Group theoretical aspects of Gödel's cosmological model

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About thirty years after Einstein had found his equations for gravitational fields Gödel published a solution of these equations, which, when interpreted as a cosmological model, allowed travelling into the past (cf. [Gö49]). It was the first model in which this property is not a consequence of trivial topological reasons. Even though Gödel's model has been rejected as a realistic one for various physical reasons there is a continuing interest in it (cf. [MaS7], [Le90]). The aim of the present article, which is based on a lecture I gave at the Gödel-Gesellschaft in December 1990, is to provide an elementary description of this model and a geometric explanation of the possibility of time travel.

Einsteins equations describe the connection of the geometry of the four dimensional space time continuum M and the distribution of mass therein. Here geometry essentially means the way of measuring "distances" in the space-time continuum. If two points x and y are close together, choosing appropriate coordinates $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$, one approximately has

$$dist(x,y)^2 = (x_0 - y_0)^2 - \sum_{I=1}^{3}(x_i - y_i)^2.$$

This shows that the relativistic "distance" does not have the properties of the usual (euclidean) distances. The following figure illustrates the existence of points with negative or zero square distance from a given point.

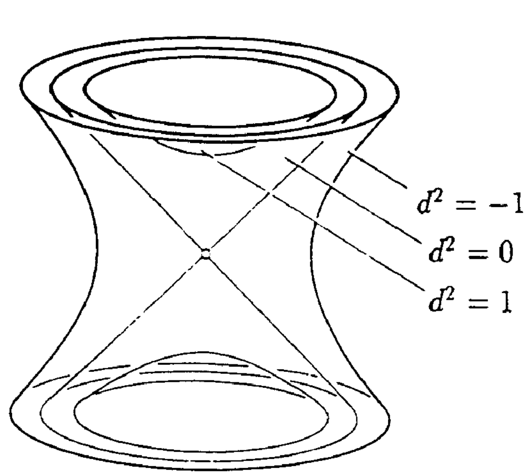

Figure 1

If one collects all these "infinitesimal" pictures for all points of the space-time one obtains a family of double cones D_x, one in each tangent space T_xM, given as

$$D_x = \{v \in T_xM = \mathbb{R}^4 : v^t\beta_x v = 0\},$$

where β_x is a symmetric 2×2-matrix depending on $x \in M$ (cf. Figure 2).

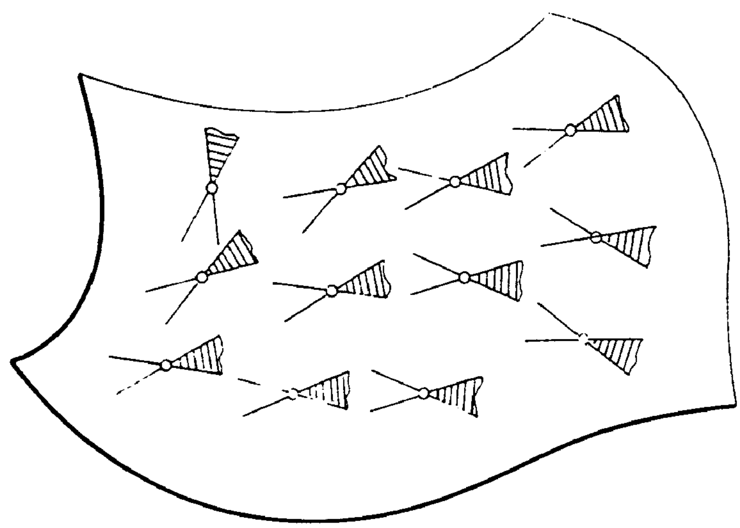

Figure 2

In mathematical terms: we have a four dimensional manifold with a Lorentzian metric.

According to the physical interpretation light moves in M along lines with tangents of zero length, i.e. pointing in the direction of the double cone and no particle can be accelerated to more then speed of light. This means that the worldline of a particle can only have tangents within the double cone.

