Applications of Lie semigroups in analysis

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This survey is an attempt to point out how the Lie theory of semigroups occurs in more classical parts of analysis. This means that I want to describe situations where semigroups and/or their tangent objects have actually been used to solve problems which *did not arise* in the semigroup context. The diversity of those applications made it necessary that I restricted myself to presenting the semigroup tools together with a short indication how they are applied rather than the application in full detail. To compensate for this I have included a fairly long list of references.

Analytic continuation of unitary representations

The abstract setting for this section will be the following. Let G be a Lie group and $\pi: G \to U(\mathcal{H})$ a unitary representation, i.e., a group homomorphism into the group of unitary operators $U(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ such that the map $G \times \mathcal{H} \to \mathcal{H}$ defined by $(g, f) \mapsto \pi(g)f$ is continuous. From π we want to construct

$$\widetilde{\pi}: \Gamma \to C(\mathcal{H}),$$

where

- $C(\mathcal{H})$ is the semigroup of contractions, i.e., norm decreasing maps $\mathcal{H} \to \mathcal{H}$
- Γ is a complex manifold and $\widetilde{\pi}$ is holomorphic as a vector valued map
- G is the Shilov boundary of Γ and $\widetilde{\pi}$ is an analytic continuation of π
- Γ is a semigroup and $\widetilde{\pi}$ is a representation of Γ.

1. The metaplectic and the oscillator semigroup

Before we go into more theoretical aspects of the analytic continuation we describe a few relevant examples. Our first example has been studied by various physicists in the context of nuclear models. The objects of interest for these physicists were a class of integral operators on the Bargman-Fock space $\mathcal{F}_n$ of entire functions on $\mathbb{C}^n$ with the L^2-norm given by the measure $d\mu(\zeta) = \pi^{-n} e^{\overline{\zeta}^t \zeta} d\zeta$. The key observation was that these integral operators formed a semigroup, so the determination of a matrix semigroup for which the integral operators form a representation would be useful in replacing calculations with operators by simple matrix calculations. At least up to constants one could do that for a subsemigroup of the complex symplectic group. In [17] KRAMER, MOSHINSKY and SELIGMAN showed that it is possible to extend the projective representation of $\mathrm{Sp}(1,\mathbb{R})$ on the Bargmann-Fock-space defined by the uniqueness of the canonical commutator relations [2] to a subsemigroup with interior in $\mathrm{Sp}(1,\mathbb{C})$ and applied this representation to the nuclear cluster model (see also [16]). The corresponding analytic extension for the symplectic groups of arbitrary dimension were described in [5]. Later BRUNET [3] proved that the projective representation can be "integrated" to a contractive representation of a double covering semigroup of the aforementioned complex semigroup. The Shilov boundary of this covering semigroup is the metaplectic group and the representation restricts to the metaplectic representation. We give the precise definitions below.

We call a function on $\mathbb{C}^n$ of *Gaussian type* if it is of the form $\zeta \mapsto e^{-\frac{1}{2}\zeta^t A\zeta}$ where A is a symmetric complex matrix. Note that such functions are holomorphic on all of $\mathbb{C}^n$. Further we call a function of Gaussian type on $\mathbb{C}^n$ a *Gaussian function* if it belongs to the Bargmann-Fock Hilbert space $\mathcal{F}_n$. Let

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$$

be an element of Ω_{2n}, the Siegel domain of complex symmetric $2n \times 2n$-matrices such that $X^*X < \mathbf{1}$. Then we set

$$K_X(\zeta,\overline{\omega}) = e^{-\frac{1}{2}(\zeta^t A\zeta + 2\zeta^t B\overline{\omega} + \overline{\omega}^t D\overline{\omega})} = e^{-\frac{1}{2}v^t X v},$$

where $v^t = (\zeta^t, \overline{\omega}^t)$. The corresponding kernel operator

$$f \mapsto \left(\zeta \mapsto \int_{C^n} K_X(\zeta,\omega) f(\omega) d\mu(\omega)\right)$$

will also denoted by K_X, i.e., we have

$$K_X f(\zeta) = \int_{C^n} e^{-\frac{1}{2}(\zeta^t A\zeta + 2\zeta^t B\overline{\omega} + \overline{\omega}^t D\overline{\omega})} f(\omega) d\mu(\omega).$$

The convergence of the integral can be shown using an isometry between $L^2(\mathbb{R}^n)$ and $\mathcal{F}_n$. The semigroup property follows from the multiplication law below ([5], 3.6).

Let $X, Y \in \Omega_{2n}$ and

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix},$$

then we have

$$K_X \circ K_Y = \frac{1}{\left(\det(\mathbf{1} - \widetilde{A}D)\right)^{\frac{1}{2}}} K_Z : \mathcal{F}_n \to \mathcal{F}_n$$

where

$$Z = \begin{pmatrix} A + \left(B(\mathbf{1} - \widetilde{A}D)^{-1}\widetilde{A}B^t\right)^s & -B(\mathbf{1} - \widetilde{A}D)^{-1}\widetilde{B} \\ -\widetilde{B}^t(\mathbf{1} - \widetilde{A}D)^{-1}B^t & \widetilde{D} + \left(\widetilde{B}^t D(\mathbf{1} - \widetilde{A}D)^{-1}\widetilde{B}\right)^s \end{pmatrix}.$$

Here $C^s = \frac{1}{2}(C + C^t)$.

In [2], Bargmann gives a realization of the projective representation of the symplectic group coming from the Stone-von Neumann Theorem via kernel operators on $\mathcal{F}_n$. He does not use $\mathrm{Sp}(n, \mathbb{R})$ but the isomorphic group $G = \mathrm{U}(n, n) \cap \mathrm{Sp}(n, \mathbb{C})$. Note that G is the set of all complex $2n \times 2n$-matrices of the form

$$g = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix},$$

where A and B are $n \times n$-block matrices, which satisfy

$$AA^* - BB^* = \mathbf{1}$$
$$A^t B = B^t A$$

or, equivalently,

$$A^* A - B^t \overline{B} = \mathbf{1}$$
$$A^t \overline{B} = B^* A.$$

From this it follows that A is invertible and that the matrices $\overline{B}A^{-1}$ and $-A^{-1}B$ are symmetric. It is shown in [2], §3 that the projective representation of G on $\mathcal{F}_n$ is given by $g \mapsto F_g(\zeta, \overline{\omega})$, where

$$F_g(\zeta, \overline{\omega}) = e^{\frac{1}{2}(\zeta^t \overline{B} A^{-1} \zeta + \zeta^t (A^{-1})^t \overline{\omega} + \overline{\omega}^t A^{-1} \zeta - \overline{\omega}^t A^{-1} B \overline{\omega})}.$$

This means that F_g is a kernel operator of Gaussian type with matrix

$$X_g = -\begin{pmatrix} \overline{B}A^{-1} & (A^{-1})^t \\ A^{-1} & -A^{-1}B \end{pmatrix}.$$

In [5] Brunet and Kramer formally extend these kernels by simply replacing $\overline{B}$ by an arbitrary C and then find conditions in which the resulting kernels yield decent operators. The result is that it makes sense to write the above formula for a subsemigroup of contractions in the complex symplectic group, where the hermitean form which is being contracted is not a positive definite one. More precisely, if V is a complex vector space and $B: V \times V \to \mathbb{C}$ a nondegenerate hermitian form, we call

$$S_B = \{g \in \mathrm{Gl}(V): B(gv, gv) \leq B(v,v) \ \ \forall v \in V\}$$

the semigroup of *B-contractions*. Its tangent wedge $\mathbf{L}(S_B) = \{x \in \mathrm{gl}(V): e^{\mathbb{R}^+ x} \subseteq S_B\}$ is then given by

$$\mathbf{L}(S_B) = \{x \in \mathrm{gl}(V): B(xv, v) + B(v, xv) \leq 0\}. \tag{1.1}$$

(See [11].) Note that the interior S_B^o of S_B is given by the above formula with $\leq$ replaced by $<$. Now we can describe our subsemigroup of $\mathrm{Sp}(n, \mathbb{C})$ and its relation to the Gaussian kernel operators (see [9]).

