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Algebraic foundation of some distribution algebras

by

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Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

1. Introduction. Since the pioneering works of L. Schwartz, J. Mikusiński, I. M. Gelfand, G. E. Shilov and others the problem of distribution multiplication has attracted the attention of the practical-minded mathematician. Distributions are a wonderful tool to work with in the theory of linear differential equations. But, alas, most of the real problems, for example, in mathematical physics are connected with interacting systems; and interaction means nonlinearity for the corresponding differential equations. So a distribution multiplication is required although everybody knows quite well that this problem cannot be solved in a general way. Numerous approaches can be found in the literature. (In our reference list we have included some of the papers dealing with this problem; [1]–[3], [5]–[7], [9]–[16], [20], [21], [23], [25]. But this list is far from being complete.) Many of these approaches are motivated by the special application the author has in mind, and therefore, they contain a large amount of mathematical arbitrariness. A special point, most authors insist upon, is that the distribution multiplication has to be commutative. But elementary calculation (Section 2) shows that noncommutative models not only make sense but, in addition, that the noncommutativity can be considered as the mathematical analogue of the fact that physical conservation laws can be violated by discontinuities (Section 3). We therefore believe that distribution multiplication should not contain any arbitrariness at all, and that the product definition should, in a canonical way, come out of elementary algebraic properties. Of course, to carry out such a program certain sacrifices, in terms of the size of the space, have to be made.

In this paper we pick up an old idea [7] from 1967. At that time we proved that in the space of the so-called almost-bounded distributions a multiplication is given in a canonical way. This result is reviewed in

Section 2. In Section 3 this algebra is applied to the most elementary problem in shock wave theory, and we demonstrate that noncommutativity corresponds to the violation of energy density conservation by going through the shock front. In the last two sections we generalize the algebraic concept, lying behind the multiplication of almost-bounded distributions, to a rather general and completely abstract situation. This process illuminates the algebraic background. Further applications of these constructions will be published in a subsequent paper.

2. The almost-bounded distributions. For many of the problems in physics, as well as in applied mathematics, which are described in terms of partial differential equations, some relevant solutions are given by discontinuous functions (e.g., in shock wave theory, quantum electrodynamics, etc.). So, it seems natural to look for distribution solutions of the corresponding differential equations. But, alas, often these problems are described by nonlinear equations, and it is well known that there is no "reasonable" distribution algebra [23]. This is easily seen. Obviously, for the δ-distribution $\delta(x)$ we have $x\delta(x) = 0$. Furthermore, $\frac{1}{x}x = 1$. Hence, because of

$$\delta(x) = \left(\frac{1}{x}x\right)\delta(x) \neq \frac{1}{x}\big(x\delta(x)\big) = 0, \tag{2.1}$$

there can be no associative algebra for the distributions.

Since ignorance is sometimes a precious asset, let us suppose that we do not know this result. For the fun of it, we calculate the product $\delta(x)\eta(x)$, where $\eta(x)$ is the jump-function $\eta(x) = -1$ for $x < 0$ and $+1$ for $x \geqslant 0$. From $\eta(x)^2 = 1$ and $\eta'(x) = 2\,\delta(x)$ we obtain, by differentiation of η^2,

$$0 = \big(\eta(x)\big)^{2'} = 2\eta(x)\,\delta(x) + 2\,\delta(x)\,\eta(x). \tag{2.2}$$

Because of

$$x\big(\delta(x)\eta(x)\big) = \big(x\delta(x)\big)\eta(x) = 0$$

we see that $\delta\eta$ is annihilated by multiplication with x.

Since the scalar multiples of the δ-distribution are the only distributions which are annihilated by multiplication with the C^∞-function x, we have

$$\delta(x)\eta(x) = a\delta(x), \qquad a \in \boldsymbol{C}.$$

And from

$$(\delta\eta)\eta = \delta(\eta\eta) = \delta\cdot 1 = \delta, \tag{2.3}$$

$$(\delta\eta)\eta = a\delta\eta = a^2\delta \tag{2.4}$$

we see that $a^2 = 1$. Hence $a = \pm 1$ and the algebra must (by virtue of (2.2)) be noncommutive.

Of course, all this only makes sense if we can justify the calculation rules we have used so far. These rules are:

(2.5.1) product rule for differentiation,

(2.5.2) associativity of the product,

(2.5.3) the distribution algebra has to be an extension of the usual function algebra.

For the sake of convenience we include the following

(2.5.4) the algebra has to be translation invariant.

