only linear operations are involved. Therefore we write

$$R^*(\tau) = \exp(\tau L_K) \tag{2.20}$$

thus obtaining a representation of the one-parameter group in terms of its infinitesimal generator. Of course, this is a highly artificial representation, since $R^*(\tau)$ and L_K act on infinite-dimensional vector spaces even when M is finite dimensional.

A consequence of these considerations is that one-parameter groups and evolution equations, whether linear or nonlinear, are more or less the same objects. To see this, let $\{R_G(\tau)|\tau \in \mathbb{R}\}$ be such a one-parameter group of diffeomorphisms on M with infinitesimal generator G. Since G is a vector field assigning to each $u \in M$ the tangent vector $G(u) \in T_u M$ we look at evolution equation

$$u_t = G(u). \tag{2.21}$$

In fact, for any initial condition $u(0)$ a solution is easily found, namely

$$u(t) = R_G(t)(u(0)) . \tag{2.22}$$

This solution for the initial value problem of (2.21) must be unique since R_G is a group: To see this take another solution $\bar{u}(t)$ fulfilling the same initial condition $\bar{u}(t = 0) = u(0)$. Then by differentiation we obtain that $R_G(-t)(\bar{u}(t))$ must be independent of t and hence equal to $u(0)$. Recalling that $R_G(-t)$ is the inverse of $R_G(t)$ we find $\bar{u}(t) = u(t)$.

This viewpoint shows that $R_G(t)$ can be understood as the flow operator of (2.21) assigning to each initial condition $u(0)$ the solution $u(t)$ at time t. Of course, not all evolution equations of the form (2.21) necessarily yield one-parameter groups, only those where every initial condition $u(t = 0) = u(0)$ have a unique solution *for all* t such that the flow operator is a C^∞-map.

This interpretation shows, that notions and methods coming from one-parameter groups must lead right-away to the crux of the algebraic aspects of evolution equations. Therein lies the problem of commutativity for nonlinear flows. It is easily seen that this important property can be expressed in terms of infinitesimal generators. Looking at the exponential form of the pull-backs for these groups one discovers the infinitesimal equivalent for commutativity:

Observation 2.2: *Let $R_K(\tau)$ and $R_G(t)$ be two one-parameter groups of diffeomorphisms with infinitesimal generators K and G. These two groups commute, i.e.*

$$R_K(\tau)^\circ R_G(t) \quad = \quad R_G(t)^\circ R_K(\tau) \text{ for all } t \text{ and } \tau \text{ in } I\!R$$

if and only if $[K, G] = 0$, i.e. their infinitesimal generators commute in the vector field Lie-algebra.

In general, it is very hard to verify whether or not a vector field is really the infinitesimal generator of a one-parameter group because usually it is difficult to see if equation (2.21) has a unique solution for every initial condition. But one of the reasons for the success of mathematical analysis is that global conditions (like existence and commutativity of groups $R_K(t)$ and $R_G(\tau)$) can be rephrased, by use of infinitesimal arguments, as local conditions. Therefore it seems natural to put the concept of symmetries onto a purely algebraic and infinitesimal basis by taking the commutativity of vector fields as definition (even in those cases where (2.21) is not the infinitesimal form of some globally defined group).

So, we define the vector field $G(u)$ to be a **symmetry** for the evolution equation (1.1) if and only if $[K, G] = 0$. Here the notion *symmetry* is used as abbreviation for what correctly should be termed as *infinitesimal symmetry-generator*.

Note that when G is a symmetry for (1.1) then K also is a symmetry for $u_t = G(u)$. Using the Jacobi identity we see that whenever the vector fields G and L are symmetries for (1.1) then $[G, L]$ is again a symmetry for this evolution equation. So, the symmetries of (1.1) are a subalgebra of the Lie algebra of vector fields.

It will turn out, that introduction of the concept of time dependent symmetries constitutes an efficient tool. A family of vector fields $G(u, t)$ depending in a C^∞-way on the parameter t is said to be a **time-dependent symmetry** of (1.1) if

$$G_t + [K, G] = 0. \tag{2.23}$$

Here, again $[K, G]$ is taken by ignoring the parameter t .

If $G(u,t)$ and $L(u,t)$ are time-dependent symmetries then $[G, L]$ is again a time-dependent symmetry. This is easily seen from (2.23) and the Jacobi identity. Hence, the time-dependent symmetries are again a subalgebra of the Lie algebra of vector fields.

The algebraic structure of time-dependent symmetries is very similar to the corresponding structure for conservation laws. For example if

$$G(u,t) \;=\; G_0(u) \;+\; G_1(u)t \tag{2.24}$$

is a time-dependent symmetry linear in t, then insertion into (2.23) and comparison of coefficients yields $G_1 + [K, G_0] = 0$. Hence $G(u,t)$ is uniquely determined by its absolute term $G_0(u)$. Furthermore, $G_1(u)$ must be a symmetry.

3   Poisson Brackets and Hamiltonian systems

If one compares equations (2.8) and (2.23) for the dynamical variables given by conserved quantities and symmetries one discovers that these equations look very similar. They both are linear evolution equations on some infinite dimensional manifold

But there is one essential difference between these two equations. A difference which is easily discovered if one looks for means of constructing new solutions. 'A priori', equation (2.23) has more structure than equation (2.8) since there is a Lie algebra involved. This is of considerable advantage because we can take the commutator of any two solutions to find a new solution. So, in order to complete the analogies between conserved quantities and symmetries it seems intriguing to look for Lie algebra structure among solutions of (2.8). Another viewpoint arises by looking at the time derivative in both cases. The time derivative is a special case of what usually is said to be a *derivation*, where derivation means the validity of the product rule (of which the Jacobi identity is a representation). Equation (2.23) tells us that this special time-derivative can be replaced by some *inner derivation*, where an inner derivation is something given by commutation with an element taken out of the structure under consideration. And inner derivations are, from the mathematical viewpoint, much nicer than outer derivations. For example, apart from the discovery that dynamical variables are operators rather than scalars, one of the reasons for the success of quantum mechanics was the ansatz that the time evolution of these operators is given by inner derivations. It is hard to imagine that quantum mechanics would have been feasible at its beginning without this assumption.

Therefore it is natural to ask whether in case of (2.8) the time derivative can be replaced by some inner derivation.

Fortunately, all these questions lead to the same structure, namely *Hamiltonian systems*. If one analyzes the situation further it all boils down to:

Problem 3.1: *Take some operator valued function* $\Theta(u)$ *mapping each manifold element* u *to some linear operator* $\Theta(u) : T_u M^* \to T_u M$. *Define a bracket among scalar fields* F_1, F_2 *by*

$$\{F_1, F_2\}_\Theta \;=\; <\nabla F_2, \Theta \circ \nabla F_1> . \tag{3.1}$$

When is this a Lie-algebra? In addition, when is $\Theta \circ \nabla$ a Lie algebra homomorphism into the vector field Lie algebra, i.e. when do we have

$$\Theta \circ \nabla \{F_1, F_2\}_\Theta = [\Theta \circ \nabla F_1, \Theta \circ \nabla F_2] \quad ? \tag{3.2}$$

We easily find the complete answer to that problem:

Theorem 3.2: *The following are equivalent:*

(1) *The bracket $\{\ ,\ \}_\Theta$ defines a Lie algebra*

(2) *The bracket $\{\ ,\ \}_\Theta$ defines a Lie algebra such that $\Theta \circ \nabla$ fulfills (3.2), i.e. $\Theta \circ \nabla$ is a Lie algebra homomorphism into the vector fields*

(3) *Θ has the following properties*

    (i) *Θ is skew-symmetric with respect to the duality between cotangent space and tangent space, i.e. $\Theta = -\Theta^+$ or $< v_1^*, \Theta v_2^* > = - < v_2^*, \Theta v_1^* >$ for all cotangent vectors $v_1{}^*, v_2{}^*$.*

    (ii) *for all cotangent vectors $v_1^*, v_2^*, v_3^* \in T_u M^*$ the following identity holds $< v_1^*, \Theta(u)'[\Theta(u)v_2^*]v_3^* > + < v_2^*, \Theta(u)'[\Theta(u)v_3^*]v_1^* > + < v_3^*, \Theta(u)'[\Theta(u)v_1^*]v_2^* >= 0$.*

Proof:

First we show the equivalence between (1) and (3).

The skew-symmetry is certainly necessary and sufficient in order to guarantee that $\{\ ,\ \}_\Theta$ is antisymmetric. Computation of the double-bracket yields

$$
\begin{aligned}
\{F_1, \{F_2, F_3\}_\Theta\}_\Theta &= \ < \nabla \{F_2, F_3\}_\Theta, \Theta \nabla F_1 > \\
&= \ < \nabla < \nabla F_3, \Theta \nabla F_2 >, \Theta \nabla F_1 > \\
&= \ F_3''[(\Theta \nabla F_2, \Theta \nabla F_1)] - F_2''[(\Theta \nabla F_3, \Theta \nabla F_1)] + < \nabla F_3, \Theta'[\Theta \nabla F_1] \nabla F_2 >
\end{aligned}
$$

Since second derivatives are symmetric with respect to their entries all second derivatives F'' cancel if $\{F_1, \{F_2, F_3\}_\Theta\}_\Theta + its\ cyclic\ permutations$ are taken. Therefore condition (3.ii) is equivalent to the Jacobi identity for $\{\ ,\ \}_\Theta$ which finishes the proof of the equivalence between (1) and (3).

Since (2) implies (1) it only remains to prove that (3.ii) implies equation (3.2). To see this take two scalar fields F_1, F_2 and some arbitrary co-vector v^*. Since the second derivatives of F_1, F_2 are symmetric they all cancel in the following computation:

$$
\begin{aligned}
< v^*, \ - \ & \Theta \nabla \{F_1, F_2\}_\Theta + [\Theta \nabla F_1, \Theta \nabla F_2] > \\
&= \ < \nabla \{F_1, F_2\}_\Theta, \Theta v^* > \ + \ < v^*, [\Theta \nabla F_1, \Theta \nabla F_2] > \\
&= \ < \nabla F_2, \Theta'[\Theta v^*] \nabla F_1 > + < v^*, \Theta'[\Theta \nabla F_1] \nabla F_2 - \Theta'[\Theta \nabla F_2] \nabla F_1 > \\
&= \ < \nabla F_2, \Theta'[\Theta v^*] \nabla F_1 > + < v^*, \Theta'[\Theta \nabla F_1] \nabla F_2 > \ + \ < \nabla F_1, \Theta'[\Theta \nabla F_2] v^* >.
\end{aligned}
$$

Now recall that in our vector space situation every fixed co-vector v^* is a gradient, a fact which is extremely easy to see: take the gradient of $< v^*, u >$ to obtain v^*. Then application of condition (3,ii) yields that the right hand side of this last equation es equal to zero. So we see that $< v^*, -\Theta\nabla\{F_1, F_2\}_\Theta + [\Theta\nabla F_1, \Theta\nabla F_2] > = 0$. Moreover, because v^* was arbitrarily chosen we obtain that the vector on the right side in this bracket is equal to zero, i.e.

$$-\Theta\nabla\{F_1, F_2\}_\Theta + [\Theta\nabla F_1, \Theta\nabla F_2] = 0$$

which shows that $\Theta\nabla$ is a Lie algebra homomorphism. ∎

Operators Θ having one of the equivalent properties of the last theorem are called **implectic operators** or **Poisson operators**, they play a fundamental role for dynamical systems. The corresponding bracket introduced in (3.1) then is termed **Poisson bracket**. The flow

$$u_t = K(u) \tag{1.1}$$

is called a **hamiltonian flow**[5] if there is some scalar field $H(u)$ and some implectic operator $\Theta(u)$ such that

$$u_t = \Theta(u)\circ\nabla H(u). \tag{3.3}$$

The scalar field H then is the so called **Hamiltonian** of the system. The Poisson brackets give a suitable frame for describing the dynamics of scalar fields with respect to the evolution given by (1.1). Using $K = \Theta\nabla H$ we find that the total time derivative of some $F = F(u(t), t)$ can be written as

$$\frac{d}{dt}F = \{\nabla H, F\}_\Theta \; . \tag{3.4}$$

Hence a scalar field is a conserved quantity if and only if it commutes (in the Lie algebra given by the Poisson brackets) with the hamiltonian of the flow. As a particular consequence of that we have that the hamiltonian H itself always is a conserved quantity. This quantity usually is called **energy**[6]. However, the most important consequence of the above theorem is that now we have a precise relation between conserved quantities and symmetries:

Theorem 3.3: *Whenever $I(u)$ is a conserved quantity for the hamiltonian flow $u_t = \Theta(u)\circ\nabla H(u)$ then $\Theta\circ\nabla I$ is a symmetry of that flow.*

Proof: By Theorem 3.2 we have $[K, \Theta\nabla I] = \Theta\nabla\{H, I\}_\theta$. This expression is equal to zero since I is a conserved quantity. ∎

This result we call *Noether's theorem* since it is a generalization of the classical result obtained by Emmy Noether ([18]). A simple exercise shows that it carries over to time-dependent conserved quantities and time-dependent symmetries as well.

