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The Leading Singularity of the Scattering Kernel for a Transparent Obstacle

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0. Introduction

Let $X_- \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $Y = \partial X_-$. Let n denote the unit normal at Y pointing to the exterior, $X_+ = \mathbb{R}^3 \setminus \bar{X}_-$. Write $X = X_+ \cap X_-$. Let α be a smooth positive function in $\bar{X}_-$, and let $\alpha = 1$ in X_+. We consider the transmission problem,

$$\begin{aligned} \Box u &= 0 \quad \text{in} \quad X \times \mathbb{R}, \\ u_+ - u_- &= 0 \quad \text{on} \quad Y \times \mathbb{R}, \\ (D_n u)_+ - (D_n u)_- &= 0 \quad \text{on} \quad Y \times \mathbb{R}, \end{aligned} \tag{0.1}$$

associated with the d'Alembertian

$$\Box = \alpha(x)(\partial/\partial t)^2 - \Delta_x .$$

Here we have set $D_n = -i\partial/\partial n$. The subscripts $+$ and $-$ indicate that boundary values are taken from X_+ and X_-, respectively.

A solution to (0.1) can be thought of as a wave propagating through space in the presence of a transparent obstacle X_- with variable index of refraction α.

The transmission problem (0.1) yields a group of unitary operators with coercive generator,

$$\begin{pmatrix} 0 & I \\ \alpha^{-1}\Delta & 0 \end{pmatrix}.$$

So, following the theory of Lax and Phillips [LP], we can associate a scattering operator with (0.1). The scattering operator is a unitary operator on $L^2(\mathbb{R} \times S^2)$ mapping the incoming translation representation (of the initial state) of a solution to (0.1), k^{in}, to its outgoing translation representation, k^{out},

$$k^{\text{out}}(T, \vartheta) = \iint_{\mathbb{R} \times S^2} S(T - T', \vartheta, \omega) k^{\text{in}}(T', \omega) dT' d_\omega S .$$

Majda and Taylor [MT] gave a time-dependent representation formula for the (Schwartz) kernel of the scattering operator,

$$S(T-T',\vartheta,\omega)=\delta(T-T')\delta(\vartheta-\omega)$$
$$-\iint_{Y\times\mathbb{R}}\langle\vartheta,n\rangle\delta''(T+t-\langle\vartheta,x\rangle)u(x,t+T')d_xSdt$$
$$-\iint_{Y\times\mathbb{R}}\delta'(T+t-\langle\vartheta,x\rangle)(\partial u/\partial n)(x,t+T')d_xSdt\,. \tag{0.2}$$

Here $u=u(\cdot\,;\omega)$ is the unique solution to (0.1) satisfying the initial condition

$$u=u^{\mathrm{in}}\quad\text{if}\quad t\ll 0,$$

where

$$u^{\mathrm{in}}(x,t)=\begin{cases}\delta(\langle\omega,x\rangle-t) & \text{if}\quad x\in X_+\\ 0 & \text{if}\quad x\in X_-\,.\end{cases}$$

We use angular brackets to denote the standard scalar product.

Fix directions ϑ, $\omega\in S^2$, $\vartheta\neq\omega$. We are interested in the singularities of $S(\cdot,\vartheta,\omega)\in\mathscr{D}'(\mathbb{R})$. It is natural to expect that $T\in\operatorname{sing\,supp}S(\cdot,\vartheta,\omega)$ only if $-T$ is the sojourn time of a ray, with initial direction ω and final direction ϑ, reflected and refracted according to the laws of geometrical optics. (See the remark at the end of Sect. 2.) Define

$$T_0=\sup\{\langle\vartheta-\omega,x\rangle;x\in X_-\}\,.$$

A ray reflected off X_- at

$$Y_0=\{x\in Y;\langle\vartheta-\omega,x\rangle=T_0\} \tag{0.4}$$

has sojourn time T_0. It is known [MT; P] that T_0 is the leading singularity of the scattering kernel, i.e.,

$$\operatorname{supp}S(\cdot,\vartheta,\omega)\subset\,]-\infty,T_0]\,,$$
$$T_0\in\operatorname{sing\,supp}S(\cdot,\vartheta,\omega)\,,$$

if the exterior normal at Y_0,

$$n_0=(\vartheta-\omega)/|\vartheta-\omega|\,, \tag{0.5}$$

is a regular value of the Gauss map, if

$$|\vartheta+\omega|<2\sqrt{\alpha_0}\,, \tag{0.6}$$

where

$$\alpha_0=\inf\{\alpha(x);x\in X\}$$

and if Y is connected. Roughly speaking, the leading singularity is, modulo distributions which are less singular than δ',

$$S(T,\vartheta,\omega)\equiv a_0(\vartheta,\omega)\delta'(T-T_0)\,,$$

in a neighbourhood of T_0. The coefficient is given by

$$a_0(\vartheta,\omega) = -(2\pi)^{-1} \sum_{\bar{x}\in Y_0} \gamma^{-1/2} \cdot \left.\frac{(a_- - 1 + \langle\vartheta, n\rangle^2)^{1/2} - \langle\vartheta, n\rangle}{(a_- - 1 + \langle\vartheta, n\rangle^2)^{1/2} + \langle\vartheta, n\rangle}\right|_{x=\bar{x}}.$$

Here $\gamma = \gamma(x)$ is the Gaussian curvature of Y. Majda and Taylor [MT] proved this for backscattering, $\vartheta = -\omega$, and in the case where $\alpha > 1$ in $\bar{X}_-$. Petkov [P] generalized their results relaxing the assumptions on α to (0.6). Assumption (0.6) has the following consequence. Among the rays with initial direction ω and final direction ϑ those penetrating the obstacle are delayed when compared to those transversally reflected at Y_0. In particular, $-T_0$ is the smallest sojourn time for this class of rays. (See also the remark at the end of Sect. 2.)

In this paper we treat the case $\alpha_- = 1$. Then the index of refraction is continuous everywhere and non-smooth only at the boundary of the obstacle. We observe that a_0 vanishes if $\alpha_- = 1$. We now state our main result.

