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The Mayer–Vietoris and the Puppe sequences in K-theory for C^*-algebras

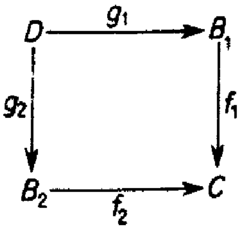
by

J. HILGERT (Darmstadt)

Abstract. We show the existence of a Mayer–Vietoris and a Puppe sequences in the K-theory for C^*-algebras. Both sequences generalize the respective sequences in the commutative case in the sense that they reduce to those sequences under the identification $K_\alpha(C_0(X)) = \tilde{K}^\alpha(X)$ if all algebras involved are chosen to be commutative, i.e. of the form $C_0(X)$ for a locally compact space X. The sequences are used to calculate the K-theory of certain bundle-C^*-algebras with continuous identity field.

0. Notation and preliminaries. For any C^*-algebra A call $SA := \{f\colon [0, 1] \to A$ continuous, $f(0) = 0 = f(1)\}$ the *suspension* of A. For two C^*-algebras A and B we say that two morphisms $\varphi_i\colon A \to B$, $i = 0, 1$, are *homotopic* if there exists a family $\Phi_t\colon A \to B$ of morphisms for $t \in [0, 1]$ such that $\Phi\colon I \times A \to B$ defined by $\Phi(t, a) = \Phi_t(a)$ is jointly continuous and $\Phi_i = \varphi_i$ for $i = 0, 1$. We write $\varphi_1 \simeq \varphi_0$. The morphism $\varphi\colon A \to B$ is called a *homotopy equivalence* if there exists a morphism $\psi\colon B \to A$ such that $\varphi \circ \psi \simeq \mathrm{id}_B$ and $\psi \circ \varphi \simeq \mathrm{id}_A$. A C^*-algebra C is called *contractible* if $\mathrm{id}_C \simeq 0\colon C \to C$. Recall (cf. [3]) that the K-functor does not distinguish homotopic morphisms. Thus homotopy equivalences induce isomorphisms and contractible C^*-algebras have vanishing K-groups.

I. Mayer–Vietoris sequence. Let B_1, B_2 and C be C^*-algebras and $f_i\colon B_i \to C$ C^*-morphisms for $i = 1, 2$. Suppose f_2 is onto. Consider the pullback



The C^*-algebra D can be written as $\{(b_1, b_2) \in B_1 \oplus B_2\colon f_1(b_1) = f_2(b_2)\}$. Then there is a natural inclusion $j\colon D \to B_1 \oplus B_2$. The map j induces group homomorphisms $j_*\colon K_*(D) \to K_*(B_1) \oplus K_*(B_2)$.

We define group homomorphisms $v_*\colon K_*(B_1) \oplus K_*(B_2) \to K_*(C)$ by $v_*\colon (f_1)_* - (f_2)_*$, where $(f_i)_*\colon K_*(B_i) \to K_*(C)$ is the group homomorphism induced by f_i for $i = 1, 2$. This means, for $\mathfrak{b}_i \in K_*(B_i)$, that $v_*(\mathfrak{b}_1 \oplus \mathfrak{b}_2) = (f_1)_*(\mathfrak{b}_1) - (f_2)_*(\mathfrak{b}_2)$.

There are two more maps which play an important role in the Mayer–Vietoris sequence. We show the construction of α_0: $K_0(C) \to K_1(D)$; the map α_1: $K_1(C) \to K_0(D)$ is constructed analogously.