Example 3.4: Pendulum

If the manifold is a vector space and Θ an antisymmetric operator which does not depend on the manifold point u then we obviously have $\Theta(u)' = 0$, hence Θ fulfills condition 3 of

---

[5] In the classical finite dimensional situation for hamiltonian flows it is usually required that Θ is invertible. But here we are mainly interested in infinite dimensional manifolds where, for topological reasons, invertibility is a little bit problematic, therefore we have skipped this restrictive condition.

[6] One has to be a little bit careful with this interpretation since it can happen, as we will see soon, that a flow has more than one hamiltonian formulation.

Theorem 3.2 and therefore must be implectic. A particular example for such an operator is the antisymmetric matrix appearing in equations (1.5) and (1.7). So, these equations give hamiltonian formulations for these systems and, as stated above, their hamiltonians are given by energy conservation.

It should be mentioned that whenever the manifold is finite dimensional and the implectic operator Θ is invertible then locally there is a coordinate transformation on the manifold such that in the new coordinates the implectic operator is an off-diagonal matrix having -1's in the upper half and $+1$'s in the lower half of the off-diagonal (see ([16, page 30])). This means that equations (1.3) represent the prototype of hamiltonian equations in finite dimension.

The importance of conserved quantities is seen from the fact that even for a nonlinear system like the pendulum knowing the energy allows to integrate that equation completely. To see this we first remark that we already know the orbits in phase space, since they are given as $H = constant$
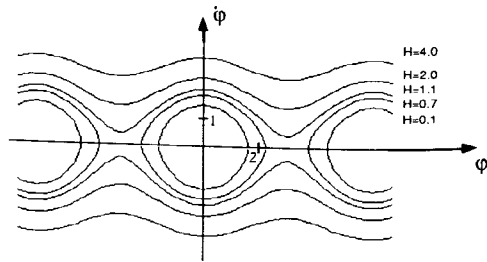

Fig. 2: Phase space orbits of the pendulum

To integrate the equation along these orbits we choose a fixed value E for this conserved quantity, then

$$H(\dot{\varphi}, \varphi) = \frac{1}{2}\dot{\varphi}^2 - \cos(\varphi) = E \tag{3.5}$$

is a differential equation of first order and separation of variables yields that

$$t - \int^{\varphi} \frac{d\alpha}{\sqrt{2E + 2\cos(\alpha)}} = constant = F \tag{3.6}$$

must be a constant. This formula obviously gives the solution of (1.7) in implicit form. Expressing E again by $H(\dot{\varphi}, \varphi)$ we obtain that

$$F = t - \int^{\varphi} \frac{d\alpha}{\sqrt{2H(\dot{\varphi}, \varphi) + 2\cos(\alpha)}} \tag{3.7}$$

must be constant along any line on which $H(\dot{\varphi}, \varphi)$ is constant. Hence F is constant along the orbits of (1.7). Rewriting, this in phase-space variables we have found a time-dependent conserved quantity for (1.8).

The pendulum provides the most simple which illustrates that knowing suitable and enough conserved quantities implies that an equation can be integrated. □

Example 3.5: Korteweg-de Vries equation

The Korteweg-de Vries equation [13(]KdV for short)

$$u_t = K(u) := 6uu_x + u_{xxx} \tag{3.8}$$

plays a central role in the history of completely integrable systems on infinite dimensional manifolds.

Usually, this equation is considered to be a flow on the space $\mathcal{S}$ of **tempered functions**. These are the C^∞-functions f in the real variable $x \in \mathbb{R}$ having the property that f and all its derivatives vanish at $\pm\infty$ faster than any rational function.

Define $\mathcal{S}^*$ to be the space of C^∞-functions in x such that all derivatives grow at most polynomially at $\pm\infty$. This space can be taken as a space of linear functionals on $\mathcal{S}$ by using an L^2 scalar product in the following way , namely

$$< U, u > = \int U(x)u(x)dx, \quad U \in \mathcal{S}^*, u \in \mathcal{S}. \tag{3.9}$$

We use the convention that if no boundaries are given then integrals always go over $\mathbb{R}$ or $\mathbb{R}^n$, respectively. As topology we take the weakest convex topology making all these functionals continuous. Then the scalar fields

$$I_0(u) = \int u(x)dx \tag{3.10}$$

$$I_1(u) = \int u(x)^2 dx \tag{3.11}$$

$$I_2(u) = \int (u^3 - \frac{1}{2}u_x{}^2)dx \tag{3.12}$$

are C^∞-functions $M \longrightarrow \mathbb{R}$. The derivative of, for example, $I_2(u)$ is computed to be

$$I_3'[v] = \frac{\partial}{\partial \epsilon}\big|_{\epsilon=0} \int \{(u + \epsilon v)^3 - \frac{1}{2}(u + \epsilon v)_x{}^2\}dx . \tag{3.13}$$

Integration by parts yields

$$I_3'[v] = \int (3u^2 + u_{xx})v dx. \tag{3.14}$$

Hence, provided the duality between tangent and cotangent space is represented by (3.9), then the gradient of I_2 can be identified with $3u^2 + u_{xx}$. The gradients of I_0, I_1 are given in the same way by the functions 1 and $2u$, respectively. We compute $I_1'[K(u)]$ for $K(u)$, the vector field given by the KdV. Integration by part (together with the fact that the function u is tempered) yields

$$I_2'(u)[K(u)] = \int 2u(u_{xxx} + 6uu_x)dx = 2\int(2u^3 + uu_{xx} - \frac{1}{2}u_x^2)_x dx. \tag{3.15}$$

The latter expression is equal to zero because the integrand is the derivative of a tempered function. So, I_1 is constant along the orbits of the KdV. The same holds true for the fields I_0 and I_2, whence all these quantities are conserved for the KdV. A time-dependent conserved quantity is for example

$$F(u,t) = \int \{xu - 3tu^2\}dx . \tag{3.16}$$

We can write the Korteweg-de Vries equation in the following way

$$u_t = \Theta^\circ \nabla H$$

where

$$\Theta = D \quad \text{differential operator with respect to } x$$

and

$$H = \int (u^3 - \frac{1}{2}u_x{}^2)dx.$$

Since Θ is an antisymmetric operator which does not depend on the manifold points, Θ must be implectic and this is a hamiltonian formulation. So one is inclined to call H the energy of the system. However, another way to write the KdV is the following

$$u_t = \Theta^\circ \nabla H$$

where

$$H = \frac{1}{2} \int u^2 dx ,$$

and where

$$\Theta = D^3 + 2(Du + uD) \tag{3.17}$$

is again antisymmetric and is shown to fulfill condition (3. ii) of Theorem 3.2 (see Section 7). Hence this is a second hamiltonian formulation for the KdV.□

This last example shows that for some systems hamiltonian formulations are not unique. It will turn out, however, that this non-uniqueness is a highly desired property which will help to construct suitably many conserved quantities and thus will enable us to integrate the equation. The main idea for generating infinitely many conservation laws from two different hamiltonian formulations goes back to F. Magri ([14]) who proposed that one hamiltonian formulation should be used for going from a conserved quantity to a symmetry and then by the second one one should go back to another conserved quantity. Thus an infinite sequence of conservation laws would be generated. There is one apparent difficulty with this concept, namely, that the map $\Theta\nabla$ is not invertible. This difficulty is overcome by transferring the result stated in Theorem 2.3 to co-vector fields instead of scalar fields. Then instead of going back to scalar fields one goes back with the hamiltonian formulation to co-vector fields instead and from there, by integration, to the corresponding potentials. Of course, for doing that one requires that the co-vector fields generated this way are closed (a notion which we will introduce in the next section). This requirement of constructing only closed co-vector fields will lead to the notion of compatibility (treated in Section 6). Another difficulty with this concept arises already for the KdV when other boundary conditions at $\pm\infty$ are considered. Then we cannot write down the hamiltonians

anymore for this equation since the integrals (3.10) to (3.12) then clearly diverge. In order to subsume even this case under a common theory we have to lift all our notions to a new level of abstraction. This new level of abstraction, which will be formulated in the next section. then provides a more transparent setup so that the necessary considerations can be carried out more easily.

4 Lie derivatives

In this section we would like to review the basics of symplectic geometry and Hamiltonian mechanics on an abstract level. This high degree of abstraction will enable us to represent the relevant results in a very concise way.

Let $\mathcal{F}$ be some commutative algebra (over $\mathbb{R}$ or $\mathbb{C}$) with identity. We now assume $(\mathcal{L}, [\ ], \mathcal{F})$ to be a Lie-module ([17]). Recall that being a **Lie module** means that $(\mathcal{L}, [\ ])$ is a Lie algebra such that a multiplication between elements of $\mathcal{L}$ and $\mathcal{F}$ is defined and that, furthermore, there is a canonical homomorphism

$$K \to L_K$$

from $\mathcal{L}$ into the derivations on $\mathcal{F}$. For $K, G \in \mathcal{L}$ and $f \in \mathcal{F}$ these derivations have to fulfill

$$[K, fG] = f[K, G] + L_K(f)G \tag{4.1}$$

$$L_K L_G - L_G L_K = L_{[K,G]} . \tag{4.2}$$

Of course, being a **derivation** on $\mathcal{F}$ means that the product rule

$$L_K(fg) = L_K(f)g + f L_K(g) \text{ for all } f, g \in \mathcal{F} \tag{4.3}$$

holds. In the following we require, for technical reasons, that the map $K \to L_K$ is *injective*.

Remark 4.1: *Lie modules are the canonical extensions of Lie algebras admitting a Lie algebra homomorphism into the derivations of $\mathcal{F}$. To be precise: Let $\mathcal{L}_1$ be some Lie algebra contained in some $\mathcal{F}$-module $\mathcal{L}$ such that $\mathcal{L}$ is the linear hull of $\{fK | f \in \mathcal{F}, K \in \mathcal{L}_1\}$. Then if a Lie algebra homomorphism $K \to L_K$ from $\mathcal{L}_1$ into the derivations of $\mathcal{F}$ is given then there is a unique extension of $(\mathcal{L}_1, [\ ])$ into a Lie module structure $(\mathcal{L}, [\ ], \mathcal{F})$ such that (4.1) and (4.2) hold.*

The proof of this remark is a simple computation. One takes $K_1, K_2 \in \mathcal{L}_1$, then makes the obvious definition

$$[f_1 K_1, f_2 K_2] := f_1 f_2 [K_1, K_2] + f_1 L_{K_1}(f_2) K_2 - f_2 L_{K_2}(f_1) K_1 . \tag{4.4}$$

and the extension to all of $\mathcal{L}$ is obtained by taking sums. ∎

If suitable topologies in $\mathcal{L}$ and $\mathcal{F}$ are given then we assume that all quantities introduced below are continuous. For $\mathcal{F}$-linear functionals $\gamma : \mathcal{L} \to \mathcal{F}$ we denote the application of γ to $K \in \mathcal{L}$ by $< \gamma, K >$. Such a functional γ is said to be **closed** if

$$L_K < \gamma, G > -L_G < \gamma, K >=< \gamma, [K, G] > \text{ for all } G, K \in \mathcal{L} . \tag{4.5}$$

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For $f \in \mathcal{F}$ we denote by ∇f the special $\mathcal{F}$-linear functional on $\mathcal{L}$ given by

$$< \nabla f, K >:= L_K f \ for \ all \ K \in \mathcal{L}. \tag{4.6}$$

Because of (4.2) all these ∇f are closed. A suitable $\mathcal{F}$-module of $\mathcal{F}$-linear maps $\mathcal{L} \to \mathcal{F}$ generated by closed $\mathcal{F}$-linear functionals $\gamma : \mathcal{L} \to \mathcal{F}$ is denoted by $\mathcal{L}^*$. We assume that $\mathcal{L}^*$ contains all ∇f, $f \in \mathcal{F}$. Elements in $\mathcal{L}^*$ which are of the form ∇f are called **gradients** and f is called the **potential** of ∇f. Observe that for $f, g \in \mathcal{F}$ elements of the form $g\nabla f$ are in general not gradients. Therefore the gradients do not form an $\mathcal{F}$-module.

An important observation is that the derivative L_K can be extended to all tensors, i.e. to all $\mathcal{F}$-multilinear forms on $\mathcal{L}^*$ and $\mathcal{L}$. This extension is obtained by defining first

$$L_K G := [K, G] \ for \ all \ G \in \mathcal{L} \tag{4.7}$$

and then by the requirement that for L_K the *product rule* holds for tensor products and for those quantities which come from inserting elements of $\mathcal{L}$ and $\mathcal{L}^*$ into $\mathcal{F}$-multilinear forms. This general extension of L_K is again called **Lie derivative** with respect to K.