Lemma 1.1. *Let $B: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the hermitian form given by the matrix*

$$L = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

and S_B the semigroup of B-contractions. Then

$$\phi(\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}) = \begin{pmatrix} -(B^t)^{-1} & -(B^t)^{-1}D \\ A(B^t)^{-1} & -B + A(B^t)^{-1}D \end{pmatrix}$$

defines a map $\phi: \mathcal{D}_\Omega \to S_B^o$ where $\mathcal{D}_\Omega = \{X \in \Omega_{2n}: \det(B) \neq 0\}$. The map ϕ is invertible with inverse $\psi: S_B^o \to \mathcal{D}_\Omega$ given by

$$\psi(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = -\begin{pmatrix} CA^{-1} & (A^t)^{-1} \\ A^{-1} & -A^{-1}B \end{pmatrix}.$$

□

Proposition 1.2. (i) *The set $S_\Omega^\sharp = \{(cK_X) \in GK_C: X \in \mathcal{D}_\Omega\}$ is a subsemigroup of GK_C of Gaussian kernel operators and the map $\phi: \mathcal{D}_\Omega \to S_B^o$ induces a semigroup homomorphism $\phi: S_\Omega^\sharp \to S_B^o$.*

(ii) *The set*

$$S_\Omega = \{(cK_X) \in S_\Omega^\sharp: c^2 = \det(-B)\},$$

where

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix},$$

is a subsemigroup of $S_\Omega^\sharp$ and the semigroup homomorphism $\phi\colon S_\Omega \to S_B^o$ is a double covering. □

The proposition above shows that the double covering of S_B^o together with its canonical map into S_Ω can be viewed as an example of our abstract setting as stated at the beginning of this section. Following Brunet we call this semigroup the *metaplectic semigroup*.

Another example of a semigroup extension, which arises in a similar way, and in fact turns out to be essentially the same, can be found in Howe's paper [14]. His goal, however, is purely mathematical, namely the proof of certain estimates for symbols of pseudo-differential operators. On the other hand the actual application of the resulting semigroups is also of a technical nature. Here, too one considers certain integral operators, this time on $L^2(\mathbb{R}^n)$, and shows that they can be viewed as a representation of a semigroup. The algebraic structure of this semigroup, or rather a simple extension of this semigroup, is then what is used in the proof of the estimates. Again we give the precise definitions.

We call a function on $\mathbb{R}^n$ a function of *Gaussian type* if it is of the form $\xi \mapsto e^{-\frac{1}{2}\xi^t A\xi}$ where A is a symmetric complex matrix. It is integrable if the real part of A is positive definite. We call a function of Gaussian type a *Gaussian function* if it is integrable or, equivalently, if the real part of A is positive definite, i.e., if A belongs the *generalized Siegel upper halfplane* $\mathcal{S}_n$.

Let $\mathcal{S}_{2n}$ be the Siegel upper halfplane of complex symmetric $2n \times 2n$-matrices with positive definite real part and let

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$$

be an element of $\mathcal{S}_{2n}$. Then we set

$$K_X(\xi, \eta) = e^{-\frac{1}{2}(\xi^t A\xi + 2\xi^t B\eta + \eta^t D\eta)} = e^{-\frac{1}{2}v^t X v},$$

where $v^t = (\xi^t, \eta^t)$. Again the corresponding kernel operator

$$f \mapsto \left(\xi \mapsto \int_{\mathbf{R}^n} K_X(\xi, \eta) f(\eta)\, d\eta\right)$$

will also be denoted by K_X, i.e., we have

$$K_X f(\xi) = \int_{\mathbf{R}^n} e^{-\frac{1}{2}(\xi^t A\xi + 2\xi^t B\eta + \eta^t D\eta)} f(\eta)\, d\eta.$$

As in the previous case one derives the semigroup property from the multiplication law of the integral operators (see [14], 3.2.2).

Let $X, Y \in \mathcal{S}_{2n}$

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, Y = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{B}^t & \widetilde{D} \end{pmatrix},$$

then we have

$$K_X \circ K_Y = \frac{(2\pi)^n}{\det(D+\widetilde{A})^{\frac{1}{2}}} K_Z \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

where

$$Z = \begin{pmatrix} A - B(D+\widetilde{A})^{-1}B^t & -B(D+\widetilde{A})^{-1}\widetilde{B} \\ -\widetilde{B}^t(D+\widetilde{A})^{-1}B^t & \widetilde{D} - \widetilde{B}^t(D+\widetilde{A})^{-1}\widetilde{B} \end{pmatrix}.$$

As in in the case of the metaplectic semigroup it is possible to find a smaller semigroup by restricting the scalars to certain square roots depending on the kernel. More precisely, let

$$\mathcal{D}_o = \{X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{S}_{2n} \colon \det B \neq 0\}.$$

Then the sets $S_o^\sharp = \{(cK_X) \in GK_{\mathbb{R}} \colon X \in \mathcal{D}_o\}$ and

$$S_{tw} = \{(cK_X) \in S_o^\sharp \colon c^2 = \det(-\frac{B}{2\pi})\},$$

where

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix},$$

are subsemigroups of the semigroup of Gaussian kernel operators $(GK_{\mathbb{R}}, \circ)$. Yet Howe does not use this semigroup directly, but instead uses the *Weyl-transform*, a partial Fourier transform, and in this way replaces the Gaussian functions, viewed as integral operators, by Gaussian functions together with a *twisted convolution*.

The *Weyl transform* maps Schwartz functions on $\mathbb{R}^{2n}$ to kernel operators on $L^2(\mathbb{R}^n)$ via

$$\rho(F) = K_{\rho(F)}$$

where

$$K_{\rho(F)}(\xi, \eta) = \int_{\mathbb{R}^n} F(\xi - \eta, \tau) e^{\pi i (\xi+\eta)^t \tau} d\tau.$$

Proposition 1.3. ([14], 13.2) *Let* $v^t = (\xi^t, \eta^t)$ *and*

$$X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \in \mathcal{S}_{2n}.$$

Then

$$K_{\rho(F_X)} = \frac{(2\pi)^{\frac{n}{2}}}{(\det D)^{\frac{1}{2}}} K_{\widetilde{X}}$$

with

$$\widetilde{X} = \begin{pmatrix} A - (B - i\pi)D^{-1}(B^t - i\pi) & -A + (B - i\pi)D^{-1}(B^t + i\pi) \\ -A + (B + i\pi)D^{-1}(B^t - i\pi) & A - (B + i\pi)D^{-1}(B^t + i\pi) \end{pmatrix} \in \mathcal{S}_{2n}. \quad \Box$$

We denote the map $X \mapsto \widetilde{X}$ by $\widetilde{\rho}: \mathcal{S}_{2n} \to \mathcal{S}_{2n}$.