Note first that there is a natural isomorphism between $\ker f_2$ and $\ker g_1$. Let l: $\ker f_2 \to D$ be the inclusion induced by that isomorphism. Note also that the surjectivity of f_2 implies that g_1 is onto. Thus we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f_2 & \xrightarrow{l} & D & \xrightarrow{g_1} & B_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \longrightarrow & \ker f_2 & \xrightarrow[i]{} & B_2 & \xrightarrow[f_2]{} & C & \longrightarrow & 0 \end{array}$$

This diagram induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \xrightarrow{(g_1)_*} & K_0(B_1) & \xrightarrow{\partial_g} & K_1(\ker f_2) & \xrightarrow{l_*} & K_1(D) & \xrightarrow{(g_1)_*} & K_1(B_1) & \xrightarrow{\partial_g} \\ & \downarrow{\scriptstyle (f_1)_*} & & \| & & \downarrow{\scriptstyle (g_2)_*} & & \downarrow{\scriptstyle (f_1)_*} & \\ \xrightarrow[(f_2)_*]{} & K_0(C) & \xrightarrow[\partial_f]{} & K_1(\ker f_2) & \xrightarrow[i_*]{} & K_1(B_2) & \xrightarrow[(f_2)_*]{} & K_1(C) & \xrightarrow[\partial_f]{} \end{array}$$

Now we define α_0: $K_0(C) \to K_1(D)$ by $\alpha_0 := l_* \circ \partial_f$.

THEOREM (Mayer–Vietoris sequence, cf. [2], [6]). *Let B_1, B_2 and C be C^*-algebras, f_i: $B_i \to C$ be C^*-morphisms for $i = 1, 2$ and let D be the pullback over f_1 and f_2. Moreover, assume that f_2 is surjective. Then the following sequence is exact:*

$$\begin{array}{ccccc} K_0(D) & \xrightarrow{j_*} & K_0(B_1) \oplus K_0(B_2) & \xrightarrow{v_*} & K_0(C) \\ \uparrow{\scriptstyle \alpha_1} & & & & \downarrow{\scriptstyle \alpha_0} \\ K_1(C) & \xleftarrow[v_*]{} & K_1(B_1) \oplus K_1(B_2) & \xleftarrow[j_*]{} & K_1(D) \end{array}$$

Proof. First we show that $\operatorname{im} j_* \subset \ker v_*$. For $\mathfrak{d} \in K_*(D)$ we have

$$\begin{aligned} v_*(j_*(\mathfrak{d})) &= v_*((g_1)_*(\mathfrak{d}) \oplus (g_2)_*(\mathfrak{d})) \\ &= (f_1)_*((g_1)_*(\mathfrak{d})) - (f_2)_*((g_2)_*(\mathfrak{d})) = (f_1 \circ g_1)_*(\mathfrak{d}) - (f_2 \circ g_2)_*(\mathfrak{d}) = 0. \end{aligned}$$

The reverse inclusion is obtained by a diagram chase in the above diagram. Let $\mathfrak{b}_1 \in K_*(B_1)$ and $\mathfrak{b}_2 \in K_*(B_2)$ be such that $(f_1)_*(\mathfrak{b}_1) = (f_2)_*(\mathfrak{b}_2)$. Then $\partial_g(\mathfrak{b}_1) = \partial_f \circ (f_1)_*(\mathfrak{b}_1) = \partial_f \circ (f_2)_*(\mathfrak{b}_2) = 0$ whence there exists $\mathfrak{d} \in K_*(D)$ such that $(g_1)_*(\mathfrak{d}) = \mathfrak{b}_1$. We have $(f_2)_*(\mathfrak{b}_2 - (g_2)_*(\mathfrak{d})) = 0$ so that there exists $\mathfrak{a} \in K_*(\ker f_2)$ with $i_*(\mathfrak{a}) = \mathfrak{b}_2 - (g_2)_*(\mathfrak{d})$. Now we set $\mathfrak{d}' = \mathfrak{d} + l_*(\mathfrak{a})$ and obtain

$(g_1)_*(\mathfrak{d}') = (g_1)_*(\mathfrak{d}) + 0 = \mathfrak{b}_1$ and $(g_2)_*(\mathfrak{d}') = (g_2)_*(\mathfrak{d}) + (g_2 \circ l)_*(\mathfrak{a}) = (g_2)_*(\mathfrak{d}) + i_*(\mathfrak{a}) = \mathfrak{b}_2$. Thus we have proved that $\operatorname{im} j_* = \ker v_*$.