Recall that $\mathcal{F}$−multilinear forms are maps from $(\odot \mathcal{L}^*)^r \odot (\odot \mathcal{L})^n$, $n, r \subset \mathbb{N}$, into $\mathcal{F}$ which are $\mathcal{F}$ linear in each entry. These multilinear forms are called **tensors** (n-times **covariant** and r-times **contravariant**). Elements of $\mathcal{L}$ and $\mathcal{L}^*$ are special tensors which are 1-times contravariant and covariant, respectively. In the following we do not distinguish between an $\mathcal{F}$-linear operator $\Theta : \mathcal{L}^* \to \mathcal{L}$ and the tensor $\tilde{\Theta} : \mathcal{L}^* \otimes \mathcal{L}^* \to \mathcal{F}$ given by $\tilde{\Theta}(\gamma_1, \gamma_2) := < \gamma_1, \Theta\gamma_2 >$. In the same way we identify operators $J : \mathcal{L} \to \mathcal{L}^*$ and $\Phi : \mathcal{L} \to \mathcal{L}$ with special tensors which are two-times covariant and once co-contravariant, respectively.

To illustrate the construction of L_K we compute Lie derivatives for 1-times covariant tensors and for 2-times contravariant tensors. First, we compute the Lie derivative for some fixed $\gamma \in \mathcal{L}^*$. We consider $< \gamma, G >$, $G \in \mathcal{L}$. The product rule applied to $< \gamma, G >$ yields

$$< L_K(\gamma), G >= L_K < \gamma, G > - < \gamma, [K, G] > .$$

i.e. the linear map $L_K(\gamma) : \mathcal{L} \to \mathcal{F}$ is

$$L_K(\gamma) = L_K \cdot \gamma - \gamma \cdot L_K \tag{4.8}$$

For later use we note that for $f \in \mathcal{F}, K \in \mathcal{L}$ the following holds

$$L_{(fK)}(\gamma) = f L_K(\gamma) + < \gamma, K > \nabla f . \tag{4.9}$$

As an additional example we take some $\mathcal{F}$-linear operator $\Theta : \mathcal{L}^* \to \mathcal{L}$. Its Lie derivative L_K we compute again by the product rule applied to $< \gamma_1, \Theta\gamma_2 >$ where γ_1, γ_2 are arbitrary chosen elements in $\mathcal{L}^*$. This yields

$$< \gamma_1, L_K(\Theta)\gamma_2 >= L_K < \gamma_1, \Theta\gamma_2 > - < L_K(\gamma_1), \Theta\gamma_2 > - < \gamma_1, \Theta L_K(\gamma_2) > \tag{4.10}$$

On the right side, the Lie-derivative of the first term is given by definition of the Lie-module. and the Lie derivatives of the γ's were already determined by (4.8). Since γ_1 and γ_2 were

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arbitrary, (4.10) defines completely the Lie derivatives for the two-times contravariant tensor θ. In the same way we can define, by induction, the derivative L_K for arbitrary tensors.

For purely covariant tensors α, i.e. multilinear forms on $(\otimes \mathcal{L})^n$ we can define a so called **exterior derivative** d, a notion which plays an important role for hamiltonian vector fields. On $\mathcal{F}$ we define this exterior derivative d to be the gradient

$$d_{|\mathcal{F}} = \nabla , \tag{4.11}$$

and when a tensor is r-times covariant ($r \geq 1$) then we define this exterior derivative by

$$(d\alpha) \bullet K := L_K(\alpha) - d(\alpha \bullet K) \ for \ all \ K \in \mathcal{L} . \tag{4.12}$$

Here $\alpha \bullet K$ means that K is inserted as the first entry in the form α, for example $\gamma \bullet K = \langle \gamma, K \rangle$ when $\gamma \in \mathcal{L}^*$. One easily sees that (4.12) is an inductive definition over the order of covariance. In the following notation we use the convention that d and L_K are more binding than $\bullet$, i.e. $d\alpha \bullet K = (d\alpha) \bullet K \neq d(\alpha \bullet K)$, and similarly for the Lie derivative. Furthermore we observe that we may use (4.12) as the definition for the exterior derivative also in case of zero-forms $f \in \mathcal{F}$ if we adopt the formal notation that for zero-forms f the expression $f \bullet K$ is equal to zero. More generally, for n-**forms** (n-times covariant tensors) α we define $\alpha \bullet K_1 \bullet \ldots \bullet K_{n+1}$ (i.e. application of $n + 1$ $\bullet$'s) to be zero. This notation will considerably shorten subsequent proofs.

Observation 4.2:

(i) Exterior derivative and Lie-derivative commute.

$$L_K d = d L_K \tag{4.13}$$

(ii) As usual we obtain that $d \cdot d = 0$, a fact which is well known for concrete situations from differential geometry.

Proof:
(i): Consider arbitrary $G, K \in \mathcal{L}$ and covariant α, then by use of (4.12) we obtain:

$$
\begin{aligned}
((L_G d - dL_G)\alpha) \bullet K &= L_G(d\alpha \bullet K) - d\alpha \bullet L_G K - L_K L_G \alpha + d(L_G \alpha \bullet K) \\
&= L_G L_K \alpha - L_G d(\alpha \bullet K) - d\alpha \bullet L_G K - L_K L_G \alpha + d(L_G \alpha \bullet K) \\
&= L_{[G,K]}(\alpha) - (L_G d - dL_G)(\alpha \bullet K) - d(\alpha \bullet [G,K]) - d\alpha \bullet L_G K \\
&= d\alpha \bullet [G, K] - (d\alpha) \bullet L_G K - (L_G d - dL_G)(\alpha \bullet K) \\
&= (dL_G - L_G d)(\alpha \bullet K)
\end{aligned}
$$

Hence $((L_G d - dL_G)\alpha) \bullet K = (L_G d - dL_G)(\alpha \bullet K)$ and the claim follows by induction over the order of covariance. Observe that the necessary beginning of our induction argument is given by the fact that $\bullet$ applied to zero-forms gives zero.

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(ii): We again use a repeated argument over the order of covariance. Consider an arbitrary covariant α and arbitrary vector fields K, G, then:

$$
\begin{aligned}
(d \cdot d\alpha) \bullet K \bullet G &= (L_K d\alpha \bullet G - d(d\alpha \bullet K)) \bullet G \\
&= L_K(d\alpha \bullet G) - d\alpha \bullet L_K G - d(d\alpha \bullet K) \bullet G \\
&= L_K(d\alpha \bullet G) - d\alpha \bullet L_K G - L_G(d\alpha \bullet K) \bullet G + d(d\alpha \bullet K \bullet G) \\
&= L_K L_G \alpha - L_K d(\alpha \bullet G) + L_{[G,K]}\alpha + d(\alpha \bullet [K,G]) - L_G L_K \alpha \\
&\quad + L_G d(\alpha \bullet K) + d(L_K \alpha \bullet G) - d(d(\alpha \bullet K) \bullet G) \\
&= d(-L_K(\alpha \bullet G) - \alpha \bullet [G,K] + L_G(\alpha \bullet K) + (L_K \alpha) \bullet G - d(\alpha \bullet K) \bullet G) \\
&= d(L_G(\alpha \bullet K) - d(\alpha \bullet K) \bullet G) \\
&= d^2(\alpha \bullet K \bullet G)
\end{aligned}
$$

Again, the beginning of our induction argument is given by the fact that $\bullet$ applied to zero-forms gives zero. ∎

Definition 4.3:

(i) A tensor T is said to be **invariant with respect to the vector field** K if $L_K T = 0$.

(ii) Observe that condition (4.5) for γ being closed can be written in terms of the exterior derivative as $d\gamma = 0$. Therefore we define a covariant tensor α to be **closed** if $d\alpha = 0$.

Remark 4.4:

(i) Gradients are closed because of $d \cdot d = 0$. If $\mathcal{L}$ is the vector field Lie algebra on some manifold M and $\mathcal{F}$ are the scalar fields, then locally the converse is also true (Poincaré Lemma ([25]).

(ii) Observe that (4.12) implies that any closed covariant tensor α is invariant with respect to K if and only if $\alpha \bullet K$ is again closed.

(iii) Let J be some invertible $\mathcal{F}$-linear operator $J : \mathcal{L} \to \mathcal{L}^*$. If $L_G J = 0$ for all those G with closed JG then J itself must be closed.

Proof of (ii) and (iii):

(ii): Direct consequence of (4.12).

(iii): Because $\mathcal{L}^*$ is generated by its closed elements and since J is invertible we have that $\mathcal{L}$ is the $\mathcal{F}$-linear hull of those G in $\mathcal{L}$ such that JG is closed. For $f \in \mathcal{F}$ we obtain $L_{(fK)}J - d(JfK) = f(L_K J - d(JK))$. Therefore $L_K J - d(JK)$ vanishes for all K if it vanishes for the subset of those G such that JG is closed. So the assumption on J shows that the right side of (4.12) vanishes. Hence we have $d(J) \bullet K = 0$ for all K. ∎

Now, let $\Theta : \mathcal{L}^* \to \mathcal{L}$ be some antisymmetric $\mathcal{F}$-linear map. We define a **bracket** in $\mathcal{L}^*$ in the following way:

$$\{\gamma_1, \gamma_2\}_\Theta := L_{(\Theta\gamma_1)}\gamma_2 - L_{(\Theta\gamma_2)}\gamma_1 + d < \gamma_1, \Theta\gamma_2 > \tag{4.14}$$

for $\gamma_1, \gamma_2 \in \mathcal{L}^*$.

Before presenting the basic result for these brackets we gather some useful identities. Using (4.12) we can rewrite (4.14) as

$$\{\gamma_1, \gamma_2\}_\Theta := L_{(\Theta\gamma_1)}\gamma_2 - (d\gamma_1) \bullet (\Theta\gamma_2) . \tag{4.15}$$

So when γ_1 is *closed* then by application of (4.12) this bracket reduces to

$$\{\gamma_1, \gamma_2\}_\Theta := L_{(\Theta\gamma_1)}\gamma_2 \ for \ closed \ \gamma_1 \in \mathcal{L}^* . \tag{4.16}$$

Furthermore we easily find with (4.9) how this bracket acts on multiplication with elements of $\mathcal{F}$

$$\{\gamma_1, f\gamma_2\}_\Theta := f\{\gamma_1, \gamma_2\}_\Theta + (L_{(\Theta\gamma_1)}f)\,\gamma_2 . \tag{4.17}$$

Theorem 4.5: *Let $\Theta : \mathcal{L}^* \to \mathcal{L}$ be $\mathcal{F}$-linear and antisymmetric, then the following are equivalent:*

(i) $\{\,,\,\}_\Theta$ *defines a Lie algebra among the closed elements of $\mathcal{L}^*$*

(ii) $\Theta\{\gamma_1, \gamma_2\}_\Theta = [\Theta d\gamma_1, \Theta d\gamma_2]$ *for all closed $\gamma_1, \gamma_2 \in \mathcal{L}^*$.*

(iii) $\{\,,\,\}_\Theta$ *defines a Lie algebra in $\mathcal{L}^*$*

(iv) $\Theta\{\gamma_1, \gamma_2\}_\Theta = [\Theta\gamma_1, \Theta\gamma_2]$ *for all $\gamma_1, \gamma_2 \in \mathcal{L}^*$*

(v) $L_{\Theta\gamma}(\Theta) = 0$ *for all closed $\gamma \in \mathcal{L}^*$.*

(vi) For arbitrary $\gamma \in \mathcal{L}^*$ we have that Θ is invariant with respect to $\Theta\gamma$ if and only if $d\gamma \bullet (\Theta\gamma_1) \bullet (\Theta\gamma_2) = 0$ for all $\gamma_1, \gamma_2 \in \mathcal{L}^*$.

Proof:
In the following we omit, for simplicity, the sub-Θ in the brackets $\{\,\}_\Theta$.
(i) $\Leftrightarrow$ (ii): Observe that by use of (4.14) then the antisymmetry of Θ implies for closed γ's that

$$\{\gamma_1, \gamma_2\} = L_{(\Theta d\gamma_1)}\gamma_2 = -L_{(\Theta d\gamma_2)}\gamma_1. \tag{4.18}$$

So, $\{\,,\,\}$ is antisymmetric anyway, and only the Jacobi identity has to be proved in order to show that this is a Lie algebra. Using (4.18) we find for the triple bracket

$$\{\gamma_3, \{\gamma_1, \gamma_2\}\} = -L_{\Theta\{\gamma_1, \gamma_2\}}\gamma_3$$
$$= L_{(\Theta\gamma_3)}L_{(\Theta\gamma_1)}\gamma_2 = -L_{(\Theta\gamma_3)}L_{(\Theta\gamma_2)}\gamma_1 .$$

Now, using for the cyclic sum suitable representations obtained from this formula we find with (4.2)

$$
\begin{aligned}
\{\gamma_3\{\gamma_1,\gamma_2\}\} + \text{ cyclic} &= L_{(\Theta d\{\gamma_1,\gamma_2\})}\gamma_3 + L_{(\Theta d\gamma_1)}L_{(\Theta d\gamma_1)}\gamma_3 - L_{(\Theta d\gamma_2)}L_{(\Theta d\gamma_1)}\gamma_3 \\
&= \{L_{(\Theta\{\gamma_1,\gamma_2\})} - [L_{(\Theta d\gamma_1)}, L_{(\Theta d\gamma_2)}]\}\gamma_3 \\
&= \{L_{(\Theta\{\gamma_1,\gamma_2\})} - L_{[\Theta d\gamma_1,\Theta d\gamma_2]}\}\gamma_3
\end{aligned}
\tag{4.19}
$$

Since γ_3 was arbitrary, the Jacobi identity for the triple product can only hold if

$$
L_{(\Theta d\{\gamma_1,\gamma_2\})} - L_{[\Theta d\gamma_1,\Theta d\gamma_2]} = 0
\tag{4.20}
$$

which is equivalent to (ii) since $K \to L_K$ was assumed to be injective. On the other hand, whenever (ii) (and (4.20) as a consequence) holds then the cyclic sum obviously must be equal to zero, which implies the Jacobi identity.