Theorem 0.1. *Let $\vartheta, \omega \in S^2$. Assume* (0.6). *Define T_0, Y_0 and n_0 as in* (0.3), (0.4), *and* (0.5). *Then*

$$\operatorname{supp} S(\cdot, \vartheta, \omega) \subset]-\infty, T_0].$$

Assume that n_0 is a regular value of the Gauss map. Then, in a neighbourhood of T_0,

$$S(T, \vartheta, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(T-T_0)\sigma} a(\sigma) d\sigma,$$

with a symbol $a \in S^1_{phg}(\mathbb{R})$. Assume that $\alpha_- = 1$. Then $a \in S^0_{phg}(\mathbb{R})$ and modulo S^{-1},

$$a(\sigma) \equiv \sum_{\bar{x}\in Y_0} (-\pi/4)\gamma^{-1/2} \langle\vartheta, n\rangle^{-3} (\partial\alpha/\partial n)_{-|x=\bar{x}}. \tag{0.7}$$

Here γ denotes the Gaussian curvature of Y.

Thus $S(\cdot, \vartheta, \omega)$ has a conormal singularity in T_0 or else $T_0 \notin \operatorname{sing\,supp} S(\cdot, \vartheta, \omega)$. In case $\alpha_- = 1$ the strongest possible singularity in T_0 is

$$S(T, \vartheta, \omega) \equiv a_1(\vartheta, \omega)\delta(T - T_0).$$

Here a_1 is defined as the right-hand side in (0.7). $a_1 \neq 0$ if, e.g., Y is connected and $(\partial\alpha/\partial n)_- \neq 0$. Hence, upon making these assumptions, we get $T_0 \in \operatorname{sing\,supp} S(\cdot, \vartheta, \omega)$. It is easy to derive the following inverse scattering result from this.

Corollary 0.2. *If $\alpha_- = 1$, $(\partial\alpha/\partial n)_- \neq 0$, and if $Y = \partial X_-$ is connected, then the convex hull of X_- can be recovered from the set of leading singularities of the scattering kernel for backscattering directions.*

Results on the singularities of the scattering kernel for hyperbolic equations have been given by several authors. The case most studied is that of an opaque obstacle (cf. Majda [M] and others). For transparent obstacles we refer to [MT] and [P].

This paper is organized as follows. In Sect. 1 we study the reflection of singularities at $Y \times \mathbb{R}$. There our main goal is to obtain the principal and the subprincipal symbol of the Neumann operators at hyperbolic points. The proof of Theorem 0.1 is given in Sect. 2. In Sects. 3 and 4 we compute the symbols of conormal distributions occurring in Sect. 2.

1. Solutions to the Transmission Problem

Let $f, g \in \mathscr{E}'(Y \times \mathbb{R})$. There exists a unique extendible distribution u in $X \times \mathbb{R}$ satisfying the homogeneous transmission problem,

$$\begin{aligned} \Box u &= 0 \quad \text{in} \quad X \times \mathbb{R}, \\ u_+ - u_- &= f \quad \text{at} \quad Y \times \mathbb{R}, \\ (D_n u)_+ - (D_n u)_- &= g \quad \text{at} \quad Y \times \mathbb{R}, \\ u &= 0 \quad \text{if} \quad t \ll 0. \end{aligned} \tag{1.1}$$

We infer this from the basic existence and regularity theory for the mixed problem.

The speed of propagation for $\Box$ is bounded by $1/\sqrt{\alpha_0}$. Hence a standard domain of dependence argument shows that

$$\begin{aligned} &(x,t) \in \operatorname{supp} u \Rightarrow \\ &\text{there exists } (\bar{x}, \bar{t}) \in (\operatorname{supp} f \cup \operatorname{supp} g) \\ &\text{with} \quad |x - \bar{x}| \leqq (t - \bar{t})/\sqrt{\alpha_0}\,. \end{aligned} \tag{1.2}$$

We now prepare for a study of the Neumann operators which relate the boundary values at hyperbolic points of a solution to (1.1).

Let us fix $(\bar{x}, \bar{t}) \in Y \times \mathbb{R}$. For simplicity we assume

$$n(\bar{x}) = (0, \ldots, 1).$$

Then, using the implicit function theorem, we can parametrize Y in a neighbourhood of $\bar{x}$ with $y \in \mathbb{R}^{3-1}$, $|y|$ small,

$$y \to \bar{x} + (y, \psi(y)).$$

Here ψ is a smooth function with $\psi(0) = 0$, $\psi'(0) = 0$. In a neighbourhood of $(\bar{x}, \bar{t})$ we now change to new coordinates $(y, z, t) \in \mathbb{R}^{3-1} \times \mathbb{R} \times \mathbb{R}$, $|y| + |z| + |t - \bar{t}|$ small,

$$(x,t) \to (y,z,t), \qquad x = \bar{x} + (y, \psi(y) + z). \tag{1.3}$$

Then $Y \times \mathbb{R}$ is given by $z = 0$. The exterior normal is

$$n(x) = J(y)^{-1}(-\psi'(y), 1)$$

where

$$J(y) = (1 + |\psi'(y)|^2)^{1/2}.$$

We shall distinguish the principal and the subprincipal symbols from the full symbols by the superscript ^{0} and ^{s}, respectively. In canonical coordinates $(x, t; \xi, \tau)$ we have

$$\Box^0 = |\xi|^2 - \alpha(x)\tau^2, \qquad \Box^s = 0.$$

The canonical transformation associated with (1.3) is given by

$$(x,t;\xi,\tau)\to(y,z,t;\eta,\zeta,\tau) \quad \text{if} \quad \xi=(\eta-\zeta\psi'(y),\zeta).$$

Therefore we have, in canonical coordinates $(y,z,t;\eta,\zeta,\tau)$,

$$\Box^0=(J\zeta-J^{-1}\langle\psi',\eta\rangle)^2-(\alpha\tau^2-|\eta|^2+J^{-2}\langle\psi',\eta\rangle^2),$$
$$\Box^s=0.$$

Here we used that the Jacobian of (1.3) equals 1. This property simplifies the transformation formula for the subprincipal symbol [H, Vol. 3, p. 83].

