Invariant Lorentzian orders on simply connected Lie groups

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0. Introduction

The study of the causal structure of space-time in the theory of relativity led a series of authors to the study of partially ordered Lie groups (cf. [Pa 81, 84], [Ol 81, 82], [Gu 76], [Le 84], [Se 76], [Vi 80]). We restrict ourselves to orders that satisfy a certain compatibility condition between the order and the algebraic structure:

Definition 0.1. Let G be a group and $\leqq$ be a partial order on G, then $\leqq$ is called an *invariant order* if the following monotonicity law holds:

(M) $\quad g_1 \leqq g_2$ then $hg_1k \leqq hg_2k$ for all $h, k, g_1, g_2 \in G$.

We can associate with an order $\leqq$ on G its *positive cone* $P_{\leqq} = \{g \in G: g \geqq 1\}$ where 1 is the identity of G. It is an easy exercise to see that an invariant order is completely determined by its positive cone:

Remark 0.2. Let G be a group and P a subset of G. Define a relation $\mathscr{R} \subset G \times G$ on G by setting $g_1 \mathscr{R} g_2$ if and only if $g_2 g_1^{-1} \in P$. Then $\mathscr{R}$ is an invariant order with $P_{\mathscr{R}} = P$ if and only if the following conditions hold:

(i) $P \cap P^{-1} = \{e\}$.
(ii) $PP \subset P$, i.e. P is a semigroup.
(iii) $gPg^{-1} \subset P$ for all $g \in G$. □

An invariant order $\leqq$ on a group G will be called *directed* if for any two g_1, $g_2 \in G$ there exist $h_1, h_2 \in G$ such that $h_1 \leqq g_1$, $g_2 \leqq h_2$. Again we can translate this property of the order $\leqq$ into a property of its positive cone $P_{\leqq}$ by a standard argument:

Remark 0.3. Let G be a group and $\leqq$ be an invariant ordering of G. Then the following statements are equivalent:

(1) The order $\leqq$ is directed.
(2) $P_{\leqq}$ generates G as a group. □

If we now let G be a Lie group the orders that are of most interest are the continuous (Vinberg's notation) or infinitesimally generated (our notation) ones. We will shortly explain what this means: If S is a subsemigroup of a Lie group G such that S generates G as a group, we may associate with S a *tangent object* $\underline{L}(S)=\{x\in L(G)\colon \exp \mathbf{R}^+ x\subset \bar{S}\}$ where $L(G)$ is the Lie algebra of G with exponential function $\exp\colon L(G)\to G$, and $\bar{S}$ is the closure of S in G. It is well known (cf. [HL 83], [Vi 80] etc.) that $\underline{L}(S)$ is a *wedge*, i.e. a closed convex set which is also closed under addition and multiplication by positive scalars. Moreover it satisfies (cf. Loc. cit.):

(L) $$e^{\operatorname{ad} x}\underline{L}(S)=\underline{L}(S) \quad \text{for all} \quad x\in \underline{L}(S)\cap(-\underline{L}(S)).$$

The semigroup S is called *infinitesimally generated* if S and the semigroup T, algebraically generated by $\exp(\underline{L}(S))$, satisfy:

(IG) $\exp \underline{L}(S)\subset S\subset \bar{T}$; $\underline{L}(S)-\underline{L}(S)$ generates $L(G)$ as Lie algebra.

Now we can state what we mean by an infinitesimally generated order:

Definition 0.4. Let G be a Lie group and $\leqq$ be a directed, invariant order on G, then $\leqq$ is called *infinitesimally generated* if $P_{\leqq}$ is an infinitesimally generated subsemigroup of G.

If $\leqq$ is a directed, infinitesimally generated, invariant order on the Lie group G then $\underline{L}(P_{\leqq})$ satisfies a condition that is even stronger than (L):

Proposition 0.5. *Let G be a Lie group and $\leqq$ be a directed, infinitesimally generated, invariant order then $\underline{L}(P_{\leqq})$ is a generating invariant cone, i.e. it satisfies*

(i) $\underline{L}(P_{\leqq})-\underline{L}(P_{\leqq})=L(G)$.
(ii) $e^{\operatorname{ad} x}\underline{L}(P_{\leqq})=\underline{L}(P_{\leqq})$ for all $x\in L(G)$.
(iii) $\underline{L}(P_{\leqq})\cap(-\underline{L}(P_{\leqq}))=\{0\}$.