The implications (iv) $\to$ (ii) and (iii) $\to$ (i) are obvious.

(ii) $\to$ (iv) and (i) $\to$ (iii) are either done by direct computation or by using remark 4.1 together with (4.17). To see this observe that (i) gives a Lie algebra $(\mathcal{L}_{cl}^*, \{\ \})$, where $\mathcal{L}_{cl}^*$ are the closed elements in $\mathcal{L}^*$ and that (4.17) gives a Lie algebra homomorphism $\gamma \to L_\gamma^* = L_{\Theta\gamma}$ from $\mathcal{L}_{cl}^*$ into the derivations of $\mathcal{F}$. Now taking the unique extension (described in remark 4.1) to the $\mathcal{F}$-module generated by $\mathcal{L}_{cl}^*$ (which by definition is equal to $\mathcal{L}^*$), one obtains the Lie algebra defined by (4.14). The fact that the homomorphism Θ extends from $\mathcal{L}_{cl}^*$ to $\mathcal{L}^*$ is due to the property that $\Theta L_\gamma^* = L_{\Theta\gamma}$.

(v) $\Leftrightarrow$ (ii): By (4.16) the condition

$$
\Theta L_{(\Theta\gamma)}\gamma_1 = L_{(\Theta\gamma)}(\Theta\gamma_1) \text{ for all closed } \gamma,\gamma_1 \in \mathcal{L}^*
\tag{4.21}
$$

is equivalent to (ii). And by the product rule this is equivalent to (v).

(vi) $\Leftrightarrow$ (iv): Using (4.15) and the antisymmetry of Θ we find for arbitrary $\gamma,\gamma_1,\gamma_2 \in \mathcal{L}^*$

$$
\begin{aligned}
< \gamma_1, (L_{(\Theta\gamma)}\Theta)\gamma_2 > &= < \gamma_1, L_{(\Theta\gamma)}(\Theta\gamma_2) > - < \gamma_1, \Theta L_{(\Theta\gamma)}\gamma_2 > \\
&= < \gamma_1, [\Theta\gamma, \Theta\gamma_2] > - < \gamma_1, \Theta\{\gamma,\gamma_2\} > + < \gamma_1, \Theta(d\gamma \bullet \Theta\gamma_2) > \\
&= < \gamma_1, [\Theta\gamma, \Theta\gamma_2] > - < \gamma_1, \Theta\{\gamma,\gamma_2\} > -d\gamma_1 \bullet (\Theta\gamma_2) \bullet (\Theta\gamma_1) \cdot
\end{aligned}
$$

Hence we obtain

$$
< \gamma_1, (L_{\Theta\gamma}\Theta)\gamma_2 > +d\gamma_1 \bullet (\Theta\gamma_2) \bullet (\Theta\gamma_1) = < \gamma_1, [\Theta\gamma, \Theta\gamma_2] - \Theta\{\gamma,\gamma_2\} >
\tag{4.22} \cdot
$$

Equating the right side of (4.22) to zero is equivalent to (iv) and equating its left side to zero is equivalent to (v). ∎

Let us have a closer look at condition (v): If Θ is invertible, then the condition therein imposed on γ is equivalent to $d\gamma = 0$, i.e. γ must be closed. Hence, for $J := \Theta^{-1}$ condition (v) means that J is K-invariant ($K \in \mathcal{L}$) if and only if JK is closed. Looking now at 4.4 (ii) and (iii) we see that for an invertible Θ one of the equivalent conditions of the theorem above is fulfilled if and only if $J = \Theta^{-1}$ is closed. In the finite dimensional theory

the antisymmetric closed invertible J are called symplectic. So, loosely speaking, Θ has the algebraic behaviour of the inverse of a symplectic operator. Therefore, if one of the conditions of Theorem 4.5 is fulfilled, Θ is said to be **implectic**, a name which stands for inverse symplectic. Sometimes instead of implectic, the name **Poisson tensor** is chosen. For reasons which will become obvious in the next section the $\{\ ,\ \}_\Theta$ are called the **Poisson brackets** with respect to Θ.

We decided not to insist on the invertibility condition because of the infinite dimensional nature of our manifolds, so we have to extend the notion *symplectic* for this more general situation: An operator $J : \mathcal{L} \to \mathcal{L}^*$ is called **symplectic** if it is antisymmetric and closed, and if in addition its kernel $ker(J) = \{G \in \mathcal{L} | JG = O\}$ is a Lie ideal in $\mathcal{L}$. Being a Lie ideal means of course that $[K,G] \in ker(J)$ for all $G \in ker(J)$ and $K \in \mathcal{L}$. This additional ideal-condition is automatically fulfilled if J is invertible because then $ker(J) = 0$.

In a analogy to implectic operators one can use symplectic operators J to construct in $J\mathcal{L}$ suitable Poisson brackets: Take $\gamma_1, \gamma_2 \in J\mathcal{L}$ and choose G_1, G_2 such that $\gamma_i = JG_i$, $i = 1,2$. Then we define

$$\{\gamma_1,\gamma_2\}^{(J)} := L_{G_1}(\gamma_2) - L_{G_2}(\gamma_1) + <\gamma_1, G_2> . \tag{4.23}$$

Rewriting that with (4.12) and using $d(J) = 0$ we obtain

$$\{\gamma_1,\gamma_2\}^{(J)} = L_{G_1}(G_2) - d(JG_1) \bullet G_2 = JL_{G_1}G_2 + d(J) \bullet G_1 \bullet G_2 = J[G_1,G_2] .$$

Since $ker(J)$ is an ideal with respect to $[\ ,\ ]$ the bracket $\{\ ,\ \}^{(J)}$ does not depend on the choice of G_1, G_2. Furthermore, because J is one-to-one from the equivalence classes modulo $ker(J)$ to $J\mathcal{L}$, the bracket $\{\ ,\ \}^{(J)}$ must be a Lie algebra such that J is a homomorphism from $\{\ ,\ \}^{(J)}$ into $[\ ,\ ]$

We may summarize this section: *An implectic operator makes out of $\mathcal{L}^*$ a Lie algebra module $(\mathcal{L}^*, \{\ ,\ \}_\Theta, \mathcal{F})$, with corresponding $\mathcal{F}$-derivations $L_\gamma^*, \gamma \in \mathcal{L}^*$ such that Θ is a homomorphism from this Lie algebra module into $(\mathcal{L}, [\ ,\ ], \mathcal{F})$. Here homomorphism means that for all $\gamma_1, \gamma_2 \in \mathcal{L}^*$ we have*

$$\Theta\{\gamma_1,\gamma_2\}_\Theta = [\Theta\gamma_1, \Theta\gamma_2] \tag{4.24}$$

and

$$\Theta L_{\gamma_1}^* = L_{\Theta\gamma_1} . \tag{4.25}$$

In general, it is highly desirable to construct for given tensors suitable Lie algebra elements which leave these tensors invariant. Indeed, as we will see, the search for symmetries, conservation laws, invariant spectral problems and the like may be subsumed under this general theme. In view of that problem it is obvious that symplectic and implectic tensors must play a fundamental role because for them the closed elements in $\mathcal{L}^*$ imediatly give rise to such invariances.

5   Hamiltonian and Bi-Hamiltonian Fields

In this section we always assume that Θ is an implectic operator. For some of the results presented here the reader is referred to the fundamental papers ([10,] [11,] [12,] [14,] [15]).

Comparing the different Poisson brackets defined in Sections 3 and 4, one discovers that they are defined on rather different spaces. However, these two notions are easily connected if, as before in the concrete situation, a *bracket* among the elements of $\mathcal{F}$ is defined as follows:

$$\{f_1, f_2\}_\Theta := < df_2, \Theta df_1 > = L_{(\Theta df_1)} f_2 \ for \ f_1, f_2 \in \mathcal{F} \ . \tag{5.1}$$

One easily sees that ∇ maps the $\mathcal{F}$-brackets into the $\mathcal{L}^*$-brackets:

$$\nabla\{f_1, f_2\}_\Theta = \{\nabla f_1, \nabla f_2\}_\Theta \ . \tag{5.2}$$

This suggests that these $\mathcal{F}$-brackets also form a Lie algebra. Indeed this is the case, the proof is literally almost the same as the proof (i) $\Leftrightarrow$ (ii) in the last theorem[7].

Definition 5.1:

(i) *Elements $K \in \mathcal{L}$ which are of the form $K = \Theta\gamma$ with closed γ and implectic Θ are* called **hamiltonian** *(with respect to Θ).*

(ii) *Given some symplectic J, then elements $K \in \mathcal{L}$ such that JK is closed are called* **inverse-hamiltonian**[8] *(with respect to J).*

(iii) *Let Θ be implectic and J be closed, and assume that J is not the inverse of Θ. Then some K is called a* **bi-hamiltonian field** *(with respect to Θ and J) if JK is closed and if there is some closed γ such that $K = \Theta\gamma$.*

The power of bi-hamiltonian fields is seen from:

Observation 5.2: *Consider a bi-hamiltonian field*

$$K = \Theta\gamma, \ JK = \gamma_1, \ \gamma \ and \ \gamma_1 \ being \ closed$$

as described above, and define $K_{n+1} = (\Theta J)^n K$. Then all the K_n and $\gamma_n := JK_n$ are invariant with respect to K.

Proof:

---

[7]An interesting question is whether or not the Jacobi identity for the $\mathcal{F}$-brackets is a sufficient condition to guarantee that Θ is implectic. In most situations this is indeed the case. One only needs some kind of Poincaré Lemma to show that.

[8]Observe that for invertible Θ or J the notions hamiltonian and inverse-hamiltonian do coincide. This is the reason why in the finite dimensional theory the notion inverse-hamiltonian does not appear, since there symplectic forms are usually assumed to be non-degenerate. In the infinite dimensional situation however, the assumption of nondegeneracy is not advised because invertibility of operators usually involves topological considerations and depends very much on the spaces under consideration.

Using the bi-hamiltonian nature of K we see from Remark 4.4 (ii) and from Theorem 4.5 (v) that Θ and J are invariant with respect to K. Hence, by the product rule, all K_n have to be invariant with respect to K. ∎

Let us now illustrate the notions and techniques of the last chapter in the light of the concrete situation which was considered in Sections 2 and 3.

Standard situation :

Let M be a C^∞-manifold, we consider C^∞-tensor fields on M. In particular let $\mathcal{L}$ be the vector fields, $\mathcal{F}$ be the scalar fields and let $\mathcal{L}^*$ be those C^∞-maps from M into the cotangent-bundle which are given by assigning to $u \in M$ a continuous linear functional on the tangent space $T_u M$ at u.

In order to carry out computations with Lie derivatives we should show what these look like in charts. As we already know, for a scalar field f the Lie derivative is the usual gradient

$$L_A f = <\nabla f, A> = f'[A] \tag{5.3}$$

and the application of L_A to a co-vector field γ is found to be

$$L_A \gamma = \gamma'[A] + A'^+ \gamma \tag{5.4}$$

where A'^+ denotes the transpose of the operator A' with respect to the duality between tangent and cotangent space.

Furthermore, if $\Theta : T^*M \to TM$ and $J : TM \to T^*M$ are two-times contravariant and two-times covariant, respectively, then their Lie derivatives are

$$L_A \Theta = \Theta'[A] - \Theta A'^+ - A'\Theta \tag{5.5}$$

$$L_A J = J'[A] + A'^+ J + J A' \tag{5.6}$$

Finally the Lie derivative for an operator $\Phi : TM \to TM$ is equal to

$$L_A \Phi = \Phi'[A] - A'\Phi + \Phi A' . \tag{5.7}$$

Fix some $K \in \mathcal{L}$, then $f \in \mathcal{F}$ and $G \in \mathcal{L}$ are invariant with respect to K if and only if f is a conserved quantity and G a symmetry group generator, respectively. Using the Lie derivatives, we see that Θ is implectic if and only if condition 3 (ii) of Theorem 3.2 is fulfilled. In the same way we obtain as equivalent condition for J being closed that

$$< J(u)'[G_1]G_2, G_3 > + < J(u)'[G_2]G_3, G_1 > + < J(u)'[G_3]G_1, G_2 >= 0 \tag{5.8}$$

must hold for all $G_1, G_2, G_3 \in \mathcal{L}$. Hence Section 4 generalizes the situation considered in Section 3.