A point $(y,t;\eta,\tau)\in T^*(Y\times\mathbb{R})\backslash 0$ is called hyperbolic for $\Box$ with respect to the $\pm$ side, $(y,t;\eta,\tau)\in H_\pm$, if the equation for ζ

$$\Box^0(y,\pm 0,t;\eta,\zeta,\tau)=0$$

has two distinct real roots or, equivalently, if

$$\alpha_\pm\tau^2-|\eta|^2+J^{-2}\langle\psi',\eta\rangle^2>0 \tag{1.4}$$

[H, Vol. 3, p. 424].

We recall a construction of the Neumann operators, $B_\pm$, on solutions to $\Box u=0$ which are microlocally outgoing near $H_\pm$.

In $\pm z>0$ and near $H_\pm$ we can factorize the principal symbol of $\Box$,

$$\Box^0=(J\zeta-\lambda_2^0)(J\zeta-\lambda_1^0),$$
$$\lambda_{1(2)}^0=J^{-1}\langle\psi',\eta\rangle_{(}\pm_{)}\varepsilon(\alpha_\pm\tau^2-|\eta|^2+J^{-2}\langle\psi',\eta\rangle^2)^{1/2}.$$

Here we take $\varepsilon=-\operatorname{sign}(z\tau)$. This choice of sign insures that t increases along the bicharacteristics of $J\zeta-\lambda_1^0$ which are issued from $H_\pm$ into $\pm z>0$. As in [H, Vol. 3, Lemma 23.2.8], we derive a factorization of $\Box$ itself,

$$\Box\equiv(JD_z-A_2)(JD_z-A_1). \tag{1.5}$$

Here A_1, A_2 are properly supported first order pseudo-differential operators which are tangential with respect to $z=0$, i.e. they contain no D_z. λ_k^0 is the principal symbol of A_k, $k=1,2$. The factorization holds modulo operators which are smoothing on solutions to non-characteristic boundary problems. A_1 and A_2 are uniquely determined modulo smoothing tangential pseudo-differential operators.

The normal derivative is, in coordinates (y,z,t),

$$D_n=JD_z-J^{-1}\langle\psi',D_y\rangle.$$

Set

$$B=A_1+D_n-JD_z.$$

B is a first order tangential pseudo-differential operator. We can form its restrictions to $z=0$,

$$B_\pm=\pm B_{|\pm z=0}.$$

$B_\pm$ are the so-called (outgoing) Neumann operators. They are only defined microlocally in $H_\pm$, i.e. $B_\pm v$ is well-defined modulo $C^\infty(Y\times\mathbb{R})$ for $v\in\mathscr{E}'(Y\times\mathbb{R})$ with $WFv\Subset H_\pm$. (We identify conic subsets of the cotangent bundle with subsets of the cosphere bundle. In this sense we use the notation $\Subset$.)

We now derive formulas for the principal symbol, $b^0_\pm$, and the subprincipal symbol, $b^s_\pm$, of $B_\pm$ in the canonical coordinates $(y, t; \eta, \tau)$.

Lemma 1.1. *In $H_\pm$ we have*

$$b^0_\pm = -\tau(\alpha_\pm - |\eta/\tau|^2 + J^{-2}\langle\psi', \eta/\tau\rangle^2)^{1/2} \tag{1.6}$$

and, at $y=0$,

$$\begin{aligned}\pm b^s_\pm = &(2i)^{-1}\Delta\psi + (4i)^{-1}(\alpha_\pm - |\eta/\tau|^2)^{-1} \\ &\times(2\langle\eta/\tau, \psi''\cdot\eta/\tau\rangle - (\partial\alpha/\partial n)_\pm).\end{aligned} \tag{1.7}$$

Proof. For the principal symbol of B we have

$$\begin{aligned}b^0 &= \lambda^0_1 - J^{-1}\langle\psi', \eta\rangle \\ &= \varepsilon(\alpha\tau^2 - |\eta|^2 + J^{-2}\langle\psi', \eta\rangle^2)^{1/2}.\end{aligned}$$

In view of the definition of ε we have $-\tau = \pm\varepsilon|\tau|$ if $\pm z > 0$. Hence (1.6) follows.

At $y=0$ the subprincipal symbol of $D_n - JD_z$ equals

$$\begin{aligned}&(i/2)\left(\sum_1^{3-1}\partial^2/\partial y_j\partial\eta_j + \partial^2/\partial t\partial\tau\right)(-J^{-1}\langle\psi', \eta\rangle) \\ &= (2i)^{-1}\Delta\psi.\end{aligned}$$

Hence the subprincipal symbol of B is

$$b^s = \lambda^s_1 + (2i)^{-1}\Delta\psi \quad \text{at} \quad y=0. \tag{1.8}$$

We apply the composition formula for subprincipal symbols to (1.5). We get

$$\begin{aligned}\square^s = &(J\zeta - \lambda^0_2)(-\lambda^s_1) + (-\lambda^s_2)(J\zeta - \lambda^0_1) \\ &+ (2i)^{-1}\{J\zeta - \lambda^0_2, J\zeta - \lambda^0_1\}.\end{aligned}$$

Recalling $\square^s = 0$ we deduce that this equation is equivalent to, at $y=0$,

$$\begin{aligned}&\lambda^s_1 + \lambda^s_2 = 0, \\ &\lambda^0_2\lambda^s_1 + \lambda^s_2\lambda^0_1 + \{\lambda^0_1 - \lambda^0_2, J\zeta - \lambda^0_1\}/2i = 0.\end{aligned}$$

With $2b^0 = \lambda^0_1 - \lambda^0_2$ and $\lambda^0_1 = J^{-1}\langle\psi', \eta\rangle + \varepsilon b^0$ it follows that

$$2b^0\lambda^s_1 = -i\{b^0, J\zeta - J^{-1}\langle\psi', \eta\rangle\} \quad \text{at} \quad y=0,$$

or, equivalently,

$$(2b^0)^2\lambda^s_1 = -i\{(b^0)^2, J\zeta - J^{-1}\langle\psi', \eta\rangle\} \quad \text{at} \quad y=0.$$

Since $J - 1 = O(y^2)$ and $\partial_y(b^0)^2 = 0$ at $y=0$ we obtain, at $y=0$,

$$\lambda^s_1 = (2b^0)^{-2}(i\tau^2(\partial\alpha/\partial z) - 2i\langle\eta, \psi''\cdot\eta\rangle).$$

(1.7) follows if we insert this into (1.8) and observe that $b^s_\pm = (\pm b^s)_\pm$. The proof is complete.