Proof. Note first that the semigroup $P_{\leqq}$ generates G as a group since $\leqq$ is directed. If $S=\bar{P}_{\leqq}$ and $x\in\underline{L}(P_{\leqq})$ then $\exp \mathbf{R}^+x\subseteq S$ so that $\exp \mathbf{R}^+e^{\operatorname{ad} y}x=(\exp y)(\exp \mathbf{R}^+x)(\exp y)^{-1}\subseteq(\exp y)S(\exp y)^{-1}\subseteq S$ for all $y\in L(G)$ by property 0.2(iii) and the continuity of inner automorphisms. Thus $e^{\operatorname{ad} y}x\in\underline{L}(P_{\leqq})$ and (ii) is satisfied. But then $\underline{L}(P_{\leqq})-\underline{L}(P_{\leqq})$ is an ideal, as we see by differentiating (ii). Hence, by (IG) it is equal to $L(G)$. Finally note that $0\neq x\in\underline{L}(P_{\leqq})\cap(-\underline{L}(P_{\leqq}))$ would imply $\exp \mathbf{R}x\in P_{\leqq}$ so that $P_{\leqq}\cap P_{\leqq}^{-1}$ couldn't be trivial. □

Naturally the question arises whether for any generating invariant cone W in a Lie algebra L there exists an invariant ordering $\leqq$ of the connected Lie group

G with $L(G)=L$ such that $L(P_{\leqq})=W$. The answer is no in general even if we assume that G is simply connected as was shown by Ol'shanskii [Ol 82b]. But if one makes additional hypotheses on the geometry of the cone W the answer turns out to be yes. The kind of cones we want to consider — and again this is motivated by the study of causal structures (cf. [Le 84]) — are those which are given by a quadratic form q in L in the sense that $\{x\in L\colon q(x,x)\leqq 0\}$ is a double cone (i.e. that q has signature $(+,\ldots,+,-)$) and W is one half of this double cone. Such cones and the corresponding forms we call *Lorentzian*. Then our main result is:

Theorem 0.6. *Let G be a simply connected Lie group and W be an invariant Lorentzian cone in $L(G)$ then there exists a directed, infinitesimally generated, invariant order $\leqq$ on G such that W is the tangent object of its positive cone $P_{\leqq}$.*

We will proceed as follows: In Section 1 we give some general lemmas which turn out to be useful later on. In Section 2 we will state a classification theorem for invariant Lorentzian cones (cf. [HH 85c]), prove the main result for important special cases like the oscillator algebra (cf. § 2 for a definition) and finally put the results together to show Theorem 0.6.

1. On the existence of semigroups with prescribed tangent objects

Let L be a Lie algebra and W be a generating invariant cone in L. If G is the simply connected Lie group with $L(G)=L$ and S is any subsemigroup of G such that $\exp W\subset S$ then S has interior points and hence generates G as a group. Therefore we may consider the tangent object $\underline{L}(S)$ of S and find that $W\subset\underline{L}(S)$. If $\underline{L}(S)=W$, then consider the semigroup T generated algebraically by $\exp W$. The inclusion $\exp W\subset T\subset\bar{S}$ implies that T generates G as a group and that $\underline{L}(T)=\underline{L}(S)$ since $\underline{L}(S)=\underline{L}(\bar{S})=W$. But now Theorem 5.9 [HL 83] implies that $T\cap T^{-1}=\{1\}$. Moreover, since $\underline{L}(\bar{T})=W$ is invariant and $\exp L=\exp(W-W)$ generates G we know that $gTg^{-1}\subset T$ for all $g\in G$. This yields

Proposition 1.1. *Let G be a simply connected Lie group and W be a generating invariant cone in $L(G)$. Then the following two statements are equivalent:*

(1) *There exist a directed, infinitesimally generated, invariant order $\leqq$ such that $\underline{L}(P_{\leqq})=W$.*

(2) *There exist a subsemigroup S of G which generates G as a group and satisfies $\underline{L}(S)=W$.* □

Proposition 1.1 reduces the problem of proving the existence of a directed, infinitesimally generated, invariant order to the problem of showing the existence of a subsemigroup with a prescribed tangent cone. This problem has been studied in [Hi 86], [Vi 80], [Ol 82a, b].