We require our model M to be *time orientable*, which simply means that we have a consistent and continuous way of choosing one half of the double cone D_x as the cone pointing into the future at x. The closed convex hull of the chosen half of D_x will be denoted by C_x. Then massive particles must have worldlines with tangents in the interior of C_x. Such curves we call *timelike future directed*.

In this context time-travel means: There exist closed timelike (future directed) curves. Such curves can be manufactured by wrapping up a space-time model, such as $\mathbb{R}^4$ with the metric given above (i.e. Minkowski space), in an appropriate way (cf. Figure 3).

This is an example of the trivial topological reasons referred to above. Gödel's model M as a manifold simply is $\mathbb{R}^4$. In the coordinates $x = (x_0, x_1, x_2, x_3)$ the metric is given by

$$ds^2 = (dx_0 + e^{x_1}dx_2)^2 - \frac{e^{2x_1}}{2}dx_2^2 - dx_1^2 - dx_3^2$$

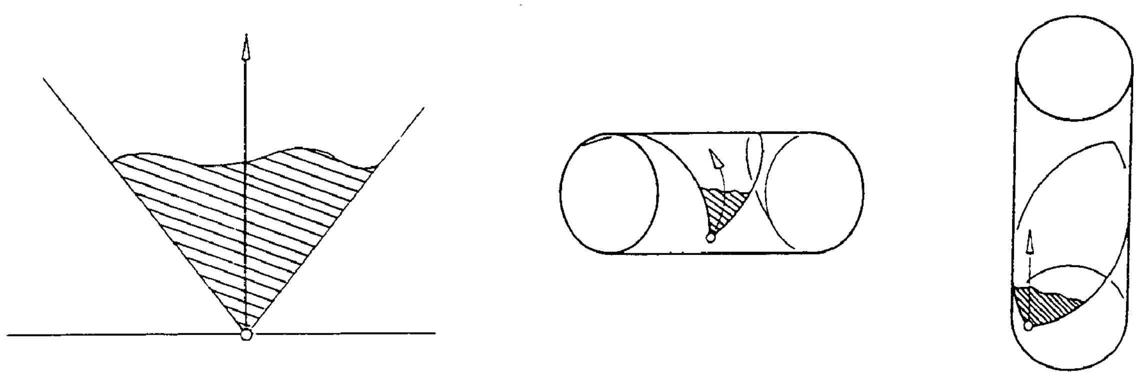

Figure 3

or, what is the same, by

$$\beta_x = \begin{pmatrix} 1 & 0 & e^{x_1} & 0 \\ 0 & -1 & 0 & 0 \\ e^{x_1} & 0 & \frac{e^{2x_1}}{2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Gödel shows the existence of closed timelike curves in this space-time using the explicit coordinate transform

$$(x_0, x_1, x_2, x_3) \mapsto (t, r, \varphi, y)$$

defined via

$$\begin{aligned} e^{x_1} &= \cosh 2r + (\cos\varphi)\sinh 2r \\ x_2 e^{x_1} &= \sqrt{2}\sin\varphi \\ \tan(\frac{\varphi}{2} + \frac{x_0 - 2t}{2\sqrt{2}}) &= e^{-2r}\tan\frac{\varphi}{2} \quad |\frac{x_0 - 2t}{2\sqrt{2}}| < \frac{\pi}{2} \\ x_3 &= 2y. \end{aligned}$$

He asserts that with this transform one obtains a new metric

$$4(dt^2 - dr^2 + (\sinh^4 r - \sinh^2 r)d\varphi^2 + 2\sqrt{2}(\sinh^2 r)d\varphi dt - dy^2),$$

from which one can easily deduce that for large enough R the circle

$$t = 0, r = R, y = 0$$

is a closed timelike curve (cf. Figure 4)