Remark. The subprincipal symbol of $B_\pm$ is not defined invariantly at $H_\pm$. However the Neumann operator acting on half-densities, $\tilde{B}_\pm$, has an invariantly defined subprincipal symbol.

In coordinates (y,t) we have $\tilde{B}_{\pm}=J^{1/2}B_{\pm}J^{-1/2}$ because $d_xSdt=Jdydt$. Since $J'(0)=0$ it follows that $\tilde{b}^s_{\pm}=b^s_{\pm}$ at $y=0$. We give an invariant meaning to the right-hand side in (1.7) and obtain

$$\pm 2i\tilde{b}^s_{\pm}=(b^0_{\pm})^{-2}(S^*(\eta)-\tau^2(\partial\alpha/\partial n)_{\pm}/2)+\mathit{trace}\ S^*\,.$$

Here S^* is the second fundamental form of the hypersurface $Y\subset\mathbb{R}^3$ viewed as a quadratic form on T^*Y.

It follows from (1.6) that B_+, B_- and B_++B_- are elliptic in H_+, H_- and $H_+\cap H_-$, respectively. So we can construct microlocal inverses to these operators in these sets.

Proposition 1.2. *Let $\Gamma\Subset H_+\cap H_-$ be closed and conic. Let $t_0\in\mathbb{R}$ be such that no bicharacteristic issued from Γ in positive time direction hits $T^*(Y\times]-\infty,t_0[)\backslash 0$. Let $E\in\Psi^{-1}(Y\times\mathbb{R})$, properly supported, such that E is an inverse for B_++B_- microlocally near Γ. Then, for every f, $g\in\mathscr{E}'(Y\times\mathbb{R})$ with $WFf\cup WFg\subset\Gamma$, the solution u to* (1.1) *satisfies modulo C^∞, on $Y\times]-\infty,t_0[$*

$$(\pm D_nu)_{\pm}\equiv B_{\pm}u_{\pm}\,, \tag{1.9}$$

$$u_+\equiv E(B_-f+g)\,,\qquad u_-\equiv E(-B_+f+g)\,. \tag{1.10}$$

Proof. Let f, $g\in\mathscr{E}'(Y\times\mathbb{R})$ with $WFf\cup WFg\subset\Gamma$. Let u be the solution to (1.1). Consider $w^{\pm}\in\mathscr{E}'(Y\times\mathbb{R})$,

$$w^+=E(B_-f+g)\,,\qquad w^-=E(-B_+f+g)\,.$$

It is easy to check that

$$w^+-w^-\equiv f\,,\qquad B_+w^++B_-w^-\equiv g\,, \tag{1.11}$$

hold modulo $C^\infty(Y\times\mathbb{R})$. Using Fourier integral operators we can construct a distribution v solving in $X\times]-\infty,t_0[$ the following problem:

$$\begin{aligned}&\Box v\in C^\infty\\ &v_{\pm}\equiv w^{\pm}\ \mathrm{mod}\,C^\infty\\ &v\in C^\infty\quad\text{if}\quad t\ll 0\,.\end{aligned} \tag{1.12}$$

In fact, near Γ we construct v as a solution to

$$\begin{aligned}&(JD_z-\Lambda_1)v\in C^\infty\,,\\ &v_{\pm}\equiv w^{\pm}\ \mathrm{mod}\,C^\infty\,.\end{aligned}$$

Here we use that t increases along the bicharacteristics of $J\zeta-\lambda_1^0$ issued from $H_{\pm}$ into $\pm z>0$. Recalling the definition of B and of $B_{\pm}$ we obtain

$$(\pm D_nv)_{\pm}\equiv B_{\pm}w^{\pm}\quad\text{in}\quad Y\times]-\infty,t_0[\,. \tag{1.13}$$

Combining (1.1), (1.11), (1.12), and (1.13) we see that $u-v$ is, modulo C^∞, for $t<t_0$, an outgoing solution of the homogeneous transmission problem. The basic regularity theory for mixed problems implies that, modulo $C^\infty(Y\times]-\infty,t_0[)$,

$$u_{\pm}\equiv v_{\pm}\,,\qquad (D_nu)_{\pm}\equiv(D_nv)_{\pm}\,. \tag{1.14}$$

Now (1.9) and (1.10) follow from the definition of $w^{\pm}$ and (1.12), (1.13), (1.14).

The construction of the Neumann operators and the solution of the transmission problem at hyperbolic points is, of course, not new. See, e.g., [T], Chapter 9. We have not found a reference for the formula for the subprincipal symbol of the Neumann operator, however.

We remark that the results in this section are valid for any space dimension d, $d \geqq 2$, not only for $d=3$.

2. Proof of Theorem 0.1

We fix $\vartheta, \omega \in S^2$ satisfying (0.6). Note that $\alpha_0 \leqq 1$. Hence $\vartheta \neq \omega$ and the first term on the right-hand side of (0.2) vanishes. We shall write $S(T)=S(T, \vartheta, \omega)$ for short. Integrating, in (0.2), with respect to t and setting $T'=0$ we obtain

$$S(T)=\int_Y v(x, \langle \vartheta, x\rangle - T) d_x S \tag{2.1}$$

with

$$v=D_t(\langle \vartheta, n\rangle D_t - D_n) u_{|Y \times \mathbb{R}}.$$

Here u is the solution to (0.1) satisfying $u=u^{\text{in}}$ for $t \ll 0$. The operations on distributions performed here are legitimate because the wavefront sets are in favorable position, [H, Vol. 1, Chap. 8.2].

