Progress of Theoretical Physics, Vol. 78, No. 5, November 1987

Solitons in Interaction

Benno FUCHSSTEINER

Universität Gesamthochschule Paderborn
Fachbereich 17-Mathematik Informatik, 4790 Paderborn

(Received July 10, 1987)

Several new nonlinear systems are given which are completely integrable. These systems can be considered as flows describing the self-interaction of single solitons in multisoliton fields. The construction of action variables, recursion operators, bi-hamiltonian formulation and the like is performed for these nonlinear systems. Furthermore virtual solitons are introduced and it is shown that 2-solitons in general may be understood as the superposition of two pairs of interacting solitons exchanging one virtual soliton and that the interacting soliton itself can be considered as the result of a collision of a wave with a virtual soliton. In a sense, virtual solitons only pop up during the time that solitons interact with each other. In case of the KdV the details of decomposition into interacting and virtual solitons are plotted, and a qualitative analysis of interaction is given. A brief discussion is appended, how to describe multisolitons by their "trajectories".

§ 1. Introduction

We show that, for constant m, the following nonlinear homogeneous partial differential (integrodifferential) equations:

$$ss_t = ss_{xx} - 2s_x s_x + 2m\, ss_x\,, \tag{1·1}$$

$$s^2 s_t = s^2 s_{xxx} - 3ss_x s_{xx} + \frac{3}{2}s_x{}^3 + \frac{3}{2}ms^2 s_x\,, \tag{1·2}$$

$$s_t = s_{xxx} + \frac{3(ms - s_{xx})^2}{2(ms^2 - s_x{}^2)}s_x\,, \tag{1·3}$$

$$s_{xt} = \frac{1}{2}s\cos\left[\int_{-\infty}^{x} \frac{ms(\xi) - s_{\xi\xi}(\xi)}{2\sqrt{ms(\xi)^2 - s_\xi(\xi)^2}}d\xi\right], \tag{1·4}$$

$$|\Psi|^2\Psi_t = -i\Psi_{xx}|\Psi|^2 + i\Psi|m\Psi + \frac{i}{2}\Psi_x|^2 - i(m\Psi + i\Psi_x)^2\bar{\Psi} \tag{1·5}$$

are completely integrable in the sense that they have an infinite dimensional abelian symmetry group generated in the usual way by some hereditary recursion operator or strong symmetry. Equations (1·2) through (1·5) have infinitely many constants of motion which are in involution. Furthermore, these equations admit soliton solutions. Typical two-soliton solutions for (1·2) are given in Fig. 1-4.A and 1-4.B (see below).

These nonlinear equations are intimately connected to the soliton structure of well-known integrable equations, namely, Burgers, KdV, MKdV, sine-Gordon and cubic Schrödinger equation. In a sense they describe the time evolution of **solitons in interaction.**

Interaction of soliton solutions in the two-soliton case of the KdV

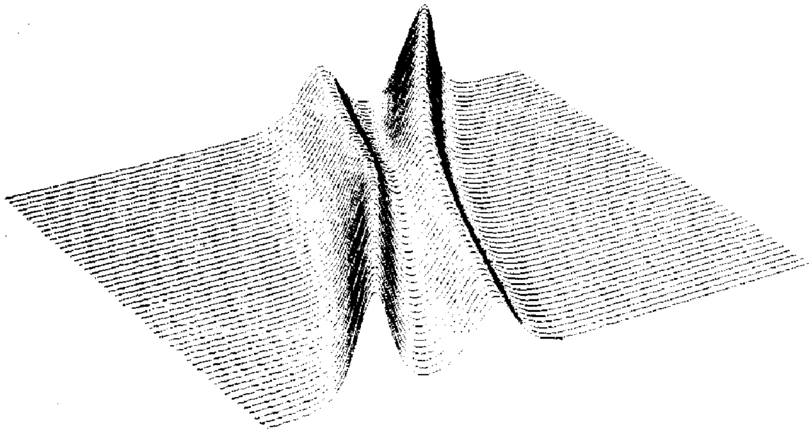

Fig. 1. Two-Soliton of the KdV. $c_1=2.56$, $c_2=1.44$.

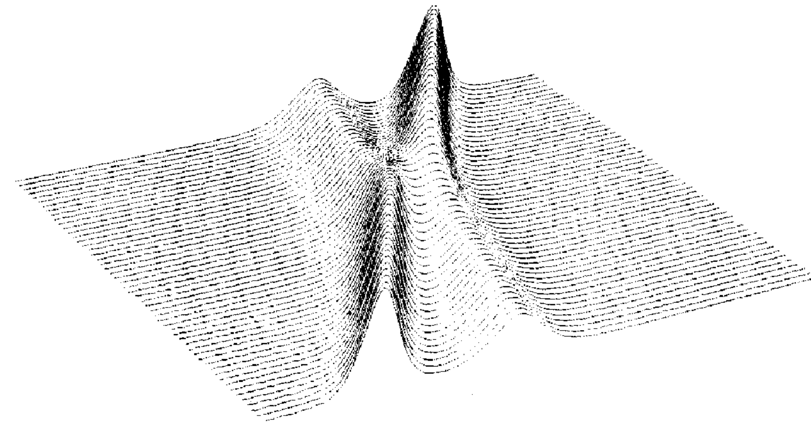

Fig. 2. Two-Soliton of the KdV. $c_1=2.56$, $c_2=1.08$.

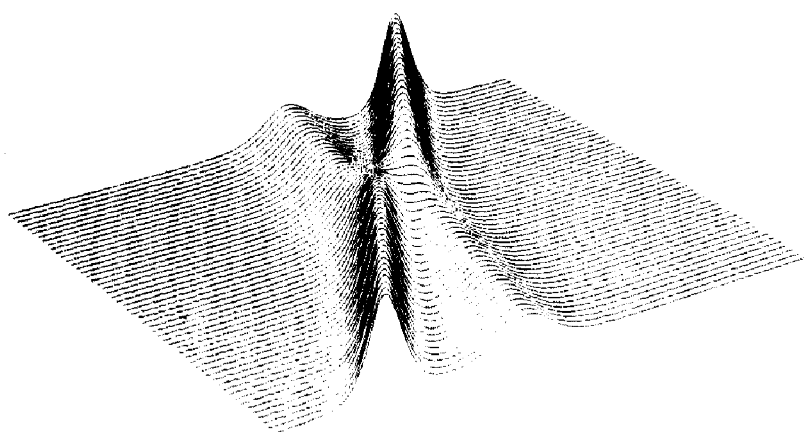

Fig. 3. Two-Soliton of the KdV. $c_1=2.56$, $c_2=0.92$.

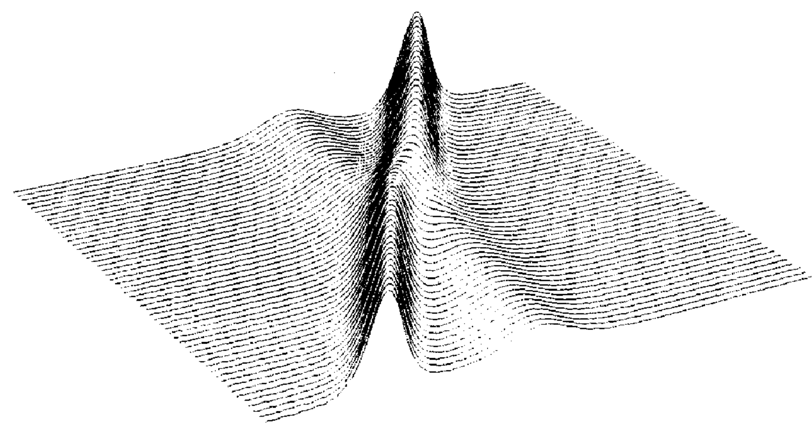

Fig. 4. Two-Soliton of the KdV. $c_1=2.56$, $c_2=0.64$.

$$u_t = u_{xxx} + 6uu_x \tag{1·6}$$

has been studied by Lax in a fundamental paper.[18] He found out that, basically, there are three different types of interaction for solitons of the KdV. These type-of-interaction depend on the ratio of speeds c_1 and c_2 of the asymptotic solitons:

$$c_1/c_2 < (3+\sqrt{5})/2 \quad \text{(case a)},$$

$$(3+\sqrt{5})/2 < c_1/c_2 < 3 \quad \text{(case b)},$$

$$3 < c_1/c_2 \quad \text{(case c)}.$$

Lax observed that in case a) "the two solitons interchange their role without passing through each other" wheareas in cases b) and c) "the big wave first absorbs, then reemits the small wave, and that in case b) the absorption of the small wave raises a secondary peak on the big wave". In order to have a better understanding for these different behaviors we have plotted all three cases in Fig. 2-4. In these figures —compared to the scaling of the variable x—the height of the solitons has been enlarged by a factor 7 and the time t has been enlarged by a factor 4. In all cases c_1 has been chosen to be

$$c_1 = 2.56$$

and c_2 has been chosen to be the following:

$$c_2 = 1.08 \quad \text{Fig. 2 (case a)},$$

$$c_2 = 0.92 \quad \text{Fig. 3 (case b)},$$

$$c_2 = 0.64 \quad \text{Fig. 4 (case c)}.$$

In addition—for reasons which become obvious later on—we have added (Fig. 1) a plot for $c_2 = 1.44$. This shows again a type-a interaction.

In this paper, by a change of viewpoint, we find that even in the KdV-case basically there is only one type of interaction.

In order to motivate our change of viewpoint let us consider as analogy for multisoliton solutions of the KdV a field consisting of several particles. Certainly, to study the interaction of these particles one would rather like to have a look at the individual particle during the interaction instead of considering the superposition of all particles. Since the multisoliton solution in itself corresponds to the superposition, one easily gets the idea that consideration of the individual soliton might give a better understanding for soliton interaction. The only problem which remains is how to "individualize" the interacting soliton because a look on the KdV-solution only gives asymptotic solitons.