Now, we consider again the dynamical system

$$u_t = K(u) . \tag{1.1}$$

In analogy to Definition 5.1 we call (1.1) a **bi-hamiltonian system** if there are closed $J(u)$, implectic $\Theta(u)$ and *closed* γ_0, γ_1 such that

$$K = \Theta\gamma_0 \ and \ JK = \gamma_1 \ . \tag{5.9}$$

If that is the case then we obtain as consequence of 5.2
Observation 5.3: *Define inductively*

$$K_1 = K \ and \ K_{n+1} = \Theta J K_n \tag{5.10}$$

$$\gamma_{n+1} = J\Theta\gamma_n \tag{5.11}$$

then all the K_n and γ_n are invariant with respect to K. Hence the K_n are symmetry group generators for (1.1) and, if the γ_n are gradients, then they are gradients of conserved quantities for the evolution equation (1.1).

Example 5.4: Korteweg-de Vries equation

Consider the situation as in Example 3.5. We observe that the inverse of D

$$(D^{-1}f)(x) := \int_{-\infty}^{x} f(\xi)d\xi$$

is a well defined operator $D^{-1} : S \to S^*$. The Korteweg-de Vries equation

$$u_t = K(u) := 6uu_x + u_{xxx} \tag{3.8}$$

is a bi-hamiltonian system since for the implectic $\Theta = D^3 + 2(Du + uD)$ and the symplectic $J = D^{-1}$ we have that (5.9) is fulfilled when γ_0 and γ_1 are taken to be the gradients of I_0 and I_1 as given in (3.10) and (3.11), respectively. Hence putting

$$K_{n+1} := (\Phi)^n K, \ where \ \Phi = D^2 + 2DuD^{-1} + 2u \ , \tag{5.12}$$

then all these K_n are symmetry group generators. Since Φ is recursively generating symmetry group generators, this usually is called a **recursion operator** [20(]sometimes also a **strong symmetry**, for example in [21,] [5,][3)]. $\square$

Although Observation 5.3 is very useful for constructing symmetry groups, there still remains a major problem is the analysis of the system (1.1). Ultimately, we are interested in constructing suitable coordinates (action variables) for the flow, so we need scalar quantities which are invariant under the flow. Certainly, action variables give rise to symmetry group generators, however the converse is not always true, because the corresponding co-vector fields may not be closed. Therefore the question whether or not the fields γ_n are closed is of great importance. If that happens then, by the Poincaré lemma, one can, at least locally, construct suitable coordinates. And if, as in the example, the manifold under consideration is a vector space then even the construction of global potentials is a simple excercise:

Lemma 5.5: *Let the manifold of all u's be a vector space. Then a co-vector field $\gamma(u)$ is the gradient of a scalar field $F(u)$ if and only if γ is closed.*

Proof:

Since gradients are closed we only have to prove the existence of F for closed γ. So let $\gamma(u)$ be closed. Consider the scalar field

$$F(u) = \int_0^1 < \gamma(\lambda u), u > d\lambda. \tag{5.13}$$

Take some arbitrary $v(u)$ in the tangent bundle. Then by (5.4) we obtain

$$
\begin{aligned}
< \nabla F(u), v > &= \int_0^1 \{\lambda < \gamma'(\lambda u)[v], u > + < \gamma(\lambda u), v >\}d\lambda \\
&= \int_0^1 \{< \lambda\gamma'(\lambda u)[u], v > + < \gamma(\lambda u), v >\}d\lambda \tag{5.14} \\
&= \int_0^1 \frac{d}{d\lambda}\{< \lambda\gamma(\lambda u), v >\}d\lambda = < \gamma(u), v >
\end{aligned}
$$

This proves $\nabla F(u) = \gamma(u)$ since $v(u)$ was arbitrary ∎
Observe that this proof can be generalized to the situation where the manifold can be parametrized by a star-shaped subset of a vector space.

6 Compatibility

In this section we introduce the essential structure which will be responsible for the fact that in most cases of bi-hamiltonian systems the invariant co-vector fields constructed by use of Observation 5.2 are indeed closed. Again we will consider the necessary arguments at a rather general level. This is not done in order to introduce an unreasonable level of abstraction but rather to get rid of unnecessary ballast. Further, it turns out that with a general formulation of **compatibility** in Lie algebras one is more flexible with respect to applications. In our presentation we follow closely [9.]

Let, $(\mathcal{L}, [\ , \ ])$ be some Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. We call $(\mathcal{L}, [\ , \ ])$ the **reference algebra**. Let furthermore Λ be a vector space over the same scalars. Now, consider a linear transformation

$$T : \Lambda \to \mathcal{L} .$$

from Λ into the reference algebra $\mathcal{L}$. We call some (bilinear) product $\{ \ , \ \}$ in Λ (not necessarily assumed to be a Lie product) a T-**product** if T is a homomorphism into $(\mathcal{L}, [\ , \ ])$, i.e. if

$$T\{a, b\} = [Ta, Tb] \ for \ all \ a, b \in \Lambda . \tag{6.1}$$

To emphasize that some product is a T-product we write $[\ , \ ]_T$ instead of $\{ \ \}$. A linear $\Phi : \mathcal{L} \to \mathcal{L}$ is said to be **hereditary** if

$$[a, b]_\Phi := [\Phi a, b] + [a, \Phi b] - \Phi[a, b] \tag{6.2}$$

defines a Φ-product in $\mathcal{L}$. Using then $\Phi[a, b]_\Phi = [\Phi a, \Phi b]$, we see that Φ is hereditary if and only if:

$$\Phi^2[a, b] + [\Phi a, \Phi b] = \Phi\{[\Phi a, b] + [a, \Phi b]\} \ for \ all \ a, b \in \mathcal{L} . \tag{6.3}$$

Rephrasing a notion introduced earlier we call a linear map $\Phi : \mathcal{L} \rightarrow \mathcal{L}$ **invariant** with respect to $k \in \mathcal{L}$ if

$$\Phi[k, b] = [k, \Phi b] \ for \ all \ b \in \mathcal{L} \tag{6.4}$$

Theorem 6.1: *Let Φ be a hereditary map which is invariant with respect to k. Then $\{\Phi^n k | n \in I\!\!N_0\}$ is an abelian subset of $(\mathcal{L}, \ [\ ])$. If Φ is invertible then $\{\Phi^n k | n \in Z\!\!\!Z\}$ is abelian as well.*

For the proof we need:

Lemma 6.2: *Let Φ be hereditary and let it be invariant with respect to k. Then Φ is invariant with respect to Φk. If Φ is invertible then it is invariant with respect to $\Phi^{-1}k$ as well. Thus the set $\{k | \Phi$ invariant with respect to $k\}$ of all elements which leave Φ invariant is a subalgebra of $\mathcal{L}$ which is invariant under the application of Φ (and of Φ^{-1} if Φ is invertible).*

Proof:

Replace a by k in (6.3). Since Φ is invariant with respect to k the first and fourth term cancel and the equality reads

$$[\Phi k, \Phi b] = \Phi[\Phi k, b] \ for \ all \ b \ . \tag{6.5}$$

This clearly implies that Φ is invariant with respect to Φk. If Φ is invertible, we replace a in (6.3) by $\Phi^{-1}k$ and apply Φ^{-1} to the remaining two terms. ∎

Proof of Theorem 6.1:

From the Lemma 6.2 we obtain by induction that Φ is invariant with respect to any $\Phi^m k$ and $\Phi^n k$. Hence using antisymmetry we find

$$[\Phi^m k, \Phi^n k] = \Phi^{m+n}[k, k] = 0$$

for all m, n. For invertible Φ, in this argument Φ^{-1} has to replace Φ. ∎

Remark 6.3: *Let Φ be hereditary and let a_1 and a_2 be eigenvectors of Φ , i.e.*

$$\Phi a_i = \lambda_i a_i \ , \ \lambda_i = scalar, \ i = 1, 2 \ .$$

Then for these a_i relation (6.3) is equivalent to

$$(\Phi - \lambda_1)(\Phi - \lambda_2)[a_1, a_2] = 0 \ . \tag{6.6}$$

Hence, when an operator Φ has a spectral resolution, this operator is hereditary if and only if all the corresponding spectral projections are algebra homomorphisms.

Now, let us return to the general situation of maps from Λ into $\mathcal{L}$. Assume that in Λ we have $T-$ and $\Psi-$products $[\ . \ ]_T$ and $[\ , \ ]_\Psi$, respectively. These products are said to be **compatible** if

$$\{a, b\} := [a, b]_T + [a, b]_\Psi \tag{6.7}$$

defines a $(T + \Psi)$-product in Λ.

Lemma 6.4: *Let $[\ , \ ]_T$ and $[\ , \ ]_\Psi$ be $T-$ and $\Psi-$products, respectively. These products are compatible if and only if*

$$T[a, b]_\Psi + \Psi[a, b]_T = [Ta, \Psi b] + [\Psi a, Tb] \ for \ all \ a, b \in \Lambda \tag{6.8}$$

Proof:

Observe that $T[a,b]_T = [Ta,Tb]$ and $\Psi[a,b]_\Psi = [\Psi a, \Psi b]$. So, (6.8) is obviously equivalent to

$$(T + \Psi)\{[a,b]_T + [a,b]_\Psi\} = [(T + \Psi)a, (T + \Psi)b] ,\qquad (6.9)$$

which proves the claim. ∎

Observation 6.5: *Let λ be a scalar. Obviously, $[\ ,\ ]_{\lambda T}$ defined by $[a,b]_{\lambda T} := \lambda[a,b]_T$ is a (λT)-product whenever $[\ ,\ ]_T$ is a T-product. Now, replacing in (6.8) Ψ and $[\ ,\ ]_\Psi$ by $\lambda\Psi$ and $[\ ,\ ]_{\lambda\Psi}$, respectively, we see that (6.8) remains valid. In other words, (6.8) is linear in Ψ (as well as in T). Hence, if $[\ ,\ ]_{T_1}$ and $[\ ,\ ]_{T_2}$ are compatible , and if both are compatible with $[\ ,\ ]_T$ then $[\ ,\ ]_{\lambda T_1} + [\ ,\ ]_{\sigma T_2}$ is always compatible with $[\ ,\ ]_T$.*

Observation 6.6: *Consider the case when the reference algebra is equal to Λ, i.e. $\Lambda = \mathcal{L}$ and put $T = I, \Psi = \Phi$. Furthermore, assume that $[\ ,\ ]_T$ is the given product in $(\mathcal{L},[\ ])$, and that $[\ ,\ ]_\Psi = \{\ \}$ is a second product such that $\Phi : (\mathcal{L},\{\ \}) \to (\mathcal{L},[\ ])$ is a homomorphism. Then (6.8) holds if and only if $\{\ \}$ is the product defined in (6.3). Hence, Φ is hereditary if and only if $(\mathcal{L},\{\ ,\ \})$ and $(\mathcal{L},[\ ,\ ])$ are compatible.*

In order to shorten our notions we call Ψ and T compatible if their Ψ- and T- products, $[\ ,\ ]_\Psi$ and $[\ ,\ ]_T$ are compatible. By application of this notion to the special case of hereditary operators Φ_1, Φ_2 we see that Φ_1 and Φ_2 are compatible if and only $\Phi_1 + \Phi_2$ is again hereditary.

Theorem 6.7: *Consider maps $T, \Psi : \Lambda \to \mathcal{L}$ and their corresponding products $[\ ,\ ]_T$ and $[\ ,\ ]_\Psi$. Assume that Ψ is invertible. Then T and Ψ are compatible if and only if $\Phi = T\Psi^{-1}$ is hereditary.*

Proof:

Define a second product in $\mathcal{L}$ by

$$\{a,b\} := \Psi[\Psi^{-1}a, \Psi^{-1}b]_T \text{ for } a,b \in \mathcal{L} .$$

Then $\Phi : (\mathcal{L},\{\ \}) \to (\mathcal{L},[\ ,\ ])$ is a homomorphism. Using the definition of $\{\ .\ \}$ and (6.1) (for Ψ instead of T) we obtain

$$
\begin{aligned}
(I + \Phi)([a,b] + \{a,b\}) &= (T + \Psi)\Psi^{-1}([a,b] + \{a,b\}) \\
&= (T + \Psi)([\Psi^{-1}a, \Psi^{-1}b]_\Psi + [\Psi^{-1}a, \Psi^{-1}b]_T) .
\end{aligned}
$$

For general $a,b \in \mathcal{L}$ the right side is equal to $[(T + \Psi)\Psi^{-1}a, (T + \Psi)\Psi^{-1}b]$ if and only if T and Ψ are compatible, but this expression is equal to $[(I + \Phi)a, (I + \Phi)b]$ and equal to the left side if and only if I and Φ are compatible. Hence the compatibility of T and Ψ and that of I and Φ are equivalent. Now using Observation 6.6 we see that this is equivalent to Φ being hereditary. ∎

By similar arguments we find:

Observation 6.8: *Let Φ, Ψ be compatible hereditary operators and assume that Φ and Ψ do commute. Then $\Phi\Psi$ is hereditary. As a consequence, if Φ be hereditary, then any polynomial in Φ is hereditary.*

