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Linear Aspects in the Theory of Solitons and Nonlinear Integrable Equations

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In this survey we show how to obtain from the analytic structure of one-soliton solutions, the complete action angle variable representation of arbitrary multi-solitons. Special attention is paid to the interacting solitons and their relation to singularity analysis.

§1. Introduction

Nowadays, the field of Functional Analysis is sometimes called *Topological Linear Algebra*. Of course, this name is chosen with a certain attitude of disrespect in order to set this discipline apart from other, more interesting areas like nonlinear analysis or nonlinear differential equations. Quite often the achievements which have been made over the last two decades in the field of nonlinear integrable equations are then mentioned in the same context as a striking example for a beautiful nonlinear theory. In order to emphasize a counter point to that popular opinion this survey is devoted to the theme of showing what *linear perspectives* can achieve in the area of nonlinear integrable equations and the theory of nonlinear soliton interaction.

Most of what I have to say I will demonstrate at the example of the Korteweg de Vries equation. But everything can be applied, and if necessary generalized, to other equations as well. The Korteweg de Vries equation is chosen only because it is widely known and thus may evoke some helpful familiarity and intuition for those readers who are not to well acquainted with the field. Of course, it is well known that for this equation there exists a linearizing transform in terms of the *Inverse Scattering Transform*, but this linearization can only be seen after one has discovered the crucial Lax representation. In this paper however we exhibit arguments which can be used as a heuristic method from the beginning. Without further information, they can be used to check if there is any hope for a linearization. And if there is such a hope these methods give some help to construct the crucial quantities necessary for a further analysis.

The Korteweg de Vries equation

$$u_t = u_{xxx} + 6uu_x, \tag{1.1}$$

was found in 1895 (see ref. 19) in an attempt to explain some observations in the area of shallow water wave theory which were made in 1836 by Scott Russell (see his report from 1844[32]). At that time the solution one was mainly interested in, was the *traveling wave* coming out of the ansatz

$$u(x, t) = s(x+ct). \tag{1.2}$$

This solution is easily found in its explicit form because by this ansatz the partial differential equation reduces to an ordinary differential equation which can be easily solved. The explicit form of the solution, which we need later on in order to describe more complex phenomena, is

$$s(x+ct) = \frac{c}{2}\cosh^{-2}\left\{\frac{\sqrt{c}}{2}(x-x_0+ct)\right\}. \tag{1.3}$$

Nowadays this solution would be called a *one-soliton solution*. By the observations of Scott Russell it was clearly known that a certain *nonlinear superposition principle* must hold in the presence of several solitons.*

* In Russells colorful language: If such a heap be forced into existence, it will rapidly fall to pieces and become disintegrated and resolved into different waves, which do not move forward in company with each other,

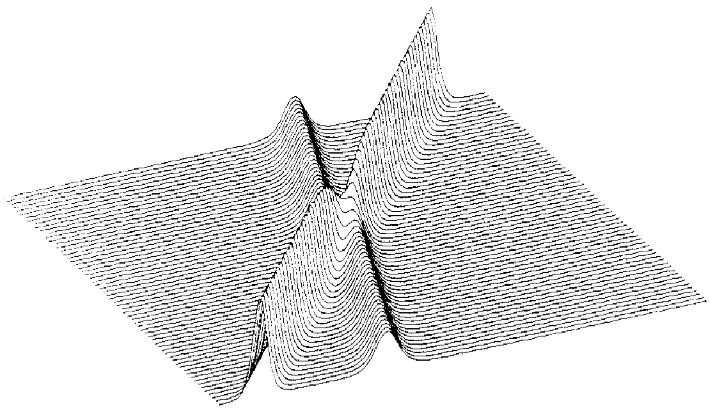

Fig. 1. Two-soliton solution of the KdV.

This superposition can be clearly seen by looking at the two-soliton solution for which I have plotted the t-slices in Fig. 1. One sees that asymptotically (i.e. for $t \to -\infty$ or $t \to +\infty$) this wave fulfills the ansatz made above for a traveling wave. But nevertheless these waves interact in a nonlinear fashion since their speeds are dependent on the height of the respective waves. Since the waves regain completely their original speed one often describes this phenomenon as an *elastic interaction*. What I will demonstrate in the following is:

- By looking on Fig. 1 and by taking into account the *explicit form* of the one-soliton solutions given in (1.3) one can derive, without any further information and in a purely deductive way, all the spectacular properties of this particular equation.**
- Furthermore it will be possible to discover by this viewpoint the necessary tools in order to give a linearization of the flow represented by eq. (1.1). To make this precise, we shall give a complete *action angle variable representation* for the interaction of an arbitrary number of solitons.
- A problem which will be addressed in particular is whether or not we can derive equations which will describe the interacting solitons individually, even during the interaction with other solitons. It will turn out that this question is intimately connected to the so called singularity analysis for completely integrable nonlinear equations.

§2. Linear Aspects

In this section we give heuristic arguments which motivate the notions introduced afterwards.

Looking at Fig. 1 we already discovered that the two-soliton solution decomposes asymptotically into traveling waves

$$u(x, t) \simeq \sum_{i=1}^{2} s_i(x + c_i t + q_i^{\pm}) \text{ for } t \to \pm\infty, \tag{2.1}$$

but move on separately, each with a velocity of its own, and each of course continuing to depart form each other. Thus a large compound—heap becomes resolved···by a species of spontaneous analysis.[32]

** When I expressed the opinion for the first time, among other workers in the field, that in principle one must be able to determine the complete integrability of the KdV, as well as that of other equations, by looking at the one-soliton they convincingly proved to me that I must be clearly out of my mind. Their argument was that everybody certainly must agree on the fact that the existence of a traveling wave solution has nothing to do with complete integrability whatsoever. Of course, they were right, with this observation, not with their more personal claim. Surely, existence of traveling wave solutions does not mean anything, however the analytical form of these traveling waves has to do a lot with complete integrability. To carry that point to the extreme, I believe it possible that one day we will adopt the viewpoint that the whole theory of completely integrable flows on infinite dimensional manifolds is solely an application of certain properties of some special functions. However I admit, that when that happens, namely that soliton theory will be considered as an application of some fancy theorems in analysis, then a lot of the fun, and especially a lot of the frontiers spirit, will have been taken away from the subject.

where the c_i are the different speeds of the asymptotically emerging solitons. The s_i are the corresponding functions given in (1.3) and the quantities $q_i^{\pm}$ describe suitable phases. Obviously, the totality of all these two-soliton solutions, for variable asymptotic speeds and phases, forms a four dimensional manifold, and we can describe this manifold conveniently by the parameters c_i and $q_i = q_i^{+}$. These parameters are scalar fields on the manifold and it is an elementary task to see how these parameters change during the flow given by eq. (1.1). Obviously, the c_i do not change at all and by definition we have for the q_i that

$$q_i^{\pm}(u(.,t)) = q_i^{\pm}(u(.,0)) + c_i t, \qquad (2.2)$$

hence they must be growing linearly with time. So using the new parametrization of the manifold given by the c_i and the q_i we can easily express the flow as

$$\frac{d}{dt}\begin{pmatrix} c_1 \\ c_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c_1 & 0 \\ 0 & 0 & 0 & -c_2 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \qquad (2.3)$$

We observe that this is a linear system, furthermore that it is a hamiltonian system because the vector on the right hand side clearly is a gradient

$$\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\,\text{grad}\,(c_1^2 + c_2^2). \qquad (2.4)$$

This is not the only hamiltonian formulation, another one is given by the second line of eq. (2.3) since the vector to which the matrix is applied is again a gradient

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \text{grad}\,(c_1 + c_1). \qquad (2.5)$$

Clearly the matrix in front of it induces a symplectic structure.

Now we take into account that the traveling waves under consideration vanish rapidly at infinity and that the nonlinear term in (1.1), in comparison to the linear terms, does not give any contribution in case of vanishing field variables. This allows us to guess the structure of solutions which arise when we start at $-\infty$ with three different traveling waves whose phases are chosen such that first two of them interact, while the other one is far away, and then, after that, the interaction with the remaining wave happens. Since we have elastic interaction between two waves this suggests that elastic interaction also happens between three waves and more. However, this may not be true in general. A detailed analysis shows that such a conclusion is only valid if there are additional conditions fulfilled. Speaking from a physical viewpoint, these conditions require that all the energy of the field is carried by the asymptotically emerging solitons, or mathematically speaking, there must be some invariant positive definite scalar field, which is additive for functions with disjoint support, and which has the property that if for finite time evaluated on the field it leads to the same value as the sum of evaluations on the asymptotically emerging traveling waves. So let us assume that such a condition holds (which can be easily shown for the KdV), then asymptotically, at $+\infty$ there are emerging three traveling waves having the same speeds as the waves we started with. These three-soliton solutions we again parametrize by their speeds and phases, and we obtain, with respect to this parametrization, a representation of the dynamics very similar to (2.3). This process can be continued, thus leading for suitable N-solitons to a parametrization in which the flow has the simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_1 \\ \vdots \\ c_N \\ q_1 \\ \vdots \\ q_N \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \operatorname{grad}(c_1^2 + \cdots + c_N^2)$$

$$= \begin{pmatrix} 0 & -\Lambda \\ \Lambda & 0 \end{pmatrix} \operatorname{grad}(c_1 + \cdots + c_N). \tag{2.6}$$

Here I denoted the $N \times N$-unit matrix and Λ is the matrix having the c_i in the diagonal

$$\Lambda = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & c_N \end{pmatrix}. \tag{2.7}$$

This again is a linear hamiltonian system and the parametrization we have been led to usually is called an action-angle representation (see ref. 1). One can draw from this observation some interesting conclusions:

- For every N there is a $2N$-dimensional manifold which is invariant under (1.1) and which is of such a nature that (1.1) defines a linear hamiltonian flow on it. If one defines the obvious Poisson brackets on that manifold one discovers easily that the c_i are conserved quantities which are in involution. So, again, (1.1) defines on these manifolds a flow for which an action-angle representation can be given.

Since we can blow up arbitrarily the dimension of the invariant manifolds on which the flow has this nice structure, we are led to the assumption that the whole flow is a linearizable hamiltonian system. However, this only is an assumption for which we have not yet rigorous arguments, at least not until we really have constructed the relevant quantities. Of course, even if we find structural arguments which show that in some sense the flow is linearizable, this only suggests a structural linearizability, it does not necessarily mean that we ever will be able to write it down explicitly. But for the moment we are happy to draw structural conclusions from that assumption, the question how to obtain concrete solutions will be postponed for a while.