Let us see if there is a canonical algebra which satisfies the rules (2.5). Of course, in virtue of (2.1), the best we can hope for is a noncommutative algebra in a subspace of distributions. And the least we have to expect, are two different algebras, because interchanging the factors yields an algebra isomorphism different from the identity (noncommutativity!).

2.1. Definition. A distribution φ is said to be *almost-bounded* if, for every $n \in \boldsymbol{N}$, its nth derivative is of the form

$$\varphi^{(n)}(x) = b(x) + \Delta(x),$$

where b is a locally bounded function and where Δ has a discrete support without accumulation point. By $B(\boldsymbol{R})$ we denote the space of almost-bounded distributions.

2.2. Theorem [7]. *For the space of almost-bounded distributions there are exactly two algebras fulfilling* (2.5). *These algebras are for* $\varphi, \tilde{\varphi} \in B(\boldsymbol{R})$ *given by*

$$\varphi(x)\tilde{\varphi}(x) \underset{\text{def}}{=} \lim_{\varepsilon\downarrow 0}\varphi(x+\varepsilon)\tilde{\varphi}(x) \tag{2.6}$$

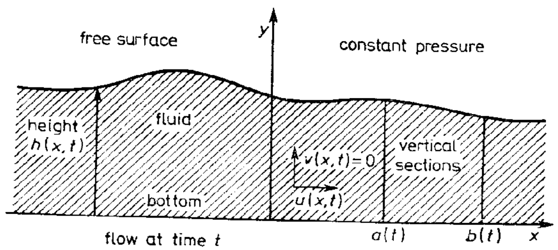
and

$$\varphi(x)\tilde{\varphi}(x) \underset{\text{def}}{=} \lim_{\varepsilon\uparrow 0}\varphi(x+\varepsilon)\tilde{\varphi}(x). \tag{2.7}$$

The products (2.6) and (2.7) do always exist since Definition 2.1 ensures that the places where the singularities of $\varphi, \tilde{\varphi}$ live are not too close together. In view of Theorem 2.2 and the fact that we have to expect at least two different algebras, we are entitled to say that the almost-bounded distributions have a canonical algebraic structure. Before we are going to make the algebraic background more transparent we like to illustrate the use of these algebras at a simple example.

3. An elementary application. In this section we describe bores (hydraulic jumps) by distribution solutions of the nonlinear partial differential equations derived from shallow water wave theory (0th order). Bores are shock waves (see the standard literature [4], [17], [24], [27]) which, over the time, have even attracted much attention in the popular scientific literature (for example [18]). This fact can be understood by everybody who ever had the opportunity to observe such a beautiful phenomenon. Mathematically speaking, bores are in complete analogy to the shock wave phenomena of nonlinear gas dynamics.

For reasons, which will become obvious later on, we like to have a look on the derivation of the relevant equations. We consider the flow of an incompressible fluid along the horizontal x-axis.



Here $u(x, y, t)$ denotes the velocity along the x-axis and $v(x, y, t)$ the velocity in the direction of the y-axis. We assume that the viscosity is zero, and that there is neither surface tension nor rotation. The only exterior force is gravitation. Then the assumptions of shallow water wave theory (of 0th order) are $v(x, y, t) = 0$ and that $u(x, y, t)$ has to be independent of y. This is the same as assuming that vertical sections remain vertical sections and that the pressure $p(x, y, t)$ in the fluid is the same as the hydrostatic pressure:

$$\varrho g\{h(x, t)-y\}-p(x, y, t) = \text{constant (exterior pressure)}.$$

Here ϱ, g are suitable physical constants. In other words, we can assume the conservation of mass between moving vertical sections $a(t)$ and $b(t)$

(3.1) $$\int_{a(t)}^{b(t)} h(x, t)\,dx = \text{constant},$$

and that the change in momentum is given by the difference of the pressure acting on these sections

(3.2) $$\varrho \frac{d}{dt}\int_{a(t)}^{b(t)} h(x, t)\, u(x, t)\,dx = \int_0^{h(a(t))} p\big(a(t), y, t\big)\,dy - \int_0^{h(b(t))} p\big(b(t), y, t\big)\,dy.$$

Elementary manipulation with these equations yield the usual nonlinear equations

(3.3.1) $$(hu)_x + h_t = 0,$$

(3.3.2) $$uu_x + gh_x + u_t = 0.$$

Now, inserting the ansatz $h(x, t) = h(x-ct)$, $u(x, t) = u(x-ct)$, one sees that no shock wave solutions do occur. So at first view it seems that our distribution algebra is of no use at all since it even cannot be used to describe the most elementary phenomena. But we completely forgot that the algebra was noncommutative, and we used commutativity freely by going from (3.1), (3.2) to (3.3). Doing this derivation again, without using commutativity, we see that the relevant equations are

(3.4.1) $$(hu)_x + h_t = 0,$$

(3.4.2) $$huu_x + \tfrac{1}{2} g(h^2)_x + hu_t = 0$$

instead of (3.3).