Now consider again the
Lie-module situation:
Let $(\mathcal{L}, [\,], \mathcal{F})$ be a Lie module as considered in Section 4 and define $\Lambda = \mathcal{L}^*$. If $\Phi : \mathcal{L} \to \mathcal{L}$ is a tensor, then condition (6.3) is easily rephrased as

$$\Phi L_A(\Phi) = L_{(\Phi A)}(\Phi) \ for \ all \ A \in \mathcal{L} \ . \tag{6.10}$$

So Φ is hereditary if and only if (6.10) is fulfilled. In case of the *standard situation* this can be expressed in charts as

$$\Phi \Phi'[A]B - \Phi[\Phi' A]B = \Phi \Phi'[B]A - \Phi[\Phi' B]A \ for \ all \ A, B \in \mathcal{L} \tag{6.11}$$

a condition which appeared in [3.] Because of Theorem 4.5 (4.14) defines a Θ-product in $\mathcal{L}^* = \Lambda$ if and only if Θ is implectic. Since Θ enters the definition (4.14) linearly we have:

Observation 6.9: *Two implectic operators Θ_1, Θ_2 are compatible if and only if $\Theta_1 + \Theta_2$ is again implectic.*

From Theorem 6.7 we obtain:
Corollary 6.10: *Let Θ_1, Θ_2 be implectic and assume that Θ_1 is invertible. Then $\Theta_1 + \Theta_2$ is implectic if and only if $\Phi = \Theta_2 \Theta_1^{-1}$ is hereditary.*

These results we apply to the
Bi-hamiltonian case: Let Θ_1, Θ_2 be compatible implectic operators such that Θ_1 is invertible. Let K be a bi-hamiltonian vector field

$$K = \Theta_1 \gamma_1 = \Theta_2 \gamma_2, \ \gamma_1, \gamma_2 \ closed \ .$$

Then define

$$\Phi := \Theta_2 \Theta_1^{-1}, \ \Phi^+ := \Theta_1^{-1} \Theta_2$$
$$J_n := \Theta_1^{-1} \Phi^n, \ \Theta_{n+1} = \Phi^n \Theta_1 \tag{6.12}$$

Theorem 6.11:

  (i) *All J_n and all γ_n are closed.*

  (ii) *All tensors $\Phi, \Phi^+, K_n, \gamma_n, J_n, \Theta_n$ are invariant with respect to every K_m, in particular all K_n, K_m commute.*

  (iii) *If, in addition, Θ_2 is invertible, then the Θ_n are implectic.*

Before we can prove this we need to introduce a canonical extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$.

The affine extension of $\mathcal{L}$:

Let $\tilde{\mathcal{L}} = \mathcal{L} \otimes \mathbb{C}[\xi]$ be the Lie algebra of formal power series in the indeterminate ξ with coefficients in $\mathcal{L}$. Extend in the same way $\mathcal{F}$ and $\mathcal{L}^*$

$$\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathbb{C}[\xi], \ \tilde{\mathcal{L}}^* = \mathcal{L}^* \otimes \mathbb{C}[\xi] .$$

Define that all Lie derivatives, and consequently the exterior derivative, ignore the variable ξ. i.e.

$$[\sum_n A_n \xi^n, \sum_m B_m \xi^m] = \sum_{n,m} [A_n, B_m] \xi^{n+m} . \tag{6.13}$$

Obviously $(\tilde{\mathcal{L}}, [\ ], \tilde{\mathcal{F}})$ is again a Lie module. We can embed the tensor structure of $\mathcal{L}$ into that of $\tilde{\mathcal{L}}$ by treating ξ as scalar. By comparison of coefficients we then obtain that a covariant tensor $T[\xi]$ with respect to $\tilde{\mathcal{L}}$

$$T[\xi] = \sum T_n \xi^n,$$

(given by a formal power series in ξ with tensors in $\mathcal{L}$ as coefficients) is closed if and only if all the T_n are closed. Now having this additional structure we present as a simple excercise:

Proof of Theorem 6.11:
Consider the affine extension $\tilde{\mathcal{L}}$ of $\mathcal{L}$. Since Θ_1, Θ_2 are compatible implectic operators we have that $\tilde{\Theta} : \tilde{\mathcal{L}}^* \to \tilde{\mathcal{L}}$ defined by $\tilde{\Theta} = \Theta_1 + \xi\Theta_2$ is again implectic (Observation 6.5). Observe that $\tilde{\Theta}$ has an inverse $\tilde{J}$.

$$\tilde{J} = \tilde{\Theta}_1^{-1} = \Theta_1^{-1} \sum_{n=0}^{\infty} (-\xi)^n (\Theta_2 \Theta_1^{-1})^n . \tag{6.14}$$

Hence $\tilde{J}$ must be closed in $\tilde{\mathcal{L}}$. Therefore every

$$J_n = \Theta_1^{-1} (\Theta_2 \Theta_1^{-1})^n = \Theta_1^{-1} (\Phi)^n$$

must be closed in $\mathcal{L}$. This is the essential step where we needed the extension of $\mathcal{L}$. From now on we argue in $\mathcal{L}$.

From the bi-hamiltonian formulation we know (Observation 5.2) that the Θ_1, Θ_2, Φ and all the K_n are K-invariant (product rule).
(i): We already know that J_n is closed and K-invariant. Hence by Remark 4.4 (ii) the $\gamma_n = J_n K$ must be closed. This holds for all n.
(ii): We know that Φ is invariant with respect to all the K_m (consequence of Lemma 6.2). Since Θ_1^{-1} and the γ_n are closed, Θ_1^{-1} must be invariant with respect to $K_n = \Theta_1 \gamma_n$ (Remark 4.4 (ii)). By the product rule we then find that $J_m = \Theta_1^{-1} \Phi^m$ and $\Theta_{m+1} = \Phi^m \Theta_1$ are K_n-invariant.
(iii): If Θ_2 is invertible as well we may interchange the role of Θ_1 and Θ_2 in order to see that $\tilde{J}_n = \Theta_2^{-1} \Phi^{-n}$ is closed. So its inverse Θ_n^{-1} must be implectic. ∎

7 Examples and Applications

Observe that when the duality between tangent and co-tangent space is represented as in Example 3.5 then the differential operator D is trivially implectic. In this section we show how, from this knowledge, new pairs of implectic operators can be constructed.

We first point out the relation between Lei-module-isomorphisms and variable transformations. This connection allows efficient use of the invariant manner in which we introduced the main notions.

Consider Lie modules $(\mathcal{L}, \mathcal{F})$ and $(\tilde{\mathcal{L}}, \tilde{\mathcal{F}})$. In both modules we denote the Lie product by [ ] since no confusion arises. A pair (S, σ) of maps $S : \mathcal{L} \to \tilde{\mathcal{L}}$ and $\sigma : \mathcal{F} \to \tilde{\mathcal{F}}$ is said to be a **Lie-module-homomorphism** if

- S and σ are homomorphisms with respect to the algebraic structures in $\mathcal{L}$ and $\mathcal{F}$, respectively,

- $S(fK) = \sigma(f)S(K)$ for all $f \in \mathcal{F}$ and $K \in \mathcal{L}$.

- $S \cdot L_K = L_{(S(K))}$ for all $K \in \mathcal{L}$.

If S and σ are invertible then this is called a **Lie-module-isomorphism**. Isomorphisms allow us to carry over the whole tensor structure from $(\mathcal{L}, \mathcal{F})$ to $(\tilde{\mathcal{L}}, \tilde{\mathcal{F}})$. To do this we first define for $\gamma \in \mathcal{L}^*$ the corresponding $\tilde{\gamma} \in \tilde{\mathcal{L}}^*$ by

$$< \tilde{\gamma}, \tilde{K} > := \sigma(< \gamma, S^{-1}\tilde{K} >) . \tag{7.1}$$

The map $S^* : \gamma \to \tilde{\gamma}$ is called the **reciprocal image**. Let Ψ be some $\mathcal{L}$-tensor (r-times contravariant and n-times covariant), then the corresponding $\tilde{\mathcal{L}}$-tensor $\tilde{\Psi}$ is defined by

$$\tilde{\Psi}(S^*\gamma_1, ..., S^*\gamma_r, SK_1, ..., SK_n) := \sigma \cdot \Psi(\gamma_1, ..., \gamma_r, K_1, ..., K_n) \; for \; \gamma_i \in \mathcal{L}^* \; and \; K_i \in \mathcal{L} . \tag{7.2}$$

For example, if a two-times contravariant tensor Θ is taken in operator notation then (7.2) means that

$$< S^*\gamma_1, \tilde{\Theta}S^*\gamma_2 > = \sigma(< \gamma_1, \Theta\gamma_2 >) \; for \; all \; \gamma_1, \gamma_2 \in \mathcal{L}^* \tag{7.3}$$

where $< . >$ denote the respective dualities in $\mathcal{L}$ and $\tilde{\mathcal{L}}$. This yields

$$\tilde{\Theta} = S\Theta S^T \tag{7.4}$$

where $S^T : \tilde{\mathcal{L}}^* \to \mathcal{L}^*$ is the transpose of S given by

$$\sigma(< S^*\tilde{\gamma}, K >) = < \tilde{\gamma}, S^*K > .$$

Observe that in this formula we used $S^{*T} = S^{-1}$. With the same ease other transformation formulas may be explicitly determined (see [4]). As a consequence of our invariant definitions we find

Remark 7.1: *The notions implectic, symplectic, closed and hereditary are invariant under Lie-module isomorphisms.*

The most important Lie-module isomorphisms are given by variable transformations and these constitute an efficient tool for the construction of new compatible pairs of implectic operators.

Variable transformations:

Let M and $\tilde{M}$ be C^∞-manifolds and denote the respective manifold variables by u and $\tilde{u}$. Assume that there is a function $u \to \tilde{u} = T(u)$ such that T and its inverse (denoted by $\tilde{T}$) are C^∞. Let the meaning of $\mathcal{L}$ and $\mathcal{F}$ be as in the *standard situation* and consider the corresponding Lie module with respect to M. Then vector fields from M to $\tilde{M}$ are transformed by the variational derivative of T:

$$K(u) \to \tilde{K}(\tilde{u}) := T'(\tilde{T}(\tilde{u}))[K(\tilde{T}(\tilde{u}))] \tag{7.5}$$

and for scalars we define

$$f(u) \to \tilde{f}(\tilde{u}) := f(\tilde{T}(\tilde{u})) . \tag{7.6}$$

These transformations define a Lie-module isomorphism.

The most simple example for that isomorphism is when M is a vector space and

$$\tilde{u} = \lambda u + a , \tag{7.7}$$

where λ is some scalar and a some constant vector in M. Then we have that $S := T' = \lambda \, Id$ and we obtain

Remark 7.2: *Under the substitution $u \to \lambda u + a$ (a a constant vector and λ some scalar) in each tensor the properties: implectic, symplectic, closed and hereditary are preserved.*

Example 7.3: Construction of compatible pairs

For technical reasons, we now consider the space $\mathcal{S}_-$ of C^∞-functions f in the real variable $x \in \mathbb{R}$ having the property that f and all its derivatives vanish at $-\infty$ faster than any rational function and that at $+\infty$ all derivatives grow at most polynomially. As $\mathcal{S}_+$ we denote the corresponding space where the role of $-\infty$ and $+\infty$ has been interchanged. Between $\mathcal{S}_+$ and $\mathcal{S}_-$ we introduce an L^2 scalar product as in (3.9)

$$< U, u > = \int U(x)u(x)dx, \quad U \in \mathcal{S}_+ , u \in \mathcal{S}_- . \tag{7.8}$$

As before, D denotes differentiation with respect to x and D^{-1} denotes integration from $-\infty$ to x. Let the manifold under consideration be $M = \mathcal{S}_-$. Observe that D is implectic and invertible. We consider the variable transformation $\tilde{u} = T(u) := u^2 + u_x$ on M. Observe that, because of the boundary conditions we have chosen, this is one to one by the Implicit Function Theorem. To see that, compute for $T'(u) = 2u + D$ the inverse of $T'(u)$ by solving the linear differential equation $T'(u)z = g$ for given g and unknown z. On $\mathcal{S}_-$ this has a unique solution and the operator $T'(u)^{-1}$ mapping g into z is

$$T'(u)^{-1} = \exp\left(-2(D^{-1}u)\right) D^{-1} \exp\left(+2(D^{-1}u)\right) . \tag{7.9}$$

Starting with the implectic operator D (with respect to the manifold variable u) we find by variable transformation (formula (7.4)) that

$$\begin{aligned}
\tilde{\Theta}(\tilde{u}) &= T'(u)DT'(u)^T \\
&= (2u + D)D(2u - D) \\
&= D^3 + 2D(u^2 + u_x) + 2(u^2 + u_x)D
\end{aligned} \tag{7.10}$$

Using the relation between u and $\tilde{u}$ we see that

$$\tilde{\Theta}(\tilde{u}) = D^3 + 2D\tilde{u} + 2\tilde{u}D \tag{7.11}$$

is implectic as it was claimed already in Example 3.5. Now performing the substitution (for $a(x) = 1$) as described in Remark 7.1 we find that

$$\tilde{\Theta}(\tilde{u}+1) \;=\; D^3 \;+\; 2(D\tilde{u} \;+\; \tilde{u}D) + 4D = \tilde{\Theta}(\tilde{u}) + 4D \tag{7.12}$$

is again implectic. Because D is already known to be implectic we have that D and $\tilde{\Theta}$ are compatible. Hence by Corollary 6.10

$$\tilde{\Phi} = \tilde{\Theta}D^{-1} = D^2 \;+\; 2D\tilde{u}D^{-1} \;+\; 2\tilde{u} \;, \tag{7.13}$$

must be hereditary.