This is what has been done in this paper. As usual we identify the eigenvectors of the recursion operator with the x-derivatives of the interacting solitons. But then using the structure of this operator we are able to formulate a nonlinear evolution equation for the time evolution of these interacting solitons. Surprisingly, the only field-variable entering in these equations is the soliton in itself, not the variable corresponding to the superposition of solitons. The interaction is reflected by the nonlinear terms of the equation. Thus each interacting soliton is described by a nonlinear equation where only self-interaction appears. And the information about other solitons in the field is hidden in the initial data. Thus small variations of initial data characterize different states of these interacting solitons.

To be more explicit: Each N-soliton solution of, for example, the KdV is a superposition of N solutions of different equations of type (1·2). These equations differ only by the masses m_i. These masses m_i appear in the corresponding N-soliton solution as asymptotic speeds of emerging solitons.

The same holds for other equations where a recursion operator is known. The evolution equations for single solitons in interaction for

$$u_t = u_{xxx} + 6u^2 u_x \,, \qquad \text{(modified-KdV)}$$

$$u_t = \frac{1}{2}\sin(D^{-1}u) \qquad \text{(potential sine-Gordon)}$$

and

$$\phi_t = -i\phi_{xx} + 2i\phi|\phi|^2 \qquad \text{(cubic Schroedinger equation)}$$

are given by Eqs. (1·3) to (1·5). The corresponding equation for

$$u_t = u_{xx} + 2uu_x \qquad \text{(Burgers equation)}$$

is given by (1·1). If no recursion operator is known, the situation is more complicated. These cases will be treated in a subsequent paper.

Thus a decomposition of N-soliton solutions into solutions of the more fundamental "single-soliton" equations has been performed for most of the popular completely integrable equations in 1+1 dimensions. For the two-solitons of the KdV, which were given in Fig. 1~4, this decomposition has been plotted. Figure n.A gives the time-evolution of the larger soliton appearing asymptotically in Fig. n, whereas Fig. n.B gives the time evolution of the smaller soliton.

Looking at these pictures one observes that basically there are only one, or, depending on the viewpoint,at most two different types of interaction. In any way the bounds given in 18) for the quotients of celerities, do not seem to have a meaning any more if interacting solitons are considered. Generally speaking, the solitons are always attracted by each other. The attraction becomes very strong when the positions of the solitons are close. Each soliton is stretched in shape under the attraction given by the other soliton. However, if the number of maxima is counted, then there still is a difference between the larger soliton and the smaller one. And for some choices of parameters, one may detect a difference in the behavior of the larger soliton. The smaller soliton always develops a second maximum (which becomes exponentially small with the distance of the other soliton) whereas the larger soliton has a second maximum if characteristic speeds of both humps are close together ($c_1/c_2 < 2$). This situation is illustrated by the two-soliton solution given in Fig. 1 and its decomposition into interacting solitons.

The development of second maxima can be understood by considering a simple analogy where the soliton is replaced by some elastic compound of material being

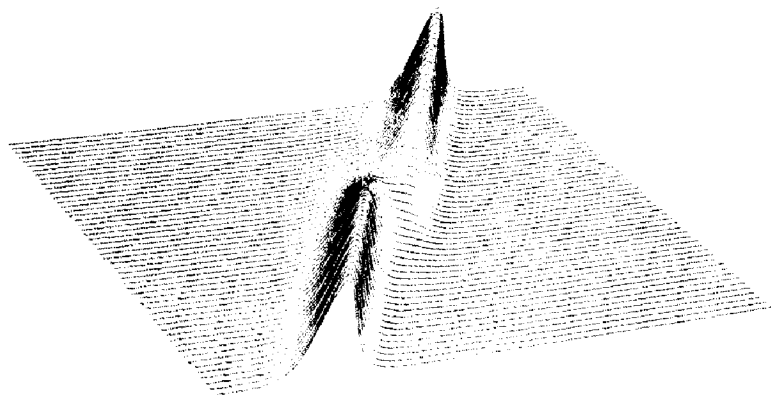

Fig. 1.A. Larger Soliton. $c_1=2.56$, interacting with $c_2=1.44$.

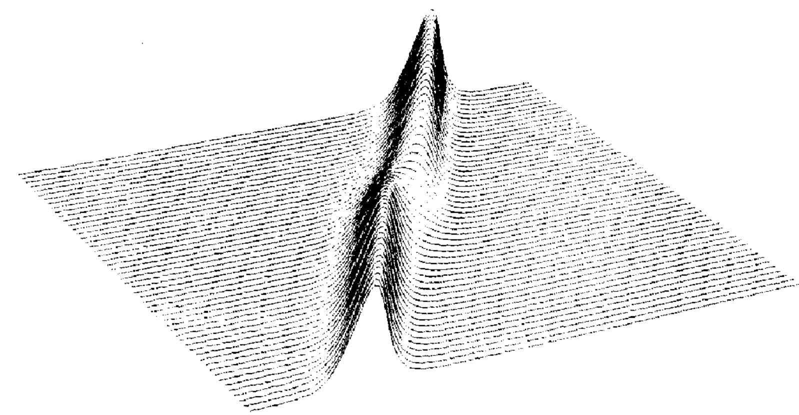

Fig. 2.A. Larger Soliton. $c_1=2.56$, interacting with $c_2=1.08$.

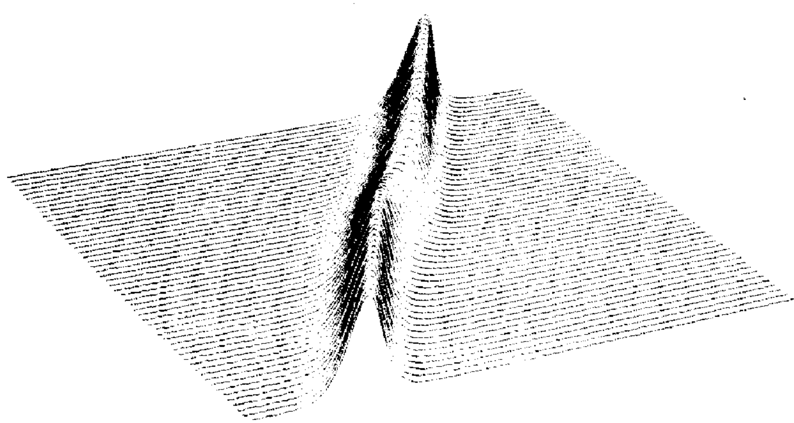

Fig. 3.A. Larger Soliton. $c_1=2.56$, interacting with $c_2=0.92$.

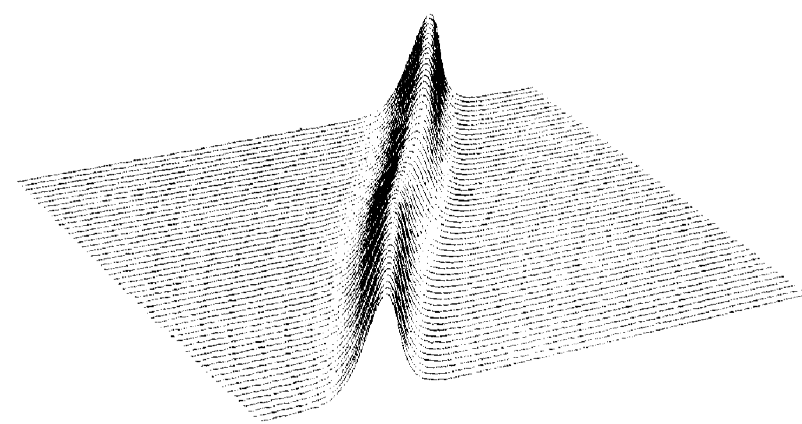

Fig. 4.A. Larger Soliton. $c_1=2.56$, interacting with $c_2=0.64$.

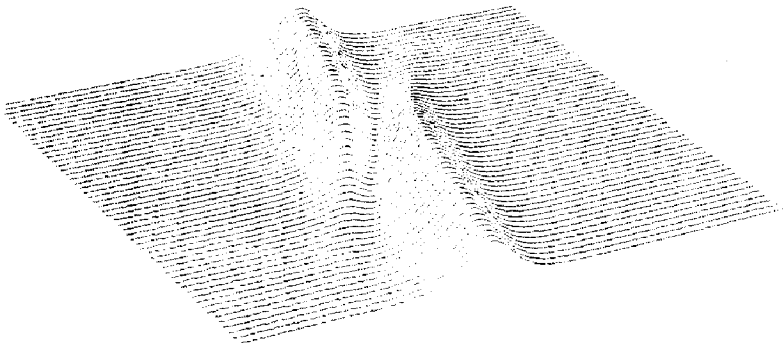

Fig. 1.B. Smaller Soliton, $c_2=1.44$, interacting with $c_1=2.56$.

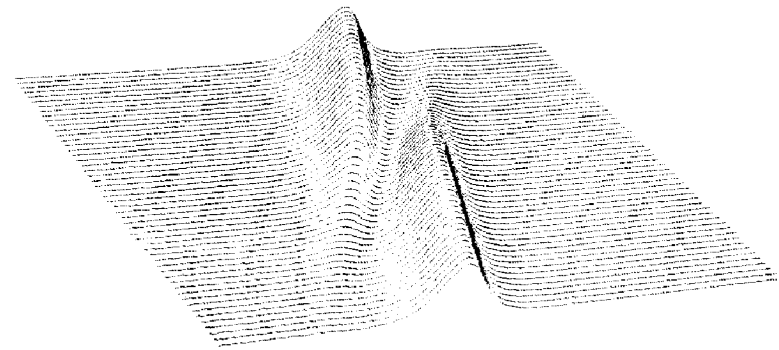

Fig. 2.B. Smaller Soliton. $c_2=1.08$, interacting with $c_1=2.56$.

