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Solution of a Hyperbolic Inverse Problem by Linearization

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1 Introduction

Waves travelling through a medium carry information about the medium to distant locations. This fact is fundamental to many methods in science and engineering for exploring structural properties and material parameters of media. In mathematical formulations of such situations one considers the map which sends the coefficients of a hyperbolic differential equation to the boundary values of its solutions. The inversion of this non-linear map is desired. In this paper we use the inverse of the formal derivative of the coefficients-to-solutions map to partially invert the latter.

The problem we study is suggested by seismic exploration of the earth. We shall work with a simplified mathematical model of the earth. We assume that the earth $X \subset \mathbb{R}^3$ consists of two layers, the upper layer X_+ and the lower layer X_-. The layers are separated by a smooth interface Z. $Y = \partial X$ is the surface of the earth. The surface Y and the interface Z do not intersect. One may, for example, think of X_- as the earth's core, of X_+ as the earth's mantle and crust, and of Z as the core-mantle boundary. A wave $u(t, x)$ travelling through the earth is governed by the acoustic wave equation

$$\Delta u - \alpha \partial_t^2 u = 0 \tag{1}$$

where $\alpha(x) > 0$ is the square of the slowness. α is assumed to be smooth except for a jump discontinuity across Z. We wish to determine the location of the reflector Z and the size of the jump of α at Z from traces of waves measured at the surface Y.

We sketch a linear inversion procedure for α. Assume there is given a smooth and positive function β on X. β represents a model of the earth, the background model. It is smooth at Z. β is regarded as a first approximation to α. Associated with β is the wave equation

$$\Delta v - \beta \partial_t^2 v = 0. \tag{2}$$

Associated with β and a solution v of this wave equation is the perturbation equation

$$\Delta \dot{v} - \beta \partial_t^2 \dot{v} = \dot{\beta} \partial_t^2 v. \tag{3}$$

This equation is arrived at by requiring that (2) vanishes at least formally to second order in ϵ as $\epsilon \to 0$ if β and v are replaced by $\beta + \epsilon\dot{\beta}$ and $v + \epsilon\dot{v}$, respectively. Given a source distribution f on $\Upsilon = \mathbb{R} \times Y$ and a perturbation $\dot{\beta}$ on X we get causal solutions u, v, and $\dot{v}$ of (1), (2), and (3), respectively, such that the normal derivatives of u, v, and $v + \dot{v}$ are equal to f on the space-time boundary Υ. We fix a source f, say a Dirac distribution. The linear map L that sends $\dot{\beta}$ to $\dot{v}|\Upsilon$ is similar to a Radon transformation. We assume that an approximate backprojection operator for L exists. We apply such a backprojection operator to $\dot{v}|\Upsilon = u|\Upsilon - v|\Upsilon$. In this way we obtain a perturbation $\dot{\beta}$. We take $\beta + \dot{\beta}$ as a new approximation to α.

The purpose of this paper is to explain in what sense $\beta + \dot{\beta}$ is a better approximation to α than β is. This task is made difficult by the fact that the inversion procedure involves two unrelated approximations: a formal linearization of the coefficients-to-solutions map, and a high-frequency approximation. (The latter enters through the backprojection operator which is a microlocal inverse of L.) However, we can show that the singularities of $\dot{\beta}$ are directly related to Z and to the size of the jump of α across Z. More precisely, assuming that the background model and the actual model coincide in the upper layer, i.e., $\beta = \alpha$ in X_+, we recover at $z \in Z$ the number

$$\alpha_+(z) \cos^2 \theta_+ \; r \tag{4}$$

Here α_+ is the boundary value of α at the positive (= upper) side of Z. θ_+ and θ_- are the angles at z of reflection and refraction, respectively.

$$r = \frac{\cot \theta_+ - \cot \theta_-}{\cot \theta_+ + \cot \theta_-}$$

is the reflection coefficient associated with the jump of α across Z. Using concepts of microlocal analysis we give a precise statement for a rather general setting in Theorem 1 of section 4. The result provides a rigorous justification for using this particular linearization approach in the hyperbolic inverse problem.

If β is only close to α in $\overline{X_+}$ then, in addition, a stability result for rays associated with (2) is needed to solve the inverse problem. We do not deal with this aspect of the hyperbolic inverse problem.

There is a large body of literature on the linearization approach to inversion for high-frequency waves. In geophysics this method in sometimes called ray-theoretic Born migration (or inversion). In recent years the linearized problem was studied with techniques from microlocal analysis by Beylkin [1], Rakesh [16], and Beylkin and Burridge [2]. The problem we investigate was studied before in the geophysics literature by Bleistein et al. [3]. They evaluated, using classical geometrical optics, the application of linearized inversion operators to Kirchhoff (high-frequency) data. (4) occurs as a weight factor in their singular function of the reflector Z. Our Theorem 1 contains a rigorous derivation of the formulas obtained in [3], and it contains a precise definition of the singular function.

The plan of the paper is as follows. In section 2 we define the boundary trace of waves in a two-layered acoustic medium and describe its singularities. In section 3 the formal linearization of the coefficient-to-solutions map is defined and its singularities are analysed. In section 4 we state and prove our main theorem on linearized inversion.

2 Primary Reflections

We study wave propagation in a two-layered acoustic medium. Let X be an open and bounded subset of $\mathbb{R}^n$ with C^∞ boundary $Y = \partial X$. Let $Z \subset X$ be a compact hypersurface. Assume that Z separates X into two disjoint open subsets X_+ and X_- with $X \setminus Z = X_+ \cup X_-$, $\partial X_+ = Y \cup Z$, and $\partial X_- = Z$. We shall call Y the surface, Z the reflector, and X_+ (resp. X_-) the upper (resp. lower) layer of the medium X. Z is oriented by X_+ and X_-. Throughout the paper we use the abbreviations $\Omega = \mathbb{R} \times X$, $\Omega_\pm = \mathbb{R} \times X_\pm$, $\Upsilon = \mathbb{R} \times Y$, and $\Gamma = \mathbb{R} \times Z$. We let $\partial_n u$ and $\partial_{n_\pm} u$ denote the normal derivative of u at Y and Z, with normal vectors pointing into X and $X_\pm$, respectively. Let α be a positive function in $C^\infty(\overline{X_+}\dot{\cup}\overline{X_-})$. $C^\infty(\overline{X_+}\dot{\cup}\overline{X_-})$ denotes the space of smooth functions on $X_+ \cup X_-$ which extend smoothly to the disjoint union of $\overline{X_+}$ and $\overline{X_-}$. In other words, this is the space of functions on $\overline{X}$ which are smooth on $\overline{X}$ except for a jump discontinuity at Z.

Waves are solutions u of a boundary problem with transmission conditions at the reflector:

$$\begin{array}{rcll} \Delta u - \alpha \partial_t^2 u & = & 0 & \text{in } \Omega_+ \cup \Omega_- \\ \partial_n u & = & f & \text{on } \Upsilon \\ u_+ & = & u_- & \text{on } \Gamma \\ (\partial_{n_+} u)_+ + (\partial_{n_-} u)_- & = & 0 & \text{on } \Gamma \end{array} \tag{5}$$

(The subscripts $\pm$ indicate that the boundary values at Z are to be taken from the $\pm$ side.)