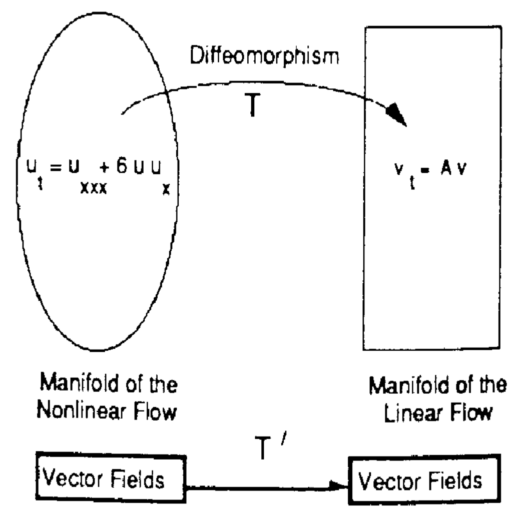

Fig. 2.

As the main outcome of our considerations we keep in mind Fig. 2 to represent the linearizability of eq. (1.1). However, all this abstract insight does not yet assist us to discover what the N-solitons look like in real-life coordinates. Certainly eq. (2.6) has a structure of extreme simplicity but might not be of great value to somebody who is interested in concrete solutions and the physical interpretation of these solutions.

§3. Group Structure and Hereditariness

Here we exhibit how the consideration of one-parameter diffeomorphisms on some infinite dimensional manifold helps in the analysis of our nonlinear system. Sometimes it is argued that diffeomorphism-groups on manifolds have nothing to do with linear considerations. However, to my opinion that is not quite true and only a matter of the viewpoint one adopts. Consider for example an entire function in the complex plane. Then going with the function value from one point in the plane to another certainly is not a linear operation, wheras shifting all these functions by this difference of the two points is certainly a linear operation on that function space. These linear operations form a one-parameter diffeomorphism group having the differntial operator as infinitesimal generator. Thus solving the corresponding linear differential equation on a suitable infinite dimensional vector space gives that this one-parameter diffeomorphism group must be represented by the exponential of the differential operator, an exponential which is Taylors formula. Hence, the question whether or not an operation is linear may sometimes only be depending on whether one is willing to blow up the dimensiom of the problem considerably. Certainly, for diffeomorphism groups on manifolds this is true, because if one represents such a diffeomorphism group by the induced action it has on the scalar fields on this manifolds, then this action is the exponential of a linear operator on that space of scalar fields.

Let us return to our linear problem. Here, even the simple question how to find the N-soliton solutions is not yet answered by knowing that these solutions can be represented as solutions of a linear equation on some *abstract* manifold. So our first problem is the characterization of these special solutions. For this we need the notion of *symmetry group*.

Take another evolution equation

$$u_\sigma = K(u), \tag{3.1}$$

and define for it the resolvent map which assigns to the initial value condition $u(x) = u(x, \sigma=0)$ the solution $u(x, \sigma)$. This map we denote by $R_K(\sigma)$. Similarly we define the resolvent for (1.1) which we denote by $R_G(t)$, letting G stand for the vector field on the right side of (1.1). In case that the initial value problem on the manifold under consideration can be suitably solved for arbitrary σ, the map $\sigma \to R(\sigma)$ defines a one parameter group of diffeomorphisms on the corresponding manifolds. Equations (1.1) and (3.1) are said to *commute* if these resolvent maps $R_K(\sigma)$ and $R_G(t)$ do commute for all t and σ. In that case $R_K(\sigma)$ is said to be a *one parameter symmetry group* of (1.1). Of course, such a condition cannot be checked by considering the quantities R_K directly since the maps R_K and R_G are rarely accessible in their explicit forms. Therefore an infinitesimal version of this notion has to be considered. One easily shows that the resolvents commute if and only if the corresponding vector fields $K(u)$ and

$$G(u) = u_{xxx} + 6uu_x, \tag{3.2}$$

do commute in the vector field Lie algebra. So we require

$$[K, G] := G'[K] - K'[G] = 0. \tag{3.3}$$

In case that we have a parametrization of the manifold given by a vector space, the bracket can be defined via the variational derivatives. Here $K'[G]$ denotes the *variational derivative* of K in direction G, i.e.

$$K'[G] = \frac{\partial}{\partial \varepsilon} K(u + \varepsilon G(u))|_{\varepsilon=0}. \tag{3.4}$$

In case of a manifold which is not a vector space one better uses Lie derivatives instead. To this we come later on. The quantities K and G are called *infinitesimal generators* of the corresponding groups R_K and R_G. Now, invariant manifolds, and especially the manifolds of N-soliton solutions, can be described as group invariant manifolds. Actually, this description we already used implicitly in case of the one-solitons. There we made use of the most simple symmetry group given by translation of the x-variable. Let us see this: The translation group

$$u(x) \to u(x+\sigma), \tag{3.5}$$

has as infinitesimal generator the vector field

$$K_0(u) = u_x, \tag{3.6}$$

furthermore an obvious symmetry group generator is given by the field $G(u)$ itself. Hence $cK_0 - G$ must again be a symmetry group generator, and our ansatz (1.2) was the same as the requirement that the solution has to be invariant under the group given by that generator. This is easily seen because (1.2) is the same as saying that u has to be an element out of the manifold

$$M = \{u \mid cK_0 - G = 0\}, \tag{3.7}$$

being invariant under that group. This suggests that other interesting solutions can be found by taking more symmetry group generators

$$K_2, K_3, K_4, \cdots, \tag{3.8}$$

into account. We can define special interesting solutions by requiring that the initial condition has to be out of the following invariant manifold

$$M_N=\left\{u \mid \text{there are } C_n \text{ such that } \sum_{n=1}^{N} C_n K_n(u)=0\right\}. \tag{3.9}$$

It turns out that these are the N-soliton solutions. The only problem is to find these generators. This is a difficult question since, in contrast to linear systems, nonlinear systems may not have any more symmetry group generators than the obvious ones.* Even the verification that a given quantity is indeed a symmetry group generator in general is not at all easy. For example, the necessary computation for showing that

$$K_3(u)=u_{xxxxxxx}+14u_{xxxxx}u+42u_x u_{xxxx}+70u_{xxx}(u_{xx}+u^2)+280u_{xx}u_x u+70u_x(u_x^2+u^3), \tag{3.10}$$

must be a symmetry group generator certainly takes some time and skill. In this situation the notion of *hereditary symmetry* offers welcome help. A linear operator Φ, mapping vector fields into vector fields, is said to be a hereditary symmetry if the following

$$\Phi^2[A, B]+[\Phi A, \Phi B] = \Phi[\Phi A, B]+\Phi[A, \Phi B], \tag{3.11}$$

holds for all vector fields A, B in the vector field Lie algebra $\mathscr{L}$. Having such a hereditary symmetry one easily finds

Theorem 1: *Let Φ be a hereditary symmetry and K be a special vector field such that the following*

$$\Phi[K, A]=[K, \Phi A], \tag{3.12}$$

holds for all vector fields A then the elements of the following sequence of fields

$$K, \Phi K, \Phi^2 K, \Phi^3 K, \cdots, \tag{3.13}$$

all commute.

Proof: Using (3.12) one checks that (3.11) implies

$$\Phi[\Phi K, A]=[\Phi K, \Phi A], \tag{3.14}$$

and so forth. Hence the statement is an immediate consequence of $[K, K]=0$ because

$$[\Phi^n K, \Phi^m]=\Phi^{n+m}[K, K]. \tag{3.15}$$

EXAMPLE: Let D^{-1} denote integration from $-\infty$ to x. By direct verification** one shows that the linear operator

$$\Phi(u)=D^2+2DuD^{-1}+2u, \tag{3.16}$$

has the property required in (3.11), hence must be hereditary. Furthermore the vector field $K(u)=u_x$ clearly has the property required in (3.12) because Φ does not depend explicitly on x. Hence the vector fields formed as in the theorem do all commute. One should observe that the second one of these vector fields is the right side of KdV equation (1.1). Thus we have found infinitely many symmetry group generators for the Korteweg de Vries equation.***

Although this elementary result seems to be rather smooth and useful, it somehow dropped out of the sky and leaves some questions unanswered. Apart from the fact that we want to understand the condition (3.11) a little bit better we have no idea how to find such a quantity for a given equation. This brings us back to our starting point where we claimed that all interesting quantities and structures for the KdV can be discovered by looking at Fig. 1 and the explicit form of the one-solitons.

In order to analyze the situation let us look at eq. (2.3) or (2.6). For this equation a suitable sequence of symmetry generators is easily found. The flow itself is generated by the vector field

* In case of a linear system $v_t=Av$ all operators B commuting with A are defining symmetry generators Bv. Thus finding the spectral resolution of A is, in most cases, the same as finding all symmetry group generators.

** Quite often the property of being a hereditary symmetry is confused with the property of being a recursion operator (see ref. 27) for some K. It should be noted that the requirements for being a hereditary symmetry are much stronger than that for being a recursion operator since hereditariness implies abelian structure for a sequence of vector fields whereas the recursion property only implies commutativity of a sequence with one of its members.

*** Implicitly the operator given in (3.16) already can be found in the fundamental paper[21] of Peter Lax where this operator got its name of "Lenard operator" from.

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix}, \quad (3.17)$$

which certainly commutes with all fields of the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1^m \\ \vdots \\ c_N^m \end{pmatrix}. \quad (3.18)$$

These fields are generated in a recursive way out of the starting field (3.17) by application of the operator

$$\Psi = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad (3.19)$$

having the matrix Λ (2.7) as entries. One should note that this operator is the product of the two different operators given by the two different hamiltonian formulations of the equation. This is no coincidence but a general consequence of such a bi-hamiltonian formulation as can be concluded from the fundamental contributions of Emmy Noether (see the textbook[28]) or the original paper[25]). But since we do not need this in the following we skip these aspects. Anyway, we have seen that the symmetry group generators for (2.6) are generated in a recursive way by application of the linear operator Ψ.

For our N-soliton solutions this viewpoint already shows that these were indeed generated by symmetry group generators as described in (3.9), because clearly the solutions of (2.6) can be considered as elements of the invariant submanifold

$$M_N = \left\{ v \mid \text{there are } c_N \text{ such that } \prod_{n=1}^{N} (\Psi - c_n) v = 0 \right\}. \quad (3.20)$$

Expansion of the product into a sum yields exactly the condition (3.9) if the quantities are pulled back by the diffeomorphism depicted in Fig. 2.