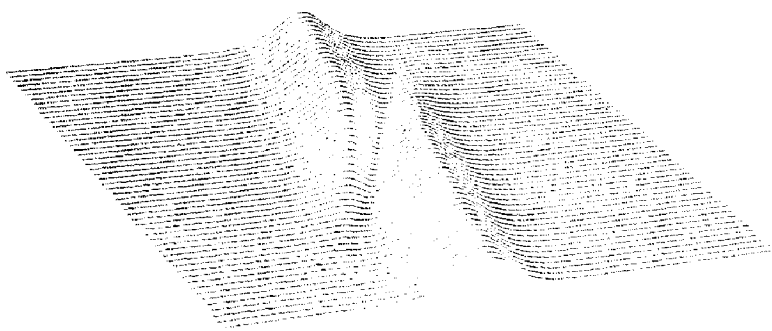

Fig. 3.B. Smaller Soliton. $c_2=0.92$, interacting with $c_1=2.56$.

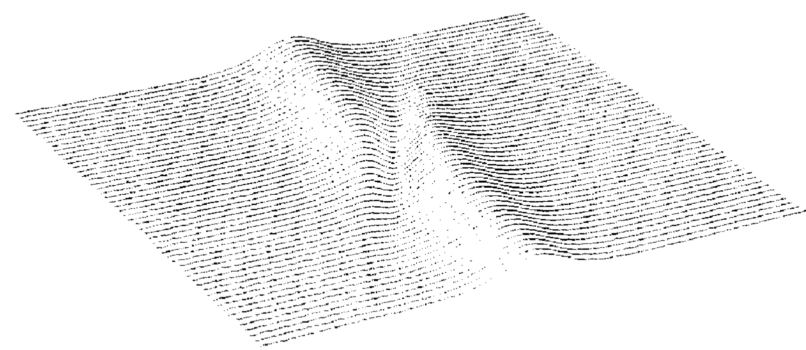

Fig. 4.B. Smaller Soliton. $c_2=0.64$, interacting with $c_1=2.56$.

under the influence of an attraction becoming exponentially small with the relative distance of the attractor. The more extended this compound is, then, when the source of the attracting force is close by, the difference between the force acting on the material on the left and the right side of the compound becomes so considerable that the hump is stretched so much that a girth develops, hence a second maximum becomes visible. In any case, the less localized a soliton is, the more probable a second maximum will develop.

One might argue that the consideration of maxima of field functions is not all that is essential for a qualitative description of interaction. This certainly is true from a physical viewpoint. And from the mathematical viewpoint this is true for the system consisting of the superposition of interacting solitons, where the maxima do not have an essential meaning (apart from giving a rough idea of what a function might look like). However, the maxima of the interacting solitons are important insofar as they give rise to the systematic derivation of classical particle movements connected to the original system. In the literature very many such connections between finite-dimensional flows (particle systems) and pure soliton solutions of infinite-dimensional flows can be found (see 16), 24), 4), 3), 2), 1), only to name a few). Sometimes, however, the disturbing fact is observed[21),22)] that the trajectories of the particles cannot be identified completely with the solitons, since the humps given by the solitons seem to jump from the trajectory of one particle to that of another particle. This phenomenon can be understood by looking at the maxima of the "interacting solitons". Trajectories mostly describe positions of maxima, but this in such a way that while the interaction takes place, the trajectory changes from the big to the small maximum. Since the small maximum becomes exponentially small with the distance of the two solitons, these trajectories look as if they were changing from one soliton to the other. A typical picture of trajectories of this kind would be obtained by taking the positions of the obvious maxima in Fig. 1. We explain these connections with particle systems in more detail in a subsequent paper.

In § 2 we review the principal spectral properties of the recursion operator which are needed for our analysis. Furthermore, there the relevant notions of different solitons are introduced. In § 3 the equations of motion for interacting solitons are derived, in § 4 their complete integrability is shown and their recursion operators are given. In § 5 the different states of interacting solitons are characterized in terms of "virtual" solitons. In the last chapter we compare the work done in this paper with the work of others.

Plots for different solutions and different soliton-types are solely given for the KdV. Of course, once the construction of these solutions is known, these plots can be given for other standard types of completely integrable systems as well.

§ 2. Solitons, or the spectral properties of the recursion operator

We give a short review on the relation between soliton solutions and spectral properties of the recursion operator (for more details see 10)).

We are interested in

SITUATION 1:
On some infinite-dimensional manifold of suitable functions in the real variable x we consider an equation

$$u_t = K(u)\,,\qquad u = u(x, t)\,, \tag{2·1}$$

such that there is a hereditary operator[7),9)] $\Phi(u)$ generating the vector field $K(u)$ out of the generator of the translation group, i.e.,

$$K(u) = \Phi(u) u_x\,. \tag{2·2}$$

The property of hereditariness then implies that the vector fields

$$K_n(u) = \Phi(u)^n u_x\,,\quad n = 0, 1, \cdots \tag{2·3}$$

do commute in the Lie-algebra of vector fields. Because of $K(u) = K_1(u)$ we then have constructed infinitely many generators of one-parameter symmetry groups for (2·1).

This situation applies to all known completely integrable differential equations in one space variable. An exception seems to be the Benjamin-Ono equation, but this is an integro-differential equation. However, in several space variables the situation is different. Nevertheless most of the techniques developed in the following also apply to these more complicated situations, although some arguments have to be changed considerably.

Let us give some of those examples which we shall need later on anyway.

EXAMPLE 1:
The following operator is hereditary[7)]

$$\Phi(u) = D^2 + 2u + 2DuD^{-1}\,, \tag{2·4}$$

where D denotes the operator of taking the x-derivative and D^{-1} stands for

$$(D^{-1}f)(x) = \int_{\infty}^{x} f(\xi)\,d\xi\,.$$

The Korteweg-de Vries equation is of the form

$$u_t = \Phi(u) u_x = u_{xxx} + 6uu_x\,. \tag{2·5}$$

The operator

$$\Phi(u) = D + DuD^{-1} \tag{2·6}$$

is hereditary[7)] and

$$u_t = \Phi(u) u_x = u_{xx} + 2uu_x \tag{2·7}$$

yields Burgers equation. Other hereditary operators are

$$\Phi(u) = D^2 + 4DuD^{-1}u\,, \tag{2·8}$$

$$\Phi(\phi) = -iD + 4i\phi D^{-1}\mathrm{Re}(\bar{\phi}\,\cdot)\,, \tag{2·9}$$

which generate the modified-KdV (mKdV) and the nonlinear Schrödinger equation, respectively. In (2·9) the operator $\mathrm{Re}(\bar{\phi}\cdot)$ stands for taking the real part of its entry multiplied by $\bar{\phi}$, i.e.,

$$\mathrm{Re}(\bar{\phi}w)=\frac{1}{2}(\bar{\phi}w+\phi\bar{w}), \qquad (2\cdot10)$$

where the bar denotes complex conjugation. Other examples for hereditary operators which are far more complex can be found in 9), 12) and 13).□

Apart from Burgers equation in all these cases the function-space under consideration is $S(\boldsymbol{R})$, the space of C^∞-functions vanishing with all their derivatives rapidly at $\pm\infty$. For Burgers equation we choose $S_-(\boldsymbol{R})$, the space of C^∞-functions vanishing with all their derivatives rapidly at $-\infty$.

It is interesting to note that (2·8) also generates the potential sine-Gordon equation

$$u_t=\Phi(u)^{-1}u_x=\frac{1}{2}\sin(D^{-1}u). \qquad (2\cdot11)$$

A decisive role in characterizing soliton solutions is played by the linear hull of the symmetry generators $K_n(u)$, $n=0, 1, \cdots$. By $\mathbf{K}$ we denote the nontrivial (not all coefficients equal to zero) linear combinations of the $K_n(u)$, $n=0, 1, \cdots$. If $L(u)=\sum a_nK_n(u)$ is an element of $\mathbf{K}$ then the polynomial $P_L(\xi)=\sum a_n\xi^n$ is called its **characteristic polynomial.** Observe that $L(u)=P_L(\Phi(u))u_x$. L is said to be **nondegenerate** if the zeros of its characteristic polynomial have only multiplicity 1. Given some $L\in\mathbf{K}$ then the set of those u such that $L(u)=0$ or, equivalently,

$$P_L(\Phi)u_x=0 \qquad (2\cdot12)$$

is said to be a **multisoliton manifold.**

Since being a multisoliton solution should not change with time evolution of (2·1), it is important that all the submanifolds determined by (2·12) are invariant under (2·1). In other words: Whenever the initial condition $u(t=0)$ fulfills one of these conditions, then the solution $u(t)$ fulfills the same condition for all time t. The proof of this is a simple application of the fact that the $K_n(u)$ are infinitesimal generators of one parameter symmetry groups.

As observed in 7), 9), 10) for multisoliton solutions we have a decomposition of u_x in terms of eigenvectors of Φ. To be precise:

THEOREM I.

The following are equivalent:

(i) *There is a nondegenerate* $L=\sum a_mK_m$ *such that* $L(u)=0$.

(ii) *The generator of the translation group can be decomposed*

$$u_x=\omega_0+\omega_1+\cdots+\omega_N \qquad (2\cdot13)$$

into eigenvectors of $\Phi(u)$.

Since we need some parts of the argument leading to this decomposition later on anyway, we may as well give a proof of this.

Proof of the theorem

(i)⇒(ii): Let $\lambda_0, \cdots, \lambda_N$ be the set of zeros of the characteristic polynomial of L. Then, obviously,

$$(\Phi-\lambda_0)(\Phi-\lambda_1)\cdots(\Phi-\lambda_N)u_x=0\,.$$

Consider the polynomial $P(\xi)=(\xi-\lambda_0)(\xi-\lambda_1)\cdots(\xi-\lambda_N)$ and define the polynomials $\Pi_n(\xi)$, $n=0, \cdots, N$ by $(\xi-\lambda_n)\Pi_n(\xi)=P(\xi)$. Since, all zeros of $P(\xi)$ have multiplicity 1 we know from elementary calculus that

$$1=\sum_{n=0}^{N} a_n\Pi_n(\xi)\,,$$

where the a_n are given by

$$a_n=(P'(\xi)_{|\xi=\lambda_n})^{-1}\,.$$

Hence

$$u_x=\sum_{n=0}^{N} a_n\Pi_n(\Phi)u_x\,. \tag{2·14}$$

Now, introducing $\omega_n=a_n\Pi_n(\Phi)u_x$ we see from

$$(\Phi-\lambda_n)\omega_n=P(\Phi)u_x=0$$

that ω_n must be an eigenvector of Φ with eigenvalue λ_n and that (2·14) is the desired decomposition.