We say that a distribution $u(t,x)$ vanishes initially if there exists $T \in \mathbb{R}$ such that $u(t,x) = 0$ if $t < T$. It follows from the existence and regularity theory for hyperbolic mixed problems that there is a continuous map $\mathcal{E}'(\Upsilon) \to \mathcal{D}'(\overline{\Omega_+ \cup \Omega_-})$ which maps f to the unique solution of (5) that vanishes initially. (See Massey and Rauch [9] for mixed initial-boundary problems with distributional data.) Since Υ is non-characteristic with respect to the wave equation we can define a continuous map

$$U : \mathcal{E}'(\Upsilon) \to \mathcal{D}'(\Upsilon), \quad Uf = u\,|_{\Upsilon}.$$

We are interested in the singularities of the Schwartz kernel of U.

Our basic reference for wavefront sets and for the symbol calculus of pseudo-differential operators and Fourier integral operators is monograph [8] of Hörmander. We identify $\frac{1}{2}$-densities and 0-densities via the Euclidean structure on $\mathbb{R}^n$. For a continuous linear operator A acting on distributions we denote its Schwartz kernel by $\mathcal{K}_A$ or, if no confusion can arise, also by A. The principal symbol of a pseudo-differential or Fourier integral operator A is denoted $\sigma(A)$. We assume all pseudo-differential operators properly supported and polyhomogeneous. For a pseudo-diffferential operator A on Ω its essential support $\mathrm{WF}(A) \subset T^*(\Omega)$ is the set obtained from the wave front set $\mathrm{WF}'(\mathcal{K}_A)$ via the diagonal map from $T^*(\Omega)$ to $T^*(\Omega) \times T^*(\Omega)$. We shall abuse standard notation and denote by A^{-1} usually only a microlocal inverse of an operator A near some point. Bicharacteristics are integral curves in $T^*(X_+)$ and $T^*(X_-)$ for the Hamilton field of the wave equation in (5). Generalized bicharacteristics are curves in

$$T^*(\Upsilon) \setminus 0 \,\dot{\cup}\, T^*(\Omega_+ \cup \Omega_-) \setminus 0 \,\dot{\cup}\, T^*(\Gamma) \setminus 0 \tag{6}$$

mainly made up of broken bicharacteristics and glancing rays. The set (6) is equipped with the topology of a compressed cotangent bundle over $\overline{X}$. (See Melrose and Sjöstrand [12], Hörmander [8, Chapter 24], and Hansen [7] for precise definitions.) We call α-rays or rays those generalized bicharacteristics on which the time variable t increases. Denote by C_α the ray relation on the set (6), i.e., $(\rho_1, \rho_0) \in C_\alpha$ if and only if there is a ray γ such that $\rho_0 = \gamma(s_0)$ and $\rho_1 = \gamma(s_1)$ where $s_0 \leq s_1$.

Proposition 1 $WF'(\mathcal{K}_U) \subset C_\alpha$

Proof. The results of [13] and [7] imply

$$\mathrm{WF}(Uf) \subset C_\alpha(\mathrm{WF}(f)), \quad f \in \mathcal{E}'(\Upsilon).$$

Testing $\mathcal{K}_U$ with exponentials we deduce

$$\mathrm{WF}'(\mathcal{K}_U) \subset C_\alpha \ \cup T^*(\Upsilon) \times 0 \ \cup 0 \times T^*(\Upsilon).$$

U is a continuous linear operator defined on the space of distributions of compact support. Hence $\mathrm{WF}'(\mathcal{K}_U)$ does not intersect $0 \times T^*(\Upsilon)$. Furthermore $UC_0^\infty \subset C^\infty$. Hence $\mathrm{WF}'(\mathcal{K}_U)$ does not intersect $T^*(\Upsilon) \times 0$. The proposition is proved.

$T^*(\Upsilon) \setminus 0$ decomposes into the sets of elliptic, hyperbolic, and glancing points with respect to the wave equation. (See Melrose [10], Hörmander [8, Chapter 24].) Bicharacteristics intersect the boundary non-tangentially at hyperbolic points. Let $\mathcal{H} \subset T^*(\Upsilon) \setminus 0$ denote the set of hyperbolic points. Similarly, let $\mathcal{H}_+, \mathcal{H}_- \subset T^*(\Upsilon) \setminus 0$ denote the set of hyperbolic points with respect to the positive and negative side of Z. At $\mathcal{H}_+ \cap \mathcal{H}_-$ rays are reflected and refracted according to the classical laws of geometrical optics. A bicharacteristic issued from a hyperbolic point is called a forward (resp. backward) bicharacteristic if time t increases (resp. decreases) along it.

To analyse reflection and refraction at Z we introduce convenient local coordinates. Fix a point in Z. Let $(x_1, \ldots, x_n)$ local coordinates on X such that this point becomes the origin, and such that $X_\pm = \{\pm x_n > 0\}$, and $Z = \{x_n = 0\}$. Furthermore assume that the metric tensor g^{ij} is Euclidean at the origin, i.e., $g^{ij}(0) = \delta_{ij}$. Write $x_0 = t$. Then $x = (x_0, \ldots, x_n) = (x', x_n)$, $x' = (x_0, x'')$, are coordinates on Ω and Γ, respectively. The induced canonical coordinates on $T^*(\Omega)$ and $T^*(\Gamma)$ are (x, ξ) and (x', ξ') where $\xi = (\xi', \xi_n)$, $\xi' = (\xi_0, \xi'')$. The principal symbols $p_\pm$ of the wave operator on the $\pm$ side of Z satisfy

$$p_\pm(x, \xi) = \alpha_\pm(x'')\xi_0{}^2 - |\xi''|^2 - |\xi_n|^2 \quad \text{if } x'' = x_n = 0.$$

Here $\alpha_\pm(x'') = \alpha(x'', \pm 0)$. We shall also write $p = p_+$. $\mathcal{H}_\pm$ is characterised by the inequality

$$\alpha_\pm(x'')\xi_0{}^2 - |\xi''|^2 > 0 \quad \text{if } x'' = 0.$$

Let $H_{p_\pm}$ denote the Hamilton field of $p_\pm$. The forward (resp. backward) bicharacteristic issued from $\mathcal{H}_+$ to $T^*(\Omega_+)$ are characterised by the inequality

$H_{p_+}x_n\, H_{p_+}x_0 > 0$ (resp. $H_{p_+}x_n\, H_{p_+}x_0 < 0$), or, in coordinates, $\xi_n\xi_0 < 0$ (resp. $\xi_n\xi_0 > 0$). The angle of reflection $\theta_\pm$ at $(x', \xi') \in \mathcal{H}_\pm$ is the angle between the normal $n_\pm(x'')$ and the tangent vector of the projection to $X_\pm$ of the bicharacteristic which issues from (x', ξ') to $T^*(\Omega_\pm)$. The tangent vector is a multiple of (ξ'', ξ_n) at $x'' = 0$. Therefore we have

$$\tan\theta_\pm = \frac{|\xi''|}{|\xi_n|} \quad \text{where } p_\pm(x,\xi) = 0, x'' = 0, x_n = 0. \tag{7}$$