Before we can describe some methods to construct semigroups with a prescribed tangent cone we have to study the situation if we already know that we have a semigroup:

Lemma 1.2. *Let S be an infinitesimally generated subsemigroup of a Lie group G such that $\underline{L}(S)$ is a generating cone in $L(G)$. Then there exist arbitrary small neighborhoods $\mathcal{U}$ of 1 in G such that $S\setminus(S\cap\mathcal{U})$ is a right semigroup ideal, i.e. $sg\in S\setminus(S\cap\mathcal{U})$ for all $s\in S\setminus(S\cap\mathcal{U})$ and $g\in S$.*

Proof. Let W_0 be a cone (note here that in our terminology a cone is a wedge containing no nontrivial subspace) in $L(G)$ such that $W_0\setminus\{0\}\subset\operatorname{int} W$, where $W=\underline{L}(S)$. Now we can find a norm $\|\ \|$ on $L(G)$ and a linear functional $w\colon L(G)\to\mathbb{R}$ such that $\|x\|=w(x)$ for all $x\in W_0$ (cf. [HL 84]).

Moreover, by [HL 83] we can find a neighborhood B_0 of zero in $L(G)$ such that $\exp|_{B_0}$ is a diffeomorphism onto its image and $\exp^{-1}(S\cap\exp B_0)\subset W_0$. Moreover since W_0 is a cone we may assume that for all $x\in B_0$ we have $d\lambda_{\exp(x)}(\mathbf{1})W\subseteq d\exp(x)W_0$ where we identify all the tangent spaces of the (flat) manifold $L(G)$ and λ_g denotes the left translation by g on G.

We call a path $\gamma\colon[0,1]\to G$ *admissible for W* if, up to C^1-parameter transformations it is of the form $\gamma\colon[0,n]\to G$ with $\gamma(t)=\prod_{k=1}^{m-1}(\exp x_k)(\exp(t-m+1)x_m)$ for $t\in[m-1,m]$, $m\leqq n$ where $x_k\in W$ for $k=1\dots n$. If now $\gamma\colon[0,1]\to G$ is admissible with $\gamma(t_0)\in\exp B_0$ and γ is differentiable at t_0 then $\gamma(t)$ and hence also $u_\gamma(t)=\exp^{-1}(\gamma(t))$ is differentiable in a neighborhood of t_0 and we obtain

$$\frac{du_\gamma}{dt}(t)=d\exp^{-1}(\gamma(t))(d\gamma(t))\in d\exp^{-1}(\gamma(t))\,d\lambda_{\gamma(t)}(\mathbf{1})(W\setminus\{0\})\subset W_0\setminus\{0\}$$

since γ is admissible and $\exp|_{B_0}$ is invertible. But then $\|u_\gamma(t)\|=w(u_\gamma(t))$ is differentiable with respect to t and we have $\frac{d}{dt}\|u_\gamma\|(t)>0$. Thus u is strictly increasing on any intervall $I\subset[0,1]$ such that $\gamma(I)\subset\exp B_0$. If now B_1 is *any* open ball in B_0 and $U_1=\exp B_1$ then any admissible $\gamma\colon[0,1]\to G$ such that $\gamma(1)\in U_1$ must satisfy $\gamma([0,1])\subset U_1$ since otherwise there exists a $t_0\in]0,1[$ such that $t_0=\sup\{t\in[0,1]\colon\gamma(t)\in G\setminus\exp B_1\}$ and $\|u_\gamma(t_0)\|>\|u_\gamma(1)\|$ which contradicts the strict monotonicity of $\|u_\gamma\|$ on $[t_0,1]$.