(ii)⇒(i): Let $\lambda_0, \cdots, \lambda_N$ be the set of different eigenvalues occurring in the decomposition (2·13) of u_x. Then, obviously

$$(\Phi-\lambda_0)(\Phi-\lambda_1)\cdots(\Phi-\lambda_N)u_x=0\,.$$

Thus, if $P(\xi)=(\xi-\lambda_0)(\xi-\lambda_1)\cdots(\xi-\lambda_N)=\sum a_n\xi^n$, we have found some

$$L(u)=\sum a_nK_n(u)$$

with $L(u)=0$ such that the zeros of the characteristic polynomial are given by the λ's.□

Assume that (2·13) holds. Introduce the quantities

$$s_i(x)=(D^{-1}w_i)(x)=\int_{-\infty}^{x} w_i(\xi)d\xi\,, \qquad i=0, \cdots, n\,.$$

Then (2·13) reads

$$u=s_0+s_1+\cdots+s_n\,. \tag{2·15}$$

If, asymptotically, the s_i become localized at different positions, then for $|t|\to\infty$ and $s=s_i$, we must have

$$\Phi(s)s_x=\lambda_i s_x$$

or

$$\lambda_i s_x=K(s)\,. \tag{2·16}$$

This means that s_i is asymptotically a one-soliton with celerity λ_i, i.e., a solution of (2·1) of the form

$$u(x, t)=s(x+\lambda_i t)\,. \tag{2·17}$$

Because of this, and for the reason that the decomposition (2·15) remains valid over the time evolution, the s_i appearing in (2·15) are called "interacting solitons". The decomposition (2·15) is the well-known decomposition into solitons emerging asymptotically (in experiments already observed by Scott Russel [23), p. 323]). Of course, this notion only makes sense if we are able to treat the s_i completely as "individuals", i.e., if we are able to describe their dynamics independently of the other members appearing in the decomposition (2·15). This will be done in the next section.

We close this section by defining different types of "interacting solitons" according to their respective spectral properties with respect to $\Phi(u)$.

DEFINITION 1:
Let u be a solution of (2·1). If w is an eigenvector of $\Phi(u)$ then $s=D^{-1}w$ is said to be an "interacting soliton" in the soliton field u. The corresponding eigenvalue is called the "celerity" of this interacting soliton. If the eigenvalue has multiplicity 1, then s is said to be a "non-degenerate soliton" otherwise we call it a "resonance soliton".

In case a soliton is asymptotically disappearing, i.e., if for its eigenvalue λ we have

$$\lim_{t\to\infty} s(x-\lambda t, t)=0 \quad \text{(pointwise)}\,,$$

then the soliton is said to be a "virtual" soliton. Otherwise we call it a "real" soliton.

Resonance solitons appear for multisolitons where the characteristic polynomial is degenerate, i.e., has multiple roots. This implies that the corresponding eigenvalues have higher multiplicity than 1. In these cases the corresponding soliton field is said to be a "resonance multisoliton".

For example, take in case of the KdV (i.e., the Φ given by (2·4)) the two-soliton manifold defined by

$$K_2(u)-2\lambda K_1(u)+\lambda^2 u_x=(\Phi-\lambda)(\Phi-\lambda)u_x=0\,. \tag{2·18}$$

Then by taking a certain limit[10)] of the well-known[17)] non-degenerate two-soliton solution one finds the following singular solution:

$$u(x, t)=16k^2\frac{2\sinh^2(\gamma)-k\Delta\sinh(2\gamma)}{(\sinh(2\gamma)-2k\Delta)^2}\,, \tag{2·19}$$

where

$$\begin{aligned}\lambda&=4k^2\,,\\ \gamma&=kx+4k^3t+\delta\,,\\ \Delta&=(x-x_0)+12k^2t\,.\end{aligned} \tag{2·20}$$

Certainly, the time evolution of this solution of the KdV is far from decomposing asymptotically into traveling waves. In the study of the KdV solutions like this one sometimes are neglected because it has a pole (of second order). Singular solutions like this have been studied for the KdV in 15). However, these kind of solutions need not to be singular in all possible cases. See, for example, the corresponding nonsingular solutions in case of the sine-Gordon equation.[25]

It should be remarked, that even in the degenerate case of resonance multisolitons there always is a decomposition corresponding to (2·13). This is easily seen by going again through the proof. If, for example,

$$P_L(\Phi)u_x=0\,,$$

then take the decomposition into fractional parts of $P_L(\xi)^{-1}$ and multiply these by $P_L(\xi)$ in order to obtain a representation of 1 in terms of a sum over factors of $P_L(\xi)$. This representation then yields a formula similar to (2·14). The only difference is that the sum not only goes over the eigenvectors but also over the principal vectors, i.e., all those w such that there is some $k\geq 1$ with $(\Phi(u)-\lambda)^k w=0$. We skip the details since in this paper we do not present completely integrable systems describing higher order parts of resonance solitons.

Examples of virtual solitons will be given in § 5.

§ 3. Dynamics of solitons in interaction

It is well known[7),9)] that the time evolution of an eigenvector w of the recursion operator $\Phi(u)$ of Eq. (2·3) may be prescribed by

$$w_t=K(u)'[w]\,, \tag{3·1}$$

where $K'(u)$ denotes the variational derivative with respect to u. This derivative is defined to be

$$K'(u)[w]=\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}K(u+\varepsilon w)\,. \tag{3·2}$$

Equation (3·1) is a direct consequence of the recursion property of $\Phi(u)$, i.e., a consequence of the equation

$$\frac{\partial}{\partial t}\Phi(u)=K'(u)\Phi(u)-\Phi(u)K'(u) \tag{3·3}$$

which holds for solutions $u(t)$ of (2·1). Rewriting (3·2) for the interacting soliton of (2·1), we find

$$s_t=(D^{-1}K'(u)D)s\,. \tag{3·4}$$

Now the procedure for obtaining the dynamics for s, which should not explicitly depend on the field u, is straightforward. We use the eigenvector equation

$$\Phi(u)s_x=\lambda s_x\,, \tag{3·5}$$

as a differential equation to determine u in terms of s. The integration constants coming up in the general solution of (3·5) will be determined by the asymptotic behavior for u and s, respectively, i.e., by the requirement that the functions u and s have to be on the prescribed manifold. Insertion of $u=u(s)$ into (3·4) then yields the desired nonlinear dynamics describing the time evolution of s.

EXAMPLE 2 (KdV):
Insertion of the recursion operator (2·4) of the KdV into (3·5) yields

$$\lambda s_x = s_{xxx} + 4us_x + 2u_x s\,. \tag{3·6}$$

Assuming s to be given, this equation can easily be solved for u since the corresponding equation

$$4us_x + 2u_x s = 0$$

with inhomogeneous part equal to zero is trivially solved. Then the full solution is found by variation of constants. We find

$$u = \frac{1}{4}s^{-2}(\lambda s^2 - 2ss_{xx} + s_x{}^2) + c_1 s^{\;2}\,. \tag{3·7}$$

Since u vanishes rapidly at infinity, we find by (3·6) that

$$s \approx e^{-\sqrt{\lambda}x} \text{ at } x = +\infty\,.$$

Under the assumption of this asymptotic behavior, (3·7) yields $u=c_1$ at $x=+\infty$. Hence, $c_1=0$ and

$$u = \frac{1}{4s^2}(\lambda s^2 - 2ss_{xx} + s_x{}^2) \tag{3·8}$$

or

$$(\sqrt{s})_{xx} + u\sqrt{s} = \frac{\lambda}{4}\sqrt{s} \tag{3·9}$$

which right away yields that the KdV is an isospectral flow for the Schrödinger operator. Using (3·2), we find for the KdV

$$K'(u) = D^3 + 6Du\,. \tag{3·10}$$

Therefore the time evolution of s is

$$s_t = s_{xxx} + 6us_x\,. \tag{3·11}$$

Replacing u by (3·8) we find Eq. (1·2) (with $m=\lambda$).

EXAMPLE 3 (Burgers equation):
The eigenvector problem for the recursion operator of Burgers equation has the form:

$$\Phi(u)w = (D + DuD^{-1})w = \lambda w$$

or

$$\lambda s_x = s_{xx} + (us)_x \tag{3·12}$$

if rewritten in terms of $s=D^{-1}w$. Solving this for u, we find for the general u

$$u=s^{-1}(\lambda s-s_x)+Cs^{-1}. \qquad (3\cdot13)$$

The requirement that s vanishes at either $+\infty$ or $-\infty$ gives $C=0$. Hence

$$u=s^{-1}(\lambda s-s_x). \qquad (3\cdot14)$$

Inserting this in the dynamics of the eigenvector

$$s_t=D^{-1}(D^2+2Du)Ds=s_{xx}+2us_x \qquad (3\cdot15)$$

yields Eq. (1·1) (again $m=\lambda$).

EXAMPLE 4:
For the mKdV equations (3·4) and (3·5) have the form

$$s_t=s_{xxx}+6u^2s_x, \qquad (3\cdot16)$$

$$\lambda s_x=s_{xxx}+4DuD^{-1}us_x. \qquad (3\cdot17)$$

Rewriting (3·17) as differential equation we obtain

$$u_xu^{-3}(s_{xx}-\lambda s)-u^{-2}(s_{xxx}-\lambda s_x)=4s_x. \qquad (3\cdot18)$$

As a solution we get

$$u=\pm\frac{1}{2}\frac{\lambda s-s_{xx}}{\sqrt{\lambda s^2-s_x^2+C}} \qquad (3\cdot19)$$

and C must be equal to zero because of the boundary condition at infinity. Using this we can rephrase (3·16) such that (1·3) is obtained for $m=\lambda$.