Now a shock wave ansatz makes sense:

(3.5.1) $$h(x, t) = H(x, t) + \alpha\eta(x-ct),$$

(3.5.2) $$u(x, t) = U(x, t) + \beta\eta(x-ct).$$

Here $H(x, t)$ and $U(x, t)$ denote the mean values of right and left hand side limits u_-, h_- and u_+, h_+, respectively:

$$H(x, t) = \tfrac{1}{2}\big(h_+(x, t) + h_-(x, t)\big),$$
$$U(x, t) = \tfrac{1}{2}\big(u_+(x, t) + u_-(x, t)\big),$$
$$\alpha = \tfrac{1}{2}\big(h_+(x, t) - h_-(x, t)\big),$$
$$\beta = \tfrac{1}{2}\big(u_+(x, t) - u_-(x, t)\big).$$

Using the algebra given by, e.g., (2.6), we obtain

(3.6.1) $$-c\alpha + \beta h_+ + \alpha u_- = 0,$$

(3.6.2) $$\beta u_+ h_+ - c\beta h_+ + g\alpha H = 0.$$

Insertion of (3.6.1) into (3.6.2) yields the usual shock wave conditions [24]

(3.7.1) $$(c-u_-)(c-u_+) = gH$$

which together with (3.6.1)

(3.7.2) $$h_+(u_+ - u_-) = (c-u_-)(h_+ - h_-)$$

determines the shock wave solutions (bores, hydraulic jumps).

Now it seems appropriate to remark that these results are completely independent of the special algebra ((2.6) or (2.7)) we have chosen. Of course, this is essential since physical results should not depend on mathematical arbitrariness.

A more interesting discovery is made if we turn our attention to the problem of energy conservation of the system. Between the moving sections $a(t)$, $b(t)$ the change in energy (kinetic and potential) and the power given by the pressure is

$$\frac{\partial E}{\partial t} = \varrho \frac{d}{dt} \int_{a(t)}^{b(t)} \tfrac{1}{2}\{u^2h + gh^2\}\,dx + $$

$$+u(b(t),t) \int_0^{h(b(t))} p(b,y,t)\,dy - u(a(t),t) \int_0^{h(a(t))} p(a,y,t)\,dy.$$

By elementary manipulation we obtain

$$\frac{1}{\varrho}\frac{\partial E}{\partial t} = \{\tfrac{1}{2}u^3h + guh^2\}_{a(t)}^{b(t)} + \tfrac{1}{2}\int_{a(t)}^{b(t)} (u^2h+gh^2)_t\,dx.$$

Hence the energy density is

$$\frac{1}{\varrho}\frac{\partial^2 E}{\partial t\partial x} = (\tfrac{1}{2}u^3h + guh^2)_x + \tfrac{1}{2}(u^2h+gh^2)_t.$$

Again by elementary calculation and (3.3) we infer from the above

$$(3.8) \qquad \frac{1}{\varrho}\frac{\partial^2 E}{\partial t\partial x} = \tfrac{1}{2}g\{[u_x, h]h + \tfrac{1}{2}[u, (h^2)_x]\},$$

where [,] denotes the usual commutator $[A, B] = AB - BA$.

The preceding calculation not only shows that the distribution algebra of Section 1 yields suitable descriptions of discontinuous solutions in terms of nonlinear differential equations but, in addition, it shows that the noncommutativity of the algebra is the mathematical analogue of the fact that physical conservation laws can be violated by discontinuities.

For an extensive investigation of shock fronts, in the context of distribution multiplication, the reader is referred to [16] and [26].

4. The algebraic background. The proof of Theorem 2.2 is lengthy and boring but not difficult at all [7]. Nevertheless, even by going through all its details, it does not become transparent from the algebraic point of view.

Furthermore, it is not clear how the proof has to be adapted for situations which are slightly different. For example, for higher dimensions, or if the convergence in (2.6) or (2.7) is replaced by convergence along a suitable ultrafilter. Completely in the dark remains the problem how

to extend the algebra under consideration by introducing new artifical elements in order to have products like $(1/x)\delta(x)$, etc.

