The Tangent Bundle for Multisolitons: Ideal Structure for Completely Integrable Systems

B. Fuchssteiner

University of Paderborn, D-4790 Paderborn, Fed. Rep. of Germany

Abstract: Multisoliton manifolds are characterized as symplectic prime ideals of the symplectic Lie algebra module generated by symmetries and mastersymmetries. This identification allows an explicit construction of the tangent bundle of the multisoliton manifolds.

1 Introduction

We consider a so called completely integrable evolution equation $u_t = K(u)$ on some manifold M of functions $u = u(\vec{x}), \vec{x} \in \mathbb{R}^n$. Then, as usual, [13] a multisoliton manifold is defined by reduction with respect to a sequence of symmetry generators $K_0(u), K_1(u), K_2(u), \ldots$. That means the submanifold

$$M_N = \{ u \mid \text{there are } \alpha_n \text{ such that } \sum_{n=0}^{N} \alpha_n K_n(u) = 0\} . \tag{1.1}$$

is the N-soliton manifold. This manifold is invariant with respect to $u_t = K(u)$. In order that this definition makes sense we require for the manifold M that a nondegeneracy condition, for a suitable symplectic form, must be fulfilled. This condition will be given later on.

This paper is concerned with the following

Problem: Find an explicit construction of the tangent bundle TM_N of M_N.

By explicit we mean that a base for the tangent vectors of M_N should be given in terms of the field variable u and not in terms of asymptotic data. This is an interesting problem because it turns out that

- in case of finite dimensional M_N the explicit knowledge of TM_N gives a direct and purely algebraic way to find the action-angle representation of $u_t = K(u)$ on M_N (see [11],[14]),
- we obtain in a simple and transparent way explicit formulas for quantities of physical relevance (see [11]),
- this yields methods of direct linearization for large classes of nonlinear equations with one independent variable less. This results from the fact that the condition for M_N is equivalent to a requirement of the following (or similar) form

$$0 = \begin{vmatrix} K_0 & K_1 & \cdots & K_N \\ K_1 & K_2 & \cdots & K_{N+1} \\ \vdots & \vdots & \cdots & \vdots \\ K_N & K_{N+1} & \cdots & K_{2N} \end{vmatrix} \tag{1.2}$$

Research Reports in Physics Nonlinear Evolution Equations and Dynamical Systems
Editors: S. Carillo · O. Ragnisco

where the $\alpha_1, \ldots, \alpha_N$ have been eliminated. Since mastersymmetries carry over ([10]) to this reduction, we easily obtain the recursive structure for the symmetry group of these equations (1.2),

- that the construction of the tangent space proves to be equivalent to a spectral resolution of the recursive structure of the symmetry group. Hence, the construction of TM_N yields a direct method to determine the eigenvectors of the recursion operator (if such an operator exists) and to find solutions of the so called interacting soliton equations (see [9]).

2 Hereditary algebra

We assume that in the Lie algebra of vector fields we have $K_0, K_1, K_2, \ldots$ and $\tau_0, \tau_1, \tau_2, \ldots$ such that

$$[K_n, K_m] = 0, [\tau_n, K_m] = (m+\rho)K_{n+m}, [\tau_n, \tau_m] = (m-n)\tau_{n+m}, \tag{2.1}$$

where ρ is some constant number. The K_n are called symmetries and the τ_n are called mastersymmetries. The algebra $\mathcal{HER}(K_n, \tau_m)$ spanned by the K_n, τ_m is called a *hereditary algebra*. This because the formal operator Φ defined by

$$\Phi K_n := K_{n+1} \text{ and } \Phi\tau_m := \tau_{m+1} \text{ for all } m, n \in \mathbb{N}. \tag{2.2}$$

is a hereditary operator as defined in [7] (see also [6]). Up to now, such a hereditary algebra is known for all completely integrable systems, even those where no local recursion operator has been found.

Remark 1 : *It should be observed that hereditary algebras have a simple structure since there is a canonical representation of such an algebra in an operator algebra of certain first order differential operators (in one variable). To see this consider*

$$\tau_m \to z^{m+1}\frac{d}{dz}, \quad K_n \to z^{n+\rho}. \tag{2.3}$$

Then this defines a Lie algebra homomorphism into the algebra given by operator commutation.