For the potential sine-Gordon equation relation (3·4) has the form

$$s_{xt}=\frac{1}{2}\cos(D^{-1}u)s. \qquad (3\cdot20)$$

Since the recursion operator for this equation is the same as the one for the mKdV we can use (3·19) to replace u in (3·20). This yields (1·4) for the evolution of the interacting soliton.

EXAMPLE 5:
Replacing in case of the cubic Schrödinger equation w by Ψ and u by ϕ we find that (3·1) has the following form:

$$\Psi_t=-i\Psi_{xx}+4i\Psi|\phi^2|+2i\phi^2\bar{\Psi}. \qquad (3\cdot21)$$

And the eigenvector problem for the recursion operator is

$$\lambda\Psi+i\Psi_x=4i\phi\rho, \qquad (3\cdot22)$$

where

$$\rho=D^{-1}\mathrm{Re}(\bar{\phi}\Psi)=\frac{1}{2}D^{-1}(\bar{\phi}\Psi+\phi\bar{\Psi}) \qquad (3\cdot23)$$

is a real function. Solving (3·22) by variation of constants, one easily finds $\Psi = \Psi(\phi, \rho)$ to be

$$\Psi = 4e^{i\lambda x}\tau, \quad \tau = D^{-1}(\phi\rho e^{-i\lambda x}). \tag{3·24}$$

The integration constant disappeared because of the boundary condition at infinity. This yields

$$\bar{\phi}\Psi = 4e^{i\lambda x}\tau\bar{\phi} = 4\rho^{-1}(\rho\bar{\phi}e^{i\lambda x})\tau = 4\rho^{-1}\tau D\bar{\tau} \tag{3·25}$$

and inserting this into (3·23) we obtain

$$\rho = 2D^{-1}\rho^{-1}D(\tau\bar{\tau}).$$

Using again the boundary condition at infinity this can be solved:

$$\rho^2 = 4|\tau|^2. \tag{3·26}$$

Hence, we obtain from (3·24) the identity

$$|\Psi|^2 = 4\rho^2. \tag{3·27}$$

Insertion of this into the square of (3·22) leads to

$$(\lambda\Psi + i\Psi_x)^2 = -16\phi^2\rho^2 = -4\phi^2|\Psi|^2,$$

which gives the desired relation $\phi = \phi(\Psi)$

$$\phi = \pm i/2(\lambda\Psi + i\Psi_x)|\Psi|^{-1}, \tag{3·28}$$

which we need in order to rewrite (3·21) in terms of Ψ alone. The resulting equation is (1·5) (for $m = \lambda$). So, Eq. (1·5) describes the time evolution of the x-derivative of the interacting soliton in case of the cubic Schrödinger equation.

§ 4. Complete integrability of interacting solitons

Essentially there are two ways to show the complete integrability of the dynamics of the interacting soliton. Both ways lead to the same recursion operator.

SITUATION 2:

As before we consider the equation

$$u_t = K(u) = \Phi(u)u_x, \qquad \Phi \text{ hereditary}, \tag{4·1}$$

and the eigenvector problem for the interacting solitons

$$(D^{-1}\Phi(u)D)s = \lambda s. \tag{4·2}$$

This leads via integration and use of the boundary condition at $|x| = \infty$ to the explicit dependence of u on s

$$u = F(s). \tag{4·3}$$

This function $F(s)$ we have computed for several examples, see Eqs. (3·8), (3·14), (3·19) and (3·28). The dynamics

$$s_t=(D^{-1}K'(u)D)s \tag{4·4}$$

for the interacting soliton can now be rephrased by use of (4·3)

$$s_t=(D^{-1}K'(F(s))D)s\,. \tag{4·5}$$

An obvious method to obtain the recursion operator for (4·5) seems to consider (4·2) or (4·3) as a Bäcklund transformation between Eqs. (4·5) and (4·1) and then using the transformation formulas for recursion operators which were given in 8). However, the transformation formulas[8)] do hold only under the assumption that the relation between u and s defines (maybe in an implicit way) a diffeomorphism between u and s. Alas, for interacting solitons this never is the case. Certainly, s uniquely defines u but u only defines s up to a multiplicative constant. This is reflected in the linearity of (4·2) with respect to s, or the fact that all equations for interacting solitons are homogeneous (i.e., they admit an additional symmetry whose infinitesimal generator is the field variable s itself). A consequence of the violation of the requirements of the implicit function theorem is that those operators which have to be inverted in the transformation formulas for recursion operators have nonempty kernels, hence are not invertible.

Knowing that the kernels of these operators are given by the additional symmetry coming out of the homogeneity of (4·5), we could try to handle this difficulty by working in the quotient space given by classes modulo this additional symmetry generator. The recursion operator then coming out of this procedure will be indefinite up to an additional multiple of the field variable. Since this indefiniteness leads anyway to a symmetry generator, we can in fact use the result of this procedure to generate the infinite-dimensional symmetry group of the evolution equation describing the interacting solitons. However, working in these equivalence classes has serious disadvantages. First, it is a somewhat cumbersome procedure. Second, we like to have an explicit recursion operator in order to be able to study its structure (hereditaryness, Hamiltonian structure, soliton solutions, angle variables and the like). Therefore we proceed in a different way by using additional information about the dynamics for interacting solitons.

The equations commuting with (4·1) are generated by the hereditary operator:

$$u_t=K_n(u)=\Phi(u)^n u_x\,. \tag{4·6}$$

Modulo an obvious symmetry, there is a one-to-one correspondence between vector fields of the s- and u-manifolds, respectively. Therefore the equations commuting with (4·5) are the evolution equations for the interacting solitons of Eqs. (4·6). By the same arguments as before these equations are

$$s_t=(D^{-1}K'_n(u)D)s\,, \qquad n=0,1,2,\cdots. \tag{4·7}$$

In (4·4) only $K(u)$ has to be replaced by $K_n(u)$. Now, using (4·3) we can express all u by s in order to rewrite (4·7) as

$$s_t=G_n(s)\,, \qquad n=0,1,2,\cdots, \tag{4·8}$$

where

$$G_n(s)=D^{-1}K'_n(u)Ds \quad \text{and} \quad u=F(s)\,. \tag{4·9}$$

Let us look for a recursion formula for the $G_n(u)$. If we use relation (4·3) in order to rewrite (4·8) in terms of u we obviously obtain (4·6). Hence, we obtain for the variational derivative of F:

$$F'(s)[G_n(s)]=K_n(u)\,. \tag{4·10}$$

With (4·9) this reads

$$K_n(u)=F'(s)D^{-1}K'_n(u)Ds\,. \tag{4·11}$$

From $K_{n+1}=\Phi K_n$ we now obtain the recursion

$$\begin{aligned} G_{n+1}(s)&=D^{-1}K'_{n+1}(u)Ds \\ &=D^{-1}(\Phi K_n)'Ds \\ &=D^{-1}\Phi'(u)[s_x]K_n+D^{-1}\Phi DD^{-1}K'_n(u)Ds\,. \end{aligned}$$

Inserting (4·11) in the first term of the last line and using (4·10) and (4·9), we obtain

$$G_{n+1}(s)=\{D^{-1}\Phi'(u)[s_x]F'(s)+D^{-1}\Phi(u)D\}G_n(s)\,. \tag{4·12}$$

Hence we have found that for u given by (4·9) the operator

$$\Psi(s)=D^{-1}\Phi'(u)[s_x]F'(s)+D^{-1}\Phi(u)D\,, \qquad \text{where } u=F(s)\,, \tag{4·13}$$

must be a recursion operator for (4·8). All these operators are hereditary (horrible explicit computation). The hierarchy of commuting flows given by (4·7) or (4·8) then can be written as

$$s_t=\Psi(s)^n s_x\,, \qquad n=0, 1, 2, \cdots. \tag{4·14}$$

EXAMPLE 6:
In the investigation of $\Psi(s)$ a crucial role is played by the operator given by the variational derivative $F'(s)$ and its left inverse. Looking at the explicit dependence of u on s one easily gets the impression that this operator is so complicated that the computation of its inverse is out of reach. Fortunately, this is not so. Here the fact that the kernel of $F'(s)$ is known helps tremendously in carrying out the necessary computations. Let us present the explicit computation of the variational derivative $F'(s)$ in case of the KdV. Instead of $F'(s)$ we compute $\{F'(s)\}^{-1}$, i.e., we compute for given $\beta(x)$ the quantity $\alpha(x)$ fulfilling

$$F'(s)[\alpha]=\beta\,. \tag{4·15}$$

This we do by solving first the homogeneous equation

$$F'(s)[\alpha_h]=0$$

and then by the method of variation of constants we solve (4·15). The solution of the homogeneous equation we know already because this must be the infinitesimal generator of the additional symmetry for s, i.e., $\alpha_h=\gamma s$, where $\gamma=$constant. Insertion of the ansatz

$$\alpha(x)=\gamma(x)s(x)\,, \qquad \text{(variation of constants)} \tag{4·16}$$

into (4·15) yields

$$\frac{1}{4}s^{-2}\{2ss_x\gamma_x-4s_xs\gamma_x-2s^2\gamma_{xx}\}=\beta\,.$$

This reduces to

$$(s\gamma_x)_x=-2s\beta\,.$$

Or, under appropriate boundary conditions at infinity

$$\gamma=-2D^{-1}s^{-1}D^{-1}s\beta\,.$$

Using (4·16) this yields

$$\alpha=\{F'(s)\}^{-1}\beta=-(2sD^{-1}s^{-1}D^{-1}s)\beta\,. \tag{4·17}$$

Hence

$$\{F'(s)\}^{-1}=-(2sD^{-1}s^{-1}D^{-1}s) \tag{4·18}$$

is the right inverse of

$$F'(s)=-\frac{1}{2}s^{-1}DsDs^{-1}\,. \tag{4·19}$$

For the investigation of structural properties of the recursion operator given by (4·13) two observations are helpful. First we obtain, because of $F'(s)s=0$ and $\Phi(u)s_x=\lambda s_x$, that s is an eigenvector of $\Psi(s)$, i.e.,