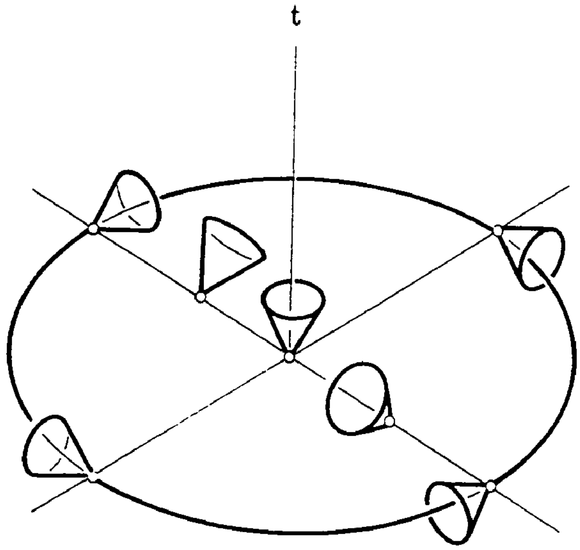

Figure 4

Apart from the technicalities in checking the above formulae there remains the question what this transformation really means. Gödel gives a geometric interpretation of the metric in the new coordinates. When reformulated in group theoretical terms it even yields an explanation of the coordinate transformation.

First we note that there exists a four dimensional group G of isometries of M. In fact, if we embed M into $\mathbb{R}^5$ via $x \mapsto (x, 1)$, then the group consisting of the matrices

$$\langle a,b,c,d\rangle := \begin{pmatrix} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & e^{-b} & 0 & c \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ acts freely on M. This means that the map $\lambda_{\langle a,b,c,d\rangle}\colon M \to M$ defined by matrix multiplication with $\langle a,b,c,d\rangle$ is fixed point free whenever $\langle a,b,c,d\rangle \neq \langle 0,0,0,0\rangle$. Moreover the action is transitive, i.e. the point $(0,0,0,0)$ can be moved to any other point in M using one of the transformations $\lambda_{\langle a,b,c,d\rangle}$. Note that these maps are differentiable as maps from $\mathbb{R}^4$ to $\mathbb{R}^4$ with derivative

$$d\lambda_{\langle a,b,c,d\rangle} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-b} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now a simple matrix multiplication shows that

$$d\lambda^t_{\langle a,b,c,d\rangle}\beta_{(0,0,0,0)}d\lambda_{\langle a,b,c,d\rangle} = \beta_{(a,b,c,d)},$$

which shows that the group acts by isometries since $\lambda_{(a,b,c,d)}(0,0,0,0) = (a,b,c,d)$.

In order to explain how group theory yields closed timelike curves we consider yet another space-time model N. As a manifold it is simply the product of $\mathrm{SL}(2,\mathbb{R})$ and $\mathbb{R}$. Here $\mathrm{SL}(2,\mathbb{R})$ denotes the group of real 2×2-matrices of determinant one. The determinant is a quadratic form on the space $M(2,\mathbb{R})$ of all 2×2-matrices which clearly is left and right inva riant under the action (matrix multiplication) of $\mathrm{SL}(2,\mathbb{R})$. Thus the determinant defines an $\mathrm{SL}(2,\mathbb{R})$-biinvariant metric on $M(2,\mathbb{R})$ via

$$\beta_A(Y,Y) = \det(Y).$$

The tangent space of $\mathrm{SL}(2,\mathbb{R})$ in the unit matrix is the set $\mathrm{sl}(2,\mathbb{R})$ of 2×2-matrices of trace zero. For such a matrix we have $\det(Y) = -\frac{1}{2}tr(Y^2)$, so that on $\mathrm{sl}(2,\mathbb{R})$ our metric takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

when expressed with respect to the basis

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The invariance now shows that the signature is the same at all points of $\mathrm{SL}(2,\mathbb{R})$ and thus we have constructed a Lorentzian metric on $\mathrm{SL}(2,\mathbb{R})$. Taking the direct product with the metric $-dy^2$ on $\mathbb{R}$ we obtain an invariant Lorentzian metric on the space-time (which at the same time is a group) N.