Furthermore, we assume that a symplectic operator J (from tangent bundle to cotangent bundle) is known such that all JK_n are closed (i.e. do have potentials). This is the same, as saying that the K_n are hamiltonian with respect to J. Recall that an operator J is said to be symplectic if the two-form ω given by $\omega(G_1, G_2) = \langle JG_1, G_2\rangle$ is closed. Here $\langle\, ,\, \rangle$ denotes the evaluation between tangent and cotangent spaces. We assume that for all $u \in M$ the form ω is nondegenerate with respect to $\mathcal{HER}(K_n, \tau_m)$, i.e. if for some fixed H we have $\omega(H, G) = 0 \forall G \in \mathcal{HER}(K_n, \tau_m)$ then $H = 0$.

In addition, and for simplicity, we require that τ_0 defines a scaling for J, i.e. $L_{\tau_0}J = \lambda J$, where L_{τ_0} is the usual Lie derivative with respect to τ_0 (see [16]). Of course, we allow the case $\lambda = 0$. Now, there are essentially two cases:

Non-hamiltonian case: For all, except at most one n, we have $L_{\tau_n}(J) \neq 0$.

Hamiltonian case: $L_{\tau_n}J = 0$ for all n.

In the first case the operators $\Phi_n = J^{-1}L_{\tau_n}(J)$ are recursion operators and the hereditary structure of the algebra implies that these operators are hereditary. Furthermore, the formal recursion operator Φ, defined in $\mathcal{HER}(K_n, \tau_m)$, then is given by one of these Φ_n. In addition, from the antisymmetry of J, we obtain that the transpose $(J\Phi)^+$ of $(J\Phi)$ is equal to $(\Phi^+ J)$. All known completely integrable systems in 1+1 dimension, apart form the Benjamin-Ono equation and the intermediate long wave equation, belong to this category. Obviously, the scaling number λ must be equal to zero in the hamiltonian case.

The second case applies to the (2+1)-dimensional cases, as well as to equations like the Benjamin-Ono equation and the intermediate long wave equation. These two cases are mutually exclusive insofar as whenever we have $L_{\tau_n} J \neq 0$ for more than one n then we may define a recursion operator and restrict our attention to the hereditary algebra generated by that operator. Examples for such hereditary algebras can be found easily in the literature ([5],[2],[8]).

We now introduce suitable scalar quantities $P_0, P_1, P_2, \ldots$. In the non-hamiltonian case we define these to be

$$P_n = \langle JK_n, \tau_0 \rangle \ . \tag{2.4}$$

And in the hamiltonian case we define the P_n to be the potentials of the covector fields JK_n.

Now, using the symmetry relation $J\Phi = \Phi^+ J$ for the recursion operator, we find in the non-hamiltonian case (see also [11])

$$P_{m+n} = \langle JK_n, \tau_m \rangle \text{ for all } m, n \in \mathbb{N} \ . \tag{2.5}$$

Then (by simple application of Lie-derivatives) one finds for the gradients ∇P_n of the P_n that, as applications on $\mathcal{HER}(K_n, \tau_m)$, we have the following equality

$$\nabla P_n = (n + \rho + \lambda) JK_n \ . \tag{2.6}$$

Hence we may assume, without loss of generality, that the K_n have been chosen in such a way that this holds. This is possible because either it is really true or we may redefine the K_n by that relation without changing the crucial commutation relations (2.1).

In the hamiltonian case, where $\nabla P_r = JK_r$, one should observe that locally all the τ_n have potentials Q_n. Furthermore the Poisson brackets

$$\{Q_m, P_n\} = \langle \nabla P_n, J^{-1}\nabla Q_m \rangle = \langle JK_n, \tau_m \rangle \tag{2.7}$$

are homomorphically mapped by $J^{-1}\nabla$ into the vector field brackets

$$J^{-1}\nabla\{Q_m, P_n\} = [\tau_m, K_n] \tag{2.8}$$

Since $[\tau_m, K_n] = (n + \rho)K_{n+m} = (n + \rho)J^{-1}\nabla P_{n+m}$ we obtain

$$\langle \nabla P_n, \tau_m \rangle = (n + \rho) P_{n+m}. \tag{2.9}$$

Hence, for both cases we have the crucial relation

$$\langle \nabla P_n, \tau_m \rangle = (n + \rho + \lambda) P_{n+m}. \tag{2.10}$$

Of course, in the hamiltonian case λ is equal to zero.