$$\Psi(s)s=\lambda s\,. \tag{4·20}$$

Furthermore, one observes that $\Psi(s)$ is homogeneous with respect to s, i.e.,

$$\Psi(\alpha s)=\Psi(s) \text{ for all } \alpha\in R\,.$$

This yields by differentiation

$$\Psi'(s)[s]=0\,. \tag{4·21}$$

We recall that an operator Ψ in some Lie algebra is said to be hereditary[7),9)] if for all A, B in the Lie algebra the following holds:

$$\Psi^2[A,B]+[\Psi A,\Psi B]=\Psi\{[\Psi A,B]+[A,\Psi B]\}\,. \tag{4·22}$$

An important consequence of this is that whenever Γ_1 up to Γ_n commute

$$[\Gamma_i,\Gamma_r]=0 \quad i,r=1,\cdots,n$$

and fulfill, for all A, the relation

$$\Psi[\Gamma_i,A]=[\Gamma_i,\Psi A]\,, \tag{4·23}$$

then the linear hull of

$$\{\Psi^n\Gamma_i|i=1,\cdots,n;\ n\in N\} \tag{4·24}$$

constitutes an abelian Lie algebra. In our case we consider as Lie algebra the algebra of vector fields. We claim that whenever Φ is hereditary, then the operator Ψ given in (4·13) again is hereditary.

We give a brief indication for the proof of this:

If there were a diffeomorphism between u and s, then this would yield right away the hereditaryness of Ψ (see 8), 9)) since this property is preserved under homomorphisms. But as we have mentioned before, this is not the case. However, this fact helps to infer that Ψ is hereditary modulo a multiple of s. Let us be more precise: For local operators (local with respect to the manifold variable, not x) we know[7] that Ψ is hereditary if for all vector fields v, w,

$$B(\Psi)(w, v)=B(\Psi)(v, w)\,, \tag{4·25}$$

where $B(\Psi)$ stands for

$$B(\Psi)(w, v)=\Psi'[\Psi w]v-\Psi\Psi'[w]v\,. \tag{4·26}$$

All variational derivatives are taken with respect to the variable s. Now, using $u=F(s)$ we find with (4·20) and (4·21) out of the hereditaryness of Φ that

$$B(\Psi)(w, v)-B(\Psi)(v, w)=\alpha s\,, \qquad \alpha\in \boldsymbol{R}\,.$$

Since $B(\Psi)(w, v)$ depends in a semilocal way on the space variable x, this only is possible for $\alpha=0$. Here "semilocal" means local with respect to the topology given on $\{x|x\in \boldsymbol{R}\}$ by the open intervals $\{(-\infty, a)|a\in \boldsymbol{R}\}$.

Now, we can use the hereditaryness of Ψ to find that the linear hull generated by s and the vector fields $\Psi^n(s)s_x$, $n\in \boldsymbol{N}$ is abelian. Hence, we have infinitely many symmetries for (4·7). But this is not so suprising since the different flows for u were already commuting.

A more important consequence of hereditaryness is that we can find infinitely many constants of motion which are in involution.

For this we need a Hamiltonian formulation. We keep the notation of 20) or 8). An evolution equation is Hamiltonian if it is of the form

$$s_t=\Theta(s)f(s) \tag{4·27}$$

with $f(s)=\nabla H(s)$ the gradient of a scalar quantity, and $\Theta(s)$ an antisymmetric linear map from cotangent space to tangent space fulfilling the following identity for all covector fields a, b, c,

$$0=\langle a, \Theta'[\Theta b]c\rangle+\langle b, \Theta'[\Theta c]a\rangle+\langle c, \Theta'[\Theta a]b\rangle\,. \tag{4·28}$$

Here $\langle\ ,\ \rangle$ denotes the duality between tangent and cotangent space. Such a map is called implectic since it behaves algebraically like the inverse of a symplectic map. If an evolution equation is Hamiltonian in this sense then the usual results can be obtained. For example,

$$\{a, b\}_\Theta=\langle\nabla a, \Theta\nabla b\rangle \tag{4·29}$$

defines Poisson brackets among the scalar quantities. Here ∇ defines the operation of taking the gradient. The time derivative of scalar quantities is expressed in the usual way by this Poisson bracket. Furthermore $\Theta(s)$ maps gradients of conserved quantities onto infinitesimal generators of one-parameter symmetry groups.

Since we were able to compute the operator $F'(s)^{-1}$, we are now able to transfer the Hamiltonian structure from Eq. (4·1) to Eq. (4·5). We write (4·1) in Hamiltonian form

$$u_t=\Omega(u)\nabla H(u)\,,\quad \Omega(u)\text{ implectic}\,,\quad H(u)\text{ scalar field.} \tag{4·30}$$

For the KdV equations this would be

$$u_t=D\nabla\Big(\int_{-\infty}^{+\infty}\Big(u(\xi)^3+\frac{1}{2}u(\xi)_\xi^2\Big)d\xi\Big)\,. \tag{4·31}$$

For other equations one easily finds their respective Hamiltonian formulation in the literature (for example, 8), 20), etc). Now

$$s_t=\Theta(s)\nabla H(u(s)) \tag{4·32}$$

with

$$\Theta(s)=\{F'(s)\}^{-1}\Omega(u)\{F'(s)^+\}^{-1} \tag{4·33}$$

is a Hamiltonian formulation for Eq. (4·5). Of course, in (4·32) the gradient has to be taken with respect to the variable s, and the notation F'^+ denotes the adjoint of F' with respect to the cotangent-tangent duality. The operator $\Theta(s)$ is implectic (the proof proceedes exactly as in 8)).

The full power of the hereditary structure now comes into the game. We observe that the covector field

$$\gamma(s)=\nabla H(u(s)) \tag{4·34}$$

is the gradient of a conservation law. Because of the hereditaryness then all the covector fields

$$\gamma_{n+1}(s)=\Psi(s)^n\gamma(s)\,,\quad n=0,1,\cdots \tag{4·35}$$

are again gradients of conserved quantities. The fact that these covector fields have potentials is a simple consequence of hereditaryness.[8] All conserved quantities constructed this way are in involution with respect to the Poisson brackets. Hence, the action variables are constructed. In fact, also the angle variables can be found by use of Ψ, or by the mastersymmetries which can be constructed out of Ψ (see 11)). But describing all the details of this procedure goes beyond the aim of this paper.

At the end of this section we like to convey (without proof) some additional information:

i) All the equations (4·7) again have Hamiltonian form.

ii) Furthermore Eq. (4·7) has n different Hamiltonian formulations. The respective implectic operators are

$$\Theta_u(s)=\Psi^n(s)\Theta(s)\,.$$

iii) All flows given by (4·7) are isospectral flows for the operator $\Psi(s)$. Hence all these flows have a Lax representation.

iv) The covector field given by the gradient of the quantity λ in Eq. (4·20) is an eigenvector (with eigenvalue λ) of the operator $\Psi^{+}(s)$.

§ 5. Multisolitons of the interacting-soliton equation and their decomposition into virtual solitons

The basic idea of the following analysis is a simple one: Consider the procedure which we described in general for systems with recursion operators, and which we applied to equations like KdV, mKdV, etc, and apply this procedure to Eqs. (1·1) to (1·5). This then leads to the discovery of new completely integrable systems which eventually describe even more elementary parts of the multisolitons of the original equations. The decomposition into these parts, we call the **second decomposition.** Although the time evolution of the components of the second decomposition looks more complicated than for the equations we started with, its solutions are, at least from the viewpoint of types of interaction, far more simple. Going through this procedure one encounters **virtual solitons.** It is a very simple exercise to compute these solutions explicitly out of the original solutions.

Let me describe the necessary procedure in all detail for the case of a two-soliton solution of some completely integrable system. We start with the system

$$u_t = K_1(u) \tag{5·1}$$

having the recursion operator $\Phi(u)$ which generates the symmetry group generators out of the generator of the translation group

$$K_n(u) = \Phi(u)^n u_x\,. \tag{5·2}$$

From § 2 we know that the two-soliton solution of (5·1), with asymptotic speeds λ_1, λ_2, is a solution of

$$K_2(u) - (\lambda_1 + \lambda_2) K_1(u) + \lambda_1 \lambda_2 u_x = 0 \tag{5·3}$$

or, equivalently,

$$(\Phi(u) - \lambda_1)(\Phi(u) - \lambda_2) u_x = 0\,. \tag{5·4}$$

Using theorem I and the coefficients given in its proof, we find the corresponding decomposition into eigenvectors of $\Phi(u)$ to be

$$u_x = w_1 + w_2\,, \tag{5·5}$$

where

$$w_1 = (\lambda_2 - \lambda_1)^{-1} (\Phi(u) - \lambda_2) u_x = (\lambda_2 - \lambda_1)^{-1} \{K_1(u) - \lambda_2 u_x\}\,, \tag{5·6}$$

$$w_2 = (\lambda_1 - \lambda_2)^{-1} (\Phi(u) - \lambda_1) u_x = (\lambda_1 - \lambda_2)^{-1} \{K_1(u) - \lambda_1 u_x\}\,. \tag{5·7}$$

Integration then yields the interacting solitons

$$s_1 = (\lambda_2 - \lambda_1)^{-1} D^{-1} \{K_1(u) - \lambda_2 u_x\}\,, \tag{5·8}$$

$$s_2=(\lambda_1-\lambda_2)^{-1}D^{-1}\{K_1(n)-\lambda_1 u_x\} \tag{5·9}$$

in terms of the symmetry generators of Eq. (5·1). This is how the functions given in Fig. n.A. and Fig. n.B. can be computed in a simple way.

