MAXIMAL SEMIGROUPS AND THE SUPPORT OF GAUSS - SEMIGROUPS

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The purpose of this note is to describe a connection between the theory of probability measures on Lie groups and the Lie theory of semigroups. The objects under consideration will be one parameter semigroups of probability measures on Lie groups and their supports. We start by giving the basic definitions.

Let G be a connected topological group. A family $(\mu_t)_{t>o}$ of probability measures on G is called a *Gauss-semigroup*, if no μ_t is a point measure, $\mu_{t+s} = \mu_t * \mu_s$ with the usual convolution and $\lim_{t\to o} t^{-1}\mu_t(G\setminus U)=0$ for every open neighborhood U of the identity in G. The Gauss-semigroup $(\mu_t)_{t>o}$ is called absolutely continuous if each μ_t is absolutely continuous with respect to a, once and for ever fixed, left Haar measure on G.

If G is a Lie group we can associate with any Gauss-semigroup $(\mu_t)_{t>o}$ an *infinitesimal generator* N of the form:

$$N = \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{r} x_i^2$$

where $\{x_1,\ 1\leq i\leq n\}$ is a basis of the Lie algebra $L(G)$ of G, viewed as left invariant first order differential operators on G. The pair (M,x_o), where M is the Lie algebra generated by $\{x_1,\dots,x_r\}$ and $x_o = \sum_{i=1}^{n} a_i x_i$ is called the *carrier* of $(\mu_t)_{t>o}$. We have (cf.[Si82]):

THEOREM 1. *Let* G *be a Lie group and* $(\mu_t)_{t>o}$ *be a Gauss-semigroup on* G *with carrier* (M,x_o). *Then*

(i) $\operatorname{Supp}\mu_t = (\bigcup_{n=1}^{\infty} (G_M \exp n^{-1} t x_o)^n)^-$, *where* $\operatorname{Supp}\mu_t$ *is the support of the measure* μ_t *and* G_M *is the analytic subgroup of* G *with Lie algebra* M.

(ii) $(\operatorname{Supp}\mu_t)(\operatorname{Supp}\mu_s) \subset \operatorname{Supp}\mu_{t+s}$ *for all* $s,t>0$. □

It is clear from this theorem that the sets $S_{\mu,a} = (\bigcup_{t>a} \operatorname{Supp}\mu_t)^-$ are semigroups for any $a\geq 0$. The semigroups $S_{\mu,a}$ will in general not contain the identity and hence are not suited too well to the Lie theory of semigroups which studies subsemigroups of Lie groups via their tangent object at the identity (see below for the precise definitions). But $S_\mu = S_{\mu,o}$ does contain the identity and it will be this semigroup we will concentrate on.

Let L be the Lie algebra in L(G) generated by $\{x_o, x_1, \ldots, x_r\}$ and G_L the corresponding analytic subgroup of G. We will call a Gauss-semigroup *generating* if $G = G_L$. By [Si82] Theorem 2, we know that $\mu_t(G\setminus G_L) = 0$ for all $t>0$. Thus, for the purpose of studying the support behaviour of Gauss-semigroups, it is no serious loss of generality to assume that $G = G_L$.

It is necessary to have some control over the interior points $int(S_\mu)$ of S_μ in order to apply the techniques developed in [HHL85] and [La86]. We find:

LEMMA 2. *Let* $(\mu_t)_{t>o}$ *be a Gauss-semigroup with carrier* (M,x_o) *then we have*

(i) *The interior* $int(S_\mu)$ *of* S_μ *is dense in* S_μ.

(ii) S_μ *is equal to the closed subsemigroup* $\overline{S}$ *of* G *generated by* exp(M) *and* $\exp(\mathbb{R}^+ x_o)$.