Using elementary geometry we rederive Snell's law

$$\sqrt{\alpha_+}\sin\theta_+ = \sqrt{\alpha_-}\sin\theta_-. \tag{8}$$

Recall the parametrices for the wave equation with Dirichlet boundary conditions at hyperbolic points. (See Chazarain [5].) Let $\rho \in \mathcal{H}$. Let E^+ and E^- be linear continuous maps $\mathcal{E}'(\Upsilon) \to \mathcal{D}'(\overline{\Omega_+})$ such that

$$(\triangle - \alpha\partial_t^2)\, E^+ \equiv 0$$

holds modulo an operator with C^∞ kernel, and $\iota^*E^\pm \in \Psi^0(\Upsilon)$ with $\rho \notin \mathrm{WF}(\iota^*E^\pm - I)$. E^+ (resp. E^-) is a *forward* (resp. *backward*) *parametrix* at ρ if $(\rho_0, \rho) \in \mathrm{WF}'(E^+)$ (resp. $(\rho_0, \rho) \in \mathrm{WF}'(E^-)$) implies that ρ_0 is on the forward (resp. backward) bicharacteristic issued from ρ. Here

$$\iota^* \in I^{1/4}(\Upsilon \times \Omega_+; N^*(\mathrm{graph}(\iota)) \setminus 0)$$

is the pullback by the natural inclusion map $\iota : \Upsilon \to \overline{\Omega_+}$ induced by the natural restriction of functions (0-densities). (See Guillemin and Sternberg [6] for pull-back and push-forward by maps.) $E^\pm$ are constructed as Fourier integral operators

$$E^\pm \in I^{-1/4}(\Omega_+ \times \Upsilon; C'_{E^\pm})$$

C_{E^+} and C_{E^-} are the canonical relations defined by the forward and backward bicharacteristics, respectively. Let

$$\Lambda^\pm = \frac{1}{i}\iota^*\partial_n E^\pm \in \Psi^1(\Upsilon)$$

Λ^+ and Λ^- are the *forward* and *backward Neumann operators* at ρ, respectively.

Similarly, forward (resp. backward) parametrices and Neumann operators E_+^+, Λ_+^+ (resp. E_+^-, Λ_+^-) are defined for the positive side of Z, and E_-^+, Λ_-^+ (resp. E_-^-, Λ_-^-) for the negative side of Z. Let us state the representation of these operators in the coordinates introduced above. The pull-back for the $\pm$-side of Z is given by

$$(\iota_{\pm}^{*}v)(x',0) = v(x',\pm 0)|dx_n|^{-\frac{1}{2}} \quad \text{if } x'=0 \tag{9}$$

if v is a smooth $\frac{1}{2}$-density on $\overline{\Omega_{\pm}}$.

The coordinates x induce coordinates (x,y) on $\Omega\times\Gamma$. In these coordinates the kernels of the parametrices are oscillatory integrals, e.g.,

$$E_{+}^{\pm}(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i\phi(x,y,\theta)}a(x,y,\theta)d\theta \tag{10}$$

where ϕ is a non-degenerate phase function satisfying $\phi(x,y,\theta) = \langle x'-y,\theta\rangle$ if $x_n=0$, $p_{+}(x,\phi_x'(x,y,\theta)) = 0$, and $\phi_{x_n}'(x,y,\theta)\phi_{x_0}'(x,y,\theta) < 0$ if $x_n \geq 0$ small, x' close to y, and $\phi_\theta'(x,y,\theta)=0$. (Angular brackets denote standard scalar product.) a is a symbol of order 0 satisfying transport equations along bicharacteristics. $a=1$ at $x_n = x'' = 0$, $x'=y$, $\phi_\theta'=0$. Observe that $\sigma(\partial_{n_\pm})(x,\xi) = \pm i\xi_n$ at $x''=x_n=0$. Therefore, if $(x',\xi')\in\mathcal{H}_+$ and $x''=0$.

$$\sigma(\Lambda_{+}^{\pm})(x',\xi') = \mp\text{sign}(\xi_0)(\alpha_{+}(x'')\xi_0^2-|\xi''|^2)^{\frac{1}{2}}$$

or, using (7),

$$\sigma(\Lambda_{+}^{\pm})(x',\xi') = \mp\text{sign}(\xi_0)|\xi''|\cot\theta_{+}.$$

Similarly, if $(x',\xi')\in\mathcal{H}_-$ and $x''=0$,

$$\sigma(\Lambda_{-}^{\pm})(x',\xi') = \mp\text{sign}(\xi_0)|\xi''|\cot\theta_{-}.$$

Let $\rho_r\in\mathcal{H}_+\cap\mathcal{H}_-$. Then $\Lambda_{+}^{+}+\Lambda_{-}^{+}$ is non-characteristic at ρ_r. If $(\Lambda_{+}^{+}+\Lambda_{-}^{+})^{-1}$ denotes a microlocal inverse of $\Lambda_{+}^{+}+\Lambda_{-}^{+}$ at (x',ξ'). We call

$$R = -(\Lambda_{+}^{+}+\Lambda_{-}^{+})^{-1}(\Lambda_{+}^{-}+\Lambda_{-}^{+})\in\Psi^0(\Gamma) \tag{11}$$

a *reflection operator* at ρ_r. From the formulas for the principal symbols of the Neumann operators given above we obtain

$$\sigma(R) = \frac{\cot\theta_{+}-\cot\theta_{-}}{\cot\theta_{+}+\cot\theta_{-}} \tag{12}$$

Let γ be a ray. Let $s_0 < s_r < s_1$ such that $\gamma(s_0),\gamma(s_1)\in\mathcal{H}$, $\gamma(s_r)\in\mathcal{H}_+\cap\mathcal{H}_-$, and such that γ restricted to the open intervals $]s_0,s_r[$ and $]s_r,s_1[$ is a bicharacteristic. Then we call γ a *primary ray from* $\gamma(s_0)$ *to* $\gamma(s_1)$ *via* $\gamma(s_r)$. $\gamma(s_r)$ is the reflection point of the primary ray. Define C_U as the set of all $(\rho_1,\rho_0)\in T^*(\Upsilon)\setminus 0\times T^*(\Upsilon)\setminus 0$ such that there is a primary ray from ρ_0 to ρ_1. C_U is locally closed canonical relation. Let W_α denote the set of $(\rho_1,\rho_0)\in T^*(\Upsilon)\setminus 0\times T^*(\Upsilon)\setminus 0$ such that there exists a ray from ρ_0 to ρ_1 which is not a primary ray.

Proposition 2 *Let $(\rho_1, \rho_0) \in C_U \setminus \overline{W_\alpha}$. Let γ be the primary ray from ρ_0 to ρ_1 via the reflection point ρ_r. Then, microlocally near (ρ_1, ρ_0),*

$$U \equiv (I - (\Lambda^+)^{-1}\Lambda^-)\iota^* E_+^+ R \iota_+^* E^+ (\Lambda^+)^{-1} \tag{13}$$

where R is the reflection operator at ρ_r. Λ^+ and Λ^- are Neumann operators, and $(\Lambda^+)^{-1}$ is a microlocal inverse of Λ^+ at ρ_1 and ρ_0. E_+^+ is a forward parametrix at ρ_r. In particular, we have $U \in I^{-1}(\Upsilon \times \Upsilon; C_U')$ near (ρ_1, ρ_0).

