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Controllability on real reductive Lie groups*

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Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\exp: \mathfrak{g} \to G$ the associated exponential function. A control system on G consists of a family $\tilde{\chi}$ of vectorfields on G. The set of endpoints of continuous paths in G, starting at the identity, which are piecewise integral curves for elements of $\tilde{\chi}$ is called the *reachable* set of $\{1\}$. The system is called *controllable* if the reachable set is all of G. We call the control system left *invariant* if it consists of left invariant vectorfields. Thus such a control system simply is a subset χ of $\mathfrak{g}$. It is well known (cf. [JS72, JK81]) that in order to study controllability properties of such systems it is enough to consider the closed convex cone generated by χ.

So let C be a closed convex cone in $\mathfrak{g}$ which we will assume to have non-empty interior. Then the associated left invariant control-system is controllable if and only if the closed semigroup $S(C)$ generated by $\exp C$ is all of G (cf. Corollary VI.1.17 in [HHL89]). In this case we call C *controllable* in G. If G is simply connected we omit the reference to G. We will give a simple geometric characterisation of controllability in the case that G is reductive, C is pointed, i.e. satisfies $C \cap -C = \{0\}$, and is invariant under the adjoint of K where NAK is an Iwasawa decomposition of G.

There are several contexts in which the controllability problem described above occurs. In [Vi80] Vinberg poses it as an open problem for simply connected simple Lie groups and Ad G-invariant cones in order to characterise what he calls continuous invariant orderings of Lie groups. Olshanskii gave a solution in [OL82]. He found a necessary and sufficient condition for C to satisfy $S(C) \neq G$. In addition he showed that for such a C one even has

$$C = \{X \in \mathfrak{g} \mid \exp \mathbb{R}^+ X \subset S(C)\}. \qquad (*)$$

In general the cones satisfying $(*)$ are exactly the cones occuring as tangent cones of subsemigroups of G. We call such cones *global* in G. Again we omit the reference to G if G is simply connected. In [Ne90b] Neeb extended Olshanskii's characterization of controllability to the case of Ad G-invariant pointed cones in semisimple Lie groups and the globality result to the case of simple

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Lie groups and Ad K invariant pointed cones. Both papers, [Ol 82] and [Ne 90 b] make heavy use of semisimple structure theory and are to a large extend computational.

Our approach is more geometric. In fact, the key idea is to describe a reductive group as a homogeneous space of a group with co-compact radical and then use the theory of maximal subsemigroups in such groups which can be reduced to simple linear algebra. Implicitly this way of viewing a reductive group is (for $G=\mathrm{SL}(2,\mathbb{R})$) contained in [Gö 49] where the first simply connected space time model violating causality was developed. It should be noted here that the study of causal structures is essentially the same as the study of control systems on Lorentzian manifolds.

Now let G be a reductive group and NAK an Iwasawa decomposition of G. If C is an Ad K-invariant pointed cone with non empty interior, averaging aver Ad K (which is compact) yields elements in (int C)$\cap Z(\mathfrak{k})$ where $Z(\mathfrak{k})$ denotes the center of $\mathfrak{k}$. Thus at least for semisimple groups with finite center C is always controllable in G. Therefore we will restrict ourselves mostly to the case of simply connected groups (cf. also [Ne 90 c]).

The basic observation in this paper is that the group $G_a = NA \times K$ acts transitively on G via $(na, k)\cdot g = nagk^{-1}$. This action allows us to identify G and G_a as manifolds and hence $\mathfrak{g}$ and $\mathfrak{g}_a = (\mathfrak{n}+\mathfrak{a})\times\mathfrak{k}$ as vector spaces.

Lemma 1 *Let G be a reductive Lie group and NAK an Iwasawa decomposition of G. Let G_a be the group $NA\times K$. If C is an* Ad(K)*-invariant cone in* $\mathfrak{g}$, *then C is global in G if and only if C is global in G_a.*

Proof. G_a operates freely and transitively on G via $(na, k)\cdot g = nagk^{-1}$. The left invariant cone field C_g on G generated by C is by hypothesis also right invariant with respect to translations from K, i.e. invariant under the action of G_a. Identifying G and G_a as manifolds we obtain a left invariant cone field $C_{(na,k)} := C_{(nak^{-1})}$ on G_a which is generated by C. Now the claim follows from Corollary VI.1.17 in [HHL 89]. □

Remark 2 The proof of Lemma 1 even shows that the closed semigroups generated by $\exp_G C$ and $\exp_{G_a} C$ coincide as sets when we identify G and G_a as manifolds.

Lemma 1 shows that in order to check the globality of an Ad(K)-invariant cone in G it suffices to check it in G_a. But G_a is compact modulo its radical and thus, for simply connected G_a, the maximal subsemigroups are known once one has determined all the hyperplanes in its Lie algebra $(\mathfrak{n}+\mathfrak{a})\oplus\mathfrak{k}$ which are subalgebras (cf. Corollary VI.5.2 and Corollary V.5.4 in [HHL 89]).