Proof. Note first that Theorem 1 implies that $\exp(\mathbb{R}^+ x_o)$ is contained in S_μ. Moreover G_M is contained in S_μ as well. In fact, let $y \in M$ then $(\exp(n^{-1}y)\exp((mn)^{-1}x_o))^n \subset \mathrm{Supp}(\mu_{1/m}) \subset S_\mu$ so that the Trotter product formula shows $\exp(y) \subset \overline{S}_\mu = S_\mu$ for y small enough. Therefore S_μ contains a neighborhood of the identity in G_M and hence all of G_M. But by [JS72] the semigroup S generated by exp(M) and $\exp(\mathbb{R}^+ x_o)$ satisfies $(int(S))^- = \overline{S}$ since M and x_o generate L(G) by our assumptions. Note finally that Theorem 1 shows that $S_\mu \subset \overline{S} = (int(S))^- \subset (int(S_\mu))^- \subset S_\mu$. □

Lemma 2 allows us to conclude that S_μ is contained in some *maximal subsemigroup* S_{max} of G unless $S_\mu = G$ (cf[La86]). Here by maximality we mean that S_{max} is no group and S_{max} and G are the only subsemigroups of G containing S_{max}.

Now suppose that $(\mu_t)_{t>o}$ is a generating Gauss-semigroup and S_μ is contained in a maximal semigroup S_{max} which is proper, i.e. $S_{max} \neq G$. Recall from Lemma 2 that G_M is contained in S_{max}. This implies that $\exp(\mathbb{R}x_o)$ can not be contained in the group of units $H = S_{max} \cap S_{max}^{-1}$ of S_{max}. Now suppose that H is normal in G then L(H) is a subalgebra of L(G) which contains M and is $ad(x_o)$-invariant. Thus the following remark, taken from [Si82], shows that $(\mu_t)_{t>o}$ can not be absolutely continuous.

REMARK 3. *A Gauss-semigroup is absolutely continuous if and only if the only* $ad(x_o)$ *- invariant subalgebra of* L(G) *containing* M *is all of* L(G). □

We collect the obtained information in

PROPOSITION 4. *If* $(\mu_t)_{t>o}$ *is a generating absolutely continuous Gauss-*

semigroup and S_μ *is a proper semigroup contained in a maximal semigroup* S_{max} *then* $H = S_{max} \cap S_{max}^{-1}$ *can not be not be normal in* G. □

Maximal subsemigroups of Lie groups may look very different and the theory describing them is by no means complete, but there are large classes of groups where they can be handled quite well (cf.[La86],[Hi86a]). The way these semigroups are described is typical for the Lie theory of semigroups in so far as it proceeds via their tangent object.

Given a closed subsemigroup S of a Lie group G we define the *tangent cone* L(S) of S by $L(S) = \{x \in L(G): \exp(\mathbb{R}^+x) \subset S\}$. It turns out (cf[HL83]) that L(S) is a closed convex cone satisfying

$$e^{ad(x)}L(S) = L(S) \quad \text{for all} \quad x \in L(S) \cap -L(S).$$

A closed subsemigroup S of a Lie group G is called a *halfspace semigroup* if L(S) is a halfspace. We give some examples:

The subsemigroup $\mathbb{R}^+$ of non-negative real numbers in $\mathbb{R}$ is a halfspace semigroup in $\mathbb{R}$. Let Aff^+ be the group of real 2 x 2 - matrices of the form

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0 \right\}$$

and

$$Aff^{++} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0,\ b \geqq 0 \right\}.$$

Then Aff^{++} is a halfspace subsemigroup of Aff^+ .

Let $Sl(2,\mathbb{R})^\sim$ be the simply connected covering group of $Sl(2,\mathbb{R})$ and Ω^+ be the closed subsemigroup of $Sl(2,\mathbb{R})^\sim$ generated by $\exp(\mathbb{R}^+u)$, $\exp(\mathbb{R}h)$ and $\exp(\mathbb{R}p)$ where

$$u = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

in $sl(2,\mathbb{R})$. Then Ω^+ is a halfspace semigroup in $Sl(2,\mathbb{R})^\sim$ (cf[HH85a]).

Note that for any half space in a Lie algebra bounded by a subalgebra there is a closed halfspace semigroup in the simply connected group corresponding to the Lie algebra whose tangent wedge is just the halfspace we started with (cf[Hi86b], [La86]). Moreover this halfspace semigroup is maximal and its group of units is the analytic subgroup corresponding to the hyperplane contained in the halfspace.