To give a satisfactory answer to these problems we have to shed some light on the algebraic background of Theorem 2.2.

Let us try to pin down what the real problem is. It is well known, although sometimes forgotten, that distributions were invented to enable applied mathematicians to consider derivatives of uncontinuous functions. But taking derivatives of, lets say, functions in the space of almost-bounded distributions (almost-bounded functions) is no problem at all: Just take the derivative wherever it exists and forget about the other points. This is an honest derivation in an honest algebra. Of course, the disadvantage of this approach is that then one cannot define a reasonable integration because the kernel of that derivation will be an infinite-dimensional subspace. The practical disadvantage of the large size of this kernel is that there will be no reasonable duality theory where the transpose of the differentiation will have nice properties. Thus one would obtain an algebra, where the interesting elements – like the δ-distributions – are missing.

So, from this point of view, the real question will be: How to reduce the kernel of an abstract derivation?

This question will be treated in the following.

4.I. *Evaluation operators.* Consider some associative algebra $(\mathscr{A}, \cdot)$ over the real numbers. A pair (E^+, E^-) of linear operators $\mathscr{A} \to \mathscr{A}$ is called a pair of *evaluation operators* if

(4.1.1) E^+, E^- are algebra homomorphisms, i.e.,

$$E^+(ab) = E^+(a)E^+(b) \text{ and } E^-(ab) = E^-(a)E^-(b) \text{ for all } a, b \in \mathscr{A},$$

(4.1.2) E^+, E^- are idempotent, i.e., $E^+E^+ = E^+$, $E^-E^- = E^-$,

(4.1.3) E^+, E^- are right-absorbing, i.e.,

$$E^+E^- = E^+ \quad \text{and} \quad E^-E^+ = E^-.$$

Standard example for such a pair of evaluation operators are the operations of left and right side limits in the algebra of almost-bounded functions. But many other examples can be constructed (several dimensions, convergence along filters, etc.)

Assume in the following that E^+, E^- is such a pair of evaluation operators. Then the "product" $*$ defined in $\mathscr{A}$ by

$$(4.2) \qquad a * b = (E^+a)\cdot b + a\cdot(E^-b) - (E^+a)\cdot(E^-b)$$

is called the *evaluation product.*

4.1. THEOREM. *$(\mathscr{A}, *)$ is an associative algebra.*

Proof. All properties—apart from associativity—are easily seen. Associativity is proved by direct calculation:

$$a*(b*c) = (E^+a)\cdot(E^+b)\cdot c + (E^+a)\cdot b\cdot(E^-c) + a\cdot(E^-b)\cdot(E^-c) - \\ -(E^+a)\cdot(E^-b)\cdot(E^-c) - (E^+a)\cdot(E^+b)\,(E^-c).$$

Moving the bracket from $(b*c)$ to $(a*b)$ does not change the right-hand side. Hence the operation $*$ must be associative. ∎

This algebra is henceforth called the *evaluation algebra*.

4.2. Definition. (i) $\mathscr{A}_S = \{a \in \mathscr{A} \mid E^+a = 0\}$ will be called the *singular* elements in $\mathscr{A}$.

(ii) $\mathscr{A}_c = \{a \in \mathscr{A} \mid E^+a = E^-a\}$ will be called the *continuous* elements in $\mathscr{A}$.

Now the following can be verified by direct calculations.

4.3. Remark. (i) E^+ and E^- are algebra homomorphisms from $(\mathscr{A}, *)$ into $(\mathscr{A}, \cdot)$ and from $(\mathscr{A}, *)$ into $(\mathscr{A}, *)$.

(ii) The products in $(\mathscr{A}, \cdot)$ and $(\mathscr{A}, *)$ coincide modulo singular elements, i.e., $a*b - a\cdot b \in \mathscr{A}_S$ for all, $a, b \in \mathscr{A}$.

(iii) $E^-a = 0$ for all $a \in \mathscr{A}_S$ (since E^- is right absorbing).

(iv) $\mathscr{A}_S$ is a two sided ideal in $(\mathscr{A}, *)$.

(v) $\mathscr{A}_c$ is a subalgebra of $(\mathscr{A}, *)$ as well as $(\mathscr{A}, \cdot)$.

(vi) If $(\mathscr{A}, \cdot)$ is commutative, then $(\mathscr{A}_c, \cdot)$ and $(\mathscr{A}_c, *)$ are commutative.

