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INFINITESIMALLY GENERATED SUBSEMIGROUPS OF MOTION GROUPS

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0. Introduction. Recent developments in nonlinear control theory (cf. [**2, 3**] etc.) and also in analysis (cf. [**19, 14, 15, 16, 17, 18**]) indicate that there is an increasing demand for a systematic Lie theory of semigroups. Whereas the groundworks of a local Lie theory begin to emerge (cf. [**12, 4, 5, 8**]), there is not much on the record on a global theory (cf. [**12, 6, 9**]). We will briefly outline the basic definitions and the principal difficulties.

Let G be a connected Lie group and S be a subsemigroup of G. In order to simplify matters we assume that the group generated by S in G algebraically is all of G. Then we can associate with S a tangent object $\underline{L}(S)$ by setting $\underline{L}(S) = \{x \in \underline{L}(G) : x = \lim_{n\to\infty} nx_n, \exp x_n \in S, n \in \mathbf{N}\}$, where $\underline{L}(G)$ is the Lie algebra of G and $\exp : \underline{L}(G) \to G$ is the exponential function. It turns out (cf. [**12**]) that $\underline{L}(S)$ is a wedge, i.e., that it is a closed convex set, which is also closed under addition and multiplication by positive scalars. Moreover it satisfies

$$e^{adx}\underline{L}(S) = \underline{L}(S) \quad \text{for all } x \in \underline{L}(S) \cap \ \underline{L}(S), \tag{0.1}$$

where $adx(y) = [x,y]$ with the bracket in $\underline{L}(G)$. We call a wedge satisfying (0.1) a *Lie wedge* and $\underline{L}(S)$ the *tangent wedge* of S.

It has been shown in [**8**] that, for any Lie wedge W, there exists a local semigroup S_w with $\underline{L}(S_w) = W$, i.e., there is a neighborhood $\mathcal{U}$ of the identity in G containing S_w such that $S_w S_w \cap \mathcal{U} \subset S_w$ and $W = \{x \in \underline{L}(G) : x = \lim_{n\to\infty} nx_n \exp x_n \in S_w, n \in \mathbf{N}\}$. On the other hand the examples (cf. [**8**]) show that by no means is every Lie wedge in $\underline{L}(G)$ the tangent wedge of a (global) subsemigroup S of G. Thus the principal question is: For which Lie wedges W in $\underline{L}(G)$ do there exist subsemigroups S of G such that $\underline{L}(S) = W$?

It is one basic idea of Lie theory that the tangent object should provide as much information as possible on the object under consideration.

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In the case of semigroups it is quite clear that, in contrast to the group situation, many different semigroups may have the same tangent object. Therefore one is interested especially in semigroups that are in a sense determined by their tangent object. We call a subsemigroup S of G *infinitesimally generated* if $\underline{L}(S)$ generates $\underline{L}(G)$ as a Lie algebra and we have

$$(0.2) \qquad \exp \underline{L}(S) \subseteq S \subseteq \langle \exp \underline{L}(S) \rangle^{-},$$

where $\langle \exp \underline{L}(S) \rangle$ is the subsemigroup of G generated algebraically by $\exp \underline{L}(S)$.

Let W be any Lie wedge in $\underline{L}(G)$ which generates $\underline{L}(G)$ as a Lie algebra. Suppose there exists a subsemigroup S of G such that $\underline{L}(S) = W$. Since we have

$$(0.3) \qquad \underline{L}(S) = \{x \in \underline{L}(G) : \exp \mathbf{R}^{+} x \subset \overline{S}\},$$

by [**12**] we may assume that S is closed and contains $\exp W$. If we now let T be the subsemigroup of G generated algebraically by $\exp W$, then obviously $W \subset \underline{L}(T) \subset \underline{L}(S) \subset W$ so that W is the tangent wedge of an infinitesimally generated subsemigroup of G. Thus a complete answer to our principal question would provide a classification of all infinitesimally generated subsemigroups and hence yield a general framework of a Lie theory of semigroups.

At the moment we are far from being able to give a complete answer to our principal question. If we want to derive positive results we need to restrict to certain classes of groups and to certain classes of Lie wedges. The groups we will consider here are of the type $G = CA$, where C is a compact subgroup and A is an abelian normal subgroup. Such a group we will call a *motion group*. Accordingly we call a Lie algebra L a *motion algebra* if it is of the type $L = K + I$, where I is an abelian ideal and K is compactly embedded, i.e., $\mathrm{spec}(ad x) \subseteq \mathbf{R} i$ for all $x \in K$.