Now consider the subsemigroup T of G generated by $\exp W$ and let $g_1,g_2\in T$. Suppose $g_2g_1\in U_1$ then we find an admissible path $\gamma\colon[0,1]\to G$ such that $\gamma(0)=1$ and $\gamma(1)=g_2g_1$. Moreover there exists a $t_0\in[0,1]$ such that $\gamma(t_0)=g_2$. But since $\gamma(1)\in U_1$ it follows that $g_2=\gamma(t_0)\in U_1$. Thus $s\in T\setminus T\cap U_1$ implies $sg\in T\setminus T\cap U_1$ for all $g\in T$.

Finally note that T is dense in S so that $s\in S\setminus S\cap U_1$ implies $sg\in S\setminus S\cap U_1$ for all $g\in S$ since $S\setminus(S\cap U_1)$ is closed in S. □

We will now use this lemma to show that under certain circumstances it suffices to establish the existence of subsemigroups on a quotient group of G:

Lemma 1.3. *Let G be a Lie group and N be a closed normal subgroup of G. If $\pi\colon G \to G/N$ is the canonical projection and W is a generating cone in $L(G)$ such that $d\pi(1)(W)$ is a cone in $L(G/N)$ for which there exists a subsemigroup S_π of G/N with $\underline{L}(S_\pi)=d\pi(1)(W)$, then there exists a subsemigroup S of G such that $\underline{L}(S)=W$.*

Proof. Note first that by the theory of local semigroups [HL 83] we can find a neighborhood B of 0 in $L(G)$, a relatively closed set $\Sigma \subset U = \exp B$ and a cone W_0 in $L(G)$ such that the following statements are true:

(i) $\exp|_B$ is a diffeomorphism onto its image.
(ii) $d\pi(1)W_0$ is a cone.
(iii) $W \setminus \{0\} \subset \operatorname{int} W_0$.
(iv) $\exp^{-1}\Sigma \subset W_0$.
(v) $\Sigma\Sigma \cap U \subset \Sigma$.
(vi) $W = \{x \in L(G)\colon \exp(\mathbf{R}^+ x \cap B) \subset \Sigma\}$.

Since W_0 is a cone there exists a neighborhood V_π of 0 in $L(G/N)$ such that $d\pi(1)(x) \in V_\pi$ and $x \in W_0$ imply $x \in B_1$, where B_1 is an open neighborhood of 0 in $L(G)$ such that $(\exp B_1)^2 \subset U$. Moreover we may assume that S_π is infinitesimally generated, and by Lemma 1.2 we find a neighborhood V'_π of 0 inside V_π such that $S_\pi \setminus (S_\pi \cap \exp_{G/N} V'_\pi)$ is a right semigroup ideal in S_π. Making V_π smaller we may just as well assume that $V_\pi = V'_\pi$ and that $\exp_{G/N}|_{V_\pi}$ is a diffeomorphism onto its image.

Now let S be the subsemigroup of G generated by $\exp W$ and suppose $g = \prod_{k=1}^n \exp x_k \in U_2 = \exp B_2$ where $x_k \in W \cap B_1$ for $k = 1 \ldots n$ and B_2 is a neighborhood of 0 in B_1 such that $\pi(U_2) \subset \exp_{G/N} V_\pi$. Then we calculate:

$$\pi(g) = \prod_{k=1}^n (\pi \exp x_k) = \prod_{k=1}^n (\exp_{G/N} d\pi(1)(x_k)) \in S_\pi \cap \exp_{G/N} V_\pi$$

so that for $g_m = \prod_{k=1}^m (\exp x_k)$ we have $\pi(g_m) \in S_\pi \cap \exp_{G/N} V_\pi$. Let us assume that $g_m \in \Sigma$. Then $y_m = \exp^{-1} g_m \in W_0$ and $\pi(g_m) \in S_\pi \cap \exp_{G/N} V_\pi$ imply that $y_m \in B_1$ so that $g_{m+1} = g_m \exp x_{m+1} \in (\exp B_1)^2 \subset U$ and hence $g_{m+1} \in \Sigma$. Thus we have shown that $g \in \Sigma$ since clearly $g_1 = \exp x_1 \in \Sigma$. But this means that $S \cap U_2 \subset \Sigma$ so that $\underline{L}(S) = W$ by (vi). □

One important point in the proof of Theorem 0.6 is a good knowledge of invariant cones in the so called oscillator algebra (cf. § 2 for definitions) and their images under the exponential map in the oscillator group. Even though these images can be described very explicitly we need to resort to the following general lemma which helps us to get around very messy calculations.