Proof. We construct an approximate solution to (5) microlocally along γ following the standard procedures. (See Chazarain [5] and Nosmas [15].) For brevity we leave to the reader the insertion of appropriate pseudo-differential cutoffs. Let $f \in \mathcal{E}'(\Upsilon)$. Without loss of generality we assume that $\mathrm{WF}(f)$ is contained in a neighbourhood of ρ_0 which is small enough not to contain ρ_1 and is such that the backward bicharacteristics issued from $\mathrm{WF}(f)$ are not in the wavefront set of u. Set $w_0 = E^+(i\Lambda^+)^{-1} f$ in Ω_+ and $w_0 = 0$ in Ω_-. Then $\rho_0 \notin \mathrm{WF}_b(u - w_0)$. (See Melrose [11] or Hörmander [8] for the definition of boundary wavefront sets.) Define $h_0 = \iota_+^* w_0$. Then $\rho_r \notin \mathrm{WF}_b(w_0 - E_+^- h_0)$. Let $h_+ = R h_0$ and $h_- = (I + R) h_0$. Then $\rho_r \notin \mathrm{WF}(h_0 + h_+ - h_-)$ and $\rho_r \notin \mathrm{WF}(\Lambda_+^- h_0 + \Lambda_+^+ h_+ + \Lambda_-^+ h_-)$. It follows that $w_0 + w_1$ satisfies the transmission conditions microlocally at ρ_r where $w_1 = E_+^+ h_+$ in Ω_+ and $w_1 = E_-^+ h_-$ in Ω_-. Hence $\rho_r \notin \mathrm{WF}_b(w_0 + w_1)$. Now define $h_1 = \iota^* w_1$. Then $\rho_1 \notin \mathrm{WF}_b(E^- h_1 - (w_0 + w_1))$. Let h_2 be a solution of $\Lambda^+ h_2 + \Lambda^- h_1 \equiv 0$ at ρ_1. Define $w_2 = E^+ h_2$ in Ω_+ and $w_2 = 0$ in Ω_-. Then $\rho_1 \notin \mathrm{WF}(\iota^*(\partial_n w_1 + \partial_n w_2))$. Define $w = w_0 + w_1 + w_2$. Then $\rho_1 \notin \mathrm{WF}(\iota^* \partial_n w)$. From results on propagation of singularities it now follows that $\rho_1 \notin \mathrm{WF}_b(u - w)$. Summarizing we have proved (13). The last statement follows from the calculus of Fourier integral operators.

Remark. The singularities of U corresponding to primary rays are called primary reflections in seismic exploration. They are often, but not always, identified as first arrivals in seismograms. The most notable exception to this are the head waves. These correspond to rays which partially travel as glancing rays in the interface Z. They occur when the wave speed in the lower layer is greater than the wave speed in the upper layer.

3 The Formally Linearized Inverse Problem

Let $\beta \in C^\infty(\overline{X})$ positive. We now consider the wave equation in a smooth "background" medium defined by β where the interface Z is absent.

$$\begin{aligned} \Delta v - \beta \partial_t^2 v &= 0 \quad \text{in } \Omega, \\ \partial_n v &= f \quad \text{on } \Upsilon. \end{aligned} \tag{14}$$

This problem is actually a special case of (5). Let F denote the continuous linear map $\mathcal{E}'(\Upsilon) \to \mathcal{D}'(\overline{\Omega})$ where $Ff = v$ is the unique solution of (14) which vanishes initially. We study the formal derivative at β of the map which sends β to $\iota^* F$. To define it we introduce the problem

$$\begin{aligned} \triangle \dot{v} - \beta \partial_t^2 \dot{v} &= \dot{\beta} \partial_t^2 v \quad \text{in } \Omega, \\ \partial_n \dot{v} &= 0 \quad \text{on } \Upsilon. \end{aligned} \tag{15}$$

Given $\beta \in C_0^\infty(X)$ we define $L\beta$ as the Schwartz kernel of the linear continuous map which maps $f \in \mathcal{E}'(\Upsilon)$ to the boundary value $\dot{v}|_\Upsilon$ where $\dot{v} \in \mathcal{D}'(\overline{\Omega})$ is the solution of (15) with v determined from (14). Both v and $\dot{v}$ are assumed initially vanishing. L is a continuous linear map $C_0^\infty(X) \to \mathcal{D}'(\Upsilon \times \Upsilon)$. Here we use the fact that the boundary problem

$$\begin{aligned} \triangle w - \beta \partial_t^2 w &= g \quad \text{in } \Omega, \\ \partial_n w &= 0 \quad \text{on } \Upsilon, \end{aligned} \tag{16}$$

has a unique initially vanishing solution $w \in \mathcal{D}'(\overline{\Omega})$ for every $g \in \mathcal{D}'(\Omega)$ which vanishes initially and satisfies $\operatorname{supp}(g) \subset \mathbf{R} \times K$ for some $K \subset X$ compact. Moreover the map G defined by $Gg = w$ is linear and continuous.

Given $\dot{\beta} \in C_0^\infty(X)$ let $M\dot{\beta}$ be the Schwartz kernel of the linear continuous map which maps f to $\dot{\beta}\partial_t^2 v$ where v is the initially vanishing solution of (14). M is a linear continuous map $C_0^\infty(X) \to \mathcal{D}'(\Omega \times \Upsilon)$. We have $L\dot{\beta} = \iota^* G M \dot{\beta}$, $\dot{\beta} \in C_0^\infty(X)$, where $L\dot{\beta}$ and $M\dot{\beta}$ are regarded as operators.

We define (β-)rays and a ray relation C_β for (14) in the same way as we defined rays and C_α for (5). We assume that $\beta = \alpha$ on Y. Then the hyperbolic sets in $T^*(\Upsilon)$ for (5) and (14) are equal. We make the general assumption that a ray which passes over a glancing point stays completely over the boundary. This assumption can be stated as follows:

$$(\rho_1, \rho_0) \in C_\beta \cap (T^*(\Omega) \times T^*(\Upsilon)) \Longrightarrow \rho_0 \in \mathcal{H} \tag{17}$$

When these assumptions hold we call β an *admissible background*.

Let $\rho_0, \rho_1 \in T^*(\Upsilon) \setminus 0 \cup T^*(\Omega) \setminus 0$ and $\nu \in T^*(X) \setminus 0$. Then, by definition, $(\rho_1, \rho_0, \nu) \in C_{sc}$ if there exist $\rho_+, \rho_- \in T^*(\Omega)$ such that

$$(\rho_-, \rho_0), (\rho_1, \rho_+) \in C_\beta$$

and

$$\rho_\pm = (t, x; \tau, \xi_\pm), \quad \nu = (x; \xi_+ - \xi_-). \tag{18}$$

Let γ_- and γ_+ be the associated rays from ρ_0 to ρ_- and from ρ_+ to ρ_1, respectively. We then call the pair (γ_+, γ_-) a (singly) *scattered ray* from ρ_0

to ρ_1. Let C_L be the subset of C_{sc} for which γ_- and γ_+ are bicharacteristics instead of generalized bicharacteristics. Then $\rho_0, \rho_1 \in \mathcal{H}$ because of (17). Let W_β be the set of $(\rho_1, \rho_0) \in T^*(\Upsilon) \setminus 0 \times T^*(\Upsilon) \setminus 0$ such that either $(\rho_1, \rho_0) \in C_\beta$ holds or there exists some ν with $(\rho_1, \rho_0, \nu) \in C_{sc} \setminus C_L$.