The Lie wedges we want to restrict ourselves to are the so called *semialgebras* (cf. [**5**]). These are Lie wedges W for which we can find a neighborhood B of 0 in $\underline{L}(G)$ such that the Campbell-Hausdorff multiplication $x * y = x + y + [x,y]/2 + \cdots$ is defined on B and $W \cap B$ is a local semigroup *w.r.t.* $*$, i.e., $(W \cap B) * (W \cap B) \subset W$.

Equivalently, semialgebras are the tangent wedges of divisible local semigroups. There is yet another characterization of semialgebras which we describe since we will use it in the sequel.

Let W be a wedge in a finite dimensional vectorspace L. Assume that W is *generating*, i.e., that $W \quad W = L$. For $x \in W$ we set $T_x = (W \quad \mathbf{R}^+x)^- \cap (\mathbf{R}^+x \quad W)^-$ (cf. [**5**]). A closer inspection of the definition of T_x yields (cf. [**5**]) that we may call T_x the *tangent space of* W *at* x in accordance with our geometrical intuition. With this terminology [**5**] shows that W is a semialgebra if and only if $[x, T_x] \subset T_x$ for all $x \in W$.

The paper is organized as follows: In Section 1 we provide some general facts on the existence of a semigroup with a prescribed tangent wedge. Section 2 will be devoted to the study of semialgebras in motion algebras, and the last section will contain a description of those semialgebras in motion algebras which are the tangent wedges of infinitesimally generated subsemigroups in the corresponding motion group.

1. A lemma on subsemigroups of Lie groups. In this section we prove a lemma on the existence of subsemigroups with a prescribed tangent cone which is of general interest. The idea is that if a cone K sits properly inside a wedge W that is already the tangent wedge of a semigroup S, then one can construct a semigroup S_K with tangent cone K by taking the union of a local semigroup with tangent cone K (which exists by Lie's Fundamental Theorem, cf. [**8**]) and a translate of S. We note that, using a considerable amount of machinery, it is possible to extend this result so that we no longer have to assume that K is a cone (i.e., satisfies $K \cap \quad K = \{0\}$). For a proof of the generalization we refer to [**11**] and [**10**]. Here we give a technical but elementary proof of the special case in order to make the paper as self-contained as possible. In order to state the lemma precisely we introduce the following notation: For two wedges W_1 and W_2 in a vectorspace L, we write $W_1 \subset\subset W_2$ if $W_1 \backslash (W_1 \cap \quad W_1)$ is contained in the interior of W_2.

Lemma 1.1. *Let G be a Lie group and S be a closed infinitesimally generated subsemigroup of G whose tangent wedge $\underline{L}(S) = W$ is generating in $\underline{L}(G) = L$. If K is a cone (i.e., $K \cap \quad K = \{0\}$) in*

$\underline{L}(G)$ satisfying $K \subset\subset W$, then there exists a closed subsemigroup S_K of G with $K = \underline{L}(S_K)$.

Proof. Note first that we can find a cone K' satisfying $K \subset\subset K' \subset\subset W$. Now choose a nonzero $x_0 \in K$. Then we can find a Campbell-Hausdorff neighborhood B in $\underline{L}(G)$, a compact neighborhood of zero B_0 contained in B and an $\epsilon > 0$ such that the maps $\phi_t : B_0 \to B$ defined by $\phi_t(x) = tx_0 * x$ are homeomorphisms onto $\phi_t(B_0)$ for all $t \in [\ \epsilon, \epsilon]$. Making ϵ and B_0 smaller if necessary, we may assume by [**12**] that there is a closed set $\sum \subset B_0 \cap K'$ with $(\sum * \sum) \cap B_0 \subset \sum$ and $K = \{x \in \underline{L}(G) : x = \lim nx_n, x_n \in \sum\}$. Finally we may assume that $\exp|_{B_0}$ is a homeomorphism onto its image satisfying $\exp(K' \cap B_0) \cap S^{-1} = \{1\}$. In fact $\exp(K') \subset S$ and $S \cap S^{-1}$ is a Lie subgroup with $\underline{L}(S \cap S^{-1}) = W \cap (\ W)$ by [**12**]. Hence $\exp^{-1}(\exp(K' \cap B_0) \cap S^{-1} \subset W \cap (\ W) \cap K' = \{0\}$.