The lemma can be obtained from [HH 85b] using the technique of analytic continuation. We will only sketch the proof which will appear in [HHL 87] with all details.

Lemma 1.4. *Let G be a Lie group and W be an invariant generating wedge in $L=L(G)$. Let D be an open subset of $L\times L$ with the following properties*

(i) *If $(x, y)\in D$, then $(rx, 0), (ry, 0)\in D$ for all $0\leqq r<1$ and $(x, sy)\in D$ for all $0\leqq s\leqq 1$.*

(ii) *There is an analytic function $m\colon D\to L$ such that $(x, y), (y, z), (m(x, y), z)\in D$ implies $(x, m(y, z))\in D$ and $m(m(x, y), z)=m(x, m(y, z))$, and that $\exp(m(x, y))=(\exp x)(\exp y)$ for all sufficiently small $x, y\in L$.*

If we set $D'=\{(x, y)\in D\colon (m(x, ty), 0)\in D$ for all $t\in[0, 1]\}$ then $m(W\times W\cap D')\subseteqq W$.

Sketch of proof. We remark first that this really is a lemma about semialgebras (cf. [HH 85b] for definitions). Let $E=\{x\in L\colon (x, 0)\in D\}$. For $x\in E$ we define

$$U_x=\{y\in L\colon (x, y)\in D\} \quad\text{and}\quad \Lambda_x\colon U_x\to L \quad\text{by}\quad \Lambda_x(y)=m(x, y).$$

Then $d\Lambda_x(0)$ turns out to be an analytic continuation of $g(\operatorname{ad} x)$ where g is the powerseries given by the function $g(T)=\dfrac{T}{1-e^T}$. Moreover the map $u(t)=m(x, ty)$ for $t\in\{t\in\mathbf{R}\colon (x, ty)\in D\}$ is the maximal solution of the initial value problem $u'(t)=d\Lambda_{u(t)}(0)(y)$, $u(0)=x$. Now one can show as in [HH 85b] that $d\Lambda_x(0)(W)\subseteq (W-\mathbf{R}^+x)^-$. Finally the convexity properties (i) allow us to use the methods of [HH 85b] to derive the desired result. □

2. The main results

We start by discribing the classification of invariant Lorentzian cones. Consider $h_m\cong\mathbf{R}^{2m}\times\mathbf{R}$, the $2m+1$-dimensional Heisenberg algebra with bracket

$$[(v, z), (v', z')]=(0, \langle dv|v'\rangle)$$

where $\langle|\rangle$ denotes the scalarproduct on $\mathbf{R}^{2m}$ and $d\colon \mathbf{R}^{2m}\to\mathbf{R}^{2m}$ is a skewsymmetric automorphism with spectrum $\{i, -i\}$. We denote $h_m\times\mathbf{R}$ with the bracket

$$[(v, z, r), (v', z', r')]=(rdv'-r'dv, \langle dv|v'\rangle, 0)$$

by $\mathcal{O}_m$ and call it the *oscillator algebra* of dimension $2m+2$. Then

$$q_m((v, z, r), (v', z', r'))=rz'+r'z+\langle v|v'\rangle$$

is an invariant Lorentzian form on $\mathcal{O}_m$. The following theorem is proved in [HH 85c]:

Theorem 2.1. *Let L be a finite dimensional real Lie algebra and W be an invariant Lorentzian cone in L, then W is defined by a Lorentzian form $q\colon L\times L\to\mathbf{R}$ such that (L, q) is isomorphic to the orthogonal direct sum of a compact Lie algebra (K, p) with a positive definite form p and a Lie algebra (L_1, q_1) which is isomorphic to one of the following types:*

(i) $L_1=\mathbf{R}$, $q_1(x, y)=-xy$.
(ii) $L_1=\mathrm{sl}(2, \mathbf{R})$, q_1 *is the Killing form.*
(iii) $L_1=\mathcal{O}_m$, $q_1=q_m$ *for some* $m=1, 2, 3, \ldots$.