The Gauss-algorithm shows that any element of $\mathrm{SL}(2,\mathbb{R})$ can be written in a unique way as the product of a rotation and an upper triangular matrix, that is

$$\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} e^u & v \\ 0 & e^{-u} \end{pmatrix}$$

(in Lie group theory this is called the Iwasawa decomposition). Therefore $\mathrm{SL}(2,\mathbb{R})$ can be covered by $H := \mathbb{R} \times \mathbb{R}^2 = \{(s,u,v)\}$. If we denote the group of rotations by K and the subgroup of upper triangular matrices in $\mathrm{SL}(2,\mathbb{R})$ by B then the group $K \times B$ acts on the Lorentz manifold $\mathrm{SL}(2,\mathbb{R})$ via $\big((k,b),A\big) \mapsto kAb^{-1}$ by isometries. Moreover the action is free and transitive as one sees from the above product decomposition. Note that one use the covering map

$$(s,u,v) \mapsto \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} e^u & v \\ 0 & e^{-u} \end{pmatrix}$$

to lift the Lorentzian metric of $\mathrm{SL}(2,\mathbb{R})$ to H and furthermore to obtain a free and transitive action of $\mathbb{R} \times B$ by isometries on H. If one now takes the direct product with $\mathbb{R}$ one finds $\mathbb{R} \times B \times \mathbb{R}$ acting freely and transitively by isometries on the simply connected covering space $\tilde{N} = H \times \mathbb{R}$ of N.

At this point one should note that the group $\mathbb{R} \times B \times \mathbb{R}$ is isomorphic to the group G introduced earlier. In fact, G is easily seen to be the direct product of $\mathbb{R}^2$ with a non-abelian two dimensional simply connected Lie group and there is, up to isomorphy, only one such group. Thus using such an isomorphism we can identify M and $\tilde{N}$ as manifolds. Both spaces are endowed with an invariant Lorentzian metric. Thus in order to check whether the metrics on the two spaces correspond to each other one only needs two compare them at one point. This means that we do not explicitly need the isomorphism between the groups G and $\mathbb{R} \times B \times \mathbb{R}$ but only the derivative at one point. In group theoretical terms, we need the isomorphism on the Lie algebra level. The Lie algebra $\mathfrak{g}$ of G, or what is the same, the tangent space at $\langle 0,0,0,0\rangle$ consists of the matrices

$$\langle\langle a,b,c,d\rangle\rangle := \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 & c \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

whereas the Lie algebra $\mathfrak{g}^\sharp$ of $\mathbb{R} \times B \times \mathbb{R}$ is the space of triples (s,F,y) with upper triangular matrices F of trace zero. For any pair (α,γ) of non-zero real numbers the map

$$\langle\langle a,b,c,d\rangle\rangle \mapsto (\alpha a, \begin{pmatrix} -b & \gamma c \\ 0 & b \end{pmatrix}, d)$$

is an isomorphism $\mathfrak{g} \to \mathfrak{g}^\sharp$. From the product decomposition of $\mathrm{SL}(2,\mathbb{R})$ we see that as a tangent space to $\tilde{N}$ the Lie algebra of $\mathbb{R} \times B \times \mathbb{R}$ is isomorphic to $\mathrm{sl}(2,\mathbb{R}) \times \mathbb{R}$ via

$$(s, \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix}, y) \mapsto (\begin{pmatrix} u & s+v \\ -s & -u \end{pmatrix}, y).$$

Thus the derivative of the map identifying the manifolds M and $\tilde{N}$ at $(0,0,0,0)$ is given by

$$(a,b,c,d) \mapsto (\begin{pmatrix} -b & \alpha a + \gamma c \\ -\alpha a & b \end{pmatrix}, d).$$