Note that the uniform continuity of the $*$-multiplication on B_0 and the fact that $K' \subset\subset W$ allow us to find an $\epsilon_1 > 0$ with $\epsilon_1 < \epsilon$ and open, relatively compact, neighborhoods B_1 and B_2 of zero such that $(K' \cap B_1)\backslash B_2 \subset \epsilon_1 x_0 * (W \cap B_0), B_2 * B_2 \subset B_1$ and $\epsilon_1 x_0 \in B_2$. In fact, choose B_1 and B_2 such that $B_2 * B_2 \subset B_1 \subset \overline{B}_1 \subset \operatorname{int} B_0$. Then $(K' \cap \overline{B}_1)\backslash B_2$ is a compact subset of the interior of $W \cap B_0$. Hence we may find an $\epsilon_1 > 0$ such that $\ \epsilon_1 x_0 * ((K' \cap \overline{B}_1)\backslash B_2)$ is still contained in $\operatorname{int}(W \cap B_0)$. But we may assume that $\epsilon_1 < \epsilon$ so that $\phi_{-\epsilon_1}$ is a homeomorphism onto its image and we obtain

$$\begin{aligned}(K' \cap B_1)\backslash B_2 &= \phi_{-\epsilon_1}^{-1}(\ \ \epsilon_1 x_0 * (K' \cap B_1)\backslash B_2) \subseteq \phi_{-\epsilon_1}^{-1}(W \cap B_0) \\ &= \epsilon_1 x_0 * (W \cap B_0)\end{aligned}$$

as desired (cf. Figure 1).

Now we define $\sum' = \sum \cap B_2$ and $S' = (\exp \sum')(\exp \epsilon_1 x_0)S$. Then

$$\begin{aligned}(\exp {\textstyle\sum}')(\exp {\textstyle\sum}') = \exp({\textstyle\sum}' * {\textstyle\sum}') &\subseteq \exp({\textstyle\sum} \cap B_1) \\ &\subseteq \exp({\textstyle\sum} \cap B_2) \cup \exp(({\textstyle\sum} \cap B_1)\backslash B_2) \\ &\subseteq \exp {\textstyle\sum}' \cup \exp((K' \cap B_1)\backslash B_2).\end{aligned}$$

But $\exp((K' \cap B_1)\backslash B_2) \subseteq (\exp \epsilon_1 x_0)\exp(W \cap B_0) \subseteq S'$ since $\exp W \subset S$. If we now set $S_K = \exp \sum' \cup S'$, then this shows that

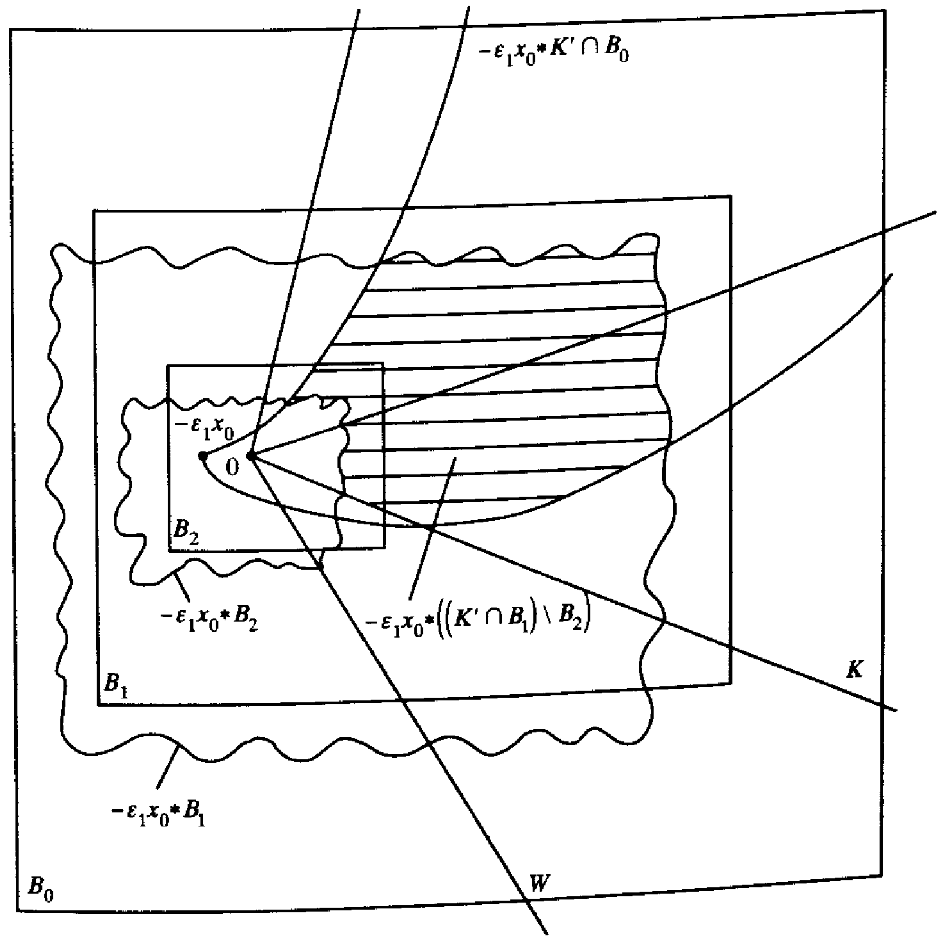

FIGURE 1a.