From Remark 4.3 we know that the quotients $(\mathscr{A}, \cdot)/\mathscr{A}_S$ and $(\mathscr{A}, *)/\mathscr{A}_S$ are equal and are constituting an associative algebra. We denote this algebra by A:

$$A = (\mathscr{A}, \cdot)/\mathscr{A}_S = (\mathscr{A}, *)/\mathscr{A}_S. \tag{4.3}$$

Since $\mathscr{A}_S$ is a two-sided ideal in $\mathscr{A}_c$, the quotient

$$A_c = (\mathscr{A}_c, \cdot)/\mathscr{A}_S = (\mathscr{A}_c, *)/\mathscr{A}_S \tag{4.4}$$

must be a subalgebra of A. We call it the *algebra of continuous elements*. Since the kernel of the homomorphism E^+ is equal to the kernel of the quotient map $q\colon (\mathscr{A}, *) \to A$, the map E^+ provides us with a monomorphism $A \to (\mathscr{A}, *)$. The same is true for E^-. Since no confusion can arise, these monomorphisms are again denoted by E^+ and E^-. We have

$$qE^+ = qE^- = I.$$

And, obviously, the algebras A, E^+A, E^-A are isomorphic.

Let us see what the algebras A and A_c look like in the case of our standard example. Recall that $\mathscr{A}$ is then equal to the almost-bounded functions. $\mathscr{A}_S$ are the functions whose supports have no accumulation points. Hence A are the almost bounded functions which are undetermined on a set without accumulation point. And A_c are exactly those functions in A where the limits from both sides coincide. Hence A_c are the functions in A which can be (uniquely) extended to continuous functions. The map $E^+\colon A \to \mathscr{A}$ is the map where each $g \in A$ is replaced by its right-side limit. For example, a derivation in A is given by taking the usual derivative wherever it exists. But this notion of derivation has the serious disadvantage that it annihilates to many functions, namely all piecewise constant functions. What we would like to do now, is to extend the algebra A and this derivation (in a canonical way) such that the only functions which are annihilated by the extended derivation are only those elements in the kernel of d which are continuous. That this can be done in all generality is the content of the next subsection.

4.II. *Derivations and trivial extensions.* Consider some algebra B (always over $\boldsymbol{R}$ or $\boldsymbol{C}$) and some B-bimodule $\mathscr{M}$. Recall that a B-bimodule $\mathscr{M}$ is a vector space such that products bm and mb, $m \in \mathscr{M}$, $b \in B$ are defined in a reasonable way (associativity, distributivity). Examples for B-bimodules are two-sided ideals in B. The set $(B \times \mathscr{M})$ can easily be made into an associative algebra by

$$(b, m)(\tilde b, \tilde m) \underset{\text{def}}{=} (b\tilde b, b\tilde m + m\tilde b), \qquad b, \tilde b \in B,\ m, \tilde m \in \mathscr{M}. \tag{4.5}$$

This algebra is called [8] the *trivial extension* (of B via the module $\mathscr{M}$) In other words, an algebra C is said to be a trivial extension of B if there are homomorphisms $\varphi\colon C \to B$ and $\psi\colon B \to C$ such that

$$\varphi\cdot\psi = \mathrm{id}|_B, \tag{4.6.1}$$

$$J\cdot J = 0 \quad \text{where} \quad J = \ker(\varphi) = \{c \in C \mid \varphi(c) = 0\}. \tag{4.6.2}$$

B itself can be considered as a trivial extension of B. For the sake of convenience we consider B as a subalgebra of C in case that $C = (B \times \mathscr{M})$ is a trivial extension of B (via the module $\mathscr{M}$). The canonical homomorphism $(B \times \mathscr{M}) \to B$ and projection $(B \times \mathscr{M}) \to \mathscr{M}$ given by $(b, m) \to b$ and $(b, m) \to m$ are denoted by φ and p, respectively, i.e., $\varphi(b, m) = b$, $p(b, m) = m$. Obviously, $\varphi + p = \mathrm{id}$.

Now consider a derivation $d\colon B \to (B \times \mathscr{M})$ from B into the trivial extension $(B \times \mathscr{M})$, that is a linear map with

$$d(c_1c_2) = d(c_1)c_2 + c_1d(c_2) \quad \text{for all } c_1, c_2 \in (B \times \mathscr{M}).$$

One easily checks then that $\varphi \circ d\colon B \to B$ is a derivation from B into B and that $p \circ d\colon B \to \mathscr{M}$ is again a derivation. The following theorem shows that there is a canonical way of extending derivations $d\colon B \to (B \times \mathscr{M})$.