In both cases we trivially have for all m and n that

$$\langle \nabla P_n, K_m \rangle = 0 \ , \tag{2.11}$$

i.e. the P_n all are conserved quantities for the flows $u_t = K_n(u)$.

Consider the algebra $\mathcal{F}$ of polynomials in the variables $< J\tau_m, \tau_n >$ and $< JK_m, \tau_n >$ (n, m arbitrary). These fields we call *scalars*. Recall that by the Lie derivative a homomorphism from $\mathcal{HER}(K_n, \tau_m)$ into the derivations on $\mathcal{F}$ is given. Hence we can make out of the Lie algebra $\mathcal{HER}(K_n, \tau_m)$ a suitable Lie algebra module by allowing that its elements are multiplied by any element in $\mathcal{F}$. This Lie algebra module we denote by $\mathcal{M}_J(K_n, \tau_m)$. The index J reminds us that this depends on the symplectic operator J. Modules constructed in this way via a symplectic form we call *symplectic Lie algebra modules.*

3 Ideals and invariant submanifolds

Since ideals play a central role in any algebraic structure we study them now for $\mathcal{M}_J = \mathcal{M}_J(K_n, \tau_m)$. Recall that a sub-Lie algebra $\mathcal{L}$ of $\mathcal{M}_J$ is said to be a *Lie ideal* (or *ideal* for short) if $[L, H] \in \mathcal{L}$ whenever $L \in \mathcal{L}$ and $H \in \mathcal{M}_J$. Invariance with respect to $\mathcal{M}_J$ is defined in terms of Lie derivatives, in directions given by elements of $\mathcal{M}_J$, so ideals of $\mathcal{M}_J$ are invariant by definition. In $\mathcal{F}$ we are also interested in ideals $\mathcal{G}$ which are invariant with respect to $\mathcal{M}_J$, i.e. if $F \in \mathcal{G}, H \in \mathcal{M}_J$ then $L_H F \in \mathcal{G}$. One of the important nontrivial observations is that the symplectic form ω makes in a canonical way out of ideals $\mathcal{L}$ in $\mathcal{M}_J$ invariant ideals $\mathcal{F}_\mathcal{L}$ in $\mathcal{F}$:

$$\mathcal{F}_\mathcal{L} \stackrel{def}{=} \{\omega(L, H) | L \in \mathcal{L}, H \in \mathcal{M}_J\} \ . \tag{3.1}$$

On the other hand, invariant ideals $\mathcal{G}$ in $\mathcal{F}$ define ideals $\mathcal{L}_\mathcal{G}$ in $\mathcal{M}_J$ by

$$\mathcal{L}_\mathcal{G} \stackrel{def}{=} \{L \in \mathcal{M}_J | \omega(L, H) \in \mathcal{G} \text{ for all } H \in \mathcal{M}_J\} \ . \tag{3.2}$$

Obviously, $\mathcal{L} \subset \mathcal{L}_{\mathcal{F}_\mathcal{L}}$ and $\mathcal{L}_{\mathcal{F}_\mathcal{L}}$ defines a completion, or hull operation, for $\mathcal{L}$. If an ideal $\mathcal{L}$ is complete, i.e. if $\mathcal{L} = \mathcal{L}_{\mathcal{F}_\mathcal{L}}$ then we call it a *symplectic ideal.* In the same way we can define for the invariant ideals in $\mathcal{F}$ a kernel operation. So, for an invariant ideal $\mathcal{G}$ in $\mathcal{F}$, we define the kernel to be $\mathcal{F}_{\mathcal{L}_\mathcal{G}}$. Again we call $\mathcal{G}$ a symplectic ideal if $\mathcal{G} = \mathcal{F}_{\mathcal{L}_\mathcal{G}}$. Obviously, there is a one-to-one correspondence between symplectic ideals in $\mathcal{M}_J$ and those in $\mathcal{F}$.

There is a maximal abelian ideal $\mathcal{A}$ in $\mathcal{M}_J$, namely the vector fields spanned by the K_n alone and admitting coefficients which are polynomials in the P_n alone. Observe that this ideal is mapped, via $\omega(\cdot, \mathcal{M}_J)$, onto those elements which depend on the P_n alone.