Proposition 1.4. ([14], §7 and [13]) *Let* $S(\mathbb{R}^{2n})$ *be the space of Schwartz functions on* $\mathbb{R}^{2n}$ *then*

$$\rho: (S(\mathbb{R}^{2n}), *_{tw}) \to (S(\mathbb{R}^{2n}), \circ)$$

is an involutive algebra isomorphism, where $*_{tw}$ *denotes twisted convolution, i.e.,*

$$F_1 *_{tw} F_2(v) = \int_{\mathbb{R}^{2n}} F_1(w) F_2(v - w) e^{-\pi i w^t J v} dw$$

with

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$$

and $\circ$ *the composition of integral operators on* $L^2(\mathbb{R}^n)$. $\Box$

This now shows that the Weyl transform yields a canonical isomorphism

$$\rho: (GK_{\mathbb{R}}, *_{tw}) \to (GK_{\mathbb{R}}, \circ).$$

For the semigroup of Gaussian functions with twisted convolution we can give a subsemigroup via squareroots and a double covering onto a semigroup of contractions in the complex symplectic group. Unfortunately the construction is not as straightforward as in the case of the metaplectic semigroup. We have to consider the *operator Cayley transform* defined by

$$c_{op}(x) = (x + \mathbf{1})(x - \mathbf{1})^{-1}$$

whenever the inverse of $x - \mathbf{1}$ exists. We note that $(x+\mathbf{1})(x-\mathbf{1})^{-1} - \mathbf{1} = (x+\mathbf{1})(x-\mathbf{1})^{-1} - (x-\mathbf{1})(x-\mathbf{1})^{-1} = 2(x-\mathbf{1})^{-1}$ so that we can apply the Cayley transform twice.

Remark 1.5. Set $D_c = \{x \in \mathrm{gl}(V): \det(x - \mathbf{1}) \neq 0\}$.
(i) $c_{op}^2: D_c \to D_c$ is the identity.
(ii) $S_B^o \subseteq D_c$.
(iii) $c_{op}: \mathbf{L}(S_B) \cap D_c \to S_B \cap D_c$ is a bijection. □

Now we consider the hermitean form $B_{\mathbb{R}}$ on $\mathbb{C}^n$ given by the matrix

$$iJ = i\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}.$$

The subsemigroup of $\mathrm{Sp}(n, \mathbb{C})$ consisting of all elements which are contractions w.r.t. $B_{\mathbb{R}}$ will be denoted by $S_{B_{\mathbb{R}}}$. Note that it follows from (1.1) that the edge of $\mathbf{L}(S_{B_{\mathbb{R}}})$ is $\mathrm{sp}(n, \mathbb{R})$. In fact we have

$$B_{\mathbb{R}}(Xv, v) + B_{\mathbb{R}}(v, Xv) = 2\,\mathrm{Re}\big(B_{\mathbb{R}}(v, Xv)\big) = 2\,\mathrm{Re}(iv^* JXv).$$

Lemma 1.6. *The map $\beta: \mathrm{Mat}(2n, \mathbb{C}) \to \mathrm{Mat}(2n, \mathbb{C})$ defined by $\beta(X) = -\frac{i}{\pi}JX$ induces a linear isomorphism $\beta: \mathcal{S}_{2n} \to \mathrm{int}\,\mathbf{L}(S_{B_{\mathbb{R}}})$ which maps the set $\mathcal{D}_{tw} = \{X \in \mathcal{S}_{2n}: \det(X + i\pi J) \neq 0\}$ onto D_c (see* Remark 1.5). □

Now we are ready to describe Howe's semigroup.

Proposition 1.7. ([14], §12) *The set $S_{tw}^\sharp = \{(cK_X) \in GK_{\mathbb{R}}: X \in \mathcal{D}_{tw}\}$ is a subsemigroup of $(GK_{\mathbb{R}}, *_{tw})$ and the map $(c, X) \mapsto c_{op}(-\frac{i}{\pi}JX)$ induces a semigroup homomorphism $S_{tw}^\sharp \to S_{B_{\mathbb{R}}}^o$. Moreover the set*

$$S_{tw} = \{(cK_X) \in S_{tw}^\sharp: c^2 = \frac{\det(X + i\pi J)}{(2\pi)^{2n}}\}$$

is a subsemigroup of $S_{tw}^\sharp$ and the semigroup homomorphism $c_{op} \circ \beta: S_{tw} \to S_{B_{\mathbb{R}}}^o$ is a double covering. □

Howe calls the semigroup S_{tw} the *oscillator semigroup* and gives various references to work which comes close to defining this semigroup. A remarkable feature of this semigroup is that it contains many classical operators. The Shilov boundary of the the oscillator semigroup is the double covering of the real symplectic group, i.e., the metaplectic group, but it is not trivial to show that the semigroup representation is an analytic continuation of the metaplectic representation. One way of doing this is to use the fact that the oscillator semigroup is isomorphic to the metaplectic semigroup. The isomorphism is given via the isometry $U\colon L^2(\mathbb{R}^n) \to \mathcal{F}_n$ given by (see [1, 2])

$$Uf(\zeta) = \int_{\mathbb{R}^n} U(\zeta, \xi) f(\xi) d\xi,$$

where

$$U(\zeta, \xi) = \pi^{-\frac{n}{4}} e^{-\frac{1}{2}(\zeta^2+\xi^2)+\sqrt{2}\zeta^t\xi}.$$

This isometry leads to a map $U_{n,n}$ from $(GK_{\mathbb{R}}, \circ)$ to $(GK_C, \circ)$ which, on the level of matrices is given by $\alpha\colon X \mapsto c_{op}(X)^{-1}$, a bijection from $\mathcal{S}_m$ to Ω_m with inverse $Y \mapsto -c_{op}(Y)$. It turns out that the maps $\alpha\colon \mathcal{S}_{2n} \to \Omega_{2n}$ and $\widetilde{\rho}\colon \mathcal{S}_{2n} \to \mathcal{S}_{2n}$ induce bijections $\alpha\colon \mathcal{D}_o \to \mathcal{D}_\Omega$ and $\widetilde{\rho}\colon \mathcal{D}_{tw} \to \mathcal{D}_o$, respectively.

Note that the map $\theta\colon S_o \to S_o$ given by

$$\theta(cK_X) = (2\pi)^{\frac{n}{2}} cK_{2\pi X}$$

is an automorphism. If we now denote by $\widetilde{\theta}$ the multiplication by 2π we can describe the interplay of the metaplectic and the oscillator semigroup by the following commutative diagram with bijective vertical maps.

$$\begin{array}{ccccccccc}
(GK_{\mathbb{R}}, *_{tw}) & \supseteq & S_{tw} & \longrightarrow & \mathcal{D}_{tw} & \xrightarrow{\beta} & \operatorname{int}\mathbf{L}(S_{B_{\mathbb{R}}}) & \xrightarrow{c_{op}} & S^o_{B_{\mathbb{R}}} \\
\downarrow \rho & & \downarrow \rho & & \downarrow \widetilde{\rho} & & & & \\
(GK_{\mathbb{R}}, \circ) & \supseteq & S_o & \longrightarrow & \mathcal{D}_o & & & & \\
\uparrow \theta & & \uparrow \theta & & \uparrow \widetilde{\theta} & & & & \downarrow c_{geo} \\
(GK_{\mathbb{R}}, \circ) & \supseteq & S_o & \longrightarrow & \mathcal{D}_o & & & & \\
\downarrow U_{n,n} & & \downarrow U_{n,n} & & \downarrow \alpha & & & & \\
(GK_C, \circ) & \supseteq & S_\Omega & \longrightarrow & \mathcal{D}_\Omega & & \xrightarrow{\phi} & & S^o_B.
\end{array}$$

The only map that has not been described before is c_{geo}, a *geometric Cayley transform*, i.e., an inner automorphism of $\mathrm{Sp}(n, \mathbb{C})$ given by $c_{geo}(g) = h_o g h_o^{-1}$ with

$$h_o = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Added in proof. The Fock realization of the oscillator semigroup has been studied independently by FOLLAND (cf. G. B. Folland, Harmonic analysis in Phase space, Ann. Math. Studies **122**, Princeton Univ. Press,1989).

2. Hardy spaces and realization of the holomorphic discrete series

Let G be a semisimple Lie group with finite center. A central object of the harmonic analysis for G is the decomposition of the (left) regular representation of G

on $L^2(G)$ into a direct integral of irreducible representations. The representations which occur in the regular representation come in finitely many series. These series are in one-to-one correspondence with the conjugacy classes of *Cartan subgroups* of G. In [8] GELFAND and GINDIKIN put forth a program to realize these representations on spaces of holomorphic functions on certain complex domains associated to the conjugacy classes of Cartan subgroups in a natural way. Their approach is as follows.