We are now ready to prove a converse to Proposition 4 in the case that G is simply connected:

PROPOSITION 5. *Suppose that* G *is simply connected and let* $(\mu_t)_{t>0}$ *be a generating Gauss-semigroup which is not absolutely continuous. Then* S_μ *is contained in a halfspace semigroup* S_{max} *whose group of units* $S_{max} \cap S_{max}^{-1}$ *is a closed normal subgroup in* G *of codimension* 1.

Proof. Note first that by hypothesis there exists an $ad(x_o)$-invariant subalgebra of $L(G)$ containing M which is not all of $L(G)$. Let P be such an algebra of maximal dimension. We claim that P must be a hyperplane in $L(G)$. In fact, suppose that $codim(P) > 1$ then $P + \mathbb{R}x_o$ is a subalgebra of $L(G)$ containing M which is $ad(x_o)^-$ invariant, but not all of $L(G)$. But on the other hand we assumed M and x_o to generate the whole algebra which contradicts our earlier statement. Thus P is a hyperplane and by the argument given above it cannot contain x_o. Therefore the $ad(x_o)$- invariance of P shows that P is an ideal in $L(G)$. Let G_P be the analytic subgroup of G corresponding to P then G_P is the group of units of a maximal halfspace semigroup S_{max} containing also $\exp(\mathbb{R}^+x_o)$. Since P is an ideal we know that G_P is normal in G. Finally we note that S_{max} contains $\exp(P)$, hence $\exp(M)$, so that Lemma 2 implies that S_{max} contains S_μ. □

Of course one wonders how serious the assumption in Proposition 5 that $G^\sim$ be simply connected is. Let G be the simply connected covering group of G and $\varphi: G^\sim \to G$ be the covering morphism. If $(\sigma_t)_{t>o}$ is a Gauss-semigroup on G with infinitesimal generator N then $(\varphi\sigma_t)_{t>o}$, consisting of the image measures, is the Gauss-semigroup on G with infinitesimal generator N. Let $Exp: L(G) \to G^\sim$ be the exponential function for $G^\sim$. Then S_σ is the closed subsemigroup of $G^\sim$ generated by $Exp(M)$ and $Exp(\mathbb{R}^+x_o)$ by Lemma 2. Therefore we get $\varphi(int(S_\sigma))$ is open dense in $S_{\varphi\sigma}$ again by Lemma 2. Thus practically all the information on the support of Gauss-semigroups we can expect to obtain via the Lie theory of semigroups, we can already get from the simply connected case.

Proposition 4 and 5 have some immediate consequences. For instance, Proposition 5 says that any generating Gauss-semigroup on $Sl(2,\mathbb{R})^\sim$ is absolutely continuous and, since the absolute continuity of a Gauss-semigroup depends only on its infinitesimal generator, the same is true for $Sl(2,\mathbb{R})$. On the other hand Proposition 4 shows that any generating absolutely continuous Gauss-semigroup on a nilpotent Lie group satisfies $S_\mu = G$ the group of units of maximal semigroups in nilpotent Lie groups contains the commutator subgroup (cf[HHL85]). Of course all of this, and more, is well known (cf[Mc84],[McW83]), but the methods given above are quite general so any kind of information one has on the maximal subsemigroups of a Lie group will yield some information on the support of Gauss-semigroups on this group.

Note that for any subsemigroup S of G containing the identity there is a largest normal subgroup contained in S (cf[La86]). It is denoted by $Core(S)$. The core of a closed semigroup S is closed, so it makes sense to talk about the *reduced pair* (G_R,S_R) where $G_R = G/Core(S)$ and $S_R = S/Core(S)$. If S is a closed halfspace semigroup then we have a complete description of (G_R,S_R):

<u>THEOREM 6</u>. (cf.[Po77]). *Let S be a closed halfspace semigroup in a connected Lie group G. Then for the reduced pair (G_R,S_R) one of the following cases occurs:*

(i) (G_R,S_R) *is topologically isomorphic to* (R,R^+)

(ii) (G_R,S_R) *is topologically isomorphic to* (Aff^+,Aff^{++})

(iii) (G_R,S_R) *is topologically isomorphic to* $(Sl(2,R)^\sim,\Omega^+)$. □

Theorem 6 tells us that the group of units of a closed halfspace semigroup S is normal if and only if the reduced pair (G_R, S_R) is equal to $(\mathbb{R}, \mathbb{R}^+)$.