4.4. THEOREM. *Let d: $B \to (B \times \mathcal{M})$ be a derivation. There is a trivial extension $(B \times \mathcal{M}^\infty)$ of B and a derivation d^*: $(B \times \mathcal{M}^\infty) \to (B \times \mathcal{M}^\infty)$ such that*

(i) $\mathcal{M}^\infty \supset \mathcal{M}$,

(ii) $d^*|_B = d$,

(iii) d^*: $\mathcal{M}^\infty \to \mathcal{M}^\infty$,

(iv) *if* $c \in (B \times \mathcal{M}^\infty)$ *with* $d^*(c) \in B$, *then* $c \in B$;

and

(v) *if $\mathcal{M}$ is generated by d, i.e.,* $\mathcal{M} = \{m \mid \exists b \in B \text{ with } d(b) = (\tilde{b}, m)\}$, *then $\mathcal{M}^\infty$ and d^* are minimal; that means any other trivial extension $(B \times \tilde{\mathcal{M}})$ with derivation $\tilde{d}$ fulfilling* (i) *to* (iv) *contains an isomorphic copy of $(B \times \mathcal{M}^\infty)$ (isomorphism τ) such that* $d^* = \tau^{-1}\tilde{d}\tau$.

Furthermore,

(vi) $\mathcal{M}^\infty$ *is the direct sum* $\mathcal{M}^\infty = \mathcal{M} \oplus d^*\mathcal{M} \oplus d^{*2}\mathcal{M} \oplus \ldots \oplus d^{*n}\mathcal{M} \oplus \cdots$

Proof. Put $\mathcal{M}^\infty$ to be the set of all finite sequences (of arbitrary length) $(s_0, s_1, \ldots, s_n)$ in $\mathcal{M}$. We embed $\mathcal{M}$ into $\mathcal{M}^\infty$ in the obvious way $s \to (s, 0, \ldots)$ and we define d^* on $(B \times \mathcal{M}^\infty)$ by:

$$d^*(b) = d(b) \quad \text{for} \quad b \in B,$$
$$d^*(s_0, s_1, \ldots, s_n) = (0, s_0, s_1, \ldots, s_n) \quad \text{for } s_0, s_1, \ldots, \in \mathcal{M}.$$

That means d^* coincides on B with d and shifts all components of elements in $\mathcal{M}^\infty$ by one place to the right. For the sake of convenience we introduce base vectors in $\mathcal{M}^\infty$, i.e., we adopt the notation

$$(s_0, s_1, \ldots, s_n) = \sum_{i=0}^{n} s_i e_i,$$

where $e_0, \ldots, e_n, \ldots$ are the obvious base vectors.

We make a B-bimodule of $\mathcal{M}^\infty$ via the following product definition for $s \in \mathcal{M}$, $b \in B$, $n \in \boldsymbol{N}$:

(4.7.1) $$(se_n)\cdot b \underset{\text{def}}{=} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \{s \cdot (d^{n-k}b)\} e_k,$$

(4.7.2) $$b \cdot (se_n) \underset{\text{def}}{=} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \{(d^{n-k}b) \cdot s\} e_k,$$

where, on the right-hand side, the multiplication in $(B \times \mathcal{M})$ is used. A direct inspection shows that $\mathcal{M}^\infty$ and d^* have the required properties (i) to (iv) and (vi).

Let us now sketch the proof for the minimality. Take $\tilde{\mathcal{M}}$ and $\tilde{d}$ fulfilling (i) to (iv). Recall that $\mathcal{M} \subset \tilde{\mathcal{M}}$. We claim that the sum

$$\mathcal{M} \oplus \tilde{d}\mathcal{M} \oplus \tilde{d}^2\mathcal{M} \oplus \cdots$$

is direct in $\tilde{\mathcal{M}}$. This is then an isomorphic copy of $\mathcal{M}^\infty$ and the minimality assertion is proved. To prove the claim assume that

$$\sum_{k=0}^{n} \alpha_k \tilde{d}^k(m_k) = 0, \qquad \alpha_k \neq 0,\ \alpha_k \in \boldsymbol{R},\ m_k \in \mathcal{M}.$$