$(\exp \sum')$ $(\exp \sum') \subseteq S_K$. Moreover, since $S' \subset S$, we have $S'S \subseteq S' \subseteq S_K$. Therefore, in order to show that S_K is a semigroup it only remains to show that $(\exp \sum')S'$ is contained in S_K.

Let $s \in S'$ and $x \in \sum'$ be fixed. Then $s = (\exp x')(\exp \epsilon_1 x_0)g$ for some $x' \in \sum'$ and $g \in S$. We have to consider two cases:

Case 1. $x*x' \in B_2$. In this case we conclude $(\exp x)(\exp x')(\exp \epsilon_1 x_0)g = \exp(x * x')(\exp \epsilon_1 x_0)g \in S'$ since $x * x' \in \sum$.

Case 2. $x * x' \in \sum \setminus B_2$. In this case we know $x * x' = \epsilon_1 x_0 * w$ for some $w \in W \cap B_0$ since $x * x' \in \sum \cap B_1 \subset K' \cap B_1$. Hence $(\exp x)(\exp x')(\exp \epsilon_1 x_0)g = (\exp \epsilon_1 x_0)(\exp w)(\exp \epsilon_1 x_0)g$ which is in S' since $\exp w \in S$.

Finally we have to show that $\underline{L}(S_K) = K$. In order to do that it suffices to show that we can find a neighborhood $\mathcal{U}$ of **1** in G such that

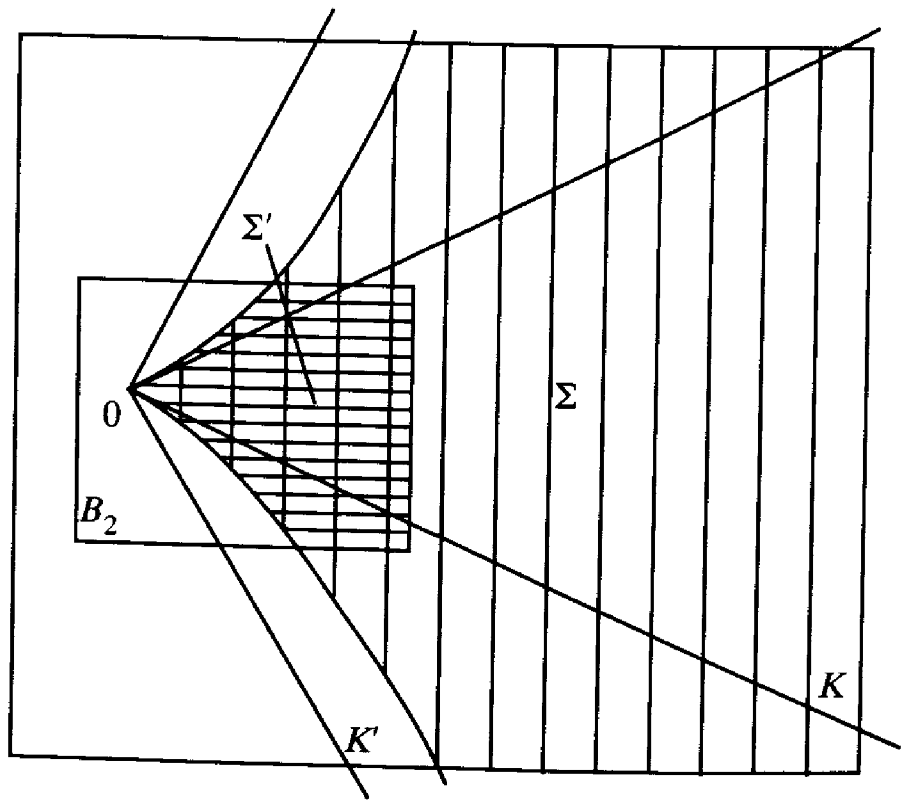

FIGURE 1b.

$\exp^{-1}(S_K \cap \mathcal{U}) \subset \sum'$ since then $L(S_K) = K$.