From (2.1) we deduce

$$\operatorname{supp} S \subset \{\langle \vartheta, x\rangle - t; (x,t) \in \operatorname{supp} v\}. \tag{2.2}$$

The scattered wave $u^{sc}=u-u^{\text{in}}$ solves (1.1) with $f=-u^{\text{in}}_+$, $g=-(D_n u^{\text{in}})_+$. The supports of f and g are contained in the compact set

$$Z=\{(x,t) \in Y \times \mathbb{R}; t=\langle \omega, x\rangle\}.$$

Clearly $\operatorname{supp} v \subset \operatorname{supp} u \cap (Y \times \mathbb{R})$. Hence, using (1.2),

$$\operatorname{supp} v \subset \{(x,t) \subset Y \times \mathbb{R}; \text{ there exists } (\bar{x}, \bar{t}) \in Z \text{ with } |x-\bar{x}| \leqq (t-\bar{t})/\sqrt{\alpha_0}\}. \tag{2.3}$$

Now, using (2.2), (2.3) and the first assertion in Lemma 2.1 below, we deduce

$$\operatorname{supp} S \subset]-\infty, T_0],$$

the first assertion in Theorem 0.1.

The following elementary geometrical fact was used by Petkov [P] to prove that T_0 is the leading singularity of S. (See the remark at the end of this section.)

Lemma 2.1. *Assume* (0.6). *Let* $(x,t) \in Y \times \mathbb{R}$ *and* $(\bar{x}, \bar{t}) \in Z$ *with* $|x-\bar{x}| \leqq (t-\bar{t})/\sqrt{\alpha_0}$. *Set* $T=\langle \vartheta, x\rangle - t$. *Then* $T \leqq T_0$. $T=T_0$ *holds if and only if* $\bar{x} \in Y_0$, $(x,t)=(\bar{x}, \bar{t})$.

Proof. With

$$t_1 = t-\bar{t}+\langle \vartheta+\omega, \bar{x}-x\rangle/2, \qquad t_2=T_0-\langle \vartheta-\omega, \bar{x}+x\rangle/2$$

we have $T_0-T=t_1+t_2$. $t_2 \geqq 0$ by definition of T_0, and $t_2=0$ holds precisely when $x, \bar{x} \in Y_0$. The assumption implies that $t_1 \geqq \varepsilon_0(t-\bar{t})$ for some $\varepsilon_0>0$. Hence $t_1 \geqq 0$, and $t_1=0$ holds if and only if $t=\bar{t}$. The proof of lemma is complete.

From now on we assume that, in addition to (0.6), $n_0=(\vartheta-\omega)/|\vartheta-\omega|$ is a regular value of the Gauss map. Then Y_0 and

$$Z_0=Z\cap(Y_0\times\mathbb{R})$$

are finite sets. Only the behaviour of v and u near Z_0 matters when studying S near T_0. To give a precise statement we choose, for every $\bar{x}\in Y_0$, $\chi_{\bar{x}}\in C_0^\infty(Y)$ with $\chi_{\bar{x}}=1$ in a neighbourhood of $\bar{x}$ and

$$Y_0\cap\operatorname{supp}\chi_{\bar{x}}=\{\bar{x}\}.$$

Let $u_{\bar{x}}$ be the solution to (1.1) with

$$f=-(\chi_{\bar{x}}u^{\mathrm{in}})_+, \qquad g=-(\chi_{\bar{x}}D_nu^{\mathrm{in}})_+,$$

and set

$$\begin{aligned}v_{\bar{x}}=[&(\chi_{\bar{x}})^2D_t(\langle\vartheta,n\rangle D_t-D_n)u^{\mathrm{in}}\\&+\chi_{\bar{x}}D_t(\langle\vartheta,n\rangle D_t-D_n)u_{\bar{x}}]_+.\end{aligned}\tag{2.4}$$

Lemma 2.2. *In a neighbourhood of* T_0

$$S(T)=\sum_{\bar{x}\in Y_0}\int_Y v_{\bar{x}}(x,\langle\vartheta,x\rangle-T)d_xS.$$

Proof. Set

$$Z_1=\{(x,t)\in Y\times\mathbb{R};\langle\vartheta,x\rangle-t=T_0\}.$$

In view of (2.1) it suffices to show that

$$Z_1\cap\operatorname{supp}\left(v-\sum_{\bar{x}\in Y_0}v_{\bar{x}}\right)=\phi.\tag{2.5}$$

Applying (1.2) to $u=u_{\bar{x}}$ and using the last part in Lemma 2.1 we deduce that Z_1 is disjoint from the supports of the following distributions,

$$[(1-\chi_{\bar{x}})D_t(\langle\vartheta,n\rangle D_t-D_n)u_{\bar{x}}]_+\quad\text{for}\quad\bar{x}\in Y_0,$$

$$\left(u^{\mathrm{sc}}-\sum_{\bar{x}\in Y_0}u_{\bar{x}}\right)_+.$$

Z_1 also does not meet the support of

$$\left[\left(1-\sum_{\bar{x}\in Y_0}(\chi_{\bar{x}})^2\right)D_t(\langle\vartheta,n\rangle D_t-D_n)u^{\mathrm{in}}\right]_+$$

Hence, if we recall the definitions of v and $v_{\bar{x}}$, we obtain (2.5). The proof of the lemma is complete.

Fix $\bar{x}\in Y_0$. We analyse $v_{\bar{x}}$. To ease the notation we shall drop the subscript $\bar{x}$, i.e., $\chi=\chi_{\bar{x}}$, $u=u_{\bar{x}}$, $v=v_{\bar{x}}$. Without loss of generality we assume

$$n_0=n(\bar{x})=(0,\ldots,1).$$

We then have $\vartheta_3=-\omega_3>0$, $\vartheta'=\omega'$, where $\vartheta'=(\vartheta_1,\vartheta_2)$, $\omega'=(\omega_1,\omega_2)$.