Proposition 3 *If $(\rho_1, -\rho_0, -\nu) \in WF(\mathcal{K}_L)$ then either $(\rho_1, \rho_0) \in C_\beta$ or $(\rho_1, \rho_0, \nu) \in C_{sc}$. Let $\chi \in \Psi^0(\Upsilon \times \Upsilon)$ compactly supported with $WF(\chi) \cap \overline{W_\beta}' = \emptyset$. Then*

$$\chi L \in I^{(n-2)/4}(\Upsilon \times \Upsilon \times X; C_L'').$$

Here C_L'' is obtained from C_L by multiplying the second and third (fiber) component by -1. Furthermore, if χ is non-characteristic at (ρ_1, ρ_0) and if there is exactly one $\nu \in T^*(X) \setminus 0$ such that $(\rho_1, \rho_0, \nu) \in C_L$ then χL is non-characteristic at $(\rho_1, -\rho_0, -\nu)$.

Proof. The kernel of M can be written

$$\mathcal{K}_M = \mu^*(\mathcal{K}_I \otimes \partial_t^2 \mathcal{K}_F) \tag{19}$$

where I is the identity operator on $\mathcal{D}'(X)$, and where μ^* is the pull-back by $\mu((t,x),(s,y),z) = (x, z, (t,x),(s,y))$, $t, s \in \mathbb{R}$, $x, z \in X$, and $y \in Y$. Recall $\mathrm{WF}'(\mathcal{K}_F) \subset C_\beta$. Since the pull-back of the wavefront set of the tensor product in (19) does not intersect the zero-section the right-hand side in (19) is indeed well-defined. If $(\rho_+, -\rho_0, -\nu) \in \mathrm{WF}(\mathcal{K}_M)$ with $\rho_+ = (t, x, \tau, \xi_+) \in T^*(\overline{\Omega}) \setminus 0$, $\rho_0 \in T^*(\Upsilon) \setminus 0$, and $\nu = (z, \zeta) \in T^*(X) \setminus 0$ then one of the following three cases must hold: (i) $\zeta \neq 0$ and $(\rho_-, \rho_0) \in C_\beta$ where $\rho_- = (t, x, \tau, \xi_-)$, $\xi_+ = \xi_- + \zeta$; (ii) $\zeta = 0$ and $(\rho_+, \rho_0) \in C_\beta$; (iii) $\rho_+ = (t, x, 0, \zeta)$ with $\zeta \neq 0$.

The kernel of L can be written

$$\mathcal{K}_L = \pi_* \delta^*(\mathcal{K}_{\iota^* G} \otimes \mathcal{K}_M) \tag{20}$$

where δ^* is the pull-back by the diagonal map

$$\delta((s,y),(t,x),(\hat{s},\hat{y}),z) = ((s,y),(t,x),(t,x),(\hat{s},\hat{y}),z)$$

and where π^* is the push-forward by the natural projection

$$\pi((s,y),(t,x),(\hat{s},\hat{y}),z) = ((s,y),(\hat{s},\hat{y}),z).$$

Let $(\rho_1, -\rho_0, -\nu) \in \mathrm{WF}(\mathcal{K}_L)$. Then there exists ρ_+ such that $(\rho_+, -\rho_0, -\nu) \in \mathrm{WF}(\mathcal{K}_M)$ and $(\rho_1, -\rho_+) \in \mathrm{WF}(\mathcal{K}_{\iota^* G})$. From results on the propagation of singularities we have $\mathrm{WF}'(\mathcal{K}_{\iota^* G}) \subset C_\beta$. In particular, ρ_+ is characteristic. This excludes (iii) of the cases stated above. If (i) holds we obtain $(\rho_1, \rho_0, \nu) \in C_{sc}$. If (ii) holds we obtain $(\rho_1, \rho_0) \in C_\beta$. The first assertion in the proposition is proved.

Now assume, in addition, $(\rho_1, \rho_0, \nu) \in C_L$ and $(\rho_1, \rho_0) \notin \overline{W_\beta}$. Then the kernels in (19) and in (20) are Lagrangian distributions near the points which contribute to $\mathrm{WF}(\mathcal{K}_L)$ near $(\rho_1, -\rho_0, -\nu)$. $\mathcal{K}_F$, $\mathcal{K}_{\iota^* G}$, μ^*, and $\pi_* \delta^*$ are Fourier integral operators of orders $-5/4$, $-5/4$, $n/4$, and 0, respectively. The composition calculus of Fourier integral operators applied microlocally to (19) and (20) implies that $\mathcal{K}_L$ is a Lagrangian distribution of order $(n-2)/4$ associated with the Lagrangian manifold C_L''.

The principal symbols of $\mathcal{K}_F$ and of $\mathcal{K}_G$ do not vanish because they are solutions to first order linear ordinary differential equations, the transport equations, with non-zero initial conditions. The last assertion of the proposition follows from this. The proof of the proposition is complete.

The locations of receivers and (point) sources often do not range over all $Y \times Y$ but only over a submanifold $S \subset Y \times Y$. The particular case where $S = \Delta(Y)$, the diagonal in $Y \times Y$, is called the zero-offset configuration in geophysics. We then define

$$L_S = \kappa_S^* L \tag{21}$$

where κ_S^* is the pull-back operator for the map

$$\kappa_S : \mathbb{R} \times S \to \Upsilon \times \Upsilon, \quad (t, (r, s)) \to (t, r, 0, s). \tag{22}$$

κ_S^* is a Fourier integral operator of order $n/4$ with canonical relation $C_{\kappa_S^*}$ the twisted conormal bundle of the graph of κ. κ_S^* is defined on half-densities. It corresponds to the restriction of functions via the identification of functions and half-densities with the induced metric on S. It follows from Proposition 3 that the dual components of time do not vanish on $\mathrm{WF}(L)$. Hence, by general results about wavefront sets and operators, the composition (21) is well-defined. Define the composition of C_L and $C_{\kappa_S^*}$ by

$$C_{L_S} = \{(\sigma, \nu);\ \exists\, (\sigma, \rho_1, \rho_0) \in C_{\kappa_S^*}, (\rho_1, -\rho_0, \nu) \in C_L\}. \tag{23}$$

We call a closed submanifold $S \subset Y \times Y$ a *source-receiver manifold* if C_{L_S} is locally the graph of a canonical transformation, and if the canonical projection $C_{L_S} \to T^*(X)$ is injective. Note that $\dim(S) = \dim(Y)$ must hold if C_{L_S} is a source-receiver manifold.