Suppose that G is contained in a complex Lie group G_C with Lie algebra $\mathfrak{g}_C$, the complexification of $\mathfrak{g} = \mathbf{L}(G)$. Further let σ be the involutive automorphism of G_C whose set of fixed points is G. Now let $H^1, \ldots, H^k$ be a set of representatives for the conjugacy classes of Cartan subgroups of G. These Cartan subgroups have complexifications H^j_C which are defined by $\mathbf{L}(H^j) = \mathbf{L}(H^j_C) \cap \mathfrak{g}$. Note that for each H^j we find a complementary group H_j in H^j_C defined by

$$H_j = \{g \in H^j_C : \sigma(g) = g^{-1}\}.$$

It turns out that $\bigcup_{j=1}^k GH_jG = G_C$. In order to obtain a *disjoint* union one has to replace H_j by certain "*Weyl chambers*". One can show [40] that these "Weyl chambers" are of the form $\exp ic_j$, where c_j is an open convex cone in the Lie algebra $\mathfrak{h}^j$ of H^j. The sets $G(\exp ic_j)G$ are open in G_C. They are the domains mentioned above. The space $\mathcal{H}(c_j)$ of holomorphic maps on the domain $G(\exp ic_j)G$ consists of *all* holomorphic maps $f: G(\exp ic_j)G \to \mathbb{C}$ which satisfy

$$\sup_{g_1 \in G, h \in \exp ic_j} \int_G |f(g_1^{-1} h g_1 g)|^2 dg < \infty. \tag{2.1}$$

GELFAND's and GINDIKIN's paper contains no proofs, and it took nine years until there appeared a paper, [40], providing proofs for some of their claims. STANTON, its author, showed that in the case that H_j is a *compact* Cartan subgroup and c_j a certain Weyl chamber, the space above is indeed non-empty. In order to show this he proved that it is possible to analytically continue the representations $\pi_\lambda: G \to U(\mathcal{H}_\lambda)$ of the holomorphic discrete series from G to $G(\exp ic_j)G$ and then calculated that all the matrix units $g \mapsto (\pi(g)v, w)_{\mathcal{H}_\lambda}$ for $v, w \in \mathcal{H}_\lambda$ are contained in $\mathcal{H}(c_j)$. Neither GELFAND and GINDIKIN nor STANTON noticed that the domain to which they analytically continued the holomorphic discrete series was a *subsemigroup* of G_C. The one who *did* notice that was G.I. OL'SHANSKIĬ. As early as 1982, he worked out a version of the Gelfand-Gindikin program which was far more complete than that of STANTON (see [31]). Unfortunately, his work only appeared in a Russian proceedings volume and therefore was virtually inaccessible. Let us describe OL'SHANSKIĬ's work (note that he uses the right regular representation). The first observation is that the domain $G(\exp ic_j)G$ can be written as $G(\exp iW^o)$ where W is an invariant (under inner automorphisms) convex cone in $\mathfrak{g}$ with interior W^o. But it is known from earlier work of OL'SHANSKIĬ's [28] that this set is an open subsemigroup of G_C. We denote it by $\Gamma^o(W)$ and its closure by $\Gamma(W)$. Now we can replace the definition of $\mathcal{H}(c_j)$ given above by saying, a holomorphic mapping $f:\Gamma^o(W) \mapsto \mathbb{C}$ belongs to $H = \mathcal{H}(W)$ if

$$\|f\|_H = \sup_{\gamma\in\Gamma^o(W)} \int_G |f(\gamma g)|^2 dg < \infty. \tag{2.2}$$

Note that in this formulation we can understand the definition of $\mathcal{H}(W)$ not only as an *analogue* of the classical Hardy spaces [41], but even as a generalization. In fact if we set $G = \mathbb{R}$ and $W = \mathbb{R}^+$ then G_C is $\mathbb{C}$ and $\Gamma^o(W)$ the upper half plane. Thus the inequality above is just the defining inequality of the classical Hardy space. We note at this point that one of the strong points of OL'SHANSKIĬ's treatment of the Gelfand-Gindikin program is its flexibility.

It is easy to see that one has a representation of $\Gamma(W)$ on $\mathcal{H}(W)$. In fact, for $f \in \mathcal{H}(W)$ and $\gamma \in \Gamma(W)$ we define a function $\mathcal{T}(\gamma)f$ on $\Gamma^o(W)$ by the formula

$$\mathcal{T}(\gamma)f(\gamma_1) = f(\gamma_1\gamma).$$

Theorem 2.1. *The following statements hold under the present circumstances:*

(i) *$\mathcal{H}(W)$ is a Hilbert space w.r.t. the norm $\|\cdot\|_H$.*

(ii) *There exists an isometry $I: H \to L^2(G)$ such that for an arbitrary function $f \in H$ and an arbitrary sequence $\gamma_1, \gamma_2, \ldots$ in $\Gamma^o(W)$ which converges to $\mathbf{1}$, the sequence $\{\gamma_j \cdot f\}$ converges to If w.r.t. the metric of $L^2(G)$.*

(iii) *I commutes with right translations from G, i.e., $I\mathcal{T}(g) = R(g)I$.*

(iv) *$\mathcal{T}(\cdot)$ is a holomorphic representation of the semigroup $\Gamma^o(W)$ on $\mathcal{H}(W)$.* □

Up to this point it is not yet clear whether all these statements are trivial, i.e., whether $\mathcal{H}(W)$ is non-empty. But OL'SHANSKIĬ also computes the image of I. We have to introduce some more notation in order to describe the result. Let π be an arbitrary unitary representation of the group G on any Hilbert space $\mathcal{H}$. To each $X \in i\mathfrak{g}$ one can associate the operator $\pi(X)$ on $\mathcal{H}$ which is determined by the condition

$$\pi(\exp itX) = \mathrm{Exp}\, it\pi(X), \quad \forall t \in \mathbb{R}.$$

Note that here *Exp* is a formal expression which really means that $i\pi(X)$ is the infinitesimal generator of the unitary one parameter group $t \mapsto \pi(\exp itX)$. We say that $\pi(X) \leq 0$ if the spectrum of the operator $\pi(X)$ is contained in the halfline $(-\infty, 0]$. Finally we say that the representation π is *W-admissible* if $\pi(X) \leq 0$ for all $X \in iW$.

Theorem 2.2. *$I(H)$ is the biggest $R(G)$-invariant subspace of L such that the corresponding unitary representation is W-admissible.* □

Ol'shanskiĭ also gave a characterization of those representations which are W-admissible with respect to some invariant cone W. They are exactly the highest weight representations (see [15], [28]). If W is the minimal invariant cone in $\mathfrak{g}$ then the W-admissible representations which occur in the regular representation are exactly the representations from the holomorphic discrete series. Analogous results are possible also for certain non Riemannian symmetric spaces (see the work of Olafsson and Ørsted [24], [25] as well as [12]).

3. The analytic continuation procedure

We now give a more detailed description of Ol'shanskiĭ's analytic continuation procedure. If G is a Lie group and S is a subsemigroup of G then the tangent object $\mathbf{L}(S)$ (according to [11]) is a Lie wedge in $\mathfrak{g} = \mathbf{L}(G)$ which means that it is a closed convex cone which is invariant under the inner automorphisms of the form $e^{\operatorname{ad} x}$ with x contained in the edge $\mathbf{L}(S) \cap -\mathbf{L}(S)$ of $\mathbf{L}(S)$. If now G is a real Lie group inside a complexification G_C and S is a subsemigroup of G_C containing G then $\mathbf{L}(S)$ is of the form $\mathfrak{g} + iW$ where W is a wedge in $\mathfrak{g}$ which is invariant under all inner automorphisms of $\mathfrak{g}$. Ol'shanskiĭ showed in [28] that for simple $\mathfrak{g}$ the semigroup generated by $\mathbf{L}(S)$ is just the product $(\exp iW)G$ where this product is even a *direct* product of topological spaces. The same result is true for solvable groups.