As in Sect. 2 we parametrize Y near $\bar{x}$ by $y\in\mathbb{R}^2$, $|y|$ small,

$$x=x(y)=\bar{x}+(y,\psi(y)).$$

we have coordinates (y, t) on $Y \times \mathbb{R}$ near $(\bar{x}, \bar{t})$. However, we shall rather use new coordinates (y, s) obtained with the change of variables

$$(y, t) \to (y, s), \qquad s = t - \langle \omega, x(y) \rangle . \tag{2.6}$$

$(\bar{x}, \bar{t})$ has new coordinates $(0, 0)$. Z_1 is given by the equation $s = 0$.

The distributions $(u^{\text{in}})_+$ and $(D_n u^{\text{in}})_+$ are conormal with respect to Z, in fact,

$$u_+^{\text{in}}(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\sigma s} d\sigma , \tag{2.7}$$

$$(D_n u^{\text{in}})_+(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\sigma s}(-\sigma\langle \omega, n \rangle) d\sigma , \tag{2.8}$$

where $x = x(y)$, $t = \langle \omega, x \rangle + s$, $|y| + |s|$ small. Observe that the canonical transformation associated with (2.6) is given by

$$(y, t; \eta - \sigma \partial_y \langle \omega, x(y) \rangle, \sigma) \to (y, s; \eta, \sigma) . \tag{2.9}$$

Lemma 2.3. *If* $\operatorname{supp}\chi$ *is sufficiently close to* $\bar{x}$ *then* $\Gamma \Subset H_+ \cap H_-$ *where*

$$\Gamma = WF(\chi u^{\text{in}})_+ \cup WF(\chi D_n u^{\text{in}})_+ .$$

Furthermore we have, modulo $C^\infty(Y \times \mathbb{R})$,

$$v \equiv P v_1 \tag{2.10}$$

with

$$P = \chi D_t(\langle \vartheta, n \rangle D_t + B_-) \in \Psi^2(Y \times \mathbb{R}) , \tag{2.11}$$

$$v_1 = E(B_+(\chi u^{\text{in}})_+ - \chi(D_n u^{\text{in}})_+) . \tag{2.12}$$

Here $E \in \Psi^{-1}(Y \times \mathbb{R})$ *is a properly supported microlocal inverse for* $B_+ + B_-$ *in a conic neighbourhood of* Γ.

Proof. The wave-front sets of u_+^{in} and $(D_n u^{\text{in}})_+$ are contained in the conormal bundle of Z, $N^*Z \backslash 0$. $N^*Z \backslash 0$ is defined by $\eta = s = 0$, $\sigma \neq 0$ in the canonical coordinates $(y, s; \eta, \sigma)$. The first assertion will follow from

$$(0, 0; 0, \sigma) \in H_+ \cap H_- \quad \text{if} \quad \sigma \neq 0 . \tag{2.13}$$

$H_\pm$ is characterized by the inequality (1.4). Transforming with (2.9) we see that (2.13) follows if

$$\alpha_\pm(\bar{x}) - |\partial_y \langle \omega, x(0) \rangle|^2 > 0$$

holds for both signs. Note that $\partial_y \langle \omega, x(0) \rangle = \omega'$. So this inequality is a consequence of (0.6).

Set $f = -(\chi u^{\text{in}})_+$ and $g = -(\chi D_n u^{\text{in}})_+$. It follows from (2.4) that

$$v = \chi D_t(\langle \vartheta, n \rangle D_t(u_+ - f) - (D_n u_+ - g)) . \tag{2.14}$$

We can now apply Proposition 1.2. We obtain, provided $\operatorname{supp}\chi$ is sufficiently close to $\bar{x}$,

$$u_+ \equiv E(B_- f + g) , \qquad (D_n u)_+ \equiv B_+ u_+ ,$$

modulo C^∞ in a neighbourhood of $\operatorname{supp}\chi$. It follows that, modulo C^∞ in a neighbourhood of $\operatorname{supp}\chi$,

$$\begin{aligned} u_+ - f &\equiv E(g - B_+ f), \\ D_n u_+ - g &\equiv B_+ u_+ - g \\ &\equiv B_+ f - g + B_+ E(g - B_+ f) \\ &\equiv -B_- E(g - B_+ f). \end{aligned}$$

We now insert these expressions into (2.14). The proof of the lemma is complete.

Corollary 2.4. *v is conormal with respect to Z if $\operatorname{supp}\chi$ is sufficiently close to $\bar{x}$,*

$$v(x,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\sigma s} c(y,\sigma) d\sigma, \tag{2.15}$$

$x = x(y)$, $t = s + \langle \omega, x \rangle$, with $c \in S^2_{phg}(\mathbb{R}^2 \times \mathbb{R})$. We have

$$c^0(0,\sigma) = -2\sigma^2 \mu(\lambda - \mu)/(\lambda + \mu), \tag{2.16}$$

where $\mu = \langle \vartheta, n_0 \rangle > 0$, $\lambda = (\alpha_-(\bar{x}) - 1 + \mu^2)^{1/2}$.

Proof. We just apply Theorem 18.2.12 in [H, Vol. 3] to (2.10) using (2.7), (2.8), and (1.6) with (2.9).

We are interested in the case where $\alpha_- = 1$. Then $c^0(0,\sigma) = 0$. We shall need a refined description of c in this case.

Lemma 2.5. *Assume that $\alpha_- = 1$. Set*

$$q(y) = 1 - |\partial_y \langle \omega, x \rangle|^2 + \langle \psi', \partial_y \langle \omega, x \rangle \rangle^2 J^{-2}.$$

Then

$$c^0(y,\sigma) = -(\sigma^2/2)(\sqrt{q} - \langle \vartheta, n \rangle)(1 - \langle \omega, n \rangle / \sqrt{q}),$$

and, modulo $S^0(\mathbb{R})$,

$$c(0,\sigma) \equiv (i\sigma/4) \langle \vartheta, n \rangle^{-2} (\partial \alpha / \partial n)_{-|x=\bar{x}}.$$

The proof of Lemma 2.5 is given in Sect. 3.