It remains to be shown that the Mayer–Vietoris sequence is exact at the corners. We show that for the right side, the left side is proved analogously. To see that $\operatorname{im} v_* \subset \ker \alpha_*$ calculate for $\mathfrak{b}_i \in K_*(B_i)$ that $\alpha_*((f_1)_*(\mathfrak{b}_1) - (f_2)_*(\mathfrak{b}_2)) = \alpha_*((f_1)_*(\mathfrak{b}_1)) - \alpha_*((f_2)_*(\mathfrak{b}_2)) = l_* \circ \partial_g(\mathfrak{b}_1) - l_* \circ \partial_f \circ (f_2)_*(\mathfrak{b}_2) = 0$.

The reverse inclusion again requires a little diagram chase. Suppose, for $\mathfrak{c} \in K_*(C)$, that $\alpha_*(\mathfrak{c}) = 0$. Then $l_* \circ \partial_f(\mathfrak{c}) = 0$ and there exists a $\mathfrak{b}_1 \in K_*(B_1)$ with $\partial_g(\mathfrak{b}_1) = \partial_f(\mathfrak{c})$. Therefore $\partial_f((f_1)_*(\mathfrak{b}_1) - \mathfrak{c}) = \partial_g(\mathfrak{b}_1) - \partial_f(\mathfrak{c}) = 0$. This in turn implies that there exists a $\mathfrak{b}_2 \in K_*(B_2)$ with $(f_2)_*(\mathfrak{b}_2) = (f_1)_*(\mathfrak{b}_1) - \mathfrak{c}$, thus $\mathfrak{c} = v_*(\mathfrak{b}_1 \oplus \mathfrak{b}_2)$.

The inclusion $\operatorname{im} \alpha_* \subset \ker j_*$ is seen from the following calculation for $\mathfrak{c} \in K_*(C)$. We have $j_*(\alpha_*(\mathfrak{c})) = j_*(l_* \circ \partial_f(\mathfrak{c})) = (g_1)_* \circ l_* \circ \partial_f(\mathfrak{c}) \oplus (g_2)_* \circ l_* \circ \partial_f(\mathfrak{c}) = 0 \oplus i_* \circ \partial_f(\mathfrak{c}) = 0$.

Finally we get the reverse inclusion again by diagram chasing. Note that $\ker j_* = \ker(g_1)_* \cap \ker(g_2)_*$. Thus for $\mathfrak{d} \in \ker j_*$ there exists an $\mathfrak{a} \in K_*(\ker f_2)$ with $l_*(\mathfrak{a}) = \mathfrak{d}$. We get $i_*(\mathfrak{a}) = (g_2)_* \circ l_*(\mathfrak{a}) = 0$ and hence there exists a $\mathfrak{c} \in K_*(C)$ with $\partial_f(\mathfrak{c}) = \mathfrak{a}$. This implies that $\alpha_*(\mathfrak{c}) = l_* \circ \partial_f(\mathfrak{c}) = l_*(\mathfrak{a}) = \mathfrak{d}$. This concludes the proof. ■

II. Puppe sequence.

Definition. Let A and B be C^*-algebras and $\varphi\colon B \to A$ a C^*-morphism. Define the *mapping cone*, denoted by C_φ, as follows:

$$C_\varphi = \{(b, f) \in B \oplus P(A)\colon \varphi(b) = f(0), f(1) = 0\},$$

where $P(A) := \{f\colon I \to A \text{ continuous}\}$ is the algebra of paths in A.

Given the map $i\colon SA \to C_\varphi$ defined by $i(f) = (0, f)$ we get a sequence of C^*-algebras which we call the *Puppe sequence*:

$$SB \xrightarrow[S\varphi]{} SA \xrightarrow[i]{} C_\varphi \xrightarrow[v_\varphi]{} B \xrightarrow[\varphi]{} A$$

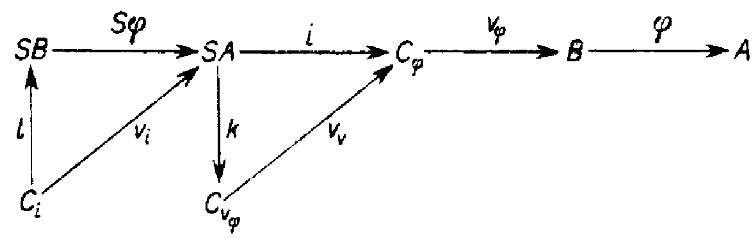
where $S\varphi(g) := \varphi \circ g$ and $v_\varphi((b, f)) := b$.