In this section we consider the following situation. Let G_C be a complex Lie group with Lie algebra $\mathfrak{g}_C$ and $\mathfrak{g}$ be a real form of $\mathfrak{g}_C$. We assume that G, the analytic subgroup of G_C with Lie algebra $\mathfrak{g}$, is closed. Let W be a proper generating invariant cone in $\mathfrak{g}$ such that the set $\Gamma(W) = (\exp iW)G$ is a closed subsemigroup of G_C. Moreover we assume that the map $G \times W \to \Gamma(W)$, defined by $(g, x) \to (\exp ix)g$ is a homeomorphism and even a diffeomorphism when restricted to $G \times W^o$. Finally we assume that there exists a (real) automorphism σ of G_C whose differential is the complex conjugation of $\mathfrak{g}_C$ with respect to the real form $\mathfrak{g}$.

Let $\pi: G \to U(\mathcal{H})$ be a unitary strongly continuous representation and $\mathcal{H}^\infty$ the set of C^∞-vectors for π. Then for any $x \in \mathfrak{g}$ the mapping $\phi: t \mapsto \pi(\exp tx)$ is a strongly continuous unitary one-parameter group in $\mathcal{H}$ so that by Stone's Theorem there is a unique, densely defined, skew adjoint, infinitesimal generator $-iH_x$ for ϕ. On $\mathcal{H}^\infty$ this generator is given by $d\pi(x)$ where $d\pi: \mathfrak{g} \to L(\mathcal{H}^\infty)$ is the Lie algebra representation associated with π. This infinitesimal representation extends to the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_C)$ of $\mathfrak{g}_C$ and we set

$$C(\pi) = \{y \in \mathfrak{g}: (id\pi(y)\xi \mid \xi) \leq 0 \ \forall \xi \in \mathcal{H}^\infty\} \tag{3.1}$$

where $(\ |\ )$ is the inner product on $\mathcal{H}$. In other words $C(\pi)$ is the set of elements of $\mathfrak{g}$ for which $id\pi(x)$ is *negative* (see [6], Section 4.1). The elements of $C(\pi)$ will be called *negative elements.*

Proposition 3.1. *$C(\pi)$ is a closed convex cone in $\mathfrak{g}$ which is invariant under* Ad G. □

For our invariant cone W in $\mathfrak{g}$ we denote the set of all unitary representations $\pi: G \to U(\mathcal{H})$ for which W consists of negative elements by $\mathcal{A}(W)$.

Definition 3.2. Let S be a semigroup with unit and $^\sharp: S \to S$ a bijection such that $s^{\sharp\sharp} = s$ and $(s_1 s_2)^\sharp = s_2^\sharp s_1^\sharp$ for all $s, s_1, s_2 \in S$. Then $^\sharp$ is called an *involution* on S and the pair $(S,^\sharp)$ is called a *semigroup with involution.*

If $\mathcal{H}$ is a (complex) Hilbert space and $C(\mathcal{H}) = \{T \in B(\mathcal{H}): ||T|| \leq 1\}$ is the set of all *contractions* on $\mathcal{H}$ then $(C(\mathcal{H}),^*)$ is a semigroup with involution, where T^* is the adjoint of T w.r.t. the inner product on $\mathcal{H}$. We provide $\Gamma(W)$ with an involution setting $g^\sharp = \sigma(g)^{-1}$ for all $g \in G_C$. The only thing we have to do in order to verify that $^\sharp$ is an involution of $\Gamma(W)$ is to show that it leaves $\Gamma(W)$ invariant. But this follows from $\big((\exp x)g\big)^\sharp = \sigma\big((\exp x)g\big)^{-1} = \big((\exp -x)g\big)^{-1} = g^{-1}(\exp x) \in \Gamma(W)$.

Now suppose that $\pi \in \mathcal{A}(W)$. Then for any $x \in W$ the operator $id\pi(x)$ generates a selfadjoint contraction semigroup (see [6], Theorem 4.6) which we denote by $t \mapsto T_x(t)$.

Definition 3.3. Let $(S,^\sharp)$ be a topological semigroup with involution, then a semigroup homomorphism $\rho: S \to C(\mathcal{H})$ is called a *contractive representation of* $(S,^\sharp)$ if it preserves the involution and is continuous w.r.t. the weak operator topology on $C(\mathcal{H})$. A contractive representation is called *irreducible* if there is no closed non-trivial subspace of $\mathcal{H}$ which is invariant under $\rho(S)$.

A contractive semigroup representation $\rho: \Gamma(W) \to C(\mathcal{H})$ is called *holomorphic* if the map $\rho\ |_{\Gamma(W)^o}$ is holomorphic (for the definition of holomorphy in Hilbert-(Banach) spaces see [42], Section 4.4).

We want to construct a holomorphic representation of $\Gamma(W)$ starting from a representation $\pi \in \mathcal{A}(W)$. The definition of the analytic continuation is straightforward. In fact, for $\gamma = (\exp tx)g \in \Gamma(W)$ we define $\tau(\gamma) = T_x(t)\pi(g)$. Then we get a strongly continuous map $\tau: \Gamma(W) \to C(\mathcal{H})$ which preserves the involutions. The problem now is to show that τ has all the desired properties. The main problem is to show the holomorphy.

First one notes that if $f: \Gamma(W) \to \mathbb{C}$ is continuous and $f\ |_{\Gamma(W)^o}$ is holomorphic such that f vanishes identically on G, then $f \equiv 0$. Then one has to use Nelson's theory of analytic vectors (see [42]).

Lemma 3.4. *There exists a neighborhood $\mathcal{U}$ of $\mathbf{1}$ in $G_{\mathbb{C}}$ and a holomorphic mapping $\widetilde{\tau}_\xi: \mathcal{U} \to \mathcal{H}$ for each $\xi \in \mathcal{H}'$, where $\mathcal{H}'$ is a dense subspace of $\mathcal{H}$ consisting of analytic vectors, such that*

$$\widetilde{\tau}_\xi(g) = \pi(g)\xi \qquad \text{for all } g \in G \cap \mathcal{U}. \qquad \square$$

Lemma 3.5. *The following assertions hold:*

(i) $\tau(\gamma)\xi = \widetilde{\tau}_\xi(\gamma)$ *for all* $\gamma \in \Gamma(W) \cap \mathcal{U}, \xi \in \mathcal{H}^\omega$.

(ii) $\gamma \mapsto \tau(\gamma)\xi$ *is holomorphic for all* $\xi \in \mathcal{H}$ *and* $\gamma \in \Gamma(W)^o \cap \mathcal{U}$.

(iii) *The map* $\gamma \mapsto \tau(\gamma)\xi$ *is real analytic on* $\Gamma(W)^o$ *for all* $\xi \in \mathcal{H}^\omega$, *where* $\mathcal{H}^\omega$ *is the set of analytic vectors in* $\mathcal{H}$ *for* π.