We now turn to the computation of the singularity of S in T_0. Set

$$\Phi(y) = \langle \vartheta - \omega, x(y) - \bar{x} \rangle.$$

$y = 0$ is a nondegenerate critical point for Φ, $\Phi(0) = 0$, $\Phi'(0) = 0$,

$$A = \Phi''(0) = 2\langle \vartheta, n_0 \rangle \psi''(0)$$

is negative definite.

Shrinking $\operatorname{supp}\chi$ still further, if necessary, we can apply the method of stationary phase to

$$a(-\sigma) := \int e^{i\sigma\Phi(y)} c(y,\sigma) J(y) dy.$$

Here c is the symbol in (2.15). It follows that $a \in S^1(\mathbb{R})$,

$$a(-\sigma) = (\det(\sigma A / 2\pi i))^{-1/2} (c(0,\sigma) + r(\sigma)), \tag{2.17}$$

$r\in S^1(\mathbb{R})$, and

$$\int_Y v(x,\langle\vartheta,x\rangle-T)d_xS=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\sigma(T-T_0)}a(\sigma)d\sigma\,.$$

In view of Lemma 2.2 this proves that S is conormal with respect to $T=T_0$ in a neighbourhood of T_0.

In Sect. 4 we prove the following result on the remainder r in (2.17).

Lemma 2.6. *Assume that* $\alpha_-=1$. *Then* $r\in S^0(\mathbb{R})$.

Assuming that $\alpha_-=1$, we can use Lemmas 2.5 and 2.6 and derive from (2.17),

$$a(\sigma)\equiv-(\pi/4)\gamma^{-1/2}\langle\vartheta,n\rangle^{-3}(\partial\alpha/\partial n)_{-|x=\bar{x}}$$

modulo $S^{-1}(\mathbb{R})$. This proves the last assertion in Theorem 0.1.

Remark. In the case $\alpha_-\neq 1$ we can insert (2.16) into (2.17) and recover the results in [MT] and [P] on the filtered scattering amplitude.

Remark. Petkov [P] uses the results of Ivrii [I] on microlocal propagation of singularities to prove that T_0 is the leading singularity of S, if (0.6) is assumed. We briefly sketch this argument. From [I] one deduces that the wave-front set of the scattered wave, u^{sc}, is a union of rays (or generalized bicharacteristics). See [Ha] for the definition of rays associated with (0.1). Using this together with the standard wave-front estimate for push-forwards applied to (2.1) one can show that $T\in\operatorname{sing\,supp} S(\cdot,\vartheta,\omega)$ only if $-T$ is the sojourn time of a ray with initial direction ω and final direction ϑ. This means that

$$T=\langle\vartheta,x_\vartheta\rangle=\langle\vartheta,x(t)\rangle-t\quad\text{for}\quad t\gg 0\,,$$

where $z(t)$ is the projection of a ray down to space satisfying

$$x(t)=\begin{cases} t\omega & \text{for}\quad t\ll 0\\ t\vartheta+x_\vartheta & \text{for}\quad t\gg 0\,.\end{cases}$$

$x(t)$ is a Lipschitz curve with $|\dot{x}(t)|\leqq 1/\sqrt{\alpha_0}$ almost everywhere. Now it follows from Lemma 2.1 that T_0 is the leading singularity of S if (0.6) is assumed.

3. Proof of Lemma 2.5

The principal and the subprincipal symbols of $B_\pm$ are given in Lemma 1.1. Using (2.9) we have in the canonical coordinates $(y,s;\eta,\sigma)$,

$$b^0_\pm=-\sigma(1-|\Xi|^2+\langle\psi',\Xi\rangle^2J^{-2})^{1/2}\,,\tag{3.1}$$

with $\Xi=\sigma^{-1}\eta-\partial_y\langle\omega,x\rangle$, and

$$\pm b^s_\pm(0,0;0,\sigma)=(2i)^{-1}\Delta\psi(0) +(i/4)\langle\vartheta,n_0\rangle^{-2}((\partial\alpha/\partial n_0)_\pm(\bar{x})-2\langle\omega',\psi''(0)\omega'\rangle)\,.\tag{3.2}$$

To derive (3.2) from (1.7) we use that the Jacobian of (2.6) equals 1.

Lemma 3.1. *Let p be the symbol of the pseudo-differential operator P in* (2.11). *We have*

$$p^0 = \sigma^2(\langle \vartheta, n\rangle - \sqrt{q}) \quad \text{if} \quad s=\eta=0, \quad |y| \text{ small}, \tag{3.3}$$

and, modulo S^0,

$$p(0,0;0,\sigma) \equiv (\sigma/4i)\langle \vartheta, n_0\rangle^{-2}(\partial\alpha/\partial n_0)_-(\bar{x}). \tag{3.4}$$

Proof. In a neighbourhood of $y=0$ where $\chi=1$ we have

$$p^0 = \sigma(\langle \vartheta, n\rangle \sigma + b^0_-).$$

(3.3) follows if we insert (3.1). The full symbol, p, is given by

$$p \equiv p^0 + p^s + \left(\sum_1^2 \partial^2 p^0/\partial y_j \partial\eta_j + \partial^2 p^0/\partial s\partial\sigma\right)\Big/ 2i$$

modulo S^0. At $y=\eta=0$, $s=0$, this simplifies to

$$p \equiv \sigma b^s_- + (\sigma/2i)\sum_1^2 \partial^2 b^0_-/\partial y_j \partial\eta_j \bmod S^0. \tag{3.5}$$

We claim that

$$\sum_1^2 \partial^2 b^0_-/\partial y_j\partial\eta_j = \Delta\psi + \langle \vartheta, n\rangle^{-2}\langle \omega', \psi''\cdot\omega'\rangle$$
$$\text{if} \quad y=\eta=s=0. \tag{3.6}$$

(3.2), (3.5), and (3.6) imply the remaining assertion (3.4). To prove (3.6) we differentiate $b^0 = b^0_\pm$. From (3.1) we derive

$$\partial b^0/\partial\eta_j = -q^{-1/2}\partial_j\langle \omega, x\rangle + 0(y^2)$$

at $\eta=0$. Here $\partial_j = \partial/\partial y_j$. Hence, at $y=\eta=0$,

$$\partial b^0/\partial\eta_j\partial y_j = -q^{-1/2}\partial_j^2\langle\omega, x\rangle$$
$$+2^{-1}q^{-3/2}(\partial_j q)(\partial_j\langle\omega, x\rangle).$$

Evaluating q, $\langle\omega, x\rangle$ and their derivatives at $y=0$ we arrive at (3.6). The proof is complete.