Note that $\exp(\overline{\sum'} * \epsilon_1 x_0)$ is compact since $\exp \overline{\sum'}$ is. But $\overline{\sum'} * \epsilon_1 x_0$ does not contain zero, for otherwise $-\epsilon_1 x_0 \in \overline{\sum'} \cap -K \subseteq K' \cap B_1 \cap -K = \{0\}$. The set $A = (\exp(\overline{\sum'} * \epsilon_1 x_0))S$ is closed since S is closed and $\exp(\overline{\sum'} * \epsilon_1 x_0)$ is compact. Moreover A contains S' but not the identity **1** since we calculate

$$\exp(\overline{\sum}' * \epsilon_1 x_0) \cap S^{-1} \subset \exp(K' \cap B_0) \cap S^{-1} \subset \{\mathbf{1}\}.$$

Thus there exists a neighborhood $\mathcal{U}$ of the identity in G such that $\mathcal{U} \cap A = \emptyset$ which implies $\mathcal{U} \cap S_K \subset \exp \sum'$. □

The following example shows that the hypothesis on K to satisfy $K \backslash \{0\} \subset \text{int } W$ cannot be dropped.

Example 1.2. Let G_1 be the Heisenberg group and K_1 be a cone in $\underline{L}(G_1)$ containing a central point in its interior. If $G = G_1 \oplus \mathbf{R}$ and

$K = K_1 \oplus \mathbf{R}^+$, then $K \subset \underline{L}(G_1) \oplus \mathbf{R}^+$ and $G_1 \oplus \mathbf{R}^+$ is a semigroup with $\underline{L}(G_1 \oplus \mathbf{R}^+) = \underline{L}(G_1) \oplus \mathbf{R}^+$. Nevertheless there cannot be a semigroup S in G with $\underline{L}(S) = K$ since $S \cap G_1$ would have to be a semigroup containing central points which is not possible by [**9**].

2. Semialgebras in motion algebras. In this section we give a complete geometric description of all generating semialgebras W in motion algebras which will enable us to decide whether we can find global semigroups S with $\underline{L}(S) = W$ or not.

First we note that motion algebras admit a sort of fitting decomposition.

Lemma 2.1. *Let $L = K + I$ be a finite dimensional Lie algebra, where K is compactly embedded and I is an abelian ideal, i.e., L is a motion algebra. Then we have the decomposition*

$$L = [K, I] + [K, K] + Z(K, L), \tag{2.1}$$

where $Z(K, L)$ is the centralizer of K in L and (2.1) is a direct decomposition of vector spaces. Moreover $Z(K, L)$ is abelian.

Proof. Since K is compactly embedded, any adx with $x \in K$ is semisimple so that L is the direct sum of the vector spaces $[K, L]$ and $Z(K, L)$. Moreover we have $K = [K, K] + Z(K)$ where $Z(K)$ is the center of K and $[K, K]$ is semisimple. Since $[K, I] \cap [K, K] = (0)$ the decomposition (2.1) follows.

Note that $Z(K, L) = Z(K) + Z(K, I)$ where $Z(K, I) = \{x \in I : [x, K] = \{0\}\}$ and hence $Z(K, L)' = Z(K, I)' = \{0\}$. □

We now are ready to give the announced description of semialgebras in motion algebras.

Theorem 2.2. *Let L be a motion algebra. Then for any generating semialgebra W in L we have*

(i) *$[L, I] \subset W$ for any abelian ideal I of L.*

(ii) *W is invariant, i.e., $e^{adx} W = W$ for all $x \in L$.*

(iii) *W is of the form $W = J + \mathbf{R}^+(a + C)$, where J is an ideal, C is a compact convex neighborhood of zero in a compactly embedded subalgebra E of L, and $a \in Z(K, L)$ where K is the maximal compactly embedded subalgebra of L containing E. Moreover C is invariant under the group generated by the e^{adk} with $k \in K$.*

Proof. (i) Let L be a counterexample to claim (i), which has minimal dimension and consider an $x \in \operatorname{int} W$. If $\mathcal{U} \subseteq W$ is a neighborhood of x then the set $[\mathcal{U}, I] = \{\sum_{i=1}^{n}[u_i, v_i] : u_i \in \mathcal{U}, v_i \in I, n \in \mathbf{N}\}$ is all of $[L, I]$ and, for any $u \in \mathcal{U}\backslash I$, the wedge $W \cap (\mathbf{R}\,u + I)$ is a generating semialgebra in $\mathbf{R}\,u + I =: A_u$. If I has codimension greater than one in L, then the minimality of the counterexample shows that $[\mathbf{R}\,u, I] \subseteq W$ for all $u \in \mathcal{U}\backslash I$ since A_u is again a motion algebra. But then $[L, I] = [\mathcal{U}, I] \subseteq W$ contrary to our assumptions. Thus I is a hyperplane in L, since L cannot be abelian.