Now, let us study the time-evolution of one of these interacting solitons say $s=s_i$. Evolution equation, symmetry generators and recursion operators can be computed as it was described in §§ 3 and 4.

Let $\Psi(u)$ and $G_n(u)$ denote the recursion operator and the symmetry generators of this dynamic system as they were given in formulas (4·9) and (4·13). Then using (4·10) we find from (5·3) that s fulfills

$$G_2(s)-(\lambda_1+\lambda_2)G_1(s)+\lambda_1\lambda_2 s_x=0 \tag{5·10}$$

or, equivalently

$$(\Psi(s)-\lambda_1)(\Psi(s)-\lambda_2)s_x=0\,. \tag{5·11}$$

Hence, from the point of view of the spectrum of $\Psi(s)$, our interacting soliton s is a two-soliton solution of equation

$$s_t=G_1(s)\,. \tag{5·12}$$

Hence, we may decompose this solution again into interacting solitons. Thus we have applied the "soliton decomposition" twice to the original equation. The asymptotic speeds for this second decomposition which can be found from formula (5·11) are again λ_1 and λ_2. But since we know that asymptotically only one soliton is "really" emerging we know that the other part in this second decomposition must be a "virtual soliton" as it was defined in § 2. So, from this point of view, the interacting solitons of the KdV are superpositions of "real solitons" and suitable "virtual solitons" of the dynamic systems given by the second decomposition. In addition one finds out that these virtual solitons only pop up when collision of the original solitons occurs.

Of course, once we have seen that an iteration of the soliton decomposition is possible, one may study further decompositions and proceed indefinitely with this procedure. But in case of the two-soliton solution this does not give a new insight into the qualitative description of interaction. In case of higher order multi-solitons this may be different.

Let me present the simple computation necessary for the second decomposition in case of the two-solitons. Application of formulas (5·8) and (5·9) to (5·12) instead of (5·1) yields for the soliton $\sigma_{i,1}$ and $\sigma_{i,2}$ of $s=s_i$.

$$\sigma_{i,1}=(\lambda_2-\lambda_1)^{-1}D^{-1}\{G_1(s)-\lambda_2 s_x\}\,, \tag{5·13}$$

$$\sigma_{i,2}=(\lambda_1-\lambda_2)^{-1}D^{-1}\{G_1(s)-\lambda_1 s_x\}\,. \tag{5·14}$$

By use of (4·9) we find

$$\sigma_{i,k}=(-1)^k(\lambda_1-\lambda_2)^{-1}D^{-1}\{D^{-1}K'_1(u)-\lambda_{k'}\}s_x\,, \tag{5·15}$$

where $k'=2$ if $k=1$ and vice versa. Insertion of (5·8) or (5·9), respectively, yields

Fig. 1.V. Second Decomposition of KdV. Virtual Soliton. $c_1=2.56$, $c_2=1.44$.

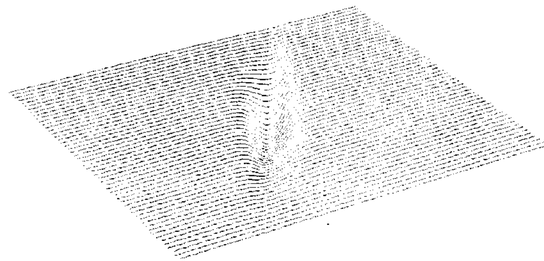

Fig. 2.V. Second Decomposition of KdV. Virtual Soliton. $c_1=2.56$, $c_2=1.08$.

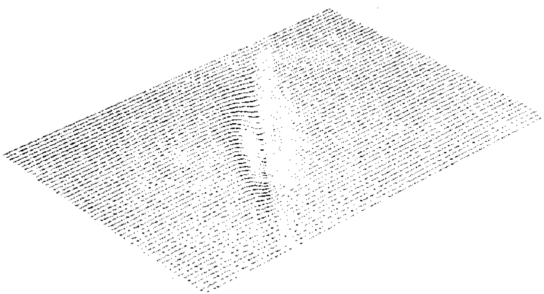

Fig. 3.V. Second Decomposition of KdV. Virtual Soliton. $c_1=2.56$, $c_2=0.92$.

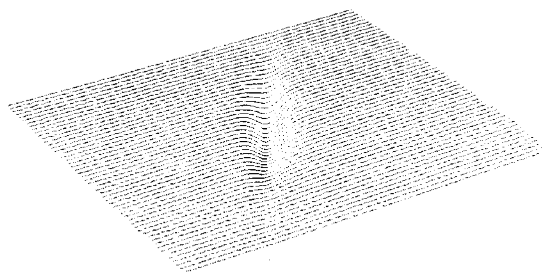

Fig. 4.V. Second Decomposition of KdV. Virtual Soliton. $c_1=2.56$, $c_2=0.64$.

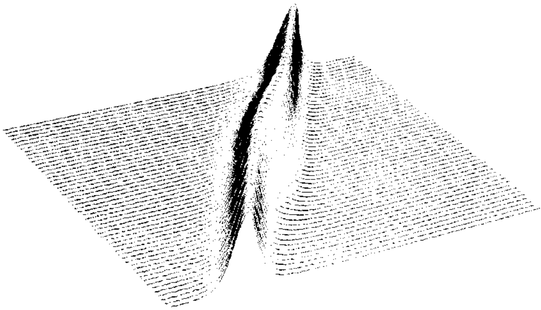

Fig. 1.Aa. Second Decomposition of KdV. Larger Soliton. $c_1=2.56$, $c_2=1.44$.

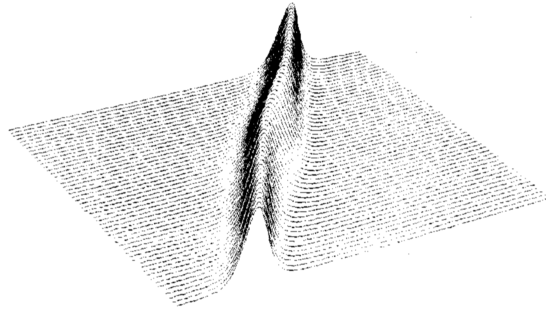

Fig. 2.Aa. Second Decomposition of KdV. Larger Soliton. $c_1=2.56$, $c_2=1.08$.

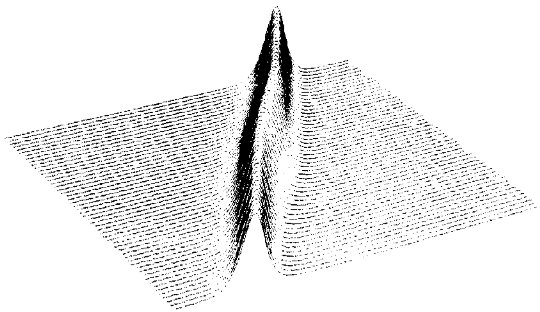

Fig. 3.Aa. Second Decomposition of KdV. Larger Soliton. $c_1=2.56$, $c_2=0.92$.

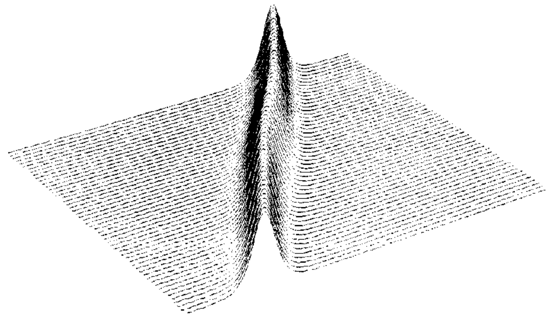

Fig. 4.Aa. Second Decomposition of KdV. Larger Soliton. $c_1=2.56$, $c_2=0.64$.

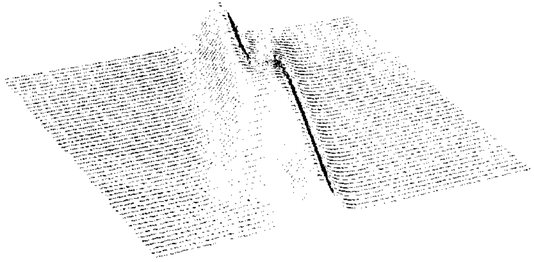

Fig. 1.Bb. Second Decomposition of KdV. Smaller Soliton. $c_1=2.56$, $c_2=1.44$.

Fig. 2.Bb. Second Decomposition of KdV. Smaller Soliton. $c_1=2.56$, $c_2=1.08$.

Fig. 3.Bb. Second Decomposition of KdV. Smaller Soliton. $c_1=2.56$, $c_2=0.92$.