We may as well restrict our considerations to abelian ideals and define an *abelian symplectic ideal* either by intersection of symplectic ideals with the maximal abelian ideal or by taking the completion only within the abelian ideals, both definitions amount to the same.

Let us now turn our attention to invariant submanifolds of M, invariance again meant with respect to $\mathcal{HER}(K_n, \tau_m)$. Since the symplectic form was assumed to be nondegenerate we have

Remark 2 : *The zero sets of symplectic ideals in $\mathcal{M}_J$ and the zero sets of their corresponding symplectic ideals in $\mathcal{F}$ coincide.*

An important role is played by the zero sets of prime ideals, these we call *soliton manifolds.*

4 Zero sets of symplectic ideals

In this section we carry out the construction of zero sets of symplectic ideals. For brevity we restrict our considerations to abelian ideals. But indeed, the construction is very similar for nonabelian ones.

First we need some considerations from linear algebra. Consider the space c_∞ of sequences in some vector space. Denote by $S : c_\infty \to c_\infty$ the *shift operator*, that is

$$(q_0, q_1, q_2, ...) \xrightarrow{S} (q_1, q_2, q_3, ...) \; . \tag{4.1}$$

We call a vector $\vec{q} = (q_0, q_1, q_2, ...)$ *N-cyclic* if there is some polynomial $Pol(S)$ of degree N such that

$$Pol(S)\vec{q} = 0 \tag{4.2}$$

and such that N is the minimal degree of a polynomial having that property. The roots of the polynomial $Pol(\xi)$ we call the *characteristic roots* of this N-cyclic vector.

We study the invariances of the manifold of N-cyclic vectors. By definition this manifold is invariant under application of any polynomial in the shift operator itself. And, let $E(\lambda), \lambda \in \mathbb{R}$ be the following group of operators

$$(q_0, q_1, q_2, ...) \xrightarrow{E(\lambda)} (q_0, \exp(\lambda)q_1, \exp(2\lambda)q_2, \exp(3\lambda)q_3, ...) \tag{4.3}$$

then $SE(\lambda) = \exp(\lambda)E(\lambda)S$. Hence, application of $E(\lambda)$ also leaves the manifold of N-cyclic vectors invariant. This because in (4.2) application of $E(\lambda)$ to $\vec{q}$ can be compensated by replacing S by $\exp(-\lambda)S$, thus amounting only in a change of the coefficients of the polynomial, or a change of the characteristic roots: $\xi_n \to \exp(\lambda)\xi_n$.

Using suitable infinitesimal generators of these invariances we find that when $\vec{q}$ is N-cyclic, then all vectors of the form

$$((k+\alpha)q_k, (k+\alpha+1)q_{k+1}, (k+\alpha+2)q_{k+2}, ...) \tag{4.4}$$

are tangential to the manifold of N-cyclic vectors at the point $\vec{q}$, and this for all α.

For the construction of the symplectic hull of some ideal $\mathcal{L}$ in $\mathcal{M}_J$ we proceed in the following way: Take some $L \in \mathcal{L}$, construct the minimal invariant ideal $\mathcal{G}_L$ in $\mathcal{F}$ such that $\mathcal{G}_L \supset \omega(L, \mathcal{M}_J)$. Then take $\mathcal{L}_{\mathcal{G}_L}$, which must be the minimal symplectic ideal in $\mathcal{M}_J$ containing L. Doing this for all L in $\mathcal{L}$ we find the smallest symplectic ideal containing $\mathcal{L}$ to be $\cup\{\mathcal{L}_{\mathcal{G}_L} | L \in \mathcal{L}\}$. The zero sets are most easily found by using *basic sets* of symplectic ideals. A subset $\tilde{\mathcal{G}}$ of some symplectic ideal is said to be basic for $\mathcal{G}$ if any u with $F(u) = 0$ for all $F \in \tilde{\mathcal{G}}$ is automatically in the zero set of $\mathcal{G}$. In the same way we define basic sets for ideals in $\mathcal{M}_J$.