(iv) *The map* $\gamma \mapsto \tau(\gamma)$ *is holomorphic on* $\Gamma(W)^o$. $\square$

Theorem 3.6. *Let $\pi \in \mathcal{A}(W)$ then π extends uniquely to a holomorphic representation τ of $\Gamma(W)$.*

Proof. We know that the map $\gamma \mapsto \tau(\gamma)$ is holomorphic on $\Gamma(W)^o$ and extends π. Moreover we know $\tau(\gamma^\sharp) = \tau(\gamma)^*$. If now $g \in G$ is fixed then $\gamma \mapsto \tau(\gamma g)$ and $\gamma \mapsto \tau(\gamma)\pi(g)$ are holomorphic on $\Gamma(W)^o$ and coincide on G. Thus we have

$$\tau(\gamma g) = \tau(\gamma)\pi(g) = \tau(\gamma)\tau(g).$$

Now we fix $\gamma_o \in \Gamma(W)$ and consider $\gamma \mapsto \tau(\gamma_o\gamma)$ as well as $\gamma \mapsto \tau(\gamma_o)\tau(\gamma)$. Again we know that both maps are holomorphic on $\Gamma(W)^o$ (since $\Gamma(W)^o$ is a semigroup ideal in $\Gamma(W)$) and by the above they agree on G. This shows that τ is a semigroup homomorphism. The uniqueness of the extension is clear. $\square$

It is also possible to prove a converse of Theorem 3.6. Note first that the restriction to G of a holomorphic representation is always a unitary representation. In fact we have:

Remark 3.7. Let $(S, ^\sharp)$ be a semigroup with involution and $\rho: S \to C(\mathcal{H})$ a contractive representation. Set $H = \{s \in S: s^\sharp s = s^\sharp s = \mathbf{1}\}$ then H is a subgroup of S and the restriction of ρ to H is a strongly continuous unitary representation. $\square$

Theorem 3.8. *Let τ be a holomorphic representation of $\Gamma(W)$ then any $x \in W$ is a negative element for $\pi = \tau\mid_G$, so that $\pi \in \mathcal{A}(W)$.*

Proof. If $x \in W$ we consider the map

$$z \mapsto \tau(\exp zix) = \tau\big(\exp(i \operatorname{Re} zx)\exp(-\operatorname{Im} zx)\big)$$

for $\operatorname{Re} z \geq 0$ and the hypothesis says that it is holomorphic for $\operatorname{Re} z > 0$. On $\operatorname{Re} z = 0$ this map is just $it \mapsto \tau(\exp -tx) = \pi(\exp -tx)$. Since $\tau(\gamma^\sharp) = \tau(\gamma)^*$ we know that $\tau(\exp tix)$ is self adjoint for $t \geq 0$. If $d\tau(ix)$ is the infinitesimal generator of $\tau(\exp tix)$ then we see that $d\tau(ix) = id\pi(x)$ and the claim follows. □

Analytic extensions of semigroup representations

OL'SHANSKIĬ's original interest in semigroups did not come from the Gelfand-Gindikin program. It rather occurred to him that he could use semigroups in the study of representations of infinite dimensional analogues of certain classical groups. A key ingredient in his "semigroup method" is again an analytic continuation procedure – this time from a representation of a local semigroup to a representation of a related Lie group. This procedure is due to two physicists, LÜSCHER and MACK [22], and can be viewed as an extension of the Hille-Phillips Theorem for one parameter semigroups.

4. Representations of infinite dimensional classical groups

The purpose of OL'SHANSKIĬ's work (see [26], [27], [32]) is to determine those unitary representations of the infinite dimensional groups $\mathrm{SO}_0(p,\infty)$, $\mathrm{U}(p,\infty)$, $\mathrm{Sp}(p,\infty)$ for $p = 0, 1, 2, \ldots$ which are admissible in a sense to be specified below. In order to simplify notation we restrict ourselves to the case $\mathrm{U}(p,\infty)$. This group can be realized on the complex Hilbert space $l^2(\mathbb{N})$ as a subgroup of the set of invertible operators preserving the (non-definite) inner product $J(\xi,\xi) = \langle \xi,\xi\rangle = -|\xi_1|^2 - \cdots - |\xi_p|^2 + |\xi_{p+1}|^2 + \ldots$ We denote by $\mathcal{G}(n)$ the group of all such operators that preserve all but the first n elements of the canonical basis of $l^2(\mathbb{N})$. Then we define

$$\mathcal{G} = \mathrm{U}(p,\infty) = \bigcup_{n\in\mathbb{N}} \mathcal{G}(n).$$

Further we let $\mathcal{G}_n$ be the subgroup of $\mathcal{G}$ consisting of all operators which fix the first n basis vectors. If now π is a unitary representation which is continuous w.r.t. the inductive limit topology, i.e., when restricted to the $\mathcal{G}(n)$, and $\mathcal{H}$ the Hilbert space on which the $\pi(g)$ operate, then we denote by $\mathcal{H}_n$ the space of vectors in $\mathcal{H}$ which are fixed by the elements $\pi(g)$ with $g \in \mathcal{G}_n$. The representation π is called *admissible* if the space $\mathcal{H}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ is dense in $\mathcal{H}$. Ol'shanskii shows that this is equivalent to continuity w.r.t. the norm topology on $\mathcal{G}$.

If one views the elements of $\mathcal{G}$ as infinitely large matrices it is clear what it means to associate to $g \in \mathcal{G}(k)$ its *left upper corner* of size n. We denote this map by θ_n and note that its image is contained in the semigroup $\Gamma(n)$ of J-contractions, where J is the hermitean form from above but restricted to the span of the first n basis vectors. More is true:

Lemma 4.1. *For k large enough we have*

(i) $\theta_n(\mathcal{G}(k)) = \Gamma(n)$.

(ii) *If $g_1, g_2 \in \mathcal{G}(k)$, then $\theta(g_1) = \theta(g_2)$ if and only if $g_1 = u g_2 v$ for some elements $u, v \in \mathcal{G}_n(k) = \mathcal{G}(k) \cap \mathcal{G}_n$.*

(iii) *The map $\theta_n: \mathcal{G}(k) \to \Gamma(n)$ is proper and open.* □

Next we assume that π is an admissible representation of $\mathcal{G}$ and n is so large that $\mathcal{H}_n$ is non-zero. Let $P_n: \mathcal{H} \to \mathcal{H}_n$ be the orthogonal projection and $C(\mathcal{H}_n)$ the semigroup of contractions on $\mathcal{H}_n$ w.r.t. the Hilbert inner product. The lemma above now guarantees the existence of a continuous map $\widetilde{\pi}_n: \Gamma(n) \to C(\mathcal{H}_n)$ defined by $\widetilde{\pi}_n(\theta(g)) = P_n \pi(g)|_{\mathcal{H}_n}$.

Theorem 4.2. *$\widetilde{\pi}_n: \Gamma(n) \to C(\mathcal{H}_n)$ is a holomorphic semigroup representation.* □

Note that we are in the following situation. $G = \mathrm{Gl}(n, \mathbb{C})$ is a Lie group with involution $\sigma(x) = -X^\sharp$, where $\sharp$ denotes the adjoint w.r.t. the form J. Let $\mathfrak{g}_\pm = \{x \in \mathfrak{g}: d\sigma(x) = \pm x\}$ and $H = \{g \in G: \sigma(g) = g\}_o$. Further $C = \{x \in \mathfrak{g}: x = x^\sharp, \langle x\xi, \xi\rangle \leq 0\ \forall \xi \in \mathbb{C}^n = l^2(1, \ldots, n)\} \subset \mathfrak{g}_-$ is an Ad H-invariant cone and $\Gamma(n)$ has $\mathfrak{g}_+ + C$ as tangent cone.

Let $\mathfrak{g}^* = \mathfrak{g}_+ + i\mathfrak{g}_-$ and suppose that G^* denotes the simply connected Lie group with $\mathbf{L}(G^*) = \mathfrak{g}^*$. In our case we have $\mathfrak{g}^* = \mathfrak{u}(p,q) \oplus \mathfrak{u}(p,q)$ so that we may view $\mathfrak{g}$ as a part, the diagonal, of $\mathfrak{g}^*$.

Theorem 4.3. (Lüscher and Mack) *Let $\rho: \Gamma(n) \to C(\mathcal{H})$ be a strongly continuous involutive representation, then there exists a representation $\pi: G^* \to U(\mathcal{H})$ such that*

$$d\pi(ix) = i d\rho(x)\ \ \forall x \in C$$

and

$$d\pi(x) = d\rho(x)\ \ \forall x \in \mathfrak{g}_+.$$

□

We ought to note here that the Lüscher–Mack Theorem is true for weaker hypotheses. In particular it would be sufficient to have a local semigroup instead of $\Gamma(n)$.