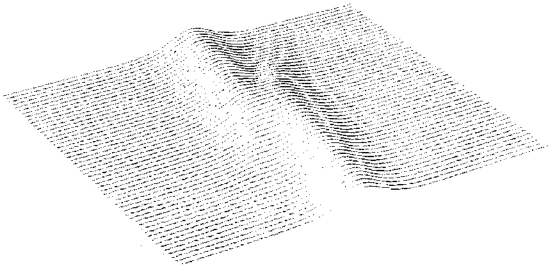

Fig. 4.Bb. Second Decomposition of KdV. Smaller Soliton. $c_1=2.56$, $c_2=0.64$.

$$\sigma_{i,k}=(-1)^{i+k}(\lambda_1-\lambda_2)^{-2}D^{-1}\{D^{-1}K'_1(u)-\lambda_{k'}\}(K_1(u)-\lambda_{i'}u_x)\,. \tag{5·16}$$

Study of asymptotic speeds yields that the $\sigma_{1,2}$ and $\sigma_{2,1}$ have to be the virtual solitons. We compute only these, because then the $\sigma_{1,1}$ and $\sigma_{2,2}$ are easily found from

$$\sigma_{1,1}=s_1-\sigma_{1,2}\,, \tag{5·17}$$

$$\sigma_{2,2}=s_2-\sigma_{2,1}\,. \tag{5·18}$$

For the virtual solitons (5·14) can be simplified by use of $K'(u)u_x=DK(u)$. We obtain

$$\sigma_{2,1}=\sigma_{1,2}=-(\lambda_1-\lambda_2)^{-2}D^{-1}\{-(\lambda_1+\lambda_2)K_1(u)+\lambda_1\lambda_2u_x+D^{-1}K'_1(u)K_1(u)\}\,. \tag{5·19}$$

Hence, the two virtual solitons occurring in the second decomposition are the same. Now, using (5·3) this reduces to

$$\sigma_{2,1}=\sigma_{1,2}=-(\lambda_1-\lambda_2)^{-2}D^{-1}\{K'_1(u)K_1(u)-K_2(u)\}\,. \tag{5·20}$$

For the KdV with

$$K(u)=u_{xxx}+6uu_x\,,$$

$$K'(u)=D^3+6Du\,,$$

this virtual soliton has the form

$$\sigma_{2,1}=\sigma_{1,2}=-(\lambda_1-\lambda_2)^{-2}\{4D^{-1}uK(u)-2uD^{-1}K(u)\}\,. \tag{5·21}$$

This quantity is plotted for the different pairs of asymptotic speeds $c_i=\lambda_i$ in Fig. n.V., $n=1,\cdots,4$. Looking at these plots, one sees that any qualitative difference in interaction has completely disappeared for the cases considered. For completeness also the quantities $\sigma_{1,1}$ (larger soliton) and $\sigma_{2,2}$ (smaller soliton) have been plotted in Fig. n. Aa and Fig. n. Bb, $n=1,\cdots,4$, respectively.

§ 6. Comparison with other work and concluding remarks

Ever since the discovery of solitons there was constant work on decomposition of a field into its soliton components. This work started with the fundamental paper,[14)] where for multisolitons the decomposition into squared eigenfunctions is given, and goes until a recent series of interesting papers on this subject 26)~29), and 19) and others.

At first glance, it looks as if all those decompositions were the same. And in fact they are, and they are the same as the one given in this paper. Nevertheless there is a fundamental difference between the present paper and the results of others.

A rather unessential difference lies in the methods applied in order to obtain the desired decomposition. Mostly, inverse scattering transform methods are used for the decomposition, as it was already the case in 14). In fact this is not necessary as it was shown already in 6). This change in method has two consequences. First, one discovers that the method must work also in cases where inverse scattering is unapplicable or only more difficult (e.g., other boundary conditions at infinity).

Secondly, this observation opens the road for the discovery, that solitons, as dynamical systems being coupled to the superposition field, also make sense in cases where apart from solitons (discrete parts of the spectrum of the recursion operator or of the scattering method) also continuous parts of the spectrum contribute.

But even this discovery is rather ancient. For example, in 7) and 9) it was stated that the eigenvector of the recursion operator can be considered as soliton in interaction and that furthermore this soliton in interaction has a well-defined dynamical behavior, namely, that of a gradient of a conserved quantity.

But our present paper, we believe, contains a completely new aspect. Namely, that in principle it is possible to find the dynamical behavior of the interacting system in such a way that no external field and no superposition with other solitons enters in the description of this dynamical behavior. Furthermore, that the dynamical system found this way also makes sense in cases which are not pure soliton solutions. So, the coupled systems which are given by other authors are decoupled, only self-interaction plays a role. To be precise: The dynamics given by other authors for interacting solitons is the one expressed by Eq. (3·1) which then in the multi-soliton case leads via (2·13) to a coupled system. In contrast to that the dynamics expressed by (4·9) even holds in the absence of the decomposition (2·13).

This decoupling is an essential prerequisite for finding a dynamical description of interacting solitons which is independent of the number of solitons present in the field.

Contrary to that, in the soliton-decompositions which can be found in the literature, one will discover that the coupled equations change with the number of solitons present.

This decoupling, which mathematically turns out to be a triviality, then allows to study the structure of the interacting solitons (i.e., show their complete integrability in the general case, and find their recursion operators). Another important consequence seems to me that very many new systems, which are completely integrable, can be constructed this way, and, if one likes it, many new Lax pairs can be found.

Without finding the dynamical behavior in a selfinteracting way the second decomposition, which to my opinion offers a better qualitative understanding of soliton interaction, would have been impossible. With the knowledge of the dynamics in terms of uncoupled equations the second decomposition, as well as the third, the fourth, etc. reduces to an observation which is more or less trivial.

There is another consequence of decoupling interacting solitons which we did not yet discuss in this paper, but which seems even more important to me. One of the ultimate aims of decomposition into solitons seems to be to find simple dynamical descriptions for the "trajectories" of solitons. That means one tries to replace the dynamics (given by some nonlinear partial differential equation) by a system of ordinary differential equations describing the positions or barycenters of the different solitons. Thus a flow on some infinite-dimensional manifold is described by a flow on a finite-dimensional manifold. Of course, such a method, if it is known on a systematic basis would provide a better understanding and a simpler description of soliton interaction. There are very many interesting contributions towards such a method (see 24), 16), 4), 1), 2), 3) and the most interesting recent papers.[21),22)]) But to my knowledge a systematic and foolproof method of how to find particle systems imitating the soliton collision is still missing.

But in principle, such a method is a consequence of the results of this paper (although the technical details may be cumbersome). Let me describe this briefly, say for the case of the KdV:

Choose a set of trajectories $y_k(t)$, $k=1, \cdots, N$ (to be specified later on) and consider a multi-soliton solution of the KdV, say some N-soliton. Define quantities $p_{i,k}^{(r)}(r=0,1,2;\ i,k=1,\cdots,N)$ to be the values which the r-th derivative of the soliton s_i attains at y_k. Then by the eigenvector equation (3·6) and the decomposition given in theorem I the values at the y_k of higher derivatives than r of the s_i can be expressed by the $p_{i,k}^{(r)}$. Now, using the dynamics which is explicitly given for the s_i we can express the time evolution of the $p_{i,k}^{(r)}$ also by these quantities and the time derivatives of the y_k. This suggests that reasonable trajectories are those, where the time derivatives can be expressed also in terms of the $p_{i,k}^{(r)}$. Then for those we have a complete description of the dynamics of the $p_{i,k}^{(r)}$. But such trajectories are easily found, for example, take y_k to be the zero of s_{kx}. Then

$$\frac{d}{dt} s_k(y_k(t), t)_x = 0$$

together with (1·2) easily gives the desired relation between y_{kt} and the $p_{i,k}^{(r)}$. Here, again the explicit knowledge of the dynamics of the interacting soliton plays an essential role.

Acknowledgements

I am indebted to K. Hering and W. Oevel for the help they gave me with the plots of this paper.

References

1) G. Bowtell and A. E. G. Stuart, Phys. Rev. **D15** (1977), 3580.
2) G. Bowtell and A. E. G. Stuart, J. Math. Phys. **24** (1983), 969.
3) F. Calogero and A. Degasperis, Lett. Nuovo Cim. **19** (1977), 525.
4) D. V. Choodnovsky and G. V. Choodnovsky, Nuovo Cim. **B40** (1977), 339.
5) A. S. Fokas and B. Fuchssteiner, Nonlinear Analysis TMA **5** (1981), 423.
6) B. Fuchssteiner, Commun. Math. Phys. **55** (1977), 187.
7) B. Fuchssteiner, Nonlinear Analysis TMA **3** (1979), 849.
8) B. Fuchssteiner and A. S. Fokas, Physica **4D** (1981), 47.
9) B. Fuchssteiner, Prog. Theor. Phys. **65** (1981), 861.
10) B. Fuchssteiner and R. N. Aiyer, J. of Phys. **20A** (1986), 377.
11) B. Fuchssteiner, *Proceedings of the 14th ICGTMP*, ed. Y. M. Choo (World Scientific Publ., Singapore, 1986), 421.
12) B. Fuchssteiner, W. Oevel and W. Wiwianka, Computer Physics Comm. **44** (1987).
13) B. Fuchssteiner and W. Oevel, Physica **A** (1987).
14) C. S. Gardner, J. M. Green, M. D. Kruskal and R. M. Miura, Comments Pure Appl. Math. **27** (1974), 97.
15) M. Jaworski, Phys. Lett. **100A** (1984), 321.
16) M. Kruskal, Lect. Appl. Math. **15** (1974), 61.
17) G. L. Lamb, *Elements of Soliton Theory, Pure and Applied Mathematics Series* (Wiley, New York-Chichester-Brisbane-Toronto, 1980).
18) P. D. Lax, Comm. Pure Appl. Math. **21** (1968), 467.
19) T. P. Moloney and P. F. Hodnett, J. of Phys. **19A** (1968), L1129.
20) C. Rebbi and G. Soliani, *Solitons and Particles* (World Scientific, Singapore, 1984).

21) S. N. M. Ruijsenaars, "Relativistic Calogero Moser Systems and Solitons" in *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations,* ed. M. Ablowitz, B. Fuchssteiner and M. Kruskal (World Scientific Publ., Singapore, 1987), p. 182.
22) S. N. M. Ruijsenaars and H. Schneider, Ann. of Phys. **170** (1986), 370.
23) J. Scott Russell, Report of the 14th Meeting of the British Association for the Advancement of Science (John Murray, London, 1844), p. 311.
24) W. R. Thickstun, J. Math. Anal. Appl. **55** (1976), 335.
25) H. Tsuru and M. Wadati, J. Phys. Soc. **53** (1984), 2908.
26) T. Yoneyama, Prog. Theor. Phys. **72** (1984), 1081.
27) T. Yoneyama, Prog. Theor. Phys. **71** (1984), 843.
28) T. Yoneyama, Phys. Soc. Jpn. **55** (1986), 3691.
29) T. Yoneyama, Phys. Soc. Jpn. **55** (1986), 3313.