Thus if we, for some reason, know that any maximal semigroup S in G has to be a halfspace semigroup with reduced pair $(G_R, S_R) = (\mathbb{R}, \mathbb{R}^+)$ then Proposition 4 tells us that for any absolutely continuous Gauss-semigroup $(\mu_t)_{t>0}$ the semigroup S_μ has to be all of G.

In this context we recall the following theorem from [La86]:

THEOREM 7. *Let G be a Lie group such that $G/\mathrm{Rad}(G)$ is compact, where $\mathrm{Rad}(G)$ is the radical of G. If S is a maximal subsemigroup of G with non-empty interior, then S is a halfspace semigroup containing every semisimple analytic subgroup and for the reduced pair (G_R, S_R) one of the following two cases occurs*

(i) (G_R, S_R) *is topologically isomorphic to* $(\mathbb{R}, \mathbb{R}^+)$

(ii) (G_R, S_R) *is topologically isomorphic to* $(\mathrm{Aff}^+, \mathrm{Aff}^{++})$. □

From this we derive

COROLLARY 8. *Let G be a Lie group such that $\mathrm{Rad}(G)$ is nilpotent and $G/\mathrm{Rad}(G)$ is compact, then for every absolutely continuous Gauss-semigroup $(\mu_t)_{t>0}$ we have $S_\mu = G$.*

Proof. It remains to show that case (ii) of Theorem 7 cannot occur. To this end note that the conjugate of a semisimple analytic subgroup is again semisimple so that the subgroup of G generated by all semisimple analytic subgroups of G is a normal subgroup and, by Theorem 7, contained in the core of any maximal semigroup. Thus G_R is nilpotent which excludes case (ii) of Theorem 7. □

COROLLARY 9. *Let G be a Lie group such that $L(\mathrm{Rad}(G)) = \mathrm{Rad}(L(G))$ carries the structure of a complex Lie algebra and $G/\mathrm{Rad}(G)$ is compact, then for every absolutely continuous Gauss-semigroup $(\mu_t)_{t>0}$ we have $S_\mu = G$.*

Proof. As in Corollary 8 we see that any Levi complement of G is contained the core C of an arbitrary maximal semigroup with nonempty interior. Thus $G_R = G/C \cong \mathrm{Rad}(G)/(\mathrm{Rad}(G) \cap C)$ and G_R contains a halfspace semigroup. Taking the inverse image in $\mathrm{Rad}(G)$ this shows that $\mathrm{Rad}(G)$ contains a halfspace semigroup. If we look at the tangent cone of this semigroup it follows from [HH85b] that it contains the commutator algebra of $\mathrm{Rad}(L(G))$ because of the complex structure. Thus $\mathrm{Rad}(G) \cap C$ contains the commutator subgroup of $\mathrm{Rad}(G)$ so that G_R is abelian which again excludes case (ii) of Theorem 7. □

Let us draw a short resumé of what has been said in this note: The supports of the measures in a Gauss-semigroup give rise to subsemigroups of the Lie groups involved. These semigroups can be studied by methods from the Lie theory of semigroups. The results will in general not be results on the supports of the single measures but on the semigroups one associates to them. In special cases, however, as in the case of decreasing supports it is possible to derive results on the supports of the single measures.

It has not been my intention to give a polished exposition of all the results that can be obtained using the methods indicated, but rather I wanted to explain the methods themselves. It is clear that one can construct many examples along these lines and it seems reasonable to believe that many related results could be obtained without a lot of extra effort.

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