Remark 3 : *Let $\mathcal{G}$ be the smallest invariant ideal containing $\tilde{\mathcal{G}}$. Then $\tilde{\mathcal{G}}$ is basic if and only if $\{u | \tilde{\mathcal{G}}(u) = 0\}$ is invariant.*

In order to treat the hamiltonian case and the non hamiltonian case jointly we introduce suitable coefficients

$$\gamma_n = \left\{ \begin{array}{cc} 1 & \text{in the non-hamiltonian case} \\ (n+\rho+\lambda)^{-1} & \text{in the hamiltonian case} \end{array} \right\}$$

Now, start with an abelian ideal $\mathcal{L}$ in $\mathcal{M}_J$and fix some element $L \in \mathcal{L}$. L can be written as

$$L = \sum_{n=0}^{N} \alpha_n \gamma_{n+m} K_{n+m} \tag{4.5}$$

where the α_n are polynomials in the P's, and where we may assume that α_0, α_N are not equal to zero. For convenience we introduce the polynomial $Pol_L(\xi) = \sum_{n=0}^{N} \alpha_n \xi^n$. The roots of this *characteristic polynomial* we call the *roots* of L. We consider $\mathcal{G}_L$ the smallest invariant ideal containing $\omega(L, \mathcal{M}_J)$ and we use $\equiv$ for equality modulo $\mathcal{G}_L$. By taking scalar products with $J\tau_r$ we obtain

$$\sum_{n=0}^{N} \alpha_n P_{n+m+r} \equiv 0 \text{ for all } r \ . \tag{4.6}$$

This we can write equivalently as

$$Pol_L(S) S^m \vec{P} \equiv 0 \tag{4.7}$$

i.e. the vector $S^m \vec{P}$ is N-cyclic modulo $\mathcal{G}_L$. This gives a linear dependence between each set of consecutive set of $N+1$ vectors out of $\{S^m \vec{P}, S^{m+1} \vec{P}, S^{m+2} \vec{P}, ...\}$ or

$$det_k^N(\vec{P}) = \begin{vmatrix} P_k & P_{k+1} & \cdots & P_{k+N} \\ P_{k+1} & P_{k+2} & \cdots & P_{k+N+1} \\ \vdots & \vdots & \cdots & \vdots \\ P_{k+N} & P_{k+1+N} & \cdots & P_{k+2N} \end{vmatrix} \equiv 0 \text{ for all } k \geq m \ . \tag{4.8}$$

Comparison between (4.6) and (4.8) shows that the $(-1)^{n+1}\alpha_n$, which have to be polynomials in the P's, must be equal, up to some factor modulo $\mathcal{G}_L$, to the subdeterminant of $det_m^N(\vec{P})$ which arises by canceling the first column and the n-th row. And this then yields that all $det_k^N(\vec{P}), k \geq m$ are elements of $\mathcal{G}_L$ and that $\mathcal{G}_L$ is the smallest ideal containing these determinants.

Observation 1 : *The set $\{det_k^N(\vec{P}) | k \geq m\}$ is basic for $\mathcal{G}_L$.*

For the proof of that statement observe that that $det_k^N(\vec{P}(u)) = 0$ for all $k \geq m$ is equivalent to the fact that $S^m \vec{P}$ is N-cyclic. So the common zero set $\tilde{M}$ of the determinants $det_k^N(\vec{P}), k \geq m$ can be represented as

$$\tilde{M} = \{u | S^m \vec{P}(u) \text{ is } N-\text{cyclic}\} \ , \tag{4.9}$$

and it remains to prove that this manifold is invariant. For that we have to show that the directional derivative of some N-cyclic vector $S^m \vec{P}(u)$ in direction of either K_j or τ_j results in a tangential vector, at the point $S^m \vec{P}(u)$, to the manifold of N-cyclic vectors. But that is trivial now, since the directional derivative in direction K_j is zero and that in direction τ_j yields a vector of the form (4.4).

Reformulation of that result in terms of linear dependence shows that any set of determinants in $\mathcal{G}_L$ which has as consequence the linear dependence of the column vectors of (4.8) (for all $k \geq m$) must be basic again. As consequence of this we obtain

Observation 2 : *The vector field*

$$\begin{vmatrix} \gamma_m K_m & P_{m+1} & \cdots & P_{m+N} \\ \gamma_{m+1} K_{m+1} & P_{m+2} & \cdots & P_{m+N+1} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{m+N} K_{m+N} & P_{m+1+N} & \cdots & P_{m+2N} \end{vmatrix} \tag{4.10}$$

is basic for the ideal $\mathcal{L}_L$ generated by (4.5).