Theorem. *The Puppe sequence induces the following exact sequence in K-theory:*

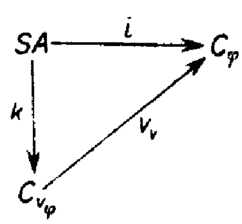
$$\begin{array}{ccccc} K_0(C_\varphi) & \longrightarrow & K_0(B) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(B) & \longleftarrow & K_1(C_\varphi) \end{array}$$

Proof. First we show that we can replace $K_*(SB)$ and $K_*(SA)$ by $K_*(C_i)$ and $K_*(C_{v_\varphi})$ respectively. In fact, we construct maps $k\colon SA \to C_{v_\varphi}$

and $l\colon C_i \to SB$ that induce isomorphisms in K-theory and give a diagram of the following kind that is commutative up to homotopy:



We have $C_{v_\varphi} = \{(b, f, g) \in B \oplus P(A) \oplus P(B)\colon \varphi(g(0)) = f(0),\ f(1) = 0,\ b = g(0),\ g(1) = 0\}$ which we can identify with $\{(f, g) \in P(A) \oplus P(B)\colon \varphi(g(0)) = f(0), f(1) = 0, g(1) = 0\}$. There is a map $k\colon SA \to C_{v_\varphi}$ defined by $k(f) := (f, 0)$. It is clearly injective. Now consider the cone $CB := \{g \in P(B)\colon g(1) = 0\}$ and the map $\mu\colon C_{v_\varphi} \to CB$ defined by $\mu((f, g)) = g$. Since $((1-t)\varphi(g(0)), g) \in C_{v_\varphi}$ for any $g \in CB$ we see that μ is surjective. Clearly $\ker \mu = k(SA)$. But the cone CB is contractible and therefore the six-term sequence associated to $0 \to SA \to C_{v_\varphi} \to CB \to 0$ shows that $k_*\colon K_*(SA) \to K_*(C_{v_\varphi})$ is an isomorphism. Moreover, the natural map $v_v\colon C_{v_\varphi} \to C_\varphi$ defined by $(f, g) \mapsto (g(0), f)$ makes the following triangle commutative:



Now consider the mapping cone $C_i = \{(f, g, F) \in SA \oplus P(C_\varphi) \subset SA \oplus P(B) \oplus \oplus P(P(A))\colon i(f) = (g(0), F(\cdot, 0)), (g(1), F(\cdot, 1)) = (0, 0)\}$. We can identify C_i with the algebra $\{(g, F) \in P(B) \oplus P(P(A))\colon \varphi \circ g = F(0, \cdot), F(1, \cdot) = F(\cdot, 1) = 0, g(0) = g(1) = 0\}$, as one easily sees, and consider the map $l\colon C_i \to SB$ given by $l(g,F) = g$. For a given $g \in SB$ set $F(s, t) := (\varphi \circ g(t))(1-s)$; then $(g, F) \in C_i$ and $l(g, F) = g$ whence l is surjective. The kernel of l is $\{(g, F) \in P(B) \oplus P(P(A))\colon F(0, \cdot) = F(1, \cdot) = F(\cdot, 1) = 0\}$ which is isomorphic to the cone $C(SA)$. Thus $l_*\colon K_*(C_i) \to K_*(SB)$ is an isomorphism. The map $v_i\colon C_i \to SA$ is given by $(g, F) \mapsto F(\cdot, 0)$. Consider the family of maps $\Phi_t\colon C_i \to SA$ defined by $\Phi_t(s) := F(s(1-t), st)$; then $\Phi_0 = v_i$, $\Phi_1(s) = F(0, s)$ and Φ_t is a homotopy. Clearly $\Phi_1 = S\varphi \circ l$.