We have to show that all the $\tilde{d}^k m_k = 0$. Take some $b \in B$ with $\tilde{d}(b) = (\hat{b}, m_0)$ (which exists since $\mathcal{M}$ is generated by d). Then

$$\tilde{d}\Big\{b + \sum_{k=1}^{n} \alpha_k \tilde{d}^{k-1}(m_k)\Big\} \in B.$$

Hence, by property (iv),

$$m_0 = 0$$

and

$$\sum_{k=0}^{n-1} \alpha_{k+1} \tilde{d}^k(m_{k+1}) = 0.$$

Induction yields that all $m_k = 0$. ■

Let us go back to the situation of the evaluation algebra $(\mathcal{A}, *)$. Let A be the quotient with respect to the singular elements $\mathcal{A}_S$. For $a \in A$ and $s \in \mathcal{A}_S$ define

(4.8.1) $$as \underset{\text{def}}{=} (E^+a) * s,$$

(4.8.2) $$sa \underset{\text{def}}{=} s * (E^-a).$$

Then $\mathcal{A}_S$ is an A-bimodule. This is easily seen from the associativity in $(\mathcal{A}, *)$ and the fact that E^+, E^-: $A \to (\mathcal{A}, *)$ are homomorphisms.

Now let $A \times \mathcal{A}_S$ denote the trivial extension. Consider a derivation D_1: $A \to A$. Obviously, this can be considered also as a derivation D_1: $A \to A \times \mathcal{A}_S$.

Our standard example suggests that the kernel of D_1 may be too large to do any reasonable analysis with this derivation. We would like to replace D_1 by a derivation D^* which coincides on A_c (continuous elements) with D_1 and which has the property that its kernel is the intersection of A_c with the kernel of D_1.

Fortunately, another derivation D_2: $A \to \mathcal{A}_S \subset A \times \mathcal{A}_S$ is given by

(4.9) $$D_2 = E^+ - E^-.$$

That this must be a derivation is the consequence of a rather general theorem ([8], Prop. 20.1.1) but it can also be checked by a simple calculation.

Now put

$$D = D_1 + D_2 = D_1 + E^+ - E^-. \tag{4.10}$$

This is a derivation $D\colon A \to A \times \mathscr{A}_S$ and certainly its kernel is the intersection of A_c with the kernel of D_1. But now we have to deal with the disadvantage that our derivation is not completely defined on all of the algebra $A \times \mathscr{A}_S$. To abolish this disadvantage we need Theorem 4.4. Taking the canonical extension provided by that theorem we obtain

4.5. Corollary. *There is a trivial extension $A \times \mathscr{A}_S^\infty$ of A and a derivation $D^*\colon A \times \mathscr{A}_S^\infty \to A \times \mathscr{A}_S^\infty$ such that*

(i) $\mathscr{A}_S^\infty \supset \mathscr{A}_S$,

(ii) $D^*|_A = D_1 + E^+ - E^-$ *and* $D^*|_{A_c} = D_1|_{A_c}$,

(iii) $D^*\colon \mathscr{A}_S^* \to \mathscr{A}_S^\infty$,

(iv) *if* $D^*(a) \in A$ *then* $a \in A$, *in particular if* $D^*(a) = 0$, *then* $a \in A_c$ *with* $D_1(a) = 0$;

and

(v) *if* $\mathscr{A}_S$ *is generated by* D_2, *i.e.*, $(E^+ - E^-)A = \mathscr{A}_S$, *then* $\mathscr{A}_S^\infty$ *and* D^* *are minimal realizations of* (i) *to* (iv).

Furthermore,

(vi) $\mathscr{A}_S^\infty$ *is the direct sum* $\mathscr{A}_S^\infty = \mathscr{A}_S \oplus D^*\mathscr{A}_S \oplus D^{*2}\mathscr{A}_S \oplus \dots$

5. Duality. In this chapter we briefly indicate what the construction of Section 4 can do for analysis in the context of duality theory.

Consider some evaluation algebra $(\mathscr{A}, *)$ and its quotient A with respect to the singular elements $\mathscr{A}_S$. Again A_c denotes the continuous elements in A. Furthermore, we assume throughout this section that $\mathscr{A}_S$ is generated by $E^+ - E^-$, i.e., $\mathscr{A}_S = (E^+ - E^-)A$.

Let an integral be given. To be precise consider

5.1. Situation: Let J be some left ideal in A and $\int \cdot$ a linear functional on J such that

(i) $D_1 J \subset J$,

(ii) $\mathscr{A}_S J = (E^+ - E^-)J$,

(iii) $\int D_1(j) = 0$ for all $j \in J_c = A_c \cap J$,

(iv) for every nonzero $a \in A$ there is some $j \in J$ with $\int aj \neq 0$.