Another consequence is that, in case we have a nonlinear equation which is suspected to be linearizable by some abstract diffeomorphism, we can expect that suitable symmetry group generators are created by application of the pull back of a linear operator. In case of the KdV this must be the pull back of the operator Ψ given above. On first view this does not seem to be a promising observation since we are far away from being able to find the explicit form of the diffeomorphism depicted in Fig. 2. But wrong, in order to find what we have to look for we only have to express the essential properties of the operator Ψ in some algebraic way such that this property then can be pulled back. Since the diffeomorphism T given Fig. 2 is far from being linear we have to be a little bit careful in that respect. To find suitable conditions we first have to express the essential properties of Ψ in some differential geometric invariant way such that it is preserved under diffeomorphisms. The right way to do this is to work with *Lie derivatives*. For those who have forgotten their differential geometry this notion is easily explained:

If one has a scalar field, say $p(u)$, and a vector field $K(u)$ on some manifold then certainly the gradient of p in direction of K is invariant against reparametrization of the manifold. So we call this the Lie derivative. However, for vector fields the derivative of the field in direction of another field is not invariant against such a reparametrization because second derivatives occur. So we have to combine directional derivatives in such a way that second derivatives cancel. The "suitable" combination is the vector field commutator, which is just the quantity given in (3.3) computed for an arbitrary parametrization of the manifold. Hence we define the Lie derivative L_K for a vector field G with respect to another vector field K by

$$L_K G = [K, G]. \quad (3.21)$$

Now having such an invariant notion for scalar and vector fields it is easy to expand that to all tensor fields by use of the product rule. For example, if $\Phi(u)$ is a (1,1)-tensor field, or a linear operator on the tangent space which depends on the manifold variable, then we define its Lie derivative by

$$L_K(\Phi G)=L_K(\Phi)G+\Phi L_K(G). \qquad (3.22)$$

We call any tensor field invariant against K if its Lie derivative with respect to K vanishes. One should observe that condition (3.12) is exactly the requirement that Φ is invariant with respect to K. Now take as manifold a linear space E, then we may identify its tangent space at each point with the space itself and a (1,1)-tensor field is just a family of linear operators $\Psi(v)$ depending on $v\in E$. Taking any parametrization we can express the Lie derivative of that field as

$$L_K(\Psi)=\Psi K'-K'\Psi+\Psi'[K], \qquad (3.23)$$

where K is an arbitrary vector field, and where Ψ' and K' denote the variational derivatives computed in that particular parametrization.

Now let us return to the special operator Ψ as defined in (3.19). The essential property obviously is that Ψ commutes with its variational derivative in the following sense

$$\Psi\Psi'[B]=\Psi'[\Psi B], \qquad (3.24)$$

i.e. the variational derivative is diagonal and constant. Hence application of Ψ to equation (3.23) yields on the right side the same as the Lie derivative of Ψ in direction of the field ΨK. This is a simple consequence of commutativity of Ψ and Ψ'. Thus:

Observation 1: *The (1,1)-tensor field Ψ given by (3.19) fulfills*

$$\Psi L_K(\Psi)=L_{\Psi K}(\Psi), \qquad (3.25)$$

for all vector fields K.

One should observe that given a parametrization then property (3.25) is equivalent to the following symmetry condition

$$\Psi\Psi'[A]K-\Psi'[\Psi A]K=\Psi\Psi'[K]A-\Psi'[\Psi K]A, \qquad (3.26)$$

for arbitrary vector fields A and K (see ref. 9). By observation 1 property (3.25) is now formulated in the language of differential geometry and can therefore be pulled back to the left side of Fig. 2. Thus a good property for Φ to look for seems to be (see ref. 9)

$$\Phi L_K(\Phi)=L_{\Phi K}(\Phi). \qquad (3.27)$$

But, surprise, writing down this property in terms of vector field commutators exactly yields condition (3.11). Hence (3.25), or equivalently (3.26), is just another equivalent condition* for hereditariness, which turns out to be the invariant formulation for the property that an operator has to have a constant diagonal variational derivative.

The considerations above gave us the properties we have to look for but they gave us no indication how to find these quantities for concrete cases. This problem will be addressed in the next section. But we do not want to conclude this section without giving an honest assessment about the present computational value of the hereditary property for complex cases.

For the KdV it can be checked in a few lines that the operator given in (3.16) has this property. But let us see how difficult it can be to check the validity of this property which has so smoothly been expressed in (3.27).

EXAMPLE (see ref. 6): Consider the so called Kawamoto equation taken out of an issue of the Journal of The Physical Society of Japan[18]

$$\varrho_t=10\varrho^4\varrho_{xx}\varrho_{xxx}+5\varrho^4\varrho_x\varrho_{xxxx}+\varrho^5\varrho_{xxxxx}. \qquad (3.28)$$

* From the viewpoint of mathematical esthetics this is the way how hereditariness should have been discovered. Of course it was discovered in a much different way, or ways, as one should say since this property, or similar conditions, came up from different directions at almost the same time (refs. 9, 17, 22). My introduction of this property was based on an analysis of the algebraic structure of the inverse scattering method, unfortunately the paper[9] I published does not give any indication how this property was found. The reason for that is that in 77 I tried to publish a paper where the analysis leading to the hereditary property was described. But since I got back the paper from the journal with the remarks that there was no new aspect whatsoever in that paper, I was so frustrated, that I decided to hide my thoughts in the subsequent papers. In this I was rather successful hence I did not have difficulties in getting that in print. The condition that the variational derivative of a hereditary operator has to commute strongly with the operator itself appears only some years later.

For this equation there is indeed a hereditary operator which generates a huge symmetry group. The explicit form of this operator is:

$$\Phi(\varrho)=\varrho^2 DJ(u)\Theta(u)D^{-1}\varrho^{-2}, \tag{3.29}$$

where the auxiliary variable u has the form

$$u=\varrho\varrho_{xx}-\frac{1}{2}(\varrho_x)^2, \tag{3.30}$$

and where the operators J and Θ are abbreviations for

$$\Theta(u)=\varrho D\varrho D\varrho u_x+3\varrho uu_x, \tag{3.31}$$

$$\begin{aligned}J(u)=&\varrho D\varrho D\varrho u_x+3(\varrho uu_x+\varrho Du^2)\\&+2[\varrho D\varrho DuD^{-1}u\varrho^{-1}+D^{-1}uDuD\varrho u]\\&+8[u^2D^{-1}u\varrho^{-1}+D^{-1}u^3\varrho^{-1}].\end{aligned} \tag{3.32}$$

The validity of (3.11) is most easily checked by computation in some chart. As seen in (3.26) this leads to the requirement that for arbitrary fields A, B the following expression has to be symmetric in A and B:

$$\Phi\Phi'[A]B-\Phi'[\Phi A]B. \tag{3.33}$$

Because of the D^{-1} dispersed in the expression for Φ it is unfortunately not at all obvious which terms in that expression cancel. This is because most terms cancel with others only after performing some tricky integrations by part. Since integration by part leads to a context sensitive language, it is anyway difficult enough to find an algorithm which checks the resulting expression for equality. So, to do that one first has to expand all derivatives by the product rule. Let us see to what that amounts. If in $\Theta(u)D^{-1}\varrho^{-2}$ all derivatives are performed we roughly obtain 30 terms since one D is standing in front of some 4th order term and another one in front of a 5th order term. Now performing the additional derivatives in J, which stand in front of 8th and 9th order terms, yields an additional factor 90. For obtaining finally Φ we have to perform one last derivative in front of some 10th order term. So right now we already have $30\times 90\times 11$ terms in Φ. But now comes the real increase in numbers: The variational derivative of a tenth order term has 10 times the size of that term. And since Φ enters quadratic in (3.32) we have to square the number of terms in Φ. So we can expect after expansion of our expression to have around

$$2\times 10\, times (30\times 90\times 11)^2=17 Billion, \tag{3.34}$$

terms in the expression which we have to check. Quite a lot and not a very simple problem!! But one should not be afraid, this operator is indeed hereditary.

§4. Phase Gauges and Bäcklund Transformations

We should observe that there is a simple *gauge invariance* for the solutions $v(t)$ of (2.6). Here we used the abbreviation

$$v(t)=\begin{pmatrix}c_1\\ \vdots\\ c_N\\ q_1\\ \vdots\\ q_N\end{pmatrix}. \tag{4.1}$$

The replacement

$$q_i\to q_i+2\lambda\varphi_i(c_1,\cdots,c_N), \tag{4.2}$$

transforms one solution v of (2.6) into another solution $\tilde{v}$. To write that explicitly, we have

$$v-\tilde{v}+\lambda\varphi=0. \tag{4.3}$$

This trivial observation is not uninteresting since on the nonlinear side of Fig. 2 the phase gauges correspond to the so called auto-Bäcklund transformations, which will provide the essential tool to compute the quantity Φ which we need so badly for our symmetry analysis. Of course, the only thing we have to do is to pull back the relation (4.3) from the right side of Fig. 2 to the left. This then yields a diffeomorphism, maybe in implicit form,

$$B(u,\tilde{u},\lambda)=0, \tag{4.4}$$

between different solutions of our nonlinear system. Doing that seems to be rather hopeless on first view. But it is not, because in reality it turns out to be a rather simple exercise in elementary calculus.

We first have to observe that the effect of (4.4) on the one-solitons only results in an x-translation because there we know that the phase shifts are nothing but x-translations, a fact which, due to the nonlinear nature of our system, is not true for higher number of solitons. Secondly we observe that the explicit form of (4.4), and (4.3) as well, is not depending on the speed of the respective solitons. Let us emphasize these two remarks:

Observation 2: *For the one-solitons*

$$s(x+ct)=\frac{c}{2}\cosh^{-2}\left\{\frac{\sqrt{c}}{2}(x-x_0+ct)\right\}, \tag{4.5}$$

there must be a family of diffeomorphisms

$$B(u, \tilde{u}, \lambda)=0, \tag{4.6}$$

between s and its suitable x-translation which is independent of the speed parameter c.