Corollary 1 *Let β be an admissible background. Let S be a source-receiver manifold. Let $\chi \in \Psi^0(\mathbb{R} \times S)$ with $WF(\chi) \cap \overline{C_{\kappa_S^*}(W_\beta')} = \emptyset$. Then $\chi L_S \in I^{(n-1)/2}(\mathbb{R} \times S \times X; C_{L_S}')$. Furthermore, $L_S^* \chi L_S$ is a pseudo-differential operator of order $n-1$ which is non-characteristic at $\nu \in T^*(X) \setminus 0$ for which there exists a unique $\sigma \in T^*(\mathbb{R} \times S) \setminus 0$ such that $(\sigma, \nu) \in C_{L_S}$ and χ is non-characteristic at σ. Here L_S^* denotes the adjoint of L_S.*

Proof. The assertions follow from Proposition 3 and the calculus of Fourier integral operators.

Beylkin [1] and Rakesh [16] proved results similar to Corollary 1. Beylkin assumed the existence of a globally defined traveltime function. This assumption implies that there are no caustics on rays over the background medium. If a globally defined traveltime function T exists then L_S can be written

$$L_S(t,r,s) = \int_{-\infty}^{\infty} e^{i\omega(T(r,x)+T(x,s))} A(r,s,x,\omega)\, d\omega \tag{24}$$

with A determined by solving transport equations along the bicharacteristics γ_- and γ_+. In [1] L_S is called a generalized causal Radon transform, and a microlocal inverse $(L_S^* L_S)^{-1} L_S^*$ is called a generalized backprojection operator. Rakesh [16] derives mapping properties of L_S and its microlocal inverses. His assumptions do not exclude caustics.

Remark. The emphasis in [1], [2], and [3] is on formulas which allow numerical computation of backprojection operators. The computation of T and A for (24) requires ray tracing. In addition, the principal symbol of $L_S^* L_S$ is needed. It is obtained from the formula for the principal symbol of a product of Fourier integral operators. For an explicit formula in the special case (24) see Beylkin [1]. The formula involves a determinant of a matrix consisting of first and second order derivatives of the traveltime function T. In seismics the algorithms of dynamic ray tracing are used to compute this type of data. (See Červený [4] for dynamic ray tracing.)

Corollary (1) essentially states that L_S is microlocally invertible if S is a source-receiver manifold. Then we have $\dim(S) = \dim(Y)$. One can ask if L_S is also invertible if $S = Y \times Y$. Beylkin and Burridge [2] study this problem for the acoustic wave equation

$$\kappa \partial_t^2 u - \nabla \cdot \sigma \nabla u = 0$$

and for the system of isotropic elastodynamics. The reason for taking a large source-receiver configuration is to have more information so that two coefficients can be recovered. The essential problem is to compute $L_S^* L_S$ or $L^* L$ symbolically. The following fact enables one to use the calculus of Fourier integral operators for clean compositions.

Proposition 4 *Assume β is an admissible background. Then $C_L^{-1} \times C_L$ intersects the diagonal*

$$T^*(X) \times \Delta(T^*(\Upsilon)) \times T^*(X).$$

cleanly with excess n. The fiber of the intersection over a point in $C_L^{-1} \circ C_L$ is diffeomorphic to $\mathbb{R}$ times the unit $n-1$ sphere.

Proof. Let $\nu = (x, \zeta) \in T^*(X) \setminus 0$ such that $(\nu, \nu) \in C_L^{-1} \circ C_L$. Then, because of (17), the fiber over (ν, ν) of the intersection is in one-to-one correspondence with the set of all (ρ_1, ρ_0) such that $(\rho_1, \rho_0, \nu) \in C_L$. Hence the fiber is diffeomorphic to the set of all (ρ_+, ρ_-) which satisfy (18). Hence it is diffeomorphic to the set $(\xi_+, \xi_-) \in \mathbb{R}^{2n}$ which satisfy for $\zeta \in \mathbb{R}^n$ fixed

$$|\xi_+|/\sqrt{\alpha_+} = |\xi_-|/\sqrt{\alpha_-} \neq 0,\ \xi_+ - \xi_- = \zeta.$$

Hence the fiber is parametrized by $t \in \mathbf{R}$ and $\xi_+ \in \mathbb{R}^n$ with $|\xi_+| = 1$.

As a consequence of proposition (4) the formula for the principal symbol of the composition $L_S^* L_S$, where $S = Y \times Y$, involves integration over $n-1$ dimensional spheres. This formula was derived, without using the calculus of Fourier integral operators, by Beylkin and Burridge [2].

4 Inversion for Jump Discontinuities

A scattered ray is determined at its scattering point by (18). When we compare this with the law of reflection for primary rays we see that both lead to the same ray relation if $\beta = \alpha$ in X_+ and $\nu \in N^*(Z)$. Therefore if $\dot\beta$ is a distribution with only conormal singularities at Z then, away from the exceptional points, U and $L\dot\beta$ are Fourier integral operators with associated canonical relation C_L. Given a source-receiver manifold S we define the sets of exceptional points with respect to S, $W_{\alpha S} = C_{\kappa_S^*}(W_\alpha{}')$ and $W_{\beta S} = C_{\kappa_S^*}(W_\beta{}')$. Furthermore, we define the *trace on* S as the distribution $u_S = \kappa_S^* U \in \mathcal{D}'(\mathbb{R} \times S)$, where κ_S is defined in (22). It is now natural to ask if a backprojection operator applied to u_S produces a perturbation $\dot\beta$ with conormal singularities at Z, and, if this is so, if the strengths of the singularities of $\dot\beta$ are related to the size of the jump of α at Z.

Functions which are smooth except for a jump discontinuity at Z are conormal at Z, more precisely, $C^\infty(\overline{X_+} \dot\cup \overline{X_-}) \subset I^{-n/4-1/2}(X; Z)$. The orientation of Z defines a decomposition of the conormal bundle of Z into a positive and a negative side, $N^*(Z) = N_+^*(Z) \dot\cup N_-^*(Z)$. If ψ is a function on X which vanishes on Z and is positive (resp. negative) on X_+ then $d\psi(z) \in N_+^*(Z)$ (resp. $d\psi(z) \in N_-^*(Z)$) at $z \in Z$.

Theorem 1 *Assume β is an admissible background with $\beta = \alpha$ in X_+. Let $S \subset Y \times Y$ be a source-receiver manifold. Let $\chi \in \Psi^0(\mathbb{R} \times S)$ compactly supported with $WF(\chi) \cap (\overline{W_{\alpha S}} \cup \overline{W_{\beta S}}) = \emptyset$. Then, if u_S is the trace on S, $L_S^* \chi u_S \in I^{3n/4-3/2}(X; Z)$ and $WF(L_S^* \chi u_S) \cap N_+^*(Z) = \emptyset$. Let $\dot\beta \in C^\infty(\overline{X_+} \dot\cup \overline{X_-})$ with $\dot\beta = 0$ in X_+ and*

$$\dot{\beta}_-(z) = -4(2\pi)^{1/4}\alpha_+(z)\cos^2\theta_+\frac{\cot\theta_+ - \cot\theta_-}{\cot\theta_+ + \cot\theta_-} \quad \text{if } z \in Z. \tag{25}$$

Here θ_+ is the angle of reflection of the primary ray reflected at $z \in Z$. The angle of refraction θ_- is given by Snell's law (8). Then $L_S^*\chi L_S\dot{\beta} \in I^{3n/4-3/2}(X;Z)$, and

$$L_S^*\chi u_S - L_S^*\chi L_S\dot{\beta} \in I^{3n/4-5/2}(X;Z). \tag{26}$$

The pseudo-differential operator $L_S^*\chi L_S \in \Psi^{n-1}(X)$ is non-characteristic at those points in $N_-^*(Z)$ which are related via C_{L_S} to non-characteristic points of χ.