Now it suffices to prove that any sequence $C_\varphi \xrightarrow{v_\varphi} B \xrightarrow{\varphi} A$, where C_φ is the mapping cone of φ and the map v_φ is the projection onto the first factor, induces an exact sequence in K-theory. But this sequence gives rise to the short exact sequence $0 \to SA \to C_\varphi \to B \to 0$ which in turn induces the exact sequence $K_n(C_\varphi) \to K_n(B) \to K_{n-1}(SA)$. Since the triangle

$$\begin{array}{ccc} K_n(B) & \longrightarrow & K_{n-1}(SA) \\ & \searrow^{\varphi_*} & \downarrow \\ & & K_n(A) \end{array}$$

where the vertical map is the suspension isomorphism commutes we deduce that $K_*(C_\varphi) \to K_*(B) \to K_*(A)$ is exact. Thus the following sequence is exact:

$$K_*(SB) \xrightarrow{S\varphi_*} K_*(SA) \to K_*(C_\varphi) \to K_*(B) \xrightarrow{\varphi_*} K_*(A)$$

and we remark that $C_{S\varphi} = S(C_\varphi)$ so that we can close the exact sequence to obtain the diagram in the statement of the theorem. ∎

III. Examples. The Mayer–Vietoris and the Puppe sequences can be applied to calculate the K-theory of C^*-algebras that are represented as section algebras of C^*-bundles. The following simple examples show how to calculate the K-groups of a section algebra from the K-groups of restrictions to smaller spaces.

Let $Y \subset X$ be compact spaces. Define a C^*-algebra D as the following pullback:

$$\begin{array}{ccc} D & \longrightarrow & M_{nk}(C(X)) \\ \downarrow & & \downarrow r \\ M_k(C(Y)) & \xrightarrow[d]{} & M_{nk}(C(Y)) \end{array}$$

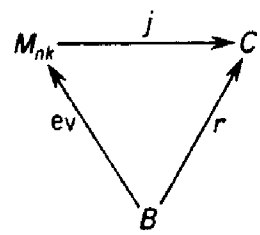
Here r simply denotes the restriction to Y and d is the map that assigns the block diagonal matrix

$$\begin{bmatrix} f & & \\ & \ddots & \\ & & f \end{bmatrix}$$

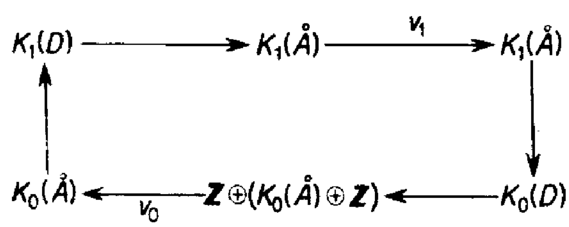
to an $f \in M_k(C(Y))$, the $k \times k$-matrix algebra over the continuous functions on Y. For the sake of brevity we define $B := M_{nk}(C(X))$, $A := M_k(C(Y))$ and $C := M_{nk}(C(Y))$. We obtain the Mayer–Vietoris sequence

$$\begin{array}{ccccc} K_1(D) & \longrightarrow & K_1(B) \oplus K_1(A) & \longrightarrow & K_1(C) \\ \uparrow & & & & \downarrow \\ K_0(C) & \longleftarrow & K_0(B) \oplus K_0(A) & \longleftarrow & K_0(D) \end{array}$$

The map d_*: $K_*(A) \to K_*(C)$ is, if we identify $K_*(A)$ with $K_*(C)$ under φ_*: $K_*(A) \to K_*(C)$ induced by the inclusion φ: $A \to C$ which maps $a \in A$ to the matrix in C that has a in the upper left corner and zeros elsewhere, just multiplication by n. If X is a contractible space and $y_0 \in Y$, the map ev: $B \to M_{nk} = M_{nk}(C)$ given as the evaluation at y_0 is a homotopy equivalence. Thus with the canonical embedding j: $M_{nk} \to C$ we get a commutative triangle up to homotopy:



Thus the triangle in K-theory induced by this one commutes, and since ev_* is an isomorphism, we can replace $K_*(B)$ by $K_*(M_{nk})$ and r_* by j_*. If we set $\mathring{A} := \{f \in A: f(y_0) = 0\}$ we get a split exact sequence $0 \to \mathring{A} \to A \to M_k \to 0$ and hence we get a split exact sequence in K-theory $0 \to K_*(\mathring{A}) \to K_*(A) \to K_*(M_k) \to 0$. Note that $K_1(M_{nk}) = K_1(M_k) = K_1(C) = 0$ and $K_0(M_{nk}) = K_0(M_k) = K_0(C) = Z$. Hence we get the following exact sequence:



where the maps v_1 and v_0 are given as follows: $v_1(\mathfrak{a}) = -n\mathfrak{a}$ for $\mathfrak{a} \in K_1(\mathring{A})$ and

$$v_0(m \oplus (\mathfrak{a} \oplus l)) = (0 \oplus m) - n(\mathfrak{a} \oplus l) = -n\mathfrak{a} \oplus (m - nl)$$

for $m \oplus (\mathfrak{a} \oplus l) \in Z \oplus (K_0(\mathring{A}) \oplus Z)$. If we assume that $K_1(\mathring{A})$ is torsion free, then v_1 is injective and therefore $K_1(D) \cong K_0(\mathring{A}) \oplus Z/\operatorname{im} v_0$. But for $\mathfrak{c}, \mathfrak{d} \in K_0(\mathring{A})$ and $m_{\mathfrak{c}}, m_{\mathfrak{d}} \in Z$ we have $\mathfrak{c} \oplus m_{\mathfrak{c}} - \mathfrak{d} \oplus m_{\mathfrak{d}} \in \operatorname{im} v_0$ if and only if there is an $\mathfrak{a} \in K_0(\mathring{A})$ and $m, l \in Z$ such that $\mathfrak{c} - \mathfrak{d} = -n\mathfrak{a}$ and $m_{\mathfrak{c}} - m_{\mathfrak{d}} = m - nl$. The condition on the integers is always satisfied, thus $K_1(D) \cong K_0(\mathring{A})/nK_0(\mathring{A}) = K_0(\mathring{A}) \otimes Z/nZ$ if $K_0(\mathring{A})$ is torsion free, too. Further, we have the exact sequence $0 \to K_1(\mathring{A})/\operatorname{im} v_1 \to K_0(D) \to \ker v_0 \to 0$. We assumed $K_0(\mathring{A})$ to be torsion free, so $\ker v_0 = \{m \oplus (\mathfrak{a} \oplus l) \in Z \oplus (K_0(\mathring{A}) \oplus Z): -n\mathfrak{a} = 0, m = nl\} \cong Z$. Thus the sequence splits and since $\operatorname{im} v_1 = nK_1(\mathring{A})$ we have $K_0(D) \cong (K_1(\mathring{A}) \otimes Z/nZ) \oplus Z$. If we now observe that $K_*(\mathring{A}) \cong K_*(C_0(Y)) = \tilde{K}^*(Y)$ we get the following

Example 1. Let $Y \subset X$ be compact spaces such that X is contractible and $\tilde{K}^*(Y)$ torsion free, and let D be the C^*-algebra of continuous functions from X into M_{nk} such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^1(Y) \otimes Z/nZ) \oplus Z$ and $K_1(D) = \tilde{K}^0(Y) \otimes Z/nZ$.

Similarly one calculates

Example 2. Let $Y \subset X$ be compact spaces with X contractible and D the algebra of continuous functions $X \to M_{nk}$ that map Y to block diagonal matrices with blocks of size $k \times k$. Then $K_0(D) = (\tilde{K}^0(Y))^{n-1} \oplus Z^n$ and $K_1(D) = (\tilde{K}^1(Y))^{n-1}$.

The assumption that X be contractible has of course been made to avoid problems in calculation which arise from the fact that we do not know the map r_*: $K_*(B) \to K_*(C)$ in general. There are some more cases where we know this map.