It will turn out that the important part in this observations the *independence with respect to c*, because it is not trivial at all that such an independence is possible at all. One has to keep in mind that the totality of translated solitons, with different speeds, is a two parameter family of functions and that in the observation we require a one-dimensional family of functions which are independent of one of the parameters but nevertheless preserve the fibers with respect to that parameter.* In fact the assumptions we made in this observation are sufficient in order to compute these diffeomorphisms. This will lead to the so called Bäcklund transformations. In order not to be misunderstood, we do not claim that whenever a solitary wave has a sech-square-profile then there is a Bäcklund transformation. What we claim is, that if all solitary waves have a sech-square-profile such that the speeds and amplitudes are related as in (4.5) then there is a Bäcklund transformation. Let us see how that is done in the KdV case:

EXAMPLE (see ref. 13): For the KdV one-solitons (1.3) we consider the following translations by $+\beta$ and $-\beta$, respectively

$$\frac{\sqrt{c}}{2}x \to \frac{\sqrt{c}}{2}x+\beta \text{ and } \frac{\sqrt{c}}{2}x \to \frac{\sqrt{c}}{2}x-\beta, \tag{4.7}$$

and denote the resulting solutions by s_β and $s_{-\beta}$. The negative translation has only been chosen in order to make the computation more symmetric. We decompose into odd and even parts

$$V_+=\frac{1}{2}(s_\beta+s_{-\beta}), \quad V_-=\frac{1}{2}(s_\beta-s_{-\beta}), \tag{4.8}$$

and, in order to abbreviate notation, we introduce

$$\xi=(x-x_0+ct) \text{ and } k=\frac{\sqrt{c}}{2}, \tag{4.9}$$

$$\begin{aligned} N(k\xi)&=\{\cosh^2(k\xi)\cosh^2(\beta)-\sinh^2(k\xi)\sinh^2(\beta)\}^{-1}\\ &=\{\cosh^2(k\xi)+\sinh^2(\beta)\}^{-1}. \end{aligned} \tag{4.10}$$

Now, consider explicitly s_β and apply the addition-theorem for the cosh-function. Splitting s_β up into odd and even parts we find

$$\begin{aligned} s_\beta&=2k^2\cosh^{-2}(k\xi+\beta) && (4.11)\\ &=2k^2\{\cosh(k\xi)\cosh(\beta)+\sinh(k\xi)\sinh(\beta)\}^{-2} && (4.12)\\ &=2k^2N(k\xi)^2\{\cosh(k\xi)\cosh(\beta)-\sinh(k\xi)\sinh(\beta)\}^2\\ &=V_++V_-, \end{aligned}$$

where now the odd part V_- and the even part V_+ are seen to be

$$V_+=2k^2N(k\xi)^2\{\cosh^2(k\xi)\cosh^2(\beta)+\sinh^2(k\xi)\sinh^2(\beta)\}, \tag{4.13}$$

$$V_-=-4k^2N(k\xi)^2\{\cosh(k\xi)\sinh(k\xi)\cosh(\beta)\sinh(\beta)\}. \tag{4.14}$$

* Actually this was the property, or rather its suitable mathematical formulation and extension to several dimensions, which I had in mind in footnote of p. 1474 when I spoke about "certain properties of special functions".

Considered as functions in the variable ξ, the even part V_+ is a polynomial of second order in the common denominator $N(k\xi)$. Furthermore V_- is, apart from multiplication with a constant, the derivative of $N(k\xi)$. Thus we obtain a relation between V_+ and V_-. Let us write this down explicitly. Simple computations give

$$V_+ + \gamma N + \delta N^2 = 0, \tag{4.15}$$

where

$$\gamma = -2k^2 \cosh(2\beta), \tag{4.16}$$

$$\delta = k^2 \sinh^2(2\beta), \tag{4.17}$$

and the derivative DN of N is

$$DN = (2k \cosh(\beta) \sinh(\beta))^{-1} V_-. \tag{4.18}$$

This gives

$$V_+ - \frac{2k\cosh(2\beta)}{\sinh(2\beta)} D^{-1}V_- + \{D^{-1}V_-\}^2 = 0. \tag{4.19}$$

Now, let β depend on k in such a way that

$$\lambda = -2k \coth(2\beta), \tag{4.20}$$

is independent of k. Then we have the following algebraic relation

$$B(s_\beta, s_{-\beta}, \lambda) := (s_\beta + s_{-\beta}) + \lambda D^{-1}(s_\beta - s_{-\beta}) + \frac{1}{2}\{D^{-1}(s_\beta - s_{-\beta})\}^2 = 0, \tag{4.21}$$

where the coefficients are independent of $k = (\sqrt{c})/2$. And by translation invariance we can now get rid of the negative translation

$$B(s, s_{-2\beta}, \lambda) = (s + s_{-2\beta}) + \lambda D^{-1}(s - s_{-2\beta}) + \frac{1}{2}\{D^{-1}(s - s_{-2\beta})\}^2 = 0. \tag{4.22}$$

For other completely integrable equations such a relation can be found in a similar fashion. Let us see what we can do now with this remarkable relation between an arbitrary soliton and its translation. We begin by considering this as an implicit function for general arguments

$$B(u, \bar{u}, \lambda) = u + \bar{u} + \lambda D^{-1}(u - \bar{u}) + \frac{1}{2}\{D^{-1}(u - \bar{u})\}^2 = 0, \tag{4.23}$$

where u and $\bar{u}$ are assumed to be on the manifold where our multisoliton solutions are taken from. That is the manifold $\mathscr{S}$ of C^∞-functions vanishing rapidly with all their derivatives at infinity. If u is fixed then the map

$$u \to \bar{u} = f(u, \lambda), \tag{4.24}$$

given implicitly by

$$B(u, \bar{u}, \lambda) = 0, \tag{4.25}$$

raelly is a function if the *implicit function theorem condition* is fulfilled. This condition requires that for $B(u, \bar{u}, \lambda)$ the kernel of the variational derivative B_u with respect to the variable u has to be trivial, i.e. for every w in the tangent space at u it is required that whenever

$$B_u[w] = 0 \textit{ and } B(u, \bar{u}, \lambda) = 0, \tag{4.26}$$

then

$$w = 0, \tag{4.27}$$

must hold. This guarantees that small changes of u cannot happen without changing $\bar{u}$. Those λ violating this condition for a certain u we call the **spectral points** of u. The other λ are said to be **non-spectral points.**

Let us see what the spectral points are in case of the one-soliton. Obviously (4.23) maps the soliton solution s onto $s_{-2\beta}$ where

$$2\beta=\operatorname{arccoth}\left(-\frac{\lambda}{\sqrt{c}}\right). \tag{4.28}$$

For reasons seen later we call this relation the **phase shift relation.** In particular s and all its translations are mapped onto the zero function for $\lambda=\pm\sqrt{c}$. Hence $\pm\sqrt{c}$ are spectral points for the one-soliton having speed c. All other λ's are non spectral because for $B(u, \bar{u}, \lambda)=0$ the operator B_u is a differential operator which is invertible on the vector space $\mathscr{S}$.

The set of spectral points is finite in general because (4.26) is an ordinary differential equation for w having only for certain λ's solutions vanishing rapidly at infinity.

After this detour we return to the special case where $u=s$ and $\bar{u}=s_{-2\beta}$ are one-solitons. Recall that the one-solitons were solutions of

$$cu_x=K(u)=u_{xxx}+6uu_x, \tag{4.29}$$

and

$$c\bar{u}_x=K(\bar{u})=\bar{u}_{xxx}+6\bar{u}\bar{u}_x. \tag{4.30}$$

Translation invariance of $B(u, \bar{u}, \lambda)=0$ yields for the variational derives the relation

$$Bu[u_x]+B_{\bar{u}}[\bar{u}_x]=0 \quad around \quad B(u, \bar{u}, \lambda)=0, \tag{4.31}$$

and insertion of (4.29) and (4.30) gives

$$B_u[K(u)]+B_{\bar{u}}[K(\bar{u})]=0 \quad around \quad B(u, \bar{u}, \lambda)=0. \tag{4.32}$$

Observe that for non spectral points λ, where $\bar{u}$ is locally uniquely given by $B(u, \bar{u}, \lambda)=0$ eq. (4.32) is a differential equation for u (or rather the integral of u). And if this equation is non-trivial then on the manifold $\mathscr{S}$ under consideration this equation has to have the same number of integration constants as eq. (4.29).*

However, there is an essential difference between (4.23) and (4.29), coming from the important fact that the relation $B(u, \bar{u}, \lambda)=0$ is independent of c:

The solutions for (4.29) are a one-parameter family (parameter given by translation) whereas the solutions for (4.32) are a two-parameter family (parameters given by translation and parameter c). Thus the system (4.32) has to many integration parameters or degrees of freedom. This can only be if the system is trivial, i.e. if

$$B_u[K(u)]+B_{\bar{u}}[K(\bar{u})]=0, \tag{4.33}$$

is identically fulfilled whenever $B(u, \bar{u}, \lambda)=0$ holds. And, since the non spectral points are dense,this must also hold for those λ which are spectral. Thus we have found for arbitrary arguments:

Observation 3: *Whenever $B(u, \bar{u}, \lambda)=0$ then*

$$B_u[K(u)]+B_{\bar{u}}[K(\bar{u})]=0 \quad around \quad B(u, \bar{u}, \lambda)=0. \tag{4.34}$$

So, whenever $\bar{u}$ is defined by $B(u, \bar{u}, \lambda)=0$ and u is a solution of the KdV (1.1) then $\bar{u}$ must be again a solution of the KdV. Such a relation sending solutions of a nonlinear equation again onto solutions is called an auto-Bäcklund transformation.

Of course, now knowing this crucial result we are able to prove (4.33).** directly. This indeed is an immediate consequence of a direct explicit computation leading to:

$$B_u[K(u)]+B_{\bar{u}}[K(\bar{u})]=(D^3+3(u+\bar{u})D)B(u, \bar{u}, \lambda). \tag{4.35}$$

* In both these equations one degree of freedom has been consumed by the requirement that the solutions have to vanish rapidly at infinity.

** In the literature this Bäcklund relation is mostly written in a different way, namely as $(u+\bar{u})+\lambda+1/2\{D^{-1}(u-\bar{u})\}^2=0$. However, there is a notable difference between this notation and our form since (4.23) is compatible with the boundary condition on the manifold $\mathscr{S}$, i.e. with the requirement that u and $\bar{u}$ vanish at infinity, whereas the usual form is not compatible with that requirement. This point will be essential in the next section where we turn our interest to the spectral points.

§5. The Spectral Problem Given by Phase Gauge

We did not abandon the problem to find the operator Φ which was the tool to generate the one-parameter symmetry groups. In this section it will turn out that this operator is given by the spectral points which we introduced in the last section for the auto-Bäcklund transformation.