To keep a concrete example in mind consider our standard example and let J be the ideal of all almost-bounded functions with compact support and put $\int$ to be $\int_{-\infty}^{+\infty} \cdot\, dx$. Then all required assumptions are fulfilled.

Let us return to the abstract situation and let us interprete what Situation 5.1 means.

Abbreviate $\langle a, j\rangle = \int aj$, $a \in A$, $j \in J$. Then the elements of A can be considered as linear functionals on J via $a \to \langle a, \cdot\rangle$. Because of (iv) this map is injective. Therefore we identify a with $\langle a, \cdot\rangle$. Now (iii) means that integration by parts is possible for continuous elements, i.e.,

$$\langle D_1 a, j\rangle = -\langle a, D_1 j\rangle \quad \forall a \in A_c,\ j \in J_c. \tag{5.1}$$

Our problem is, whether or not we can embed A into an algebra of linear functionals on J (or an enlarged ideal) such that there is a derivation D on that algebra having the property that it coincides on A_c with D_1 and such that integration by parts (with respect to D) holds in general. The answer is yes!

5.2. Theorem. *There is a trivial extension A^* of A, a derivation $D^*\colon A^* \to A^*$ and some left ideal $J^* \supset J$ of A^* and a bilinear functional $\langle \cdot, \cdot\rangle\colon A^* \times J^* \to \mathbf{R}$ such that*

(i) $D^* J^* \subset J^*$,

(ii) $D^*|_{A_c} = D_1$,

(iii) $\langle a, j\rangle = \int aj$ *for all* $a \in A$, $j \in J$,

(iv) $\langle D^* a, j\rangle = -\langle a, D^* j\rangle$ *for all* $a \in A^*$, $j \in J^*$.

Proof. Take $A^* = A \times \mathscr{A}_S^\infty$ and D^* as in Theorem 4.5. Recall that A^* is a direct sum

$$A^* = A \oplus \mathscr{A}_S \oplus D^*\mathscr{A}_S \oplus \dots \oplus D^{*n}\mathscr{A}_S \oplus \dots \tag{5.2}$$

Consider $J^* = J + A^*J + D^*A^*J + D^{*2}A^*J + \dots$ Since D^* is a derivation, this is a left ideal; furthermore, $D^*J^* \subset J^*$ and $J^* \supset J$. Again from $D_1 J \subset J$ and our product formula (4.7) we obtain $\mathscr{A}_S^\infty \cdot J = \bigcup_{n=0}^{\infty} D^{*n}\mathscr{A}_S J$. From (ii) of Situation 5.1 we have $\mathscr{A}_S J = (E^+ - E^-)J$. Hence, using $AJ \subset J$, we obtain finally

$$J^* = J \oplus (E^+ - E^-)J \oplus D^*(E^+ - E^-)J \oplus \dots \oplus D^{*n}(E^+ - E^-)J \oplus \dots \tag{5.3}$$

and this sum must be direct (see 5.2). The projection onto J we denote by π, the projection onto $(E^+ - E^-)J$ we call p. Define a linear functional on J* by

$$\int^* j \underset{\text{def}}{=} \int \pi(j) - \int D_1 \psi(j) \quad \text{for all } j \in J^*, \tag{5.4}$$

where $\psi(j) \in J$ is such that $(E^+ - E^-)\psi(j) = p(j)$.

Although $\psi(j)$ is arbitrary up to a continuous element of J the functional is well defined because $\int D_1$ vanishes on the continuous elements of J. The functional is linear. We claim that $\int^* D^* j = 0$ for all $j \in J^*$. In view of (5.3) and the fact that, by definition, $\int^*$ vanishes on $D^{*n}(E^+ - E^-)J$, $n \geqslant 1$, we have only to prove the claim for $j \in J$. So, let $j \in J$ and let

$$\int^* D^* j = \int \pi(D^* j) - \int D_1 \psi(D^* j) = \int D_1 j - \int D_1 h, \tag{5.5}$$

where h may be an element of J with $(E^+ - E^-)h = p(D^*j)$. Since $j \in J$, we have $p(D^*j) = (E^+ - E^-)j$. Therefore we may take $h = j$ and obtain in (5.5)

$$\int^* D^* j = 0 .$$

Observe (by (5.4)) that $\int^*$ is an extension of $\int$, i.e.,

$$\int^* j = \int j \quad \text{for all } j \in J. \tag{5.6}$$

Now define for $a \in A^*$, $j \in J^*$

$$\langle a, j \rangle \underset{\text{def}}{=} \int^* aj .$$

Then all the required properties are fulfilled. ■

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