Remark. Formula (25) corresponds to formula (47) in Bleistein et al. [3]. A backprojection operator for L_S applied to the trace u_S produces a modified jump discontinuity. Here modified means that only half of the wavefront set of a full jump discontinuity is present.

Proof. Let $(\rho_1, \rho_0) \in C_U$. Let γ be the primary ray from ρ_0 to ρ_1 via the reflection point $\rho_r \in \mathcal{H}_+ \cap \mathcal{H}_-$. Let $\rho_\pm = \gamma(t_r \pm 0) \in T^*(\Omega)$ where $\gamma(t_r) = \rho_r$. Throughout the proof we use local coordinates $x = (x', x_n) = (x_0, x'', x_n)$ as introduced in section 3 such that the specular point, i.e., the X-component of ρ_r, becomes the origin. Then $\rho_r = (x'; \xi')$ with $x'' = 0$, $\xi_0 > 0$, and $\rho_\pm = (x', 0; \xi', \mp\xi_n)$ where $\xi_n > 0$ and $p(\rho_\pm) = 0$. Let $\nu = (0; -2\xi_n)$. Then (18) holds and $\nu \in N^*(Z)$. Then $(\rho_1, \rho_0, \nu) \in C_L$. The associated scattered ray and γ define the same bicharacteristics. Here we use that $\beta = \alpha$ in X_+. Assume that $(\rho_1, \rho_0) \notin C_\beta$ and that $(\rho_1, \rho_0, \tilde{\nu}) \in C_L$ only if $\tilde{\nu} = \nu$.

Given b conormal with respect to Z, $b \in I^m(X;Z)$, we define the multiplication operator $B_b w = bw$ on $w \in \mathcal{D}'(\Omega)$ with $\mathrm{WF}(w) \cap N^*(\Gamma) = \emptyset$. Here we regard b as a distribution on Ω. Microlocally at (ρ_1, ρ_0), Lb is a Fourier integral operator associated with the canonical relation C_U. Arguing as in the proof of proposition 2 we obtain a factorization at (ρ_1, ρ_0)

$$Lb = (I - (\Lambda^+)^{-1}\Lambda^-)\iota^* G B_b \partial_t^2 E^+ (\Lambda^+)^{-1}. \tag{27}$$

(13) and (27) differ only by the factors $E_+^\dagger R\iota_+^*$ and $GB_b\partial_t^2$. We show that the principal symbols of the operators are equal at (ρ_1, ρ_-) if $b = \dot{\beta}$.

The coordinates x induce coordinates (x, z) on the product $\Omega \times \Omega$ and canonical coordinates $((x;\xi), (z;\zeta))$ on $T^*(\Omega) \times T^*(\Omega)$ near (ρ_+, ρ_-). Let C_1 be the H_p-flowout into $x_n \geq 0$ of the set given by the equations

$$x = z,\ x_n = 0,\ \xi' = \zeta',\ p(x, \xi) = 0. \tag{28}$$

Note that $H_p x_n > 0$ at ρ_+. C_1 is canonical relation with boundary ∂C_1 given by (28). The tangent space $\lambda_1 = T_{(\rho_+,\rho_-)}C_1$ is given by the equations

$$\lambda_1: \quad \delta x' = \delta z', \ \delta z_n = 0, \ \delta \xi' = \delta \zeta', \ p'_x \delta x + p'_\xi \delta \xi = 0. \tag{29}$$

We use the coordinates (x, ζ) on C_1.

Let μ be the Lagrangian subspace of $T_{(\rho_+,\rho_-)}(T^*(\Omega) \times T^*(\Omega))$ given by the equations

$$\mu: \quad \delta\zeta = 0, \ \delta\xi' = \delta x', \ \epsilon\delta\xi_n = \delta x_n. \tag{30}$$

μ is transversal to the fiber $\delta x = \delta z = 0$ and to λ_1 if $\epsilon > 0$ is sufficiently small.

Lemma 1 *$E_+^+ \iota_+^* \in I^0(\Omega \times \Omega; C_1' \setminus \partial C_1')$ at (ρ_+, ρ_-) and*

$$\sigma(E_+^+ \iota_+^*)(\rho_+, \rho_-; \mu) = e^{-\frac{\pi i}{4}(n+1)} |dx d\zeta|^{1/2}$$

Lemma 2 *Let $b \in I^m(X; Z)$. Then $GB_b \in I^{m+n/4-3/2}(\Omega \times \Omega; C_1' \setminus \partial C_1')$ at (ρ_+, ρ_-) and*

$$\sigma(GB_b)(\rho_+, \rho_-; \mu) = (2\pi)^{-1/4} e^{-\frac{\pi i}{4}(n-1)} \frac{\tilde{b}(\nu)}{H_p x_n(\rho_+)} |dx d\zeta|^{1/2}$$

where $\tilde{b}$ is the principal symbol of the symbol

$$\hat{b}(x'', \xi_n) = \int_{-\infty}^{\infty} e^{-ix_n\xi_n} b(x'', x_n) dx_n.$$

Strictly speaking the symbols are defined at $(\rho_+, \rho_-) \in \partial C_1$ by taking limits along γ_+.

The lemmas are proved below. Assuming them we continue with the proof of the theorem. $\dot{\beta}$ has a jump discontinuity at Z. Hence $\dot{\beta}$ is conormal of order $-n/4 - 1/2$ with respect to Z and

$$\tilde{\dot{\beta}}(0, \xi_n) = \frac{i}{\xi_n} \dot{\beta}(0, 0-).$$

Combining this with the definition of $\dot{\beta}$ we obtain, using (12), the lemmas, and $\cos^2 \theta_+ = \xi_n^2/(\alpha_+ \xi_0^2)$,

$$\sigma(E_+^+ R \iota_+^*)(\rho_+, \rho_-; \mu) = \sigma(GB_{\dot{\beta}} \partial_t^2)(\rho_+, \rho_-; \mu)$$

The principal symbols of E_+^+ and G are governed by the same transport equations on γ_+. Hence the principal symbol of $\chi(u_S - L_S\beta)$ vanishes. Recall $\nu \in N^*_-(Z)$. Observe that $L^*_S\chi L_S$ is non-characteristic at ν if χ is non-characteristic at σ where $(\sigma, \rho_1, \rho_0) \in C_{K^*_S}$. The remaining assertions of the theorem follow from this.

The formula

$$\operatorname{sgn}\begin{pmatrix} A & C \\ {}^tC & B \end{pmatrix} = \operatorname{sgn}\begin{pmatrix} A - CB^{-1}{}^tC & 0 \\ 0 & B \end{pmatrix}$$

is useful when computing signatures of symmetric matrices. It holds when B^{-1} exists. We shall use it in the proof of the lemmas to evaluate Maslov factors.