Since $B(u, \bar{u}, \lambda)=0$ is an auto-Bäcklund transformation for (1.1) the property of being a spectral point is invariant against this flow. So we have arrived at some kind of spectral problem for which (1.1) constitutes an isospectral flow. However, this is a

NONLINEAR SPECTRAL PROBLEM: Given a solution u of (1.1), find those λ's such that there is some non zero vector field ω and some $\bar{u}$ on the manifold under consideration such that

$$B_u(u, \bar{u}, \lambda)[\omega]=0 \quad when \quad B(u, \bar{u}, \lambda)=0. \quad (5.1)$$

I do not know of any criteria which give reasonable answers to the question under what circumstances such a nonlinear spectral problem is equivalent to a linear one. However, in this and other cases this problem is easily linearized (see ref. 12) in a purely algorithmic way:

Variational derivative of (4.23) with respect to u yields the operator:

$$B_u=I+(D^{-1}(u-\bar{u}))D^{-1}+\lambda D^{-1}. \quad (5.2)$$

And the spectral problem (5.1) reads as follows

$$0=\omega+(D^{-1}(u-\bar{u}))D^{-1}\omega+\lambda D^{-1}\omega. \quad (5.3)$$

This is only formally linear since $\bar{u}$ and ω are not independent. Abbreviation $D^{-1}\omega=v$ allows to write

$$D^{-1}(u-\bar{u})=-\left(\frac{v_x}{v}+\lambda\right). \quad (5.4)$$

Writing $u+\bar{u}$ as $2u-(u-\bar{u})$ and then replacing all terms $u-\bar{u}$ in (4.23) by (5.4) we obtain

$$2u+\left(\frac{v_x}{v}+\lambda\right)_x+\frac{1}{2}\left(\frac{v_x}{v}+\lambda\right)^2-\lambda\left(\frac{v_x}{v}+\lambda\right)=0, \quad (5.5)$$

which certainly is a nonlinear eigenvalue equation. By multiplication with v^2 we obtain

$$2uv^2+v_{xx}v-\frac{1}{2}v_xv_x=\frac{1}{2}\lambda^2v^2. \quad (5.6)$$

If this problem can be linearized there must be operators $A(v)$ and $\Psi(u)$ such that $A(v)v=Cv^2$ and such that $A(v)\Psi(u)v$ is equal to the left side of (5.6). Comparison of suitable terms yields in an algorithmic way:

$$D^{-1}vD\{v_{xx}+2uv+2D^{-1}(uv_x)\}=\lambda^2D^{-1}vDv. \quad (5.7)$$

Hence $A(v)=D^{-1}vD$ and $\Psi(u)=D^2+2u+2D^{-1}uD$. Going back to $\omega=v_x$ we see that ω is a solution of (5.1) if and only if ω is an eigenvector of

$$\Phi(u)=D\Psi(u)D^{-1}=D^2+2u+2DuD^{-1}. \quad (5.8)$$

And if λ is the spectral point given by (5.1) then λ^2 is the corresponding eigen-value of $\Phi(u)$.

This clearly is the operator which we already introduced (3.16) in order to give an example for a hereditary operation. That this operator is the pull back of the operator introduced in (3.19) is quite obvious. Since the spectral points are those where a soliton is annihilated the operator Φ exactly has the same eigenvectors as the matrix given in (3.19) which also does the job of annihilating solitons. That the eigenvalues are the c_i follows from the interpretation of the spectral points given before. Hence, since (3.19) was hereditary, we know that Φ again must be hereditary, at least on the N-soliton solution manifold.

One should note that the operator Φ is isospectral under the flow (1.1) since its spectrum does not change with this flow. The second element in the corresponding Lax pair in this case has a very simple form, namely it is given by the variational derivative of the right hand side of the corresponding eq. (1.1). This can be seen directly from the Bäcklund transformation as demonstrated in ref. 13, but also can be regarded as a consequence of the hereditariness of Φ. Because of translation invariance and $G(u)=\Phi(u)u_x$ we obtain from (3.12)

$$\Phi[G, A]=[G, \Phi A], \quad (5.9)$$

for all vector fields A. Expressing that by variational derivatives yields

$$\Phi'[G]=G'\Phi-\Phi G'. \quad (5.10)$$

So, when u evolves according to (1.1) we have the Lax representation

$$\frac{d}{dt}\Phi(u)=\Delta\Phi-\Phi\Delta, \quad (5.11)$$

where

$$\Delta=G'=D^3+6Du. \quad (5.12)$$

It is an interesting question how this isospectral problem is related to the well known isospectral problem given by the Schrödinger operator. The answer to that will turn out in the next section in a rather natural context.

But before we leave this section we like to demonstrate that by now we already arrived at a point where our structural considerations yield some practical results which are of importance to the physical interpretation of corresponding phenomena.

Observe that we introduced in (2.1) two different phases q^+ and q^- at $\pm\infty$. For the one-solitons these quantities of course are the same, whereas for genuine multisolitons, due to the nonlinear interaction, these quantities are different and result in a phase shift

$$\Delta_i=q^+-q^-. \quad (5.13)$$

We want to compute this phase shift.

First we emphasize the meaning of the spectral points. Since asymptotically multisolitons decompose into single solitons we may apply the phase shift relation (4.28) to multisolitons as well. Consider a N-soliton solution u of the KdV. By use of (4.28) the spectral points were demonstrated to be the $\lambda_n=\pm\sqrt{c_n}$ which correspond to annihilations of these solitons with speed c_n by translating them by

$$2\beta=\infty=\operatorname{arcoth}(\pm 1), \quad (5.14)$$

out of finite sight. Thus it is shown

- that whenever u is some N-soliton solution then for spectral λ_n the $\bar{u}$ appearing in $B(u, \bar{u}, \lambda_n)=0$ is the $(N-1)$-soliton solution where the n-th soliton with speed $c_n=\lambda_n^2$ is missing.

For explanation of the phase shifts Δ_i consider a solution $u=u(x, t)$ of the KdV such that asymptotically for $t\to\pm\infty$ a soliton with speed c_1 emerges. Let $B(u, \bar{u}, \lambda)=0$ be a Bäcklund transformation between this solution and another solution $\bar{u}$. As we have seen, the effect of this Bäcklund transformation on the emerging soliton is the same as if it were a single soliton. Hence the corresponding soliton emerging out of $\bar{u}$ has, compared to u, undergone a x-translation of the amount

$$\frac{4\beta}{\sqrt{c_1}}=\frac{2}{\sqrt{c_1}}\operatorname{arcoth}\left(-\frac{\lambda}{\sqrt{c_1}}\right). \quad (5.15)$$

The factor $2/\sqrt{c_1}$ is due to (4.7) where it is seen that $s_{-2\beta}$ is obtained out of s by exactly that translation. Now consider another soliton with speed c_2 and choose the spectral point $\lambda=\sqrt{c_2}$, i.e. the Bäcklund transformation which annihilates this second soliton with speed c_2. Assume the case $c_2>c_1$ and asymptotics at $+\infty$. Then, since the second soliton is faster than the first one, the annihilation must be done by shifting this second soliton into plus infinity. Hence we have to choose the sign in the square root in such a way that the resulting expression becomes positive. The translation for the soliton with speed c_1 is then also positive and equal to $+2/\sqrt{c_1}\operatorname{arcoth}(\sqrt{c_2/c_1})$. At $t=-\infty$ we have to reverse the direction of the translation in order to propel the soliton with speed c_2 into $-\infty$. Thus the resulting x-translation when going from u to $\bar{u}$ is

$$+\frac{4}{\sqrt{c_1}}\operatorname{arcoth}(\sqrt{c_2/c_1}). \quad (5.16)$$

In case $c_2<c_1$ we only have to change the sign of these translations.

Now, take a pure multisoliton solution and annihilate successively all emerging solitons except the one with speed c_1. Then the resulting shifts have to be added and they are positive for those solitons annihilated with higher speed and negative for those with lower speed. Since out of that annihilations results a one-soliton, i.e. a solution with no phase shift at all, we just have to change the sign of these successive shifts when we want to compute the phase shift which the soliton with speed c_1 has undergone when interacting with other solitons.

Thus the phase shift of the soliton, emerging out of u with speed c_1, compared to the corresponding single soliton, must be

$$\Delta_i = \frac{4}{\sqrt{c_1}} \sum_i \varepsilon_i \operatorname{arcoth}(\sqrt{c_i/c_1}), \tag{5.17}$$

where

$$\varepsilon_i = -1 \ \textit{if } c_i > c_1 \textit{ and} +1 \textit{ otherwise.} \tag{5.18}$$

A result well known from the literature.[20]

§6. The Interacting Soliton

There is another aspect which catches the eye when looking at eq. (2.6). Namely that this system is purely a superposition of N single soliton systems and no coupling between different solitons is visible. Thus eq. (2.6) suggests that the different solitons may be regarded *individually*. Of course, comparison with that system also suggests how this decoupling has to be done in real life coordinates. Observe that the one-soliton on the right side of Fig. 2 can be picked out by restricting the attention to the corresponding eigenvector of the operator Λ. So we can easily carry over that decomposition by considering the corresponding eigenvectors of Φ on the left side of that figure. This transfer, which leads to a complete decoupling of waves, has been discussed in all detail (see ref. 14), so we conclude this section by briefly mentioning this aspect which shows that the spectral resolution with respect to the hereditary operator Φ provides the nonlinear superposition principle for the KdV.

Guided by this comparison we define s to be an *interacting soliton* in the field u if s_x is the eigenfunction of the hereditary operator $\Phi(u)$.

This definition is also suggested for other structural reasons:

- As we have seen the flow (1.1) always is an isospectral flow for the operator Φ. Hence an eigenvalue is present for all time t if it is present for one time t_0.
- If $\Phi(u)$ is reasonably localized, and if asymptotically there emerges a soliton, then asymptotically there must be a corresponding eigenvector. So, due to the fact that (1.1) is an isospectral flow for Φ, there is always an eigenvector of $\Phi(u)$ which corresponds to this soliton.
- The dynamics of eigenvectors of the recursion operator is uniquely determined by (5.11). In fact these eigenvectors have the same dynamical behavior as infinitesimal generators of one-parameter symmetry groups. So it has to be expected that this dynamics again leads to a completely integrable flow.

Let us make the last point a little bit more precise:

If $w(t_0)$ is an eigenvector of $\Phi(u(t_0))$ with eigenvalue c then by use of (5.11) we may choose $w(t)$ such that for all t we have

$$\Phi(u(t))w(t) = cw(t), \tag{6.1}$$

$$w_t = K'[w]. \tag{6.2}$$

Combining this with the definition of the soliton

$$s_x = w, \tag{6.3}$$

we find that equations (6.1) to (6.3) completely determine the dynamics of s. Since w is a solution of the linearization (perturbation equation) of (1.1), and because these linearizations inherit, as coupled systems, the structure from (1.1), we may expect that the evolution of s has the same structural properties as the evolution of u. Indeed this can be easily proved in all detail. We get the dynamics of self interaction, i.e. the dynamics of s, in the following way:

- Consider eqs. (6.1) and (6.3) as a Bäcklund transformation between u and s. Use this to express u by s and insert then $u = F(s)$ in the evolution eq. (1.1). Thus we obtain a nonlinear evolution equation only depending on s this is the evolution for the interacting soliton.
- Using the fact that Bäcklund transformations preserve structures (Hamiltonian formulation, hereditary operators, etc.) we then can transfer the structural properties from eq. (1.1) to the evolution equation for the interacting soliton. To do that explicitly, we only need to apply the transformation formulas for the pull back.

On first view there seem to be the some minor difficulties:

- The Bäcklund transformation is an eigenvector equation and solving eigenvector equations is a difficult task.