If one applies the Lüscher Mack theorem to all the $\widetilde{\pi}_n: \Gamma(n) \to C(\mathcal{H}_n)$ then one obtains representations $\pi_n^*: \mathcal{G}^*(n) \to U(\mathcal{H}_n)$ for large n. These representations may be collected to give one representation $\pi^*: \mathcal{G}^* \to U(\mathcal{H})$, where $\mathcal{G}^* = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n^*$. It turns out that due to the holomorphy of the representations $\widetilde{\pi}_n$ one can describe π^* in terms of two sequences of natural numbers (roughly to be interpreted as highest weights for the subgroups $\mathrm{U}(\infty) \times \mathrm{U}(\infty)$ of $\mathcal{G}^*$). Finally one notes that with the embedding $\mathcal{G} \subseteq \mathcal{G}^*$ one has $\pi = \pi^*|_{\mathcal{G}}$ so that π can also be described in terms of the sequences mentioned above (see [32]).

Laplace transforms

It is well known that the Fourier transform as an integral operator on $\mathbb{R}$ has a group theoretical interpretation. The generalizations of the Fourier transform to Lie groups play an important role in (non-commutative) harmonic analysis. Whereas the Laplace transform, an integral operator on the semigroup $\mathbb{R}^+$, is also important in classical analysis, until recently there seems to have been no attempt to generalize it to a wider class of semigroups. We will describe below some work of Mizony (see [23]) and Faraut (together with the physicist Viani, see [7]) which can be viewed as first steps in this direction.

5. Semigroup theoretical interpretation of special functions

Let G/H be a symmetric space and $G = KAN$ the Iwasawa decomposition of G. Consider the boundary K/M of the Riemannian symmetric space G/K, where M is the centralizer of A in K. Note that $K/M = G/P$ with $P = MAN$ the corresponding minimal parabolic subgroup. This shows that G operates on K/M. Now let $\mathcal{O}$ be open in K/M and set

$$S(\mathcal{O}) = \{g \in G : g^{-1} \cdot (\mathcal{O}) \subseteq \mathcal{O}\}.$$

It is clear that $S(\mathcal{O})$ is a semigroup. If $\mathcal{O}$ is an H-orbit we can consider the image measure $d\mu$ on $\mathcal{O}$ of the Haar measure dh on H. The *Poisson kernel* $P_{\mathcal{O}} : S(\mathcal{O}) \times \mathcal{O} \to \mathbb{R}^+$ is defined by

$$P_{\mathcal{O}}(g, \eta) = \frac{d(g^{-1})^* \mu(\eta)}{d\mu(\eta)}$$

for $g \in S(\mathcal{O})$ and $\eta \in \mathcal{O}$.

Proposition 5.1. (a) $P_{\mathcal{O}}(g_1 g_2, \eta) = P_{\mathcal{O}}(g_1, \eta) P_{\mathcal{O}}(g_2, g_1^{-1} \cdot \eta)$
(b) $P_{\mathcal{O}}(h, \eta) = 1$ *for all* $h \in H, \eta \in \mathcal{O}$. □

This proposition shows that for any $\alpha \in \mathbb{C}$

$$\pi_\alpha(g) f(\eta) = P_{\mathcal{O}}(g, \eta)^\alpha f(g^{-1} \cdot \eta) \tag{5.1}$$

defines a representation of $S(\mathcal{O})$ on the space of functions on $\mathcal{O}$. If the function $\pi_\alpha(g)1 \in L^1(\mathcal{O}, d\mu)$ then we set

$$\Phi(\alpha, g) = \int_{\mathcal{O}} (\pi_\alpha(g)1)(\eta) d\mu(\eta). \tag{5.2}$$

Remark 5.2. Let $\pi_\alpha(g)1 \in L^1(\mathcal{O}, d\mu)$ for all $g \in S(\mathcal{O})^o$, then
(a) $\Phi(\alpha, g) = \Phi(\alpha, h_1 g h_2)$ for all $h_1, h_2 \in H$.
(b) $\Phi(\alpha, g_1)\Phi(\alpha, g_2) = \int_H \Phi(\alpha, g_1 h g_2) dh$. □

Now we assume that the open semigroup $S(\mathcal{O})^o$ admits a Cartan decomposition

$$S(\mathcal{O})^o = H S_A^o H \tag{5.3}$$

where $S_A^o \subseteq A$ and the Haar measure of G on $S(\mathcal{O})^o$ decomposes as $dg = dh d\nu(s) dh$. Then the second formula from Remark 5.2 with $s_1, s_2 \in S_A^o$ can be written as

$$\Phi(\alpha, s_1)\Phi(\alpha, s_2) = \int_{S_A^o} K(s_1, s_2, s)\Phi(\alpha, s) d\nu(s), \tag{5.4}$$

for some kernel function K. Now we can define a *convolution product* and a *spherical Laplace transform* by

$$f *_{\mathcal{O}} g(s) = \int_{S_A^o} \int_{S_A^o} K(s_1, s_2, s) f(s_1) g(s_2) d\nu(s_1) d\nu(s_2) \tag{5.5}$$

for $f, g \in C_c^\infty(S_A^o)$ and

$$\mathcal{L}_{\mathcal{O}}(f)(\alpha) = \int_{S_A^o} f(s)\Phi(\alpha, s) d\nu(s) \tag{5.6}$$

for all α with $(\pi_\alpha(s)1) \in L^1(\mathcal{O}, d\nu)$ for all $s \in S(\mathcal{O})^o$. We obtain

Proposition 5.3. *Let $f, g \in C_c^\infty(S_A^o)$ then*

$$\mathcal{L}_{\mathcal{O}}(f *_{\mathcal{O}} g)(\alpha) = \mathcal{L}_{\mathcal{O}}(f)(\alpha)\mathcal{L}_{\mathcal{O}}(g)(\alpha).$$ □

With this purely formal machinery at hand we can indicate Mizony's results. They are of a fairly technical nature, so we give just one theorem omitting some details. For the full story we refer to [23].

Theorem 5.4. *Let* $G = \mathrm{SO}_0(1,n)$ *and* $H = \mathrm{SO}_0(1,n-1)$ *then* S_A^o *can be identified with the positive reals such that*

$$d\nu(t) = \frac{1}{(\sinh t)^{n-1}} dt.$$

The function $\Phi(\alpha, t)$ *can be defined for* $\alpha = \rho - i\lambda$ *where* $\rho = \frac{n-1}{2}$ *and* $\mathrm{Im}\lambda > \rho - 1$. *The* $\Phi(\alpha, \cdot)$ *are exactly the Jacobi functions of the second kind.* □

Finally we note that Mizony also gives an inversion formula for the spherical Laplace transform in the case $G/H = \mathrm{SO}_0(1,n)/\mathrm{SO}_0(1,n-1)$.

6. Ordered spaces and partial diagonalization of integral operators

Let $X = G/H$ be a symmetric space with an invariant ordering $\leq$, i.e., $x \leq y$ implies $g{\cdot}x \leq g{\cdot}y$ for all $g \in G$. We assume that the intervals $[y,x] = \{z \in X : y \leq z \leq x\}$ are compact. A kernel $K(x,y)$ on X is called a *Volterra kernel* if it is continuous on $\Gamma = \{(x,y) : y \leq x\}$ and vanishes outside Γ. We denote the set of all Volterra kernels by $V(X)$. The product of two Volterra kernels is given by

$$K_1 \sharp K_2(x,y) = \int_{[y,x]} K_1(x,z) K_2(z,y) dz. \tag{6.1}$$

A Volterra kernel is called *invariant* if $K(g{\cdot}x, g{\cdot}y) = K(x,y)$ for all $x, y \in X$. The set of invariant Volterra kernels is denoted by $V(X)^\natural$. Note that $V(X)$ as well as $V(X)^\natural$ form an algebra under $\sharp$.