Moreover all the forms are invariant in the sense that $q([x, y], z)=q(x, [y, z])$ *for all* $x, y, z\in L$. □

Theorem 2.1 tells us that we only need to consider three types of groups in order to prove Theorem 0.6. Moreover it shows that the projection of the invariant cone along the compact subalgebra is again a proper cone. If the subgroup corresponding to this algebra is closed Lemma 1.3 applies as soon as we have established the existence of subsemigroups of the groups corresponding to $\mathbf{R}$, $\mathrm{sl}(2, \mathbf{R})$ and $\mathcal{O}_m$ with the right tangent object. For $\mathbf{R}$ and $\mathrm{sl}(2, \mathbf{R})$ we know (cf. [HH 85a]) that this can be done if and only if the corresponding groups are simply connected, i.e. $\mathbf{R}$ and $\widetilde{SL}(2, \mathbf{R})$ respectively. It remains to check the case $(\mathcal{O}_m, q_m)$:

Lemma 2.2. *Let G be a Lie group such that $L(G)=\mathcal{O}_m$ and let W be an invariant cone in $\mathcal{O}_m$ defined by q_m. Then there exists a subsemigroup S of G with $\underline{L}(S)=W$ if and only if G is simply connected.*

Proof. If G is simply connected we may view it as $\mathbf{R}^{2m}\times\mathbf{R}\times\mathbf{R}$ with the product

$$(v, z, r)(v', z', r') = \left(v+e^{rd}v', z+z'+\tfrac{1}{2}\langle dv\,|\,e^{rd}v'\rangle, r+r'\right).$$

We denote this group by O_m and call it the *oscillator group*. Note that the exponential map $\exp\colon \mathcal{O}_m\to O_m$ is given by

$$\exp(v, z, r) = \begin{cases}\left(\dfrac{1}{r}(1-e^{rd})\,dv,\ z+\dfrac{1}{2r}\|v\|^2+\dfrac{1}{2r^2}\langle dv\,|\,e^{rd}v\rangle,\ r\right) & \text{if } r\neq 0\\ (v, z, 0) & \text{if } r=0.\end{cases}$$

From this we see immediately that exp is a diffeomorphism onto its image when restricted to $B=\mathbf{R}^{2m}\times\mathbf{R}\times]-2\pi, 2\pi[$. Moreover for

$$D = \{((v, z, r), (v', z', r'))\in\mathcal{O}_m\times\mathcal{O}_m\colon -2\pi < r+r' < 2\pi\}$$

and $m(x, y)=\exp^{-1}((\exp x)(\exp y))$ we see that the hypotheses of Lemma 1.4 are satisfied. Hence we know that

$$(\exp(v, z, r)\exp(v', z', r'))\in\exp W \quad \text{for all} \quad (v, z, r), (v', z', r')\in W$$

with $-2\pi<r+r'<2\pi$. We may assume that $W\subset h_m\times\mathbf{R}^+$ replacing W by $-W$ if necessary. Now note that $(\mathbf{R}^{2m}\times\mathbf{R}\times[2\pi,\infty[)$ is a semigroup ideal in $\mathbf{R}^{2m}\times\mathbf{R}\times\mathbf{R}^+$. Hence $S=(\exp W)\cup(\mathbf{R}^{2m}\times\mathbf{R}\times[2\pi,\infty[)$ is a subsemigroup of O_m. But clearly $\underline{L}(S)=W$ so that the first implication is proved.

In order to show the converse note first that the tangent space of ∂W at any point $(0,z,0)$ for $z\neq 0$ is equal to $h_{2m}\times\{0\}$. Since $\exp|_B$ is a diffeomorphism this shows that for $\Gamma=\mathbf{Z}z$ we have

$$\{x\in\mathcal{O}_m\colon x=\lim_{n\to\infty} nx_n;\ \exp x_n\in(\exp W)\Gamma\}=\mathbf{R}^{2m}\times\mathbf{R}\times\mathbf{R}^+.$$

If now $G=O_m/\Gamma_1$ where Γ_1 is any discrete nonzero central subgroup of O_n and S is a closed subsemigroup of G with $\underline{L}(S)=W$ then $\exp W\subset S$ and by [HL 83] we have

$$\underline{L}(S)=\{x\in\mathcal{O}_m\colon x=\lim_{n\to\infty} nx_n;\ \exp_G x_n\in\bar{S}\}.$$