• The transformation formulas coming out of the pull back seem only valid for diffeomorphisms between u and s. But certainly (6.1) not even defines an honest map from u to w since, obviously, the implicit function theorem for

$$B(u, w)=(\Phi(u)-c)w=0, \tag{6.4}$$

does not hold because w itself lies in the kernel of the variational derivative B_w, i.e. eigenvectors are only determined up to a scalar factor.

Both these apparent difficulties are easily discarded for the following reasons:

• Of course, eigenvectors are difficult to find. But given an eigenvector then going the other way, namely finding the potential, often is extremely simple. And that is what only is required in our case.

• Although the implicit function theorem is violated we nevertheless can apply all the relevant transformation formulas. This because we know that the kernel of B_w consists of the function w and this function is a symmetry group generator of the equation under consideration. Hence, for the equations of the interacting soliton, we can work in the algebra modulo an additional and obvious symmetry (see ref. 14 for details).

In order to ilustrate this procedure we carry out the necessary computations in case of the KdV:

$$u_t=u_{xxx}+6uu_x \tag{6.5}$$

here eqs. (6.1) to (6.3) have the form

$$cs_x=s_{xxx}+4us_x+2u_x s, \tag{6.6}$$

$$s_t=s_{xxx}+6us_x, \tag{6.7}$$

where already w was replaced by s_x.

Now, solving (6.6) for u in terms of s we find

$$u=\frac{c}{4}-\frac{s_{xx}}{2s}+\frac{s_x^2}{4s^2}+constant\,\frac{1}{s^4}. \tag{6.8}$$

Inserting the boundary condition at infinity we find the integration constant to be zero. Hence we finally have

$$u=\frac{c}{4}-\frac{s_{xx}}{2s}+\frac{s_x^2}{4s^2}. \tag{6.9}$$

Insertion of this into (6.7) yields

$$s^2 s_t=s^2 s_{xxx}-3ss_x s_{xx}+\frac{3}{2}s_x^3+\frac{3}{2}cs^2 s_x. \tag{6.10}$$

This describes the evolution of interacting solitons for the KdV (no matter how many other solitons are present).

The same simple procedure can be applied in order to obtain the evolution of interacting solitons for other evolution equations.

At the end let us look what such an interacting soliton looks like in the two-soliton case. Here we have the spectral decomposition

$$u_x=w_1+w_2, \tag{6.11}$$

where

$$\Phi(u)w_i=c_i w_i, \quad i=1,2 \tag{6.12}$$

and where the c_i are the asymptotic speeds of these solitons. This is the same as

$$(\Phi(u)-c_1)(\Phi(u)-c_2)u_x=0, \tag{6.13}$$

and we can apply the obvious identity

$$1=\frac{1}{c_2-c_1}(\Phi(u)-c_1)+\frac{1}{c_1-c_2}(\Phi(u)-c_2), \tag{6.14}$$

in order to obtain

$$u_x=\frac{1}{c_2-c_1}(\Phi(u)-c_1)u_x+\frac{1}{c_1-c_2}(\Phi(u)-c_2)u_x, \tag{6.15}$$

or

$$u_x=\frac{1}{c_2-c_1}(G(u)-c_1u_x)+\frac{1}{c_1-c_2}(G(u)-c_2u_x), \tag{6.16}$$

where G is the right side of the KdV (1.1). Because of (6.3) the operator $(\Phi-c_2)$ cancels the first term of the decomposition (6.16) and $(\Phi-c_1)$ does the same for the second term, therefore we find

$$w_1=\frac{1}{c_1-c_2}(G(u)-c_2u_x), \tag{6.17}$$

and

$$w_2=\frac{1}{c_2-c_1}(G(u)-c_1u_x). \tag{6.18}$$

This shows that

$$s_1=\frac{1}{c_1-c_2}D^{-1}(G(u)-c_2u_x), \tag{6.19}$$

and

$$s_2=\frac{1}{c_2-c_1}D^{-1}(G(u)-c_1u_x), \tag{6.20}$$

are solutions of the nonlinear eq. (6.10) for interacting solitons.

In case, of multisolitons of higher order the same analysis goes through. Only the $G(u)$ occurring in (6.17) and (6.18) then have to be replaced by suitable sums over higher order symmetries.

For the two-soliton of the KdV, which was plotted in Fig. 1, the corresponding interacting soliton with larger speed is plotted in Fig. 3. This method of decomposing the field variable with respect to the eigenvectors of the hereditary recursion operator can again be applied to the completely integrable system (6.10) this then leads to the introduction of virtual particles which only pop up during the interaction (see ref. 14).

An additional aspect is discovered by looking at (6.9). This is a nonlinear relation between u and s. One might try to linearize that by introducing a nonlinear reparametrization on the s manifold. Insert

$$s=h(\omega), \tag{6.21}$$

to obtain out of (6.9)

$$u=\frac{c}{4}-\frac{1}{2}\frac{\omega_{xx}}{h}h'+\frac{1}{4}\left(\frac{(h')^2}{h^2}-2\frac{h''}{h}\right)\omega_x^2. \tag{6.22}$$

In order that this can become a linear relation we have to require that the coefficient in front of ω_{x^2} vanishes. This yields

$$h(\omega)=\omega^2=s, \tag{6.23}$$

and transforms (6.9) into

$$(D^2+u)\omega=\frac{c}{4}\omega, \tag{6.24}$$

which again must define an isospectral problem for (1.1). This is the well known result that the Schrödinger problem remains isospectral under the KdV, thus we have provided the link to the usual Lax pair.

§7. Action Angle Variables

We want to find the complete action angle representation for the KdV. The parametrization given by the asymptotic data (speeds c_i and phases q_i) yields such a representation, so we only have to look for a way to compute these parameters for a given field u. The idea how to do that is easily found by looking at the gradients of, say q_1 and c_1. If these quantities are mapped, via the symplectic structure, into the vector fields, they go over into

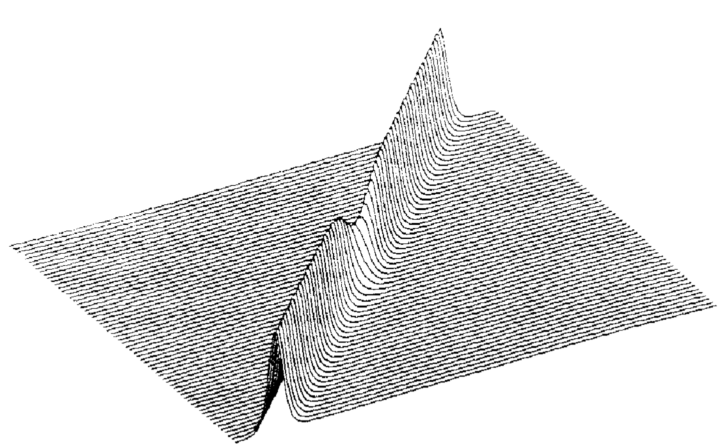

Fig. 3. Interacting soliton of the KdV.

$$\begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ +1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{7.1}$$

which are exactly the eigenvectors of the crucial operator

$$\Psi = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{7.2}$$

given in (3.19), and their corresponding eigenvalue is c_1. So all we have to do in order to obtain the action angle variables is to take suitable eigenvectors of Φ and to draw these back onto the cotangent bundle (by the Lie algebra homomorphism provided by the hamiltonian formulation). However there is a minor difficulty hidden in the word suitable since the quantities we are looking for have to be closed in order to admit, at least locally, potentials. We have to make sure that this requirement is met by chosing suitable integrating factors. For the action variables this usually does not pose any problem whereas for the angle variables finding these integrating factors is not so trivial at all. So we apply a little trick:

To find the required normalization for these eigenvectors we assume that we already have the vector fields which correspond to the gradients of the action variables c_i. Consider the time dependence of q_i given in (2.2). In terms of Poisson brackets this dependence can be written as

$$\{H, q_i\} = c_i, \tag{7.3}$$

where

$$H = \frac{1}{2}(c_1^2 + \cdots c_N^2), \tag{7.4}$$

is the scalar field which defines the flow (2.6) via its gradient. Other Poisson bracket relations which are easily seen from the hamiltonian structure of (2.6) are

$$\{q_k, q_j\} = 0, \ \{c_i, q_k\} = \delta_{ik}, \tag{7.5}$$

because of the hamiltonian structure we have a Lie algebra homomorphism from the Poisson brackets into the vector field Lie algebra. Hence the vector field W_i which represents the angle variable q_i must fulfill the following commutator relations

$$[G, W_i] = V_i, \tag{7.6}$$

and

$$[W_i, V_k] = [W_i, W_k] = 0, \tag{7.7}$$

where G is the field of the KdV-flow and the V_i are the fields corresponding to the c_i. The last sequence of brackets vanishes because of (7.5) and the fact that the gradients of the δ_{ik} are zero. These brackets determine the fields W_i uniquely up to addition of fields coming from action variables. Here we arrived at a formulation which easily can be pulled back via the unknown diffeomorphism in Fig. 2. The only additional ingredient we need is the hamiltonian formulation of the KdV in real life coordinates. This formulation is easily found. It is the well known representation

$$u_t = D \operatorname{grad} \int_{-\infty}^{+\infty} \left(\frac{1}{2} u u_{xx} + u^3 \right) dx, \tag{7.8}$$

where as duality between tangent and cotangent space we assumed

$$\langle \gamma(u), K(u) \rangle := \int_{-\infty}^{+\infty} \gamma(u(x)) K(u(x))\, dx, \tag{7.9}$$

and where the operator

$$D: \textit{cotangent space} \rightarrow \textit{tangent space} \tag{7.10}$$

induces the symplectic structure. Now, if for multisolitons we represent u_x as in (6.11) by a linear combination of eigenvectors V_i of Φ

$$u_x = \sum_{i=1}^{N} V_i, \tag{7.11}$$

then, because of the hamiltonian nature of the vector field u_x, it is guaranteed that the $D^{-1} V_i$ are closed, i.e. do have potentials. Hence they are the gradients of the action variables. So we found:

Observation 4: *Given the decomposition (7.11) (defining a multisoliton solution) then, if the second eigenvector W_i of $\Phi(u)$ are chosen such that*

$$[W_i, V_k]=[W_i, W_k]=0, \tag{7.12}$$

then the

$$D^{-1}W_i, \tag{7.13}$$

are the gradients of the corresponding angle variables.

Of course, for the application of this result an essential ingredient is still missing. Namely, that we show that we are really able to compute the second eigenvectors of Φ. For multisolitons the computation of the first eigenvector was simple, we just had to apply suitable polynomials of the operator Φ to the field u_x (as shown in (6.17) and (6.18)). But the eigenvalue problem for Φ yields a third order differential equation, hence finding any other eigenvectors explicitly by using the given one may not be possible at all. Fortunately that is not so. There are two different and simple ways to give a direct construction of these additional eigenvectors. One is given by use of partial derivatives of the field function u with respect to asymptotic data (see ref. 16). Here we present another method, a method which I chose because it leads in a natural way to some results of the next chapter.