Consider v_1 in (2.12). v_1 is conormal with respect to Z,

$$v_1(x,t) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\sigma s}c_1(y,\sigma)d\sigma,$$

$x=x(y)$, $s=t-\langle\omega, x\rangle$, with $c_1 \in S^0(\mathbb{R}^2\times\mathbb{R})$. This follows from Theorem 18.2.12 in [H, Vol. 3]. For small $|y|$

$$c_1^0 = (b^0_+ + b^0_-)^{-1}(b^0_+ \cdot 1 + \sigma\langle\omega, n\rangle)|_{\eta=s=0}$$

and thus

$$c_1^0 = (1 - \langle\omega, n\rangle/\sqrt{q})/2.$$

Here we used (2.7), (2.8), and (3.1). To compute the symbol c of v we apply formula (18.2.16) in [H, Vol. 3], to (2.10). We obtain

$$c^0(y,\sigma) = p^0(y,0;0,\sigma)c_1^0(y,\sigma) \tag{3.7}$$

and, modulo S^0,

$$c(0,\sigma)\equiv p(0,0;0,\sigma)c_1(0,\sigma) - i\left(\sum_1^2 (\partial p^0/\partial\eta_j)(\partial c_1^0/\partial y_j) - (\partial p^0/\partial s)(\partial c_1^0/\partial\sigma)\right)_{|y=\eta=s=0}. \tag{3.8}$$

Note that $p^0(0,0;0,\sigma)=0$ and $\partial c_1^0/\partial\sigma=0$. Observe that

$$1+\langle\omega,n\rangle/\sqrt{q}=0(y^2). \tag{3.9}$$

Consequently $\partial c_1^0/\partial y=0$ at $y=0$. Therefore (3.8) simplifies to

$$c(0,\sigma)\equiv p(0,0;0,\sigma)c_1^0(0,\sigma). \tag{3.10}$$

It only remains to insert the formulas for p and c_1 into (3.7) and (3.10).

4. Proof of Lemma 2.6

The remainder $r\in S^1$ in (2.17) contains the non-principal terms of the stationary phase expansion. From Theorem 7.7.5 in [H, Vol. 1], we obtain

$$i\sigma r(\sigma)\equiv\sum_{k=1}^{3} 2^{-k}\langle A^{-1}D_y,D_y\rangle^k(g^{k-1}c^0J)/(1!\dots k!)_{|y=0} \tag{4.1}$$

modulo S^0. Here $g(y)=\Phi(y)-\langle Ay,y\rangle/2$. Observe that c^0, $1-J$, and g vanish in $y=0$ to first, second, and third order, respectively. Therefore in (4.1) we can skip the last term, and we can replace J by 1. Hence to prove our claim, $r\in S^0$, it is sufficient to verify the following cancellation property.

Lemma 4.1. *At* $y=0$

$$4\langle A^{-1}D_y,D_y\rangle c^0+\langle A^{-1}D_y,D_y\rangle^2(gc^0)=0.$$

Proof. From Lemma 2.5 we derive, using (3.9),

$$c^0=-\sigma^2(\sqrt{q}-\langle\vartheta,n\rangle)+0(y^3). \tag{4.2}$$

The Taylor expansion around $y=0$ of q is

$$q(y)=\vartheta_3^2+\langle\omega',Ay\rangle-\langle Ay,Ay\rangle/4+G(y,y,\omega')/2 +(\langle\omega',Ay\rangle/2\vartheta_3)^2+0(y^3).$$

Here G is the cubic form

$$G=g^{(3)}(0)=2\vartheta_3\psi^{(3)}(0).$$

Hence

$$\sqrt{q(y)}=\vartheta_3+(2\vartheta_3)^{-1}(q(y)-\vartheta_3^2) -(2\vartheta_3)^{-3}\langle\omega',Ay\rangle^2+0(y^3). \tag{4.3}$$

Observing $\langle \vartheta, n\rangle = \vartheta_3 J^{-1} - \langle \psi', \vartheta'\rangle + 0(y^3)$ we derive

$$\langle \vartheta, n\rangle = \vartheta_3 - (2\vartheta_3)^{-1}(\langle \vartheta', Ay\rangle + \langle Ay, Ay\rangle/4 + G(y, y, \vartheta')/2) + 0(y^3). \tag{4.4}$$

Combining (4.2)–(4.4) we obtain

$$c^0 = -\sigma^2(2\vartheta_3)^{-1}(2\langle \omega', Ay\rangle + G(y, y, \omega')) + 0(y^3).$$

Hence

$$\langle A^{-1}D_y, D_y\rangle c^0 = -\sigma^2(2\vartheta_3)^{-1}\langle A^{-1}D_y, D_y\rangle G(y, y, \omega') + 0(y). \tag{4.5}$$

Next we observe that

$$gc^0 = -\sigma^2 G(y^3)\langle \omega', Ay\rangle/3!\,\vartheta_3 + 0(y^5).$$

Hence, using the symmetry of A^{-1},

$$\begin{aligned}
&\sigma^{-2}3!\,\vartheta_3\langle A^{-1}D_y, D_y\rangle(gc^0)\\
&\quad = -\langle A^{-1}D_y, D_y\rangle(G(y^3)\langle \omega', Ay\rangle) + 0(y^3)\\
&\quad = -6G(y, y, \omega') - \langle \omega', Ay\rangle\langle A^{-1}D_y, D_y\rangle G(y^3) + 0(y^3)
\end{aligned}$$

and, consequently,

$$3!\,\vartheta_3\sigma^{-2}\langle A^{-1}D_y, D_y\rangle^2(gc^0) = 12\langle A^{-1}D_y, D_y\rangle G(y, y, \omega') + 0(y).$$

We compare this with (4.5). The proof of the lemma is complete.

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