Example 3. Let $Y \subset X$ be compact spaces and Y a deformation retract of X. Let D be the C^*-algebra of continuous functions $X \to M_{nk}$ such that the values on Y are block diagonal matrices with identical blocks of size $k \times k$. Then $K_*(D) \cong K^*(Y)$. If the condition that the blocks be identical is dropped we have: $K_*(D) \cong K^*(X) \oplus (K^*(Y))^{n-1}$.

We have seen that torsion in the K-groups can cause trouble. In some cases we can get around that using the Puppe sequence.

Let X and Y be compact spaces and f: $Y \to X$ a continuous map. Consider the mapping cone C_f. We obtain a map f': $Y \to C_f$ which is the composition of f and the canonical map g: $X \to C_f$. Now consider the C^*-algebras $M_k(C(X))$ and $M_{nk}(C(Y))$. We get a map φ: $M_k(C(X)) \to M_{nk}(C(Y))$ by $\varphi(a) := d(a \circ f)$ [cf. Example 1 for the definition of d]. Consider the mapping cylinder M_φ given by the pullback

$$\begin{array}{ccc} M_\varphi & \longrightarrow & M_k(C(X)) \\ \big\downarrow & & \big\downarrow \varphi \\ P(M_{nk}(C(Y))) & \xrightarrow[\text{ev}]{} & M_{nk}(C(Y)) \end{array}$$

where ev is the evaluation at $t = 0$. Note that $P(M_{nk}(C(Y)))$ is canonically isomorphic to $M_{nk}(C(Y \times I))$ and $M_k(C(X))$ is canonically isomorphic to the algebra of maps $X \to M_{nk}$ whose values are block diagonal matrices with identical blocks of size $k \times k$. Thus we see that C_φ is the C^*-algebra of maps from C_f into M_{nk} whose values on $g(X)$ are block diagonal matrices with identical blocks of size $k \times k$ and which vanish on $y_0 \in C_f$, the vertex of the cone. Now it is easy to get

Example 4. Let X and Y be compact spaces and f: $Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C^*-algebra of continuous functions from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with identical blocks of size $k \times k$. Let $\mathring{D}$ be the subalgebra of D consisting of those maps that vanish on $y_0 \in C_f$, the vertex of the cone. Then $K_*(D) \cong K_*(\mathring{D})$ and we get the following exact sequence:

$$\begin{array}{ccccc}
K_1(\mathring{D}) & \longrightarrow & K^1(X) & \xrightarrow{nf^1} & K^1(Y) \\
\uparrow & & & & \downarrow \\
K^0(Y) & \xleftarrow[nf^0]{} & K^0(X) & \longleftarrow & K_0(\mathring{D})
\end{array}$$

Finally, if we drop the condition on the blocks, we get

EXAMPLE 5. Let X and Y be compact spaces and $f\colon Y \to X$ a continuous function. Let C_f be the mapping cone of f and D the C^*-algebra of maps from C_f into M_{nk} whose values on the canonical image of X in C_f are block diagonal matrices with blocks of size $k \times k$. Let $\mathring{D} := \ker \mathrm{ev}$, where ev is the evaluation at the vertex $y_0 \in C_f$. Then $K_1(D) \cong K_1(\mathring{D})$ and $K_0(D) \cong \mathbf{Z} \oplus K_0(\mathring{D})$. Moreover, we have the following exact sequence:

$$\begin{array}{ccccc}
K_1(\mathring{D}) & \longrightarrow & \bigoplus_1^n K^1(X) & \xrightarrow{\Sigma f^1} & K^1(Y) \\
\uparrow & & & & \downarrow \\
K^0(Y) & \xleftarrow[\Sigma f^0]{} & \bigoplus_1^n K^0(X) & \longleftarrow & K_0(\mathring{D})
\end{array}$$

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FACHBEREICH MATHEMATIK, TECHNISCHE HOCHSCHULE
Schlossgartenstr. 7, 6100 Darmstadt, West Germany

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