This is a trivial consequence of the fact that $\omega(\cdot, \mathcal{M}_J)$ maps this vector field into a basic set of the corresponding symplectic ideal in $\mathcal{F}$. Looking now for prime ideals, we find by some elementary considerations that their zero sets are given by the basic fields of the form (4.10) in case $m = 0$. Hence these are the 'multisoliton manifolds' defined in (1.1). However, it should be observed that at that point it was neither clear that these sets are invariant with respect to the τ's nor that they are zero sets of ideals, even prime ideals.

Consider again the L given by (4.5), then our representation by N-cyclic vectors yields in addition a complete description of the dynamics of the roots of L. Observe that these roots are equal to the characteristic roots of the polynomial $Pol_L(S)$ (in the shift operator) and that in addition we have given a representation of the directional derivatives with respect to K_j, τ_j in terms of infinitesimal invariances of the manifold of N-cyclic vectors, and that we know the effect these invariances have on the characteristic roots of the corresponding shift operator. Gathering all this we obtain

Observation 3 : *Let u be in the zero set of the determinant (4.10) and consider the flows*

$$u_{t_1} = K_j(u), \quad u_{t_2} = \tau_j(u) \tag{4.11}$$

then for the roots of u we have the dependence

$$\xi_n(t_1, t_2) = t_2 \xi_n(0,0) \ . \tag{4.12}$$

So, whenever the ξ's are the spectral points of some spectral problem, then the K's are defining isospectral flows and the τ's are non-isospectral (in the sense of [3],[4] or [12]).

5 Applications

Here we sketch some applications. Consider the multisoliton manifold M_N as defined in (1.1). Then we know that the K_n and τ_m are tangential to that manifold. Since the whole ideal generated by the linear dependence as required in (1.1) vanishes on M_N we obtain that the vector fields given by the columns of

$$det^N(\vec{K}) = \begin{vmatrix} \gamma_0 K_0 & \gamma_1 K_1 & \cdots & \gamma_N K_N \\ \gamma_1 K_1 & \gamma_2 K_2 & \cdots & \gamma_{N+1} K_{N+1} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_N K_N & \gamma_{N+1} K_{N+1} & \cdots & \gamma_{N+N} K_{N+N} \end{vmatrix} \tag{5.1}$$

are linear dependent. Hence, the $u \in M_N$ are solutions of the the equation given by $det^N(\vec{K}) = 0$. If that is a differential equation with respect to some independent variable, say x, then we may rewrite it as a several-component evolution equation (with the evolution parameter now x instead of t). In general the τ's are fields depending on x explicitly, hence they give rise to symmetries now depending explicitly on the evolution parameter, this because they are tangential at M_N. So they may serve as mastersymmetries for these newly formulated evolutions and we can carry over the whole hereditary structure to these equations. How that has to be done for the vector fields is described in [10]. In addition we need a hamiltonian formulation for these equations, but this is easily found from the fact that these new flows follow from a variational principle (for the determinants in (4.8)). So the flows, obtained by putting (5.1) (or similar determinants) equal to zero, are again completely integrable and their recursive structure can be computed from the information of this paper. Since these then give additional tangent fields (by iterating the arguments of the present paper) we find the whole tangent bundle for M_N.

Another application is the following:
Take the multisoliton manifold M_N as described herein by an abelian ideal $\mathcal{L}$. Then consider the corresponding (eventually nonabelian) ideal generated by taking all of its symplectic hull $\mathcal{L}_{\mathcal{F}_\mathcal{L}}$. In the non-hamiltonian case this really is a bigger ideal than $\mathcal{L}$ whereas it is equal to $\mathcal{L}$ in the hamiltonian case. But in any case it must be zero on M_N since ω is nondegenerate (remark 2). In the non-hamiltonian case we obtain from that an explicit linear dependence on M_N between the τ's. In the non-hamiltonian case this yields [11] for all $m \in \mathbb{N}$

$$\begin{vmatrix} \tau_m & P_{m+1} & \cdots & P_{m+N} \\ \tau_{m+1} & P_{m+2} & \cdots & P_{m+N+1} \\ \vdots & \vdots & \cdots & \vdots \\ \tau_{m+N} & P_{m+1+N} & \cdots & P_{m+2N} \end{vmatrix} = 0 \text{ on } M_N \ . \tag{5.2}$$

This relation indeed has a couple of nontrivial consequences. It yields the complete spectral resolution of the recursion operator on M_N (see [11] and [17]).