But since $\exp_G$ is just exp followed by the quotient map $\pi\colon O_m\to\mathcal{O}_m/\Gamma$ this means $\underline{L}(S)=\{x\in\mathcal{O}_m\colon x\in\lim_{n\to\infty} nx_n;\ \exp\ x_n\in\pi^{-1}(S)\}\supsetneqq\{x\in\mathcal{O}_m\colon x\in\lim_{n\to\infty} nx_n;\ \exp\ x_n\in (\exp W)\Gamma\}=\mathbf{R}^{2m}\times\mathbf{R}\times\mathbf{R}^+$.

This contradiction shows that $\Gamma_1=\{0\}$, i.e. that G is simply connected. □

Thus we have proved the following result which together with Proposition 1.1 implies Theorem 0.6.

Proposition 2.3. *Let G be a Lie group and W be an invariant Lorentzian cone in $L(G)$. If $L(G)=L_1\oplus L_2$ is the decomposition provided by Theorem 2.1 and $G=G_1\oplus G_2$ where G_1 and G_2 are the analytic subgroups of G corresponding to L_1 and L_2 then the following statements are equivalent:*

(i) *G_1 is simply connected.*

(ii) *There exists a subsemigroup S of G such that $\underline{L}(S)=W$.* □

References

Gu 76. Guts, A., Invariant orders of three-dimensional Lie groups *Siber. math. J.* **17** (1976), 731—736.

Hi 86. Hilgert, J., Infinitesimally generated subsemigroups of motion groups, *Preprint THD* Nr. 970 (1986), to appear in Rocky Mountain J. Math.

HH 85a. Hilgert, J. and Hofmann, K. H., Lie semialgebras are real phenomena. *Math. Ann.* **270** (1985), 97—103.

HH 85b. Hilgert, J. and Hofmann, K. H., Semigroups in Lie groups, semialgebras in Lie algebras. *Trans. AMS* **288** (1985), 481—503.

HH 85c. Hilgert, J. and Hofmann, K. H., Lorentzian cones in real Lie algebras, *Monatshefte Math.* **100** (1985), 183—210.

HHL 87. HILGERT, J., HOFMANN, K. H. and LAWSON, J. D., The Lie theory of semigroups. Monograph (in preparation).

HL 83. HOFMANN, K. H. and LAWSON, J., Foundations of Lie semigroups. *Springer Lecture Notes in Math.* **998** (1983), 128—201.

HL 84. HOFMANN, K. H. and LAWSON, J. D., On Sophus Lie's Fundamental Theorems II, *Indagationes Math.* **46** (3), (1984), 255—265.

Le 84. LEVICHEV, A., Some applications of Lie semigroup theory in relativity. *SLS-Memo* 10—05—84 (1984).

Ol 81. OL'SHANSKIĬ, G., Invariant cones in symmetric Lie algebras, Lie semigroups, and the holomorphic discrete series. *Funct. Anal. and Appl.* **15** (1981), 275—285.

Ol 82a. OL'SHANSKIĬ, G., Convex cones in symmetric Lie algebras, Lie semigroups, and invariant causal (order) structures on pseudo-Riemannian symmetric spaces. *Dokl. Sov. Math.* **26** (1982), 97—101.

Ol 82b. OL'SHANSKIĬ, G., Invariant orderings in simple Lie groups. The solution to E. B. Vinberg's problem. *Funct. Anal. and Appl.* **16** (1982), 311—313.

Pa 81. PANEITZ, S., Invariant convex cones and causality in semisimple Lie algebras and groups. *Journal of Funct. Anal.* **43** (1981), 313—359.

Pa 84. PANEITZ, S., Classification of invariant convex cones in simple Lie algebras, *Arkiv för Mat.* **21** (1984), 217—228.

Se 76. SEGAL, I. E., *Mathematical Cosmology and Extragalactic Astronomy*, Acad. Pres, New York 1976.

Vi 80. VINBERG, E., Invariant cones and orderings in Lie groups, *Funct. Anal. and Appl.* **14** (1980), 1—13.

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