Using Theorem 6.11 we find that

$$\tilde{\Theta}_2(\tilde{u}) = \tilde{\Phi}(\tilde{u})\tilde{\Theta}(\tilde{u}) \tag{7.14}$$

again is implectic. We transform that back from the $\tilde{u}$-variable to the u-variable in order to obtain with (7.10) the following implectic operator:

$$\begin{aligned}
\Theta(u) &= T'^{-1}\tilde{\Theta}_2(T'^{-1})^T \\
&= T'^{-1}\tilde{\Theta}D^{-1}\tilde{\Theta}(T'^{-1})^T \\
&= T'^{-1}T'D(T')^T D^{-1}T'D(T')^T(T'^{-1})^T \\
&= D(T')^T D^{-1}T'D \\
&= D(2u - D)D^{-1}(2u + D)D \\
&= -D^3 + 4DuD^{-1}uD
\end{aligned}$$

Now using $u \to iu$ we find with Remark 7.1 that

$$\Theta(u)_{mKdV} = D^3 + 4DuD^{-1}uD \tag{7.15}$$

is implectic. Using Remark 7.2 and $u \to (1/4)\sqrt{2}(2u + 1)$ we find that

$$\Theta_{Gardner} = \frac{1}{2}(\Theta(u)_{mKdV} + \Theta(u) + D)$$

is also implectic. here $\Theta(u)$ is the implectic operator given in (7.11) (only $\tilde{u}$ replaced by u). Since $\Theta(u)$ and D are compatible $\Theta(u) + D$ is again implectic and Θ_{mKdV} and $\Theta(u) + D$ must be compatible. Taking now $1/4\Theta_{mKdV} + 3/4(\Theta(u) + D)$ (which is implectic by Observation 6.5) and substituting $u \to (2u - 3)$ we find that $\Theta_{mKdV} + (3/4)D$ is implectic. Hence Θ_{mKdV} and D are compatible.

Observation 7.4: *If the duality between function spaces is taken to be (3.9) then the differential operators*

$$\Theta_0 = D$$
$$\Theta_{KdV} = D^3 + 2Du + 2uD$$
$$\Theta_{mKdV} = D^3 + 4DuD^{-1}uD$$

are implectic. Every two of these are compatible.

Example 7.5: Conserved quantities for the KdV

Consider the situation of Example 3.5. where the operator

$$\Theta_{KdV} = D^3 + 2Du + 2uD$$

was introduced. Taking into account the compatibility between the implectic operators D and Θ_{KdV} (as just proved) we get from Corollary 6.10 that

$$\Phi = \Theta D^{-1} = D^2 + 2DuD^{-1} + 2u , \qquad (bihamKdV)$$

is hereditary. As a consequence (Theorem 6.11) we have that the

$$\gamma_{n+1} = D^{-1}K_{n+1} = D^{-1}\Phi(u)^n K(u), \; where \; K(u) = 6uu_x + u_{xxx} \qquad (7.16)$$

are closed co-vector fields. These fields are invariant for any of the flows $u_t = K_n(u)$. Hence (Lemma 5.5) all

$$I_n(u) = \int_0^1 < \gamma_n(\lambda u), u > d\lambda = \int_0^1 \int_{\mathbb{R}} \gamma_n(\lambda u(x))u(x)d\lambda dx \qquad (7.17)$$

are conserved quantities for every one of these flows, especially for the KdV. In addition, one easily sees that all these quantities commute with respect to the Poisson brackets defined by either Θ_{KdV} or D. □

Example 7.6: Further systems

Using the compatible pairs we have up to now, and more which we generate easily by variable transformations, we can generate new and nontrivial hereditary operators. For example

$$\Phi_{mKdV} = \Theta_{mKdV} D^{-1}$$

$$\Phi_{Gardner} = \Theta_{Gardner} D^{-1}$$

$$\Phi_{sineGordon} = 2\Phi_{mKdV}^{-1}$$

$$\Phi_{BBM-like}(v) = (I - D^2)^{-1}\Phi_{KdV}(v - v_{xx})$$

$$\Phi_{sG-KdV} = \Phi_{KdV}^{-1}\Phi_{sineGordon}$$

The hereditary nature of these operators is easily seen from compatibility of known pairs. For example, take Φ_{sG-KdV} then this can be written as $\Theta_{KdV}\Theta_{mKdV}^{-1}$, and must be hereditary since these two are compatible. For $\Phi_{BBM-like}(v)$ we use the compatibility of Θ_{KdV} and of $D - D^3$, and then we performed a variable transformation $u = v - v_{xx}$.

Since none of these operators depend explicitly on x we find that they are all invariant with respect to the special vector field u_x. So for the equations

$$u_t = \Phi(u)u_x$$

we find (by application of Theorem 6.1) infinitely many symmetry group generators

$$K_n(u) = \Phi^n(u)u_x \tag{7.18}$$

Let us list these equations (following the above order)

$$u_t = u_{xxx} + 6u^2 u_x \qquad\qquad (modified\ KdV)$$

$$u_t = u_{xxx} + 6u^2 u_x + 6uu_x + u_x \qquad\qquad (Gardner\ eq.)$$

$$u_t = \sin\left(2\int_{-\infty}^{x} u(\xi)d\xi\right) \qquad\qquad (potential\_sG\ eq.)$$

$$v_t - v_{xxt} = v_{xxx} - 2vv_{xxx} - 4v_x v_{xx} + 6vv_x \qquad\qquad (BBM\_like\ eq.)$$

$$v_{xt} = 2v_{xx}\cos(2v) + 4(v_x - v_x^2)\sin(2v) + 2v_{xx}\int_{-\infty}^{x}\sin(2v(\xi))d\xi \qquad (KdV\_sG\ eq.)$$

In the case of the last equation (KdV_sG), we have performed an additional variable transformation $u = v_x$. The equation (potential_sG) is connected to so called **sine_Gordon** equation since substituting first $u = v_x$ and performing then a 45-degree rotation in the space of independent variables yields

$$v_{\xi\xi} - v_{\eta\eta} = \sin(2v)\ . \qquad\qquad (sine\_Gordon\ equation)$$

Most of these equations are well known from the literature. They all have a bi-hamiltonian formulation if the solution manifold is suitably chosen such that these operators are well defined. The invariant co-vector fields generated by this recursion mechanism are all closed because they are then generated by a hereditary operator, which stems from a compatible pair of implectic operators. For example, in case of the modified Korteweg-de Vries equation (modifiedKdV) the compatible pair of implectic operators is D and Θ_{mKdV}. The equation itself has the form

$$u_t = u_{xxx} + 6u^2 u_x = \Theta_{mKdV}(u)\nabla\frac{1}{2}\int_{\mathbb{R}} u^2(\xi)d\xi\ . \tag{7.19}$$

Furthermore

$$D^{-1}(u_{xxx} + 6u^2 u_x) = \nabla\int_{\mathbb{R}}(-u_x^2(\xi) + \frac{1}{2}u^4(\xi))d\xi\ . \tag{7.20}$$

Application of Theorem 6.11 then shows that

$$I_n(u) = \int_0^1\int_{\mathbb{R}}\gamma_n(\lambda u(x))u(x)dxd\lambda = \int_0^1\int_{\mathbb{R}}(D^{-1}\Theta_{mKdV}(\lambda u(x)))^n u(x)d\lambda dx \tag{7.21}$$

are conserved quantities for the modified Korteweg-de Vries equation. Observe that these are also conserved quantities for the potential sine-Gordon equation (potential_sG eq.) since that equation is generated by the inverse of the operator Φ_{mKdV}. Similar arguments go through for the other equations given above.

We like to mention that the applications of the theory presented in this paper are in no way exhausted by these examples. There are many more such as: the nonlinear Schrödinger equation, two-component systems, Spin chains, and the bi-hamiltonian formulations given by Fokas and Santini ([2,] [24,] [23,] [22]) for equations in two independent variables. These last examples are interesting in so far as they require the full generality of notions and methods as introduced in Sections 4,5 and 6.

8 Integrability and Solitons

Let us briefly review:

Complete integrability in the finite dimensional case:
The hamiltonian flow

$$u_t = \Theta(u)\nabla H(u).$$

(3.3)

on a $2N$-dimensional manifold M with invertible implectic operator Θ is called completely integrable if it admits N conserved quantities $I_1 := H, I_2, ..., I_N$ such that the corresponding symmetry group generators $\Theta\nabla I_1, \Theta\nabla I_2, ..., \Theta\nabla I_N$ commute. Furthermore these fields are required to be linearly independent at each manifold point. These $I_1, I_2, ..., I_N$ are called action variables.

Observation 8.1:
In this case one can find N closed and pairwise commuting vector fields $A_1, ..., A_N$ such that

$$L_{A_i} I_j = \delta_{ij} I_i$$

(8.1)

or, by use of (4.16) and Theorem 4.5

$$[A_i, \Theta\nabla I_j] = \delta_{ij}\Theta\nabla I_i .$$

(8.2)

The $\Theta\nabla I_i$ are called **action fields** and the A_i are called the conjugate **angle fields**).

The proof of this statement is technically involved, so we will only give a brief sketch. The arguments are an adaption of [16, page 28].

Proof :
STEP 1:
First one shows that around each manifold point u_0 coordinates $\{I_1, .., I_N, Q_1, ..., Q_N\}$ can be chosen such that $J(u) := \Theta(u)^{-1}$ is constant in that chart. This is done in the following way:
Represent the manifold around u_0 by some open ball in a vector space E with coordinates $\{I_1, ... I_N, \tilde{Q}_1, ... \tilde{Q}_N\}$. Then consider the operators $J(u)$ and $J_0(u) := J(u_0)$ and take a deformation $J_t(u) := J(u) + t(J_0(u) - J(u))$, $1 \ge t \ge 0$ from J to J_0. Observe that $J_t(u_0) = J_0(u_0)$ is invertible for all t with $1 \ge t \ge 0$. The openness of the set of isomorphisms shows that there is a ball B around u_0 such that $J_t(u)$ is invertible for all $u \in B$ and $1 \ge t \ge 0$. The inverse of $J_t(u)$ we denote by $\Theta_t(u)$. Since $(J_0(u) - J(u))$ is closed we find by the Poincaré lemma a 1-form γ in B such that $(J_0(u) - J(u)) = d\gamma$ and $\gamma(u_0) = 0$. Now take the t-dependent vector field $K(t, u) = \Theta_t(u)\gamma$ and consider the equation

$$v_t = K(t, v)$$

(8.3)

in B. Since $K(t = 0, u) = 0$ we can assume (by eventually restricting B again) that (8.3) has a unique solution for all $1 \ge t \ge 0$ and all $u \in B$. Define $\varphi(u)$ to be the solution of (8.3) for $t = 1$ and initial condition $v(t = 0) := u$ and let $\tilde{I}_i(u) = I_i(\varphi(u))$, $\tilde{Q}_i(u) = Q_i(\varphi(u))$, $i =$

$1, ..., N$. Observe that $\tilde{I} = I$ and that $J(u)$ now is constant in the chart given by these new coordinates.

STEP 2:

Consider the constant $J(u)$ as constructed above in the vector space given by the coordinates $\{I_1, .., I_N, Q_1, ..., Q_N\}$ and endow that space with the usual Euclidean metric of $\mathbb{R}^{2N}$. Since $J(u)$ is invertible and antisymmetric its matrix representation must be of the form

$$\begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$$

where S is invertible and symmetric. Hence by a change of basis among the Q's we can assume that S is the $N \times N$ identity matrix. Finally, taking $A_i = -\Theta\nabla(I_iQ_i)$ we locally find the desired vector fields.