To carry over this method, to obtain a similar result for the hamiltonian case, one needs an additional assumption, namely the existence of a mastersymmetry of second order. Such a higher order mastersymmetry is given in all hamiltonian cases known to the author. To be explicit: In the hamiltonian case one usually finds a hamiltonian field Z fulfilling (in case $\rho \in \mathbb{N}$)

$$[Z, K_n] = (n+\rho)(n+\rho-1)K_{n-\rho} + (n+\rho)\tau_n \ . \tag{5.3}$$

Finding such a field one easily constructs an extension of the ideal $\mathcal{G}$ to some ideal $\mathcal{N}$ (nonabelian) such that $\mathcal{N}$ vanishes on the zero set of $\mathcal{G}$. This is done by taking the scalar products between the abelian ideal $\mathcal{L}_\mathcal{N}$ and the field Z. These scalar products obviously vanish on M_N. Then the smallest invariant ideal is taken which contains these fields, and from there one goes to the corresponding nonabelian symplectic ideal in $\mathcal{M}_J$, which by remark 2 vanishes again on the manifold $\mathcal{M}_N$.

Other applications of the theoretical foundations presented in this paper are the algorithms for finding symmetry groups (see [18]) and maps between the action variables and the angle variables of the restrictions to these invariant manifolds (see [1]).

6 References

[1] S. Carillo and B. Fuchssteiner : this volume

[2] H.H. Chen, Y.C. Lee, J.E. Lin : On a new Hierarchy of Symmetries for the Integrable Nonlinear Evolution Equations, preprint, University of Maryland (1982)

[3] Y. Cheng: On Hierarchies of equations for the Benjamin-Ono Equation, Phys. Lett., 127 A, 205-208, (1988)

[4] Y. Cheng: Hierarchies of equations and their symmetries for the 2+1 dimensional nonlinear evolution equations: preprint Hefei, China, (1989)

[5] A. S. Fokas, B. Fuchssteiner: The Hierarchy of the Benjamin-Ono Equation, Phys. Lett., 86 A, 341-345, (1981)

[6] B. Fuchssteiner : Application of Hereditary Symmetries to Nonlinear Evolution equations, Nonlinear Analysis TMA 3, 849-862 (1979)

[7] B. Fuchssteiner : The Lie-Algebra Structure of Nonlinear Evolution Equations admitting infinite dimensional Abelian Symmetry Groups, Prog. Theor. Phys. 65, 861-876 (1981)

[8] B. Fuchssteiner : Mastersymmetries, Higher order time-dependent Symmetries and Conserved Densities of Nonlinear Evolution Equations, Prog. Theor. Phys. 70, 1508-1522 (1983)

[9] B. Fuchssteiner: Solitons in Interaction, Progress of Theoretical Physics, 78, 1022-1050,(1987)

[10] B. Fuchssteiner, W. Oevel: New Hierarchies of nonlinear completely integrable Systems related to a change of variables for evolution parameters, Physica, 68A, 67-95 (1987)

[11] B. Fuchssteiner, Gudrun Oevel : Geometry and Action-Angle Variables of Multi Soliton Systems, preprint, University of Paderborn (1989)

[12] P. G. Grinevich, A. Y. Orlov: Virasoro action on moduly space and higher symmetries of the KP-equation, Krichever-Novikov Problem, preprint (1989)

[13] S. P. Novikov : The periodic problem for the Korteweg-de Vries equation, Funct. Anal. Appl. 8, 236-246 (1974)

[14] Gudrun Oevel, B. Fuchssteiner, M. Blaszak : this volume

[15] Gudrun Oevel, B. Fuchssteiner, M. Blaszak : Action-angle representation of Multisolitons by potentials of mastersymmetries, Univ. of Paderborn (1989)

[16] H.M.M. TenEikelder : Symmetries for Dynamical and Hamiltonian Systems, CWI Tract, 17, CWI Amsterdam (1985)

[17] H.M.M. TenEikelder : On the local structure of Recursion operators for Symmetries, Indag. Math., 89, 386-403 (1986)

[18] W. Wiwianka and B. Fuchssteiner : this volume