Proof of Lemma 1. We rewrite (9) as

$$(\iota^*_+ v)(z', 0) = (2\pi)^{-1}\int e^{-i\omega z_n} r(z')v(z)dz_n d\omega |dz_n|^{-1/2}$$

where $r(z') = 1$ if $z'' = 0$. In combination with (10) this yields

$$(E^+_+\iota^*_+)(x, z) = (2\pi)^{-1-n}\int e^{i\Phi}a(x, z', \vartheta)r(z')d\vartheta$$

where

$$\Phi(x, z, \vartheta) = \phi(x, z', \theta) - z_n\omega, \ \vartheta = (\theta, \omega) \in \mathbb{R}^{n+1}.$$

The principal symbol of this Lagrangian distribution is (cf. [8, Chapter 25])

$$\sigma(E^+_+\iota^*_+)(\rho_+, \rho_-; \mu) = ar\, d^{-1/2}e^{\frac{\pi i}{4}s}|dxd\zeta|^{1/2}$$

where $d = |D(x, \Phi'_z, \Phi'_\vartheta)/D(x, z, \vartheta)|$, and

$$s = \operatorname{sgn}\begin{pmatrix} \Phi''_{xx} - Q & \Phi''_{xz} & \Phi''_{x\vartheta} \\ \Phi''_{zx} & \Phi''_{zz} & \Phi''_{z\vartheta} \\ \Phi''_{\vartheta x} & \Phi''_{\vartheta z} & \Phi''_{\vartheta\vartheta} \end{pmatrix}$$

where

$$Q = \begin{pmatrix} I_n & 0 \\ 0 & 1/\epsilon \end{pmatrix}. \tag{31}$$

We have $ar = 1$ if $x = z$, $x_n = 0$, $x'' = 0$, and $\phi'_\theta = 0$. This really is a consequence of $\iota^*_+E^+_+ = I$ at ρ_r. A straightforward evaluation of determinants gives $d = 1$. The signature is $s = -\operatorname{sgn} Q = -(n+1)$. Here we have to choose $\epsilon > 0$ sufficiently small so that terms composed of derivatives of Φ with respect to x_n are suitably dominated. The proof of Lemma 1 is complete.

Proof of Lemma 2. Let C_0 be the canonical relation given by the equations $x = z$, $x_n = 0$, $\xi' = \zeta'$. The tangent space $\lambda_0 = T_{(\rho_+, \rho_-)}C_0$ is

$$\lambda_0: \quad \delta x = \delta z, \ \delta x_n = 0, \ \delta\xi' = \delta\zeta'. \tag{32}$$

(x', ξ_n, ζ) are coordinates on C_0. We first show that B_b is a Fourier integral operator near (ρ_+, ρ_-) and compute its principal symbol. The multiplication by b can be written as the inverse Fourier transform of a convolution:

$$(B_b w)(x) = (2\pi)^{-2} \int e^{i\tau x_n} \hat{b}(x'', \sigma) e^{-i(\tau-\sigma)z_n} w(x', z_n) dz_n d\sigma d\tau.$$

Now we insert for $w(x', z_n)$ its Fourier inversion formula and obtain a representation of the Schwartz kernel of B_b

$$B_b(x, z) = (2\pi)^{-2-n} \int e^{i\Phi} \hat{b}(x'', \sigma) d\sigma d\tau d\theta$$

where

$$\Phi(x, z, \theta, \tau, \sigma) = \langle \theta, x' - z' \rangle + \tau(x_n - z_n) + \sigma z_n.$$

The non-degenerate phase function Φ parametrizes C_0' near $(\rho_+, -\rho_-)$. $\hat{b}$ is a symbol of order $m + n/4 - 1/2$. Hence $B_b \in I^{m+n/4}(\Omega \times \Omega; C_0')$. The principal symbol of B_b is

$$\sigma(B_b)(\rho_+, \rho_-; \mu) = (2\pi)^{-1/2} e^{\frac{\pi i}{4} s} d^{-1/2} \hat{b}(\nu) |dx' d\xi_n d\zeta|^{1/2} \tag{33}$$

where $d = |D(x', \Phi'_{x_n}, \Phi'_z, \Phi'_\vartheta)/D(x, z, \vartheta)|$, $\vartheta = (\theta, \tau, \sigma)$, and where s is the signature of Φ'' with the block Φ''_{xx} replaced by $\Phi''_{xx} - Q$. (Q is the same matrix as in (31).) Straightforward computations show $d = 1$ and $s = -n$.

To compute GB_b we use the calculus of Melrose and Uhlmann [14]. (C_0', C_1') are an intersecting pair of Lagrangian manifolds with intersection $C_0' \cap C_1' = \partial C_1'$. We solve

$$(\Delta - \beta\partial_t^2) GB_b = B_b$$

with $GB_b \in I^{m+n/4-3/2}(\Omega \times \Omega; C_0', C_1')$ using the constructions of section 6 in [14]. The notations f, g, and h in section 4 of [14] correspond to x_n, p, and (x', ζ) here. We write

$$\begin{aligned} \sigma(B_b) &= r|dx' d\zeta dp|^{1/2} && (34) \\ &= r(H_p x_n)^{1/2} |dx' dz d\xi_n|^{1/2}. && (35) \end{aligned}$$

The symbol calculus of [14] gives

$$\begin{aligned} \sigma(GB_b) &= \frac{r}{p} |dx' d\zeta dp|^{1/2} \quad \text{on } C_0' \setminus \partial C_1', \\ \sigma(GB_b) &= (2\pi)^{1/4} e^{\frac{\pi i}{4}} r (H_p x_n)^{-1/2} |dx' d\zeta dx_n|^{1/2} \quad \text{on } C_1' \setminus \partial C_1' \end{aligned} \tag{36}$$

The Maslov factors are correctly accounted for if we show that μ is transversal to the natural connecting path between λ_0 and λ_1. We check this. The connecting path consists of the Lagrangian subspaces W_s, $0 \leq s \leq 1$, which are spanned by $\lambda_0 \cap \lambda_1$ and $sH_p + (1-s)H_{x_n}$. Let $(\delta x, \delta\xi, \delta z, \delta\zeta) \in W_s \cap \mu$. Then $\delta x' = \delta z'$, $\delta z_n = 0$, and $\delta\xi' = \delta\zeta'$ because $W_s \subset \lambda_0 + \lambda_1$, and therefore, using (30), we have $\delta\zeta = \delta z = 0$, $\delta\xi' = \delta x' = 0$. Furthermore,

$$sp'_{\xi_n}\delta\xi_n + (sp'_{x_n} + (1-s))\delta x_n = 0.$$

and

$$-\epsilon\,\delta\xi_n + \delta x_n = 0.$$

Since $p'_{\xi_n} > 0$ also δx_n and $\delta\xi_n$ vanish if $\epsilon > 0$ is sufficiently small.

The symbol formula stated in the lemma now follows from (33), (34), (35), and (36). The proof of Lemma 2 is complete.

Remark. An inspection of the proof of Theorem 1 shows that there is no difficulty in generalizing it to non-smooth background media provided the non-smoothness arises only as jump discontinuities across smooth hypersurfaces.

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