Actually, we already know, more or less, that given one eigenvector V_i of Φ (eigenvalue c_i) then we can compute *explicitly* the other solutions of

$$\Phi\varphi=c_i\varphi. \tag{7.14}$$

This observation indeed is a direct consequence of the linearization between u and the interacting soliton which we gave at the end of §6. The content of that linearization was that when replacing the interacting soliton s by the square of the second eigenvector of the Schrödinger operator then eq. (6.9) is left invariant. Hence this replacement yields up to a derivative a second solution of (7.14). Unfortunately that is not the eigenvector we are looking for since this function clearly violates the boundary condition at infinity. But having the second solution of a third order problem then the remaining last solution can be found by variation of constants. So we are sure that we can determine the second eigenvector explicitly, and we see that we even can make a short-cut thus avoiding the "variation of constants" for the third order problem.

Let ω_1 be a solution of

$$(D^2+u)\omega_1=\frac{c_i}{4}\,\omega_1, \tag{7.15}$$

then by the Wronski determinant, or variation of constants, we easily find the second solution of that equation as

$$\omega_2=\omega_1 D^{-1}\omega_1^{-2}. \tag{7.16}$$

Thus, as we have seen by linearization of the interacting soliton, for arbitrary constants α and β the function

$$s=(\alpha\omega_1+\beta\omega_2)^2, \tag{7.17}$$

fulfills

$$(\Phi D)s=c_i s_x. \tag{7.18}$$

Since the linear superposition principle is valid for solutions of that equation obviously differentiation with respect to α or/and β again yields solutions. Using this we find the third solution for that equation as $\omega_1\omega_2$. Thus the three eigenvectors of Φ are

$$(\omega_1^2)_x,\ (\omega_2^2)_x,\ (\omega_1\omega_1)_x. \tag{7.19}$$

And simple analysis shows that when $(\omega_1^2)_x$ fulfills the required boundary conditions at infinity then $(\omega_1\omega_1)_x$ does the same.

§8. Singularity Analysis

We first give a very brief account of the usual Painlevé test introduced by Weiss *et al.* (see for example ref. 35). This test provides a high probability for complete integrability and was motivated by the Painlevé conjecture of Ablowitz, Segur and Ramani based on the well known results about Painlevé transcendents. There are numerous excellent surveys on that subject (see for example ref. 24 or 29). In the presentation of the subsequent results I follow closely the joint work with Sandra Carillo[15]).

We consider solutions of the evolution equation

$$u_t=G(u), \tag{8.1}$$

where $G(u)$ is a polynomial in u and its derivatives. We make the ansatz

$$u=\sum_{n=0}^{\infty}\psi^{-n}F_{(n)}(\psi_x, \psi_{xx}, \cdots), \tag{8.2}$$

in negative descending powers of a quantity ψ. We say that eq. (8.1) passes the expansion test if this ansatz is always compatible with a time evolution of ψ given by

$$\psi_t=\sigma(\psi), \tag{8.3}$$

where σ is only depending on ψ and its spatial derivatives and where it is assumed that in the Laurent expansion of σ with respect to its dependence on ψ only negative powers occur. These requirements of course are severe restrictions since in general an ansatz like (8.2) when introduced into (8.1) yields the equation

$$G\left(\sum_{n=0}^{\infty}\psi^{-n}F_{(n)}\right)=\sum_{n=0}^{\infty}\psi^{-n}\{-n\psi^{-1}F_{(n)}+F'_{(n)}\}\times\sigma(\psi), \tag{8.4}$$

where very many mixed derivatives occur. As before, in this formula the prime in F' stands for the variational derivative of F with respect to ψ. Formula (8.4) gives, by comparison of powers in ψ, rise to an algorithmic determination of the $F_{(n)}$ once the starting point $F_{(0)}$ has been chosen. There are two important cases for this algorithmic procedure.

- **Case 1** (*Painlevé test*):

We assume that $\sigma(\psi)$ is only depending on the spatial derivatives of ψ, i.e. no ψ without x-derivative occurs. This corresponds to saying that the t-derivatives affect the singularities given by the zeros of ψ in the same way as x-derivatives. This, of course, is the requirement one usually tacitly assumes in the Painlevé test. Hence this assumption singles out the so called Painlevé expansion. Looking at lowest order in ψ^{-1} in the expression (8.4) one discovers in this case as a consequence

$$F_{(0)_t}=G(F_{(0)}), \tag{8.5}$$

which proves that

$$\tilde{u}=F_{(0)}, \tag{8.6}$$

again has to be a solution of (8.1). Looking at the next order one finds the well known fact that

$$G'(\tilde{u})[F_{(1)}]=F_{(1)}(\psi)'[\sigma]=F_{(1)_t}, \tag{8.7}$$

which implies that $F_{(1)}$ is a symmetry generator for (8.1) around the manifold point $\tilde{u}$.
If one considers the expansion singled out by that case one obtains for the KdV

$$u=-2\frac{\phi_x^2}{\phi^2}+2\frac{\phi_{xx}}{\phi}+\left(-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}+\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}\right). \tag{8.8}$$

The term without negative powers in ϕ

$$\tilde{u}(\phi)=\left(-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}+\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x}\right), \tag{8.9}$$

is called the *constant level term*. There are a number of remarkable interrelations between the interacting soliton equation, the singularity equation and the Painlevé expansion. For example, using the substitution

$$s=\phi_x, \tag{8.10}$$

the constant level term can be rewritten as

$$\tilde{u}=-\frac{1}{2}\frac{s_{xx}}{s}+\frac{1}{4}\frac{s_x^2}{s^2}. \tag{8.11}$$

Formally this corresponds to the interacting soliton representation of the field variable u given in (6.9) (for eigenvalue 0). Another surprising fact is that

$$\tilde{u}\left(D^{-1}\left(\frac{\phi_x}{\phi^2}\right)\right)=u. \tag{8.12}$$

Which means that a replacement of ϕ in (8.9) leads to a recovery of the Painlevé series (8.8) from its constant level term. All these observations suggest that there is a close relation between the interacting soliton theory and the Painlevé analysis. An explanation of these and similar phenomena is provided by the discovery that usually the representation of u in terms of eigenvectors of the hereditary recursion operator yields an expansion of an algorithmic nature very similar to the Painlevé test. Let us summarize this as

- **Case 2** (*constant highest order case, soliton test*):

In order to distinguish the field variable in this case from the previous one we use s instead of ψ. If $u=\lambda$ (λ a constant) does satisfy eq. (8.1) then we put $\sigma(s)=D^{-1}G(u)D$ and $F_{(0)}=\lambda$ where we let vary the value of the constant λ. This is possible since any constant obviously satisfies the condition on the highest order term because the variational derivatives of $F_{(0)}$ disappears. In this case it does not matter at all whether or not negative powers of ψ appear explicitly in (8.3).

The reason for calling that the soliton test is that the representation of u in terms of the interacting soliton, as given in (6.9) fulfills the requirements of that case.

Before we now go into a comparison of these two cases we need the fact that the KdV· has negative scaling $\alpha=-2$. Here scaling α means that the replacement

$$x \to mx,\ u \to m^{\alpha}u,\ t \to m^{\beta}t\ (some\ \beta), \tag{8.13}$$

leaves the equation invariant. As a consequence of negative scaling we obtain that an expansion of u in terms of powers of the interacting soliton s must be of the form

$$u=c+\frac{1}{s}\Gamma_{(1)}+\frac{1}{s^2}\Gamma_{(2)}\cdots,\quad c=constant \tag{8.14}$$

with $\Gamma_{(i)}$ polynomials in s_x, s_{xx} etc. This is the form which was required in case 2 of the expansion test. Since the members of the expansion test are uniquely and algorithmically determined we know that this expansion is the unique representation for case 2.

Now, we compare the Painlevé series for (8.1) with this soliton expansion. We look at the constant level term $\tilde{u}=F_{(0)}(\phi)$ which is a homogeneous rational function in the derivatives of ϕ. We consider an expansion in negative powers of the lowest derivative $\phi_{(N)}$ in the denominator of that expression

$$\tilde{u}=\sum_{n=1}^{\infty}\phi_{(N)}^{-n}G^{(n)}(\phi_{(N+1)},\phi_{(N+2)},\cdots). \tag{8.15}$$

The zero order term of that has to disappear because it only could be a constant (independent of ϕ) which would be in contradiction to negative scaling degree. Obviously, that expansion again is a case 2-expansion for $\tilde{u}$ (instead of u), since only ϕ_N has to be renamed by s. Since the case 2 expansions were unique we know that (8.15) must coincide with the special expansion for $\tilde{u}$. It remains to compute the N (order of derivative). For this we look at the next order term in the Painlevé series. This term is of the form

$$F_{(1)}(\phi_x,\phi_{xx},\cdots)/\phi. \tag{8.16}$$

Since any rescaling $\phi\to a\phi$ leaves the whole Painlevé series invariant we know that $F_{(1)}$ is of first order, and because of the scaling degree it has to bear $-\alpha$ derivatives. Hence we obtain

$$F_{(1)}=C\phi_{(-\alpha)}, \tag{8.17}$$

where, without loss of generality, $C=1$ can be chosen. Since we know that $F_{(1)}$ is a symmetry around $\tilde{u}$ and that the eigenvector $\tilde{s}_x$ of $\Phi(\tilde{u})$ has the same dynamic as a symmetry generator around $\tilde{u}$ we find the crucial identity

$$\phi_{(-\alpha)}=\tilde{s}_x. \tag{8.18}$$

Hence we found the N to be

$$N=-\alpha-1. \tag{8.19}$$

Let us resume these results in a somewhat more systematic way. Again we consider

$$u_t=G(u), \tag{8.20}$$

with negative scaling $\alpha<0$. We write the Painlevé series and the special expansion series for u respectively as

$$u=PE(\phi)\quad (Painleve\ expansion), \tag{8.21}$$

$$u=SE(s)\ (special\ expansion). \tag{8.22}$$

If $\tilde{u}$ denotes the constant level term in the Painlevé expansion for u then we also consider the expansions for this solution $\tilde{u}$ of (8.1).

$$\tilde{u}=PE(\tilde{\phi}), \tag{8.23}$$

$$\tilde{u}=SE(\tilde{s}). \tag{8.24}$$

The map going from u, s, ϕ to $\tilde{u}$, $\tilde{s}$, $\tilde{\phi}$ we denote by $SoSi(\ \ )$, i.e.