STEP 3:

We observe that different local realizations of (8.1) differ on the overlap of their domain only by a suitable combination of the $\Theta\nabla I_i$, hence by globally defined vector fields. This property allows us to patch the A_i from one chart to the next so that they coincide on the overlap. Hence we can define them globally. ∎

Observe that locally potentials for the A_i exist, let us call them Q_i. Then (8.1) implies that for every of the flows $u_t = \Theta(u)\nabla I_n$ the $Q_m, n \neq m$ are conserved quantities, whereas Q_n changes with t but is the absolute part of the time-dependent conserved quantity $Q_n - tI_n$. So taking these coordinates we arive at

Observation 8.2: *On some $2N$-dimensional manifold, endowed with the invertible implectic operator Θ, let there be given N pairwise commuting scalar fields $I_1, ..., I_N$ (commuting with respect to the corresponding Poisson brackets). Then around each point there are local coordinates $\{I_1, ..., I_N, Q_1, ..., Q_N\}$ such that the hamiltonian flows $u_t = \Theta(u)\nabla I_n$ are linear in these coordinates so that all but Q_n are invariant and that the action on the Q_n is such that this grows linear with t.*

Now we use the vector fields from Observation 8.1 to define the following operator $J_2\mathcal{L} \to \mathcal{L}^*$

$$J_2 = \sum_{i=1}^{N} \Theta^{-1}A_i \otimes dI_i - dI_i \otimes \Theta^{-1}A_i . \tag{8.4}$$

Obviously this is an antisymmetric tensor and it is closed because both dI_i and $\Theta^{-1}A_i$ are closed. So it may serve as a symplectic operator and one easily finds that

$$\Phi := \Theta J_2 \tag{8.5}$$

is hereditary. Using the relations (8.1) we see right away that

$$J_2\Theta\nabla I_i = I_i\nabla I_i \tag{8.6}$$
$$J_2 A_i = -I_i\Theta^{-1}A_i$$

and that therefore the eigenvalues of Φ are given by the action variables. So we have

Observation 8.3: *For a finite dimensional completely integrable system in $2N$-dimensional space, with action variables $\{I_1, \ldots, I_N\}$ there always exists a hereditary operator Φ such that its spectrum is doubly degenerate. The eigenvalues are given by the action variables and the corresponding eigenvectors are the action fields and their conjugate angle fields.*

Of course, the converse is also true: Whenever on a $2N$-dimensional manifold we have a hereditary operator Φ with doubly degenerate spectrum and some implectic operator Θ such that $\Theta\Phi$ is closed, then when the the gradients of the eigenvalues are mapped with Θ onto vector fields they form a commuting algebra of vector fields (consequence of Remark 6.3). Hence any dynamic of system given by any linear combination of these vector fields must be completely integrable.

Now let us return to the general situation of infinitely dimensional manifolds. Here the notion of complete integrability has not yet been definitely defined in the literature. Often such a system is called completely integrable if it admits an infinite dimensional abelian symmetry group of hamiltonian fields. This of course, is a somehow loose definition since it is easy to construct situations where such a symmetry group does not suffice to guarantee a parametrization by action and angle variables. Instead of attempting here a general definition we shall show that, under reasonable conditions, the existence of a compatible bi-hamiltonian pair leads to complete integrability on finite dimensional reductions given by the corresponding symmetry group generators. We will give a survey on the results which can be obtained in that direction, for details the reader is referred to [8].

Let us recall the situation we considered before. On a suitable manifold M the evolution equation $u_t = K_1(u)$ where $u = u(x,t) \in M$ was considered. We assume that there is a hereditary recursion operator $\Phi(u)$ generated out of a compatible hamiltonian pair

$$\Phi(u) = \Theta_2(u)\,\Theta_1^{-1}(u) =: \Theta_2(u)\,J(u) \ .$$

As shown the operator Φ then generates a hierarchy of pairwise commuting infinitesimal symmetry group generators

$$K_{n+1}(u) := \Phi^n(u)K_1(u)$$

for the evolution equation under consideration.

In addition to what we assumed until now we require furthermore the existence of a scaling symmetry $\tau_0(u)$. By that we mean:

$$[\tau_0, K_1] = (\varrho + 1)K_1 \tag{8.7}$$

and

$$L_{\tau_0}\Phi = \Phi \ . \tag{8.8}$$

As a consequence of the scaling property the recursive application of Φ on τ_0 produces a second hierarchy of vector fields, the so-called **mastersymmetries** [6] $\tau_n = \Phi^n\tau_0$ such that the following commutator relations hold between the symmetries K_n and the mastersymmetries τ_n

$$[K_n, K_m] = 0 \ , \quad [\tau_n, K_m] = (m + \varrho)K_{n+m} \ , \quad [\tau_n, \tau_m] = (m - n)\tau_{n+m} \ . \tag{8.9}$$

Indeed, these commutator relations are a simple consequence of the hereditary property of Φ. One should observe that the relation $[\tau_n, K_m] = (m + \varrho)K_{n+m}$ is equivalent to the fact that $\tau_n + (m + \varrho)K_{n+m}$ is a time-dependent symmetry group generator of $u_t = K_m(u)$. A Lie algebra consisting of τ's and K's fulfilling (8.9) is called a **hereditary algebra**. These scaling symmetries exist for almost all popular soliton equations, and even in those cases where a scaling symmetry or a hereditary cannot be found one can nevertheless construct a suitable hereditary algebra. For example, in the KdV case $\tau_0(u) = \frac{1}{2}xu_x + u$ is the scaling symmetry, and for the mKdV and the potential sine-Gordon one finds $\tau_0(u) = xu_x + u$.

From the invariance of the symmetry group generators one finds that the submanifold (see for example [19])

$$M_N = \{ u \mid \text{there exists } \alpha_n \text{ such that } \sum_{n=0}^{N} \alpha_n K_n = 0 \} \qquad (8.10)$$

is invariant under any of the flows $u_t = K_n(u)$, in particular under $u_t = K_1(u)$. This manifold is called the manifold of N-**soliton solutions**. For the KdV typical two- and three-solitons are given in figures 3 and 4[9] .
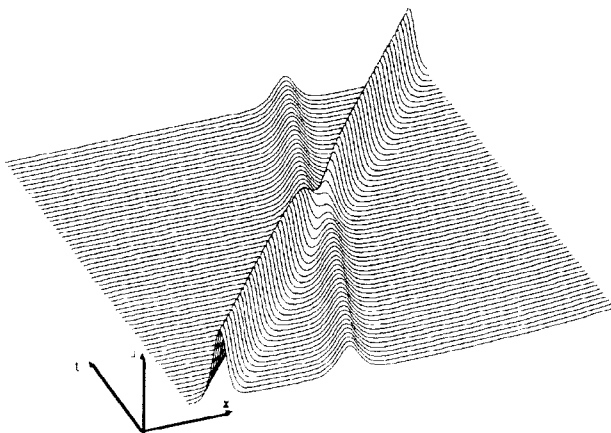

Fig. 3: Two-soliton of the KdV

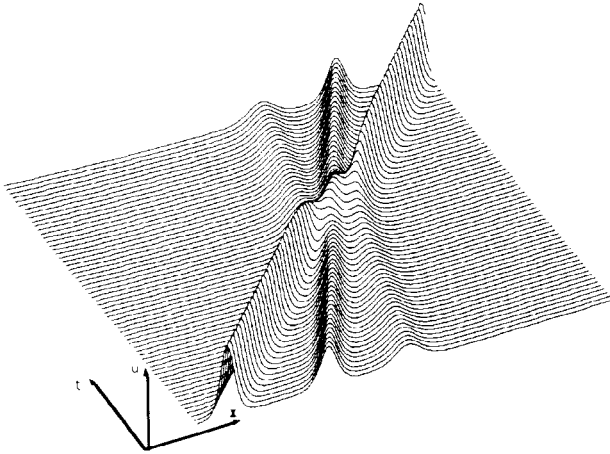[9]I am indebted to Thorsten Schulze for plotting these figures.

Fig. 4: Two-soliton of the KdV

In case when the boundary conditions at infinity for possible solutions u are chosen in such a way that the resulting manifold M_N has dimension $2N$ then by a lengthy but simple analysis [8] (mainly of the hereditary structure of Φ) one obtains from this structure the following result

Theorem 8.4:

(1) For all $r, p \in \mathbb{N}_0$ we have the following representation of the tangent space $T_u M_N$ of M_N at the point u

$$T_u M_N = \text{span} \{ K_r, K_{r+1}, ..., K_{r+N-1}, \tau_p, \tau_{p+1}, ..., \tau_{p+N-1} \} \ .$$

(2) Whenever the α_n are the coefficients given by (8.10) (to define the manifold point u) then the following hold

(i) For all $r \in \mathbb{N}_0$ we have the following identities on M_N:

$$\sum_{n=0}^{N} \alpha_n K_{n+r} = 0 \quad \text{and} \quad \sum_{n=0}^{N} \alpha_n \tau_{n+r} = 0 \ .$$

(ii) The discrete eigenvalues $c_1, ..., c_N$ of Φ are given as the zeros of the characteristic polynominal $P(\xi) = \sum_{n=0}^{N} \alpha_n \xi^n$.

(iii) The corresponding eigenstates are $\tilde{V}_i = \Pi_i(\Phi) K_0$ and $\tilde{W}_i = \Pi_i(\Phi) \tau_0$, where $\Pi_i(\xi) = P(\xi)/(\xi - c_i)$.

As a direct consequence of Theorem 8.4 we obtain that the recursion operator Φ leaves the tangent space $T_u M_N$ of the reduced manifold invariant. Hence, the restriction $\tilde{\Phi} := \Phi_{|red}$ of Φ to M_N is a linear operator on a finite dimensional space. This operator $\tilde{\Phi}$ has the properties listed below in:

Observation 8.5:

(1) *$\tilde{\Phi}$ is invertible and can be written as $\tilde{\Phi} = \tilde{\Theta}_2 \tilde{\Theta}_1^{-1}$ where $\tilde{\Theta}_1, \tilde{\Theta}_2$ are a compatible pair of implectic operators.*

(2) *The eigenvalues $c_1, ..., c_N$ of $\tilde{\Phi}$ are doubly degenerated.*

(3) *Renorming the eigenstates $\tilde{V}_i, \tilde{W}_i$ leads to eigenstates V_i and W_i which are hamiltonian vector fields w.r.t. $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$.*

(4) *The eigenstates V_i and W_i fulfil the commutator relations*

$$[V_i, V_j] = 0 = [W_i, W_j] \quad , \quad [V_i, W_j] = \delta_{ij} V_j \ . \tag{8.11}$$

These last two results show that the finite dimensional reductions, given by those members of the abelian symmetry group which is generated by the bi-hamiltonian formulation is, under suitable boundary conditions at infinity, the same situation as we found in the completely integrable finite dimensional case.

Since the eigenstates V_i, W_i are hamiltonian vector fields and since they fulfill the canonical commutator relations (8.9), their potentials can be interpreted as action/angle variables for the flow induced by (1.1) on M_N.

Although all our considerations were of a purely algebraic nature we should remark that in most cases which are relevant from the physical viewpoint the N-soliton solutions (with vanishing boundary conditions at infinity) are those solutions which decompose into N single waves for $t \to \pm\infty$

$$u_N \cong \sum_{i=1}^{N} s_i(x + c_i t + q_i) \ .$$

This can be seen for the KdV from Figures 4 and 5. By comparison with the asymptotic data we get in these case we get a simple method for finding the eigenstates of the recursion operator.

Observation 8.6: *Taking the partial derivatives of u_N w.r.t. the asymptotic data*

$$\frac{\partial u_N}{\partial q_i} \quad \text{and} \quad \frac{\partial u_N}{\partial c_i}$$

one obtains eigenstates of the recursion operator $\tilde{\Phi}$ for the eigenvalue c_i. The function $\partial u_N / \partial q_i$ then is the vector field corresponding to the action variable and this is called the

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interacting soliton *[7].*

In case of the KdV equation a plot of such quantities is easily obtained (see Figure 5)
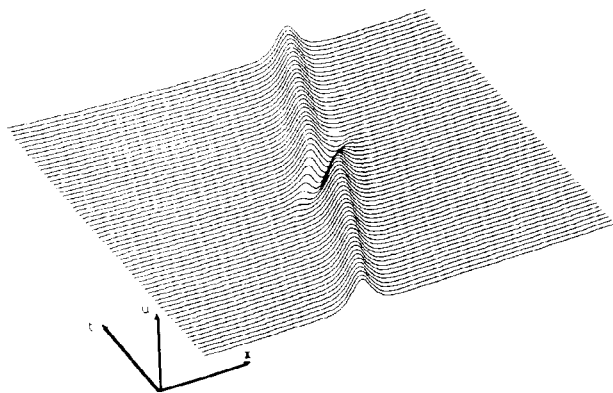

Fig. 5: Interacting soliton of the KdV

A corresponding conjugate eigenstate for the KdV is obtained by taking the derivative of the field function u_N with respect to the parameters given by an eigenvalue of the recursion operator.
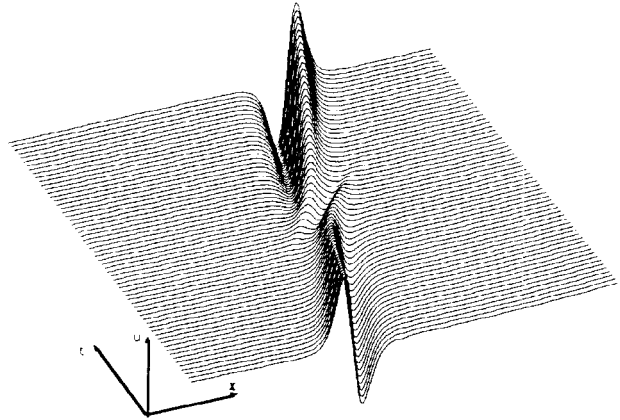
Fig. 6: Derivative of Angle-variable density of the KdV

It should be remarked that the field functions given by these plots themselves satisfy nonlinear equations which have a compatible bi-hamiltonian formulation [7]

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