Let x_o be a base point of X fixed by H. Then we can identify an invariant Volterra kernel with a function $f : G \to \mathbb{C}$ which is continuous on $S = \{g \in G : g{\cdot}x_o \geq x_o\}$, vanishes outside S and is bi-invariant under H. Using this one finds

Proposition 6.1. $V(X)^\natural$ *is commutative.* □

Now we suppose that S is of the type $S(\mathcal{O})$ satisfying the various hypotheses from the previous paragraph. Then the f_K coming from invariant Volterra kernels can be viewed as functions on S_A^o.

Proposition 6.2. *Let* K_1, K_2 *be Volterra kernels and* f_{K_1} *and* f_{K_2} *the associated functions on* S_A^o. *Suppose that* f_{K_1} *and* f_{K_2} *lie in the domain of the spherical Laplace transform, then*

$$\mathcal{L}_{\mathcal{O}}(f_{K_1}) \mathcal{L}_{\mathcal{O}}(f_{K_2}) = \mathcal{L}_{\mathcal{O}}(f_{K_1 \sharp K_2}).$$ □

Now suppose that G acts on $\mathbb{R}^k$ and that $\mathbb{R}^k$ is ordered by a proper generating cone. Consider the integral equation

$$A(u,v) = B(u,v) + \int_{\mathbb{R}^k} N(u,u')A(u',v)du' \tag{6.2}$$

where A, B and N are Volterra kernels with respect to the ordered space $\mathbb{R}^k$. We make the following assumptions:

1) The equation (6.2) is G-invariant, where G acts simultaneously on the two variables.
2) For all u and v under consideration (this may be less than $\mathbb{R}^k$) the interval $[v,u]$ is contained in a union of G-orbits $\bigcup_{p\in P} G\cdot u_p$ such that the isotropy group of u_p is H for all $p \in P$.
3) As a measure space we can write

$$\bigcup_{p\in P} G\cdot u_p = P \times G/H$$

with a finite measure dp on P.

4) The orderings of $\bigcup_{p\in P} G\cdot u_p$ and G/H are compatible, i.e., if $u = (p,x) \geq (q,y) = v$ then $x \geq y$.

Then we can write

$$\begin{aligned}
\int_{\mathbb{R}^k} N(u,u')A(u',v)du' &= \int_P \int_{G/H} N((p,x),(p',x'))A((p',x'),(q,y))dx'dp' \\
&= \int_P \int_{[y,x]} N((p,x),(p',x'))A((p',x'),(q,y))dx'dp' \\
&= \int_P N_{p,p'} \natural A_{p',q}(x,y)dp'
\end{aligned}$$

where $N_{p,p'}(x,x') = N((p,x),(p',x'))$ and $A_{p',q}(x',y) = A((p',x'),(q,y))$. Note that $N_{p,p'}$ and $A_{p',q}$ are G-invariant kernels. They are Volterra type kernels because of the compatibility (the continuity condition has to be weakened a little since $N_{p',p}$ and $A_{p',q}$ may jump to zero within Γ). Thus we finally may rewrite (6.2) as

$$\mathcal{L}_{\mathcal{O}}(f_{A_{p,q}}) = \mathcal{L}_{\mathcal{O}}(f_{B_{p,q}}) + \int_P \mathcal{L}_{\mathcal{O}}(f_{N_{p,p'}})\mathcal{L}_{\mathcal{O}}(f_{A_{p',q}})dp' \tag{6.3}$$

which is the anounced partial diagonalization. For a concrete example involving $G/H = \mathrm{SO}_0(1,n)/\mathrm{SO}_0(1,n-1)$ we refer to [7].

Differential equations and causality

Semigroups can be used in the study of the causal structure of homogeneous manifolds. Viewing causal (or timelike) paths as solutions of certain differential inclusions this can certainly be viewed as a part of analysis. On the other hand the methods involved are more geometrical than analytical in nature and moreover quite similar to the methods which come from the interplay between Lie semigroups and geometric control theory. Thus for this topic we only refer to the original papers [19, 20], [10], [18] and KUPKA's article in these proceedings.

7. Stability of causal differential equations

In this section we describe a theorem of PANEITZ on the asymptotic behaviour of certain differential equations. It has been used in the context of quantization for curved space-times (see [35], [36], [38] and [39]). We include this theorem because the infinite dimensional analogue of the invariant cone in $\mathrm{sp}(n, \mathbb{R})$ plays a decisive role in it and note in passing that also the invariant cones in $\mathrm{su}(2,2)$ show up in the further development of PANEITZ' theory as Killing vectors for positive conserved (Noether) quantities (see [38]).

Let $\mathcal{H}$ be a *real* Hilbert space with inner product $(\cdot, \cdot)$. For a given complex structure J on $\mathcal{H}$, i.e $J = J^t$ and $J^2 = I$, we set

$$\omega_J(v, w) = (v, Jw)$$

and

$$\langle v, w\rangle_J = (v, w) + i\omega_J(v, w)$$

so that $(\mathcal{H}, \langle\cdot, \cdot\rangle_J)$ is a *complex* Hilbert space with respect to the complex structure J. Note that ω_J is a symplectic form on $\mathcal{H}$. We set

$$\mathrm{Sp}(\mathcal{H}) = \{g \in \mathrm{Gl}(\mathcal{H}) \colon g^t Jg = J\}$$

and

$$\mathrm{sp}(\mathcal{H}) = \{X \in \mathrm{B}(\mathcal{H}) \colon X^t J + JX = 0\}.$$

Consider the equation

$$\frac{d}{dt}v(t) = A(t)v(t)$$

where $A(t) \in \mathrm{Sp}(\mathcal{H})$ and $v(t) \in \mathcal{H}$, and the corresponding operator equation

$$\frac{d}{dt}S(t) = A(t)S(t). \tag{7.1}$$

The following theorem of Paneitz gives sufficient conditions for a unique solution of (7.1) to exist and show a decent asymptotic behaviour.

Theorem 7.1. *Let* $A: \mathbb{R} \to C = \{X \in \mathrm{sp}(\mathcal{H}): \omega_J(Xv, v) \geq 0 \ \forall v \in \mathcal{H}\}$ *be a strongly continuous and norm bounded map such that*

$$\int_{-\infty}^{\infty} ||A(t)||dt < 2.$$

Then (7.1) *has a unique solution* $S: [-\infty, \infty] \to \mathrm{Sp}(\mathcal{H})$ *with* $S(-\infty) = I$ *and* $(S(t) + I)$ *is invertible for all* t. □

Note that C is invariant under the adjoint action of $\mathrm{Sp}(\mathcal{H})$. The operator $S = S(\infty)$ is interpreted as a scattering operator. The proof of Theorem 7.1 shows that it is the Cayley transform of an element $Y \in C$. It is important for the applications to physics to know whether S can be viewed as a unitary operator with respect to some $\langle \, , \, \rangle_{J'}$, i.e., whether S commutes with some complex structure J' in the $\mathrm{Sp}(\mathcal{H})$-orbit of J. In view of the ambiguity of the various quantization procedures one also wants to know the degree of uniqueness of such a J'.

Theorem 7.2. *Let* A, S, Y *be as above and assume that*

$$\int_{-\infty}^{\infty} A(t)dt \in C^o = \{X \in \mathrm{sp}(\mathcal{H}): \omega_J(Xv, v) \geq k||v||^2 \ \forall v \in \mathcal{H} \text{ for some } k > 0\}.$$

Then $Y \in C^o$ *and* S *commutes with a unique* J' *in the* $\mathrm{Sp}(\mathcal{H})$*-orbit of* J. □

The last statement of Theorem 7.2 is related to the fact that each element of C^o is conjugate under $\mathrm{Sp}(\mathcal{H})$ to a skew hermitean operator.

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