Now let $x \in L\backslash I$ be such that $\mathbf{R}\,x$ is compactly embedded and consider the operator $D = adx : L \to L$ and its dual $\hat{D} : \hat{L} \to \hat{L}$. Both D and $\hat{D}$ are semisimple with purely imaginary spectrum. If $y \in W\backslash I$ such that T_y is a tangent hyperplane of W in y, then $T_y \cap I$ is a hyperplane in I which is invariant under ady, hence also under D. Therefore any nonzero linear form $\omega \in \hat{I}$ with $\ker\omega = T_y \cap I$ is an eigenvector of $\hat{D}$ and thus contained in $\ker\hat{D}$. We conclude that $D(I) \subseteq \ker\omega \subseteq T_y$. Since y was an arbitrary point on $W\backslash I$ which defines a tangent hyperplane Lemma 1.2 [**4**] implies that $[L, I] = D(I) \subseteq \cap_{y\in W} T_y \subseteq W$ (cf. also [**4**]).

This final contradiction to our assumptions proves part (i) of the theorem.

In order to prove parts (ii) and (iii) we note that Lemma 2.1 implies $L = [L, I] + [K, K] + Z(K, L)$. By part (i) we know that $[L, I] \subseteq W$ and hence $W_1 = W \cap ([K, K] + Z(K, L))$ is a generating semialgebra in the compact Lie algebra $[K, K] + Z(K, L)$. Thus [**7**] shows that $W_1 = J_1 + \mathbf{R}^+(a + C)$, where J_1 and $E = \mathbf{R}\,C$ are ideals in $[K, K] + Z(K, L)$ and C is a compact convex neighborhood of 0 in E, which is invariant under the group generated by the e^{adx} with $x \in [K, K] + Z(K, L)$. Moreover $a \in Z(K, L)$ and W_1 is invariant in $[K, K] + Z(K, L)$. But since $[L, I]$ is an ideal, $W_1 + [L, I]$ is invariant. Finally we have

$L = [L, I] + [K, K] + Z(K, L)$ and $[L, I] \subseteq W$ so that $W = W_1 + [L, I]$ and the proof is finished if we set $J = J_1 + [L, I]$. □

Note that the results of Theorem 2.2 can be extended to a certain degree:

Remarks 2.3. Let L be the sum of a compactly embedded subalgebra K and a nilpotent ideal N, then any generating semialgebra W in L is invariant.

Proof. Let L be a counterexample of minimal dimension. By [**7**] we can assume that $N \neq \{0\}$. If $x \in \operatorname{int} W$ then $\mathbf{R}\,x$ is compactly embedded in $\mathbf{R}\,x + Z(N)$, where $Z(N)$ is the center of N, since, for $x = k + n, k \in K, n \in \mathbf{N}$, we have $adx|_{Z(N)} = adk|_{Z(N)}$. Therefore 2.2 implies that $[x, Z(N)] \subset W$ and thus $[L, Z(N)] \subset W$ since x was chosen arbitrarily in int W. Set $J = [L, Z(N)]$ and note that J is an ideal in L. Since L is a counterexample of minimal dimension to our claim we have $J = 0$ which means that $Z(N) \subset Z(L)$. Let now $x \in C'(W)$, i.e., such that T_x is a hyperplane and $0 \neq y \in Z(N)$. Then by the minimality of L we know that $(W + \mathbf{R}\,y)$ $/\mathbf{R}\,y$ is invariant in $L/\mathbf{R}\,y$, hence we know also that $(W + \mathbf{R}\,y)$ is invariant in L. There are two cases possible:

Case 1. $y \in T_x$. In this case T_x is also a tangent hyperplane of $(W + \mathbf{R}\,y)$ so that by [**7**] we have $[x, L] \in T_x$.

Case 2. $y \notin T_x$. In this case we have $[x, T_x] \in T_x$ since W is a semialgebra and also $[x, y] = 0$ since $y \in Z(N) \subset Z(L)$. Thus we again conclude $[x, L] \subset T_x$ which shows that W is invariant, contradicting our assumptions. This proves the claim. □

3. Subsemigroups in motion groups. The problem to decide whether there exists a subsemigroup S of a Lie group G with a prescribed tangent wedge W may be decomposed into two separate problems. The first is to find a subsemigroup $\tilde{S}$ of the universal covering group $\tilde{G}$ of G with tangent wedge W and the second is to decide whether it is possible to project $\tilde{S}$ down to G without enlarging $\underline{L}(\tilde{S})$. We have