$$\tilde{u}=SoSi(u), \tag{8.25}$$

$$\tilde{s}=SoSi(s), \tag{8.26}$$

$$\tilde{\phi}=SoSi(\phi). \tag{8.27}$$

The fundamental result is the following **DUALITY:**

$$\tilde{s}_x=\phi_{(-\alpha)},\ s_x=\tilde{\phi}_{(-\alpha)}. \tag{8.28}$$

So *SoSi* changes, up to derivatives solitons in singularities and vice versa. This explains the name, since *SoSi* is meant to be an abbreviation for *Soliton-Singularity transform.*

Interchanging the role between u and $\tilde{u}$ in the respective series shows that *SoSi* is an involutive map i.e.

$$SoSi^2=I\quad (identity). \tag{8.29}$$

Now we want to compute *SoSi* explicitly, at least the effect it has on s and ϕ. Then its effect on u can be computed by considering the series obtained from one of the tests. In order to compute $SoSi(\ \ )$ we make the ansatz

$$s=\frac{v}{\phi^k}, \tag{8.30}$$

where v is of the form

$$v=v_0(\phi_x, \phi_{xx}, \cdots)+\phi v_1(\phi_x, \phi_{xx}, \cdots)+\phi^2\cdots. \tag{8.31}$$

Insertion in $u=SE(s)$ yields

$$u=SE(v/\phi^k). \tag{8.32}$$

Since the special expansion series was homogeneous in s we see that only those parts of the derivatives of s contribute to powers of zero order in ϕ where the ϕ's are left uneffected by the derivatives. Since for those terms the powers of ϕ cancel, we easily obtain the constant level term $\tilde{u}$ as

$$\tilde{u}=SE(v). \tag{8.33}$$

Comparison with the special expansion for $\tilde{u}$ leads to

$$v=\beta\tilde{s},\ \beta\ constant, \tag{8.34}$$

or

$$s=SoSi(\tilde{s})=\beta\frac{\tilde{s}}{\phi^k}=\beta\frac{\tilde{s}}{(D^{\alpha+1}(\tilde{s}))^k}. \tag{8.35}$$

Since $SoSi(\cdot)$ has to be reciprocal we find that only the values $\alpha=-2$ and $\alpha=-1$ are possible, because for other values this transformation never can be an involutive map. In both cases we find for ϕ that

$$\tilde{\phi}=SoSi(\phi)=1/\phi. \tag{8.36}$$

Hence the substitution $\phi\to 1/\phi$ is a generic invariance for the equation of the singularity manifold. Since only derivatives occur in this equation another invariance must be

$$\phi\to\phi+c,\ c\ constant. \tag{8.37}$$

These two invariances combined yield the Möbius group

$$\phi\to\frac{a\phi+b}{c\phi+d},\quad ad-bc\neq 0. \tag{8.38}$$

A consequence of this is that the full Painlevé series is always obtained from its constant level term. This is seen from

$$u=SE(s)=SE(SoSI(\tilde{s}))=SE(SoSi(D^{-1-\alpha}\phi))$$
$$=PE(\phi). \tag{8.39}$$

So substitution of

$$s=SoSi(D^{-1-\alpha}\phi), \tag{8.40}$$

into the special expansion $u=SE(s)$ yields the Painlevé expansion $u=PE(\phi)$ for u.

We like to point out that we already know other obvious invariances for the singularity equation (and the interacting soliton equation as well). These are related to our explicit construction of the angle variables of the KdV via second eigenvectors of Φ. What we used there, and what we proved, was that the interacting soliton equation is invariant under the transformation

$$\sqrt{s_2}=a\sqrt{s}+\sqrt{s}(D^{-1}(1/s)). \tag{8.41}$$

This yields the invariance

$$\phi\to a^2\phi+2aD^{-1}(\phi_x D^{-1}(1/\phi_x))$$
$$+D^{-1}(\phi_x(D^{-1}(1/\phi_x))^2), \tag{8.42}$$

for the singularity equation. In other words: We have shown that the transformation from the action to the angle variables for the KdV is (up to trivial transformations) given by a natural invariance of the singularity manifold of this equation.

§9. Concluding Remarks

Of course, very many methods and problems have been left out in this brief survey. We just mention some of them:

- All achievements given by the beautiful Inverse Scattering method have been completely neglected in this paper. The same holds true for all results concerning the direct linearization transform. The obvious reason for omitting these aspects is that there are much better sources than I could provide. The reader interested in these aspects should consult the work of Ablowitz and Fokas, and of course, if he does not know it in detail already, he should read the classic.[2)] Some survey papers are appended in the reference list. For further work on the direct linearization transform one should also consult the work of the Dutch school around Capel (see for example ref. 5).

• There is an important extension of hereditary operators to multi-dimensions. This allows similar ideas in order to investigate equations like the Kadomtsev-Petviashvili equation and the Benjamin Ono equation (see for example ref. 33). With respect to multidimensions one also should observe that Inverse Scattering has contributed to that subject a wealth of remarkable results throughout the last years. Surveys can be found in the reference list.

• The angle variables we only characterized in terms of the spectral resolution of the operator Φ. It turns out that indeed for the N-soliton case a description in terms of local densities is possible, but that this description becomes more difficult if continuous parts of the recursion operator become important. Altogether the analysis is involved and tedious (see ref. 16 for details, see also ref. 34). The essential tool for carrying out this analysis are the so called mastersymmetries (see ref. 11).

• Other interesting aspects are discovered if one looks further into the Painlevé analysis and its relation to the interacting soliton equation. For the singularity equation many results are known how this equation is connected to other well known equations via reciprocal transformations (for example to the Harry Dym equation (refs. 30 and 31). These connections can be used to derive direct transformations between interacting solitons and solutions of other equations. These aspects have been systematically investigated in joint work with Sandra Carillo (see ref. 15). But we are far away from understanding all details of these aspects.

• We have left out completely all computational and algorithmic approaches to complete integrability. For example the ugly size of the hereditary operator of the Kawamoto equation suggests that there must be strategies to perform such computations. Since these strategies require formula manipulation by computer and parallel processing, their explanation goes beyond the aims of this paper. Another question is how to find equations like the KdV in the first place. We are developing right now in Paderborn algorithms which determine whether a given equation is completely integrable. These algorithms are already implemented for certain classes of nonlinear equations, however they are far from being perfect. Detailed reports about this work will be given soon. For other algorithmic approaches compare for example ref. 23.

This paper is dedicated to Professor Heinz König on the occasion of his 60th birthday.

References

1) V. I. Arnold: *Mathematical Methods of classical Mechanics, Graduate Texts in Mathematics* (Springer-Verlag, New York-Heidelberg-Berlin, 1978) Vol. 60.
2) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: Stud. Appl. Math. **53** (1974) 249.
3) M. J. Ablowitz and H. Segur: Solitons and the Inverse Scattering Transform, Siam studies in applied Mathematics, SIAM, Philadelphia, (1981).
4) M. J. Ablowitz: *Topics in Soliton Theory and Exactly solvable Nonlinear equations* (World Scientific Publishers, Singapore, 1987) p. 20.
5) H. W. Capel: *Topics in Soliton Theory and Exactly solvable Nonlinear equations* (World Scientific Publishers, Singapore, 1987) p. 125.
6) S. Carillo and B. Fuchssteiner: J. Math. Phys., **30** (1989) 1606.
7) A. S. Fokas: Stud. Appl. Math. **77** (1987) 253.
8) A. S. Fokas: *Topics in Soliton Theory and Exactly solvable Nonlinear equations* (World Scientific Publishers, Singapore, 1987) p. 45.
9) B. Fuchssteiner: Application of Hereditary Symmetries to Nonlinear Evolution equations, Nonlinear Analysis TMA, 3, 849–862 (1979).
10) B. Fuchssteiner: Prog. Theor. Phys. **65** (1981) 861.
11) B. Fuchssteiner: Prog. Theor. Phys. **70** (1983) 1508.
12) B. Fuchssteiner and R. N. Aiyer: J. Phys. A: Math. Gen. **20** (1987) 375.
13) B. Fuchssteiner: *Topics in Soliton Theory and Exactly solvable Nonlinear equations* (ed. M. Ablowitz, B. Fuchssteiner and M. Kruskal (World Scientific Publ. Singapore, 1987) p. 230.
14) B. Fuchssteiner: Prog. Theor. Phys. **78** (1987) 1022.
15) B. Fuchssteiner and S. Carillo: Physica A **152** (1989) 467.
16) B. Fuchssteiner and Gudrun Oevel: Rev. Math. Phys. **1** (1990) 415.
17) I. M. Gel'fand and I. Y. Dorfman: Funct. Analys. i. Eqs. Priloz. **13** (1979) 248.
18) S. Kawamoto: J. Phys. Soc. Jpn. **54** (1985) 2055.
19) J. Korteweg and G. De Vries: Philos. Mag. Ser. 5 **39** (1895) 422.
20) G. L. Lamb: *Elements of Soliton theory, Pure and Applied Mathematics series* (Wiley, New York-Chichester-Brisbane-Toronto, 1980).
21) P. D. Lax: Commun. Pure Appl. Math. **21** (1968) 467.

22) F. Magri: *Lecture Notes in Physics* (Springer, Berlin, 1980) Vol. 120, p. 233.
23) A. V. Mikhailov, A. B. Shabat and R. I. Yamilov: Russ. Math. surveys **42** (1987) 1.
24) A. C. Newell, T. Ratiu, M. Tabor and Z. Yunbo: Soliton Mathematics, Séminaire de Mathématiques Supérieures, Université de Montreal, Vol. 103, 1986.
25) A. E. Noether: Invariante Variationsprobleme, Nachr.v.d. Gesellschaft d.Wiss. zu Göttingen, 235–257, 1918.
26) Walter Oevel: Rekursionsmechanismen für Symmetrien und Erhaltungssätze in integrablen Systemen, Dissertation, Universität Paderborn (1984).
27) P. J. Olver: J. Math. Phys. **18** (1977) 1212.
28) P. J. Olver: *Applications of Lie Groups to Differential Equations*, Springer Verlag, Graduate texts in Mathematics (New York-Berlin-Heidelberg-Tokyo, 1986) Vol. 107.
29) A. Ramani, B. Grammaticos and T. Bountis: The Painlevé property and singularity analysis of integrable and non-integrable systems, preprint, 1988.
30) C. Rogers and M. C. Nucci: Phys. Scr. **33** (1986) 289.
31) C. Rogers: Phys. Lett. **120** (1987) 15.
32) A. J. Scott Russell: *Report on Waves, Report of the 14th Meeting of the British Association for the Advancement of Science* (John Murray, London, 1844) p. 311.
33) P. M. Santini and A. S. Fokas: Commun. Math. Phys. **115** (1988) 375.
34) H. M. M. TenEickelder: Indag. Math. **89** (1986) 386.
35) J. M. Weiss, M. Tabor and G. Carnevale: J. Math. Phys. **24** (1983) 522.