Lemma 3 *Let* $\mathfrak{g}$ *be a reductive Lie algebra with Iwasawa decomposition* $\mathfrak{n}+\mathfrak{a}+\mathfrak{k}=\mathfrak{g}$ *and* $\mathfrak{g}_a=(\mathfrak{n}+\mathfrak{a})\oplus\mathfrak{k}$. *Then the intersection* $\Delta(\mathfrak{g}_a)$ *of all hyperplanes in* $\mathfrak{g}_a$ *which are subalgebras is equal to* $\mathfrak{n}'\oplus\mathfrak{k}'$ *where ′ denotes the commutator.*

Proof. We use notation of [Hof 90] and note first that the s-radical of $\mathfrak{g}_a$ is all of $\mathfrak{g}_a$ since $\mathfrak{g}_a$ does not contain an isomorphic copy of $\mathrm{sl}(2,\mathbb{R})$. We note further that $\mathfrak{g}_a' = \mathfrak{n}\oplus\mathfrak{k}'$ and consequently $\mathfrak{g}_a'' = \mathfrak{n}'\oplus\mathfrak{k}'$. The subalgebra $\mathfrak{n}$ is the sum of one dimensional $\mathfrak{a}$-modules, hence $\mathfrak{g}_a'/\mathfrak{g}_a''$-modules. Finally we remark that the image of $\mathfrak{a}\oplus Z(\mathfrak{g})$ in $\mathfrak{g}_a/\mathfrak{g}_a''$ is a Cartan algebra of $\mathfrak{g}_a/\mathfrak{g}_a''$ which has trivial intersection with $(\mathfrak{g}_a/\mathfrak{g}_a'')'$. Here $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$ which is contained in $\mathfrak{k}$. Now the claim follows from Recipe (a) and Proposition 6 in [Hof 90]. □

According to Theorem 20 in [Hof90] there are two families of hyperplane subalgebras in $\mathfrak{g}_a$. The first family consists of all hyperplanes containing $\mathfrak{n}+\mathfrak{k}'$. The second consists of hyperplanes which contain the preimage of a Cartan algebra of $\mathfrak{g}_a/\mathfrak{g}_a''$. Note that $Z(\mathfrak{k})=Z(\mathfrak{k},\mathfrak{g})$ is contained in any such preimage. If now C is an $\mathrm{Ad}(K)$-invariant cone in $\mathfrak{g}$ then averaging over K shows that $Z(\mathfrak{k})\cap \operatorname{int} C \neq \emptyset$ provided that $\operatorname{int} C$ is non-empty which we always assumed. Thus the only hyperplane subalgebras which can possibly miss $\operatorname{int} C$ are the ones containing $\mathfrak{n}$. Now Corollary VI.5.2 in [HHL89] (cf. also Theorem 1.18 in [HH90]) together with Remark 2 yields

Theorem 4 *Let G be a simply connected reductive Lie group with Iwasawa decomposition* NAK *and C an $\mathrm{Ad}(K)$-invariant convex cone in $\mathfrak{g}$ satisfying $C\cap -C=\{0\}$ and $\operatorname{int} C\neq\emptyset$. Let $S(C)$ be the closed semigroup generated by* $\exp C$. *Then the following assertions hold.*

(i) *If $(\operatorname{int} C)\cap(\mathfrak{n}+\mathfrak{k}')\neq\emptyset$ then $S(C)=G$.*

(ii) *If $C\cap(\mathfrak{n}+\mathfrak{k}')=\{0\}$ then $S(C)$ satisfies*

$$C=\{X\in\mathfrak{g} \mid \exp \mathbb{R}^+ X\subset S(C)\},$$

i.e. C is global.

(iii) *If $\emptyset\neq(C\cap(\mathfrak{n}+\mathfrak{k}'))\setminus\{0\}\subset\partial C$ then $S(C)\subsetneqq G$.*

The Controllability Theorem III.5 of [Ne90b] is a consequence of our theorem since it says – for a special Iwasawa decomposition – that $S(C)=G$ if and only if the dual cone C^* of C intersects the annihilator $(\mathfrak{k}'+\mathfrak{a}+\mathfrak{n})^\perp$ of $\mathfrak{k}'+\mathfrak{a}+\mathfrak{n}$ trivially which by the Hahn Banach theorem is equivalent to $\operatorname{int} C\cap(\mathfrak{k}'+\mathfrak{a}+\mathfrak{n})\neq\emptyset$. But in the special situation of [Ne90b] this is equivalent to $\operatorname{int} C\cap(\mathfrak{k}'+\mathfrak{n})\neq\emptyset$ (cf. Theorem III.7 of [Ne90b]).

It is easy to write down examples of cones which are neither global nor controllable. Take for instance a product of a global and a controllable cone. On the other hand their are situations in which $S(C)\neq G$ automatically implies the globality of C. This is for instance the case if $\mathfrak{g}$ is simple as we will now explain.

Note that in the situation of Theorem 4 $S(C)\neq G$ implies $(\operatorname{int}(C+\mathfrak{k}'))\cap(\mathfrak{n}+\mathfrak{k}')=\emptyset$ so that Remark 2 applied to $C+\mathfrak{k}'$ shows that $S(C+\mathfrak{k}')\neq G$. If now $\mathfrak{g}$ is simple an easy argument using a Cartan decomposition of $\mathfrak{g}$ (cf. [Ne90b], proof of Theorem III.7) shows that $C+\mathfrak{k}'$ is global. This in turn is equivalent to C being global as is shown by the very general Proposition III.5 from [Ne90a]. Thus our theorem also yields a characterisation of the global invariant cones in simple Lie algebras.

Finally we remark that Theorem 1.18 in [HH90] can be used to rule out non-global cones which are not controllable provided one imposes additional conditions on the geometry of the cone. So for instance any Ad K-invariant Lorentz cone C in a semisimple Lie algebra without $\mathrm{sl}(2,\mathbb{R})$-factors is either global or controllable.

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