Proposition 3.1. *Let G and H be Lie groups and $q : G \to H$ a quotient map. If S is a subsemigroup of G generating G as a group, then $L(q)(\underline{L}(S)) \subset \underline{L}(q(S))$ where $L(q) : L(G) \to L(H)$ is the morphism associated with q. The converse need not be true. If T is a subsemigroup of H generating H as a group and containing the identity then $\underline{L}(q^{1}\ (T)) = (L(q))^{1}\ \underline{L}(T)$.*

Proof. Note first that we may assume that S is closed since $q(\overline{S}) \subset q(S)^{-}$ so that $L(q)(\underline{L}(\overline{S})) \subset \underline{L}(q(\overline{S}))$ implies $L(q)\underline{L}(S) = L(q)(L(\overline{S})) \subseteq \underline{L}(q(\overline{S})) \subset \underline{L}(S)\ \ ) = \underline{L}(q(S))$. If now $\exp_H : L(H) \to H$ and $\exp_G : L(G) \to G$ are the respective exponential functions, then $\exp_G \mathbf{R}^+ x \subseteq S$ implies $\exp_H \mathbf{R}^+ L(q)(x) = q(\exp_G \mathbf{R}^+ x) \subseteq q(S) \subset q(S)\ $, hence, by (0.3), $x \in \underline{L}(q(S))$. To see that the converse is not true consider an ice-cream cone W in $\mathbf{R}^3$ and factor a discrete subgroup of a line whose intersection with W is a halfline in the boundary of W; then W is a semigroup with $\underline{L}(W) = W$, whereas the quotient semigroup has a halfspace as tangent wedge.

To see the last statement note first that $q^{1}\ (T)$ generates G as a group since T generates H and $\ker q \subset q^{1}\ (T)$ so that $\underline{L}(q^{1}\ (T))$ makes sense. Moreover $q(q^{1}\ (T) = T$ so that the inclusion $\underline{L}(q^{1}\ (T)) \subset L(q)^{1}\ \underline{L}(T)$ follows from the first part. Conversely if $x \in L(q)^{1}\ (\underline{L}(T))$, then $\exp_H \mathbf{R}^+ L(q)x \subseteq \overline{T}$ so that $\exp_G \mathbf{R}^+ x \subseteq q^{1}\ (\overline{T})$. But since H is metrizable [**1**; Cap. IX, § 2, Prop. 1.8] implies that $q^{1}\ (\overline{T}) \subset (q^{1}\ (T))\ $, since any Cauchy sequence in $\overline{T}$ can be lifted to a Cauchy sequence in $q^{1}\ (\overline{T})$. In fact, for any $s \in q^{1}\ (\overline{T})$ we find a sequence h_n in T converging to $q(s)$ and hence a sequence $s_n \in q^{1}\ (h_n) \subseteq q^{1}\ (T)$ converging to s, i.e., $s \in (q^{1}\ (T))\ $. Thus $\exp_G \mathbf{R}^+ x \subseteq (q^{1}\ (T))\ $ or, by (0.3), $x \in \underline{L}(q^{1}\ (T))$. □

Proposition 3.1 shows that, for invariant wedges, our problems are reduced to the case of proper cones:

Corollary 3.2. *Let G be a Lie group and W a generating invariant wedge in $L(G)$. Then there exists a subsemigroup S of G with $L(S) = W$ if the analytic group A associated with $H(W) = W \cap -W$ is closed and there exists a subsemigroup T of G/A such that $\underline{L}(T) = W/H(W)$.*

Proof. Note first that $H(W)$ is an ideal since W is invariant. Therefore A is a closed normal subgroup of G and we may consider the quotient map $q : G \to G/A$. But $W/H(W)$ is a generating invariant wedge in $L(G)/L(A)$ so that $\overline{T}$ generates G/A as a group and contains the identity. Thus Proposition 3.1 implies that $q^{1}\ (\overline{T})$ is a semigroup with $L(q)^{1}\ (W/H(W)) = W$ as tangent wedge, which proves our claim. □

We can also handle the case where L is a compact Lie algebra.

Lemma 3.3. *Let G be a connected Lie group whose Lie algebra L is compact, and let W be a generating invariant cone in L. Then, for the maximal compact subgroup K of G, the following statements are equivalent:*

(1) *There exists a subsemigroup S of G such that $\underline{L}(S) = W$.*

(2) $W \cap L(K) = \{0\}$.

Proof. (1) $\Rightarrow$ (2). Let $x \in W \cap L(K)$. Then we may assume that $\exp x \in S$ since $\underline{L}(S) = \underline{L}(\overline{S})$. Since $(\exp \mathbf{R}\,x)^{-}$ is compact, this implies $(\exp \mathbf{R}\,x)^{-} \subset \overline{S}$ so that $\mathbf{R}\,x \subset W$, whence $x = 0$.