If we now calculate the metric induced on $\mathbb{R}^4$ by the metric on $\mathrm{sl}(2,\mathbb{R}) \times \mathbb{R}$ via this derivative we find

$$(\alpha a + \frac{\gamma}{2}c)^2 - \frac{\gamma^2}{4}c^2 - b^2 - d^2,$$

which for $\alpha = \frac{\sqrt{2}}{2}$ and $\gamma = \sqrt{2}$ reduces to

$$\frac{1}{2}(a+c)^2 - \frac{1}{2}c^2 - b^2 - d^2.$$

On the other hand Gödel's metric on $\mathbb{R}^4$ is given by

$$(a+c)^2 - \frac{1}{2}c^2 - b^2 - d^2.$$

This means that instead of considering the metric on $\tilde{N}$ defined using the determinant and given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to basis of sl(2, $\mathbb{R}$) described above, we have to use the metric which is given by the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

on sl(2, $\mathbb{R}$). Note that the cone defined by this metric is invariant under rotations of the form

$$\begin{pmatrix} u & v \\ w & -u \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Therefore our given metric on N is not only invariant under the action of $K \times B \times \mathbb{R}$ but also under left multiplication with elements from SL(2, $\mathbb{R}$).

What we have achieved by now is the following: we have identified Gödel's model with a covering of SL(2, $\mathbb{R}$) $\times$ $\mathbb{R}$ equipped with a metric which is invariant under left multiplication with elements from SL(2, $\mathbb{R}$). But for the elements of SL(2, $\mathbb{R}$) we have a polar decomposition: each matrix can be written as the product of a rotation from K and a positive definite symmetric matrix (this is called the Cartan decomposition in Lie group theory). On the other hand any positive symmetric matrix takes on the form

$$A_{r,\varphi} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

The decomposition

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} A_{r,\varphi}$$

then yields new coordinates (t, r, φ, y) for $\tilde{N}$, and hence for M, in which the circles $t = 0, r = R, y = 0$ for large enough R are closed timelike curves. To prove this one only needs to calculate the derivative $\frac{\partial}{\partial\varphi} A_{r,\varphi}$ of such a circle, transport it bac k to the identity via $A_{r,\varphi}^{-1} \frac{\partial}{\partial\varphi} A_{r,\varphi}$ and then calculate the square length of the result with respect to our metric on sl(2, $\mathbb{R}$). One obtains

$$(r^2 + \frac{1}{r^2})(1 - \frac{\cos^4(\varphi) + \sin^4(\varphi)}{2}) - 2(1 + \frac{\cos^4(\varphi) + \sin^4(\varphi)}{2})$$

which is positive for large enough r.

We have thus given a group theoretical construction of a coordinate transformation which enables us to explicitly write down timelike closed curves for Gödel's space-time model. But still this construction was particular for the special given model. What is remarkable is that the group theoretical approach can be used to find a testing method for the existence of closed timelike curves in much more general models, provided there is a group G acting on them freely and transitively by isometries.

The strategy is as follows (cf. [HHL89], [HH90]). One identifies M with the group G and constructs a subsemigroup S of G as the endpoints of piecewise differentiable timelike (or lightlike) future directed curves. It is then possible to show that there exist closed timelike curves if and only if $S = G$. Thus one needs a criterion for the semigroup S to be contained in a proper maximal subsemigroups of G. For a certain class of groups (the ones which are compact modulo its radical) one can read this off from the position of the cone C inside the Lie algebra $\mathfrak{g}$ of G. To be precise, G admits closed timelike curves if and only if the interior of C intersect each hyperplane in $\mathfrak{g}$ which is a subalgebra. For the Gödel space-time this criterion can be applied and shows the existence of closed timelike curves since each hyperplane in $\mathfrak{g}$ which is a subalgebra contains either $\mathbb{R}x_0 + \mathbb{R}x_3$ or $\mathbb{R}x_2$. Thus we can assert the possibility of time travel without resorting to the $\mathrm{SL}(2, \mathbb{R})$ version of the model. On the other hand the space-time $\tilde{N}$ with the original metric coming from the determinant does *not* admit closed timelike curves.

Finally we note that the analysis carried out in this article can be reversed and then together with the general method outlined above yields results about the controllability of systems in reductive Lie groups.

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Received Dec 12, 1990

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