(2) $\Rightarrow$ (1). Note first that $G \cong K \oplus V$, where V is a vector-group. Let L_M be a hyperplane in $L(G)$ containing $L(K)$ and satisfying $L_M \cap W = \{0\}$. This is possible by (2). Then L_M is an ideal in $L(G)$ whose corresponding analytic subgroup M is closed and contains K. Now consider $G/M \cong \mathbf{R}$ and the cone $(W + L_M)/L_M$ in $L(G/M)$. Identifying G/M with $L(G/M)$ we see that $(W + L_M)/L_M$ is a subsemigroup of G/M, so that Proposition 3.1 shows that there is a subsemigroup S_1 of G with $\underline{L}(S_1) = L_M + \mathbf{R}^{+}x$ where $0 \neq x \in W$. But since $L_M \cap W = \{0\}$ we have $W \subset\subset L_M + \mathbf{R}^{+}x$ so that Lemma 1.1 yields the existence of the desired S. □

Using this result we obtain

Theorem 3.4. *Let G be a motion group and W be a generating semialgebra in $L(G)$. If A is the analytic subgroup corresponding to $H(W)$ and K is a maximal compact subgroup of G, then the following statements are equivalent.*

(1) *There exists an infinitesimally generated subsemigroup S of G such that $\underline{L}(S) = W$.*

(2) *The group A is closed and $W \cap L(K) \subset H(W)$.*

Proof. (1) $\Rightarrow$ (2). Recall first that $A \subset S$ since S is infinitesimally generated. If $x \in L(\overline{A})$ then $\exp \mathbf{R}x \subseteq \overline{A} \subset \overline{S}$ so that $\mathbf{R}x \in \underline{L}(S)$, i.e., $x \in H(W)$. Hence $L(\overline{A}) = L(A)$ and thus $A = \overline{A}$. Now consider the quotient map $q : G \to G/A; S = q^{-1}(S))$ since $A \subset S$ so that $W = \underline{L}(S) = L(q)^{-1}(\underline{L}(q(S)))$ by Proposition 3.1. But then $\underline{L}(q(S)) = W/H(W)$. Note that Theorem 2.2 implies that W is invariant and $L(G)/H(W)$ is a compact Lie algebra. Thus we may apply Lemma 3.3 to $W/H(W)$ in $L(G/A)$ and find $(W/H(W)) \cap L(K_1) = \{0\}$ in $L(G/A) = L(G)/H(W)$ where K_1 is the maximal compact subgroup of G/A.

Note that $q(K)$ is compact, hence contained in K_1. Therefore $L(K) \subset L(q)^{-1}L(K_1)$, whence

$$\begin{aligned} W \cap L(K) \subset L(q)^{-1}(W/H(W)) \cap L(q)^{-1}(L(K_1)) \\ = L(q)^{-1}(W/H(W) \cap L(K_1)) \\ = L(q)^{-1}(\{0\}) \\ = H(W). \end{aligned}$$

(2) $\Rightarrow$ (1). Conversely, if A is closed we can consider G/A and find, again by Theorem 2.2, that $L(G/A)$ is compact and $W/H(W)$ is a generating invariant cone in $L(G/A)$. Let K_1 again denote the maximal compact subgroup of G/A and $q : G \to G/A$ the quotient map. Then $K \subset q^{-1}(K_1)$ and, by [**13**], even $q^{-1}(K_1) = KA$ since K is also a maximal compact subgroup of $q^{-1}(K_1)$. Hence $L(q)^{-1}L(K_1) = L(KA) = L(K) + H(W)$ and $L(q)^{-1}(W/H(W) \cap L(K_1)) = W \cap (L(K) + H(W)) = W \cap L(K) \subset H(W)$ by (2). Thus Lemma 3.3 applies to $W/H(W)$ and yields a subsemigroup S_1 of G/A such that $\underline{L}(S_1) = W/H(W)$. But then Proposition 3.1 shows that $S = q^{-1}(S_1)$ has tangent wedge W so that the Theorem is proven in view of the introductory remarks. □

We conclude with

Corollary 3.5. *Let G be a simply connected motion group and W be* *a generating semialgebra in $L(G)$, then there exists an infinitesimally generated subsemigroup S of G such that $\underline{L}(S) = W$.*

Proof. If G is simply connected it is of the form $V \times K$, where K is a semisimple compact group and V is a vectorgroup. But then $W \cap L(K) \subset H(W)$ by [**7**]. □

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