Radon transform on halfplanes via group theory

Joachim Hilgert*

1 A suitable double fibration

Consider the halfplane $X = \{(a,b) \in \mathbf{R}^2 | a > 0\}$ as a subset of $\{(a,b,1) \in \mathbf{R}^3\}$ and the group

$$G = \left\{ (\alpha,\beta,\gamma) := \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in GL(3,\mathbf{R}) | \alpha > 0 \right\}$$

which acts transitively on X via

$$(\alpha,\beta,\gamma) \odot (a,b) = (\alpha a, a\beta + b + \gamma).$$

The stabilizer of $x_0 := (1,0)$ is

$$H_X = \{(1,\beta,-\beta) \in G | \beta \in \mathbf{R}\} \cong \mathbf{R}.$$

Let Ξ be the set of halflines in X which end in $\partial X \cong \{0\} \times \mathbf{R}$. Such a line is uniquely determined by its intersection with ∂X and its slope. More precisely, for $v, w \in \mathbf{R}$ we consider the halfline $L_{v,w} = \{(t, v + tw) | t > 0\}$. Then we can identify Ξ with $\mathbf{R} \times \mathbf{R}$ via $L_{v,w} \longleftrightarrow (v,w)$. Note that the affine action of G on X induces a transitive action of G on Ξ. It is given by

$$(\alpha,\beta,\gamma) \diamond (v,w) = (v + \gamma, \frac{w + \beta}{\alpha}).$$

This time we set $\xi_0 := (0,0) \in \Xi$ and note that the stabilizer ξ_0 is

$$H_\Xi = \{(\alpha,0,0) \in G | \alpha > 0\}.$$

Now we have the desired double fibration, which will give us the Radon transform

$$\begin{array}{ccc} & G & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_\Xi} \\ G/H_X = X & & \Xi = G/H_\Xi, \end{array}$$

where $\pi_X(\alpha,\beta,\gamma) = (\alpha, \beta + \gamma)$ and $\pi_\Xi(\alpha,\beta,\gamma) = (\gamma, \frac{\beta}{\alpha})$.

*Supported by a DFG Heisenberg-grant.

E. A. Tanner and R. Wilson (eds.), Noncompact Lie Groups and Some of Their Applications, 141–146.

2 The Radon transform

We use the following left invariant Haar measures on G, H_X and H_Ξ:

$$d\mu_G(\alpha,\beta,\gamma) = \frac{d\alpha}{\alpha}d\beta d\gamma, \quad d\mu_{H_X}(1,\beta,0) = d\beta, \quad d\mu_{H_\Xi}(\alpha,0,0) = \frac{d\alpha}{\alpha}$$

and define the Radon transform and its dual as

$$\mathcal{R}f(gH_\Xi) := \int_{H_\Xi} f(ghH_X)d\mu_{H_\Xi}(h) \quad \forall f \in C_c(X)$$

and

$$\check{\mathcal{R}}\phi(gH_X) := \Delta_G(g)\int_{H_X} \phi(ghH_\Xi)d\mu_{H_X}(h) \quad \forall \phi \in C_c(\Xi).$$

Explicitly we find

$$\mathcal{R}f(v,w) = \int_{\mathbf{R}_+^\times} f(a, v+aw)\frac{da}{a}$$

and

$$\check{\mathcal{R}}\phi(a,b) = \frac{1}{a}\int_{\mathbf{R}} \phi(v, \frac{b-v}{a})dv = \int_{\mathbf{R}} \phi(b-av, v)dv.$$

Note that $\mathcal{R}$ and $\check{\mathcal{R}}$ are dual to each other:

$$\int_{\mathbf{R}}\int_{\mathbf{R}_+^\times} f(a,b)\check{\mathcal{R}}\phi(a,b)\frac{da}{a}db = \int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{R}f(v,b)\phi(v,b)dvdb.$$

3 Plane waves and a Fourier transform

A *horocycle* in X is an orbit of a conjugate of H_Ξ. The set of horocylces coincides with Ξ. For each $v \in \partial X$ we define a pencil $\mathcal{P}_v$ of horocyles via

$$\xi \in \mathcal{P}_v \quad \text{if and only if} \quad \xi = (v,w).$$

For $x \in X$ and $\xi \in \mathcal{P}_v$ we define the *complex distance* $d_v(x,\xi)$ to be the (unique) element $(1,\beta,0) \in A := \{(1,\beta,0) \in G\}$ such that $(1,\beta,0)\odot x \in \xi$. Let $x = (a,b) \in X$ and $v \in \mathbf{R}$ then there exists a unique $\xi_{x,v} \in \mathcal{P}_v$ such that $x \in \xi_{x,v}$. It satisfies

$$d_v(x_0, \xi_{x,v}) = (1, v + \frac{b-v}{a}, 0).$$

We define our plane waves associated to a $\lambda \in \mathbf{a}_{\mathbf{C}}^*$, where $\mathbf{a}$ is the Lie algebra of A, which can be identified with $\mathbf{R}$, and v in $\mathbf{R}$:

$$e_{\lambda,v}(a,b) := e^{\langle \lambda, v + \frac{b-v}{a}\rangle}.$$

Proposition.

1. $e_{\lambda,v}$ *is a joint eigenfunction for all* $D \in \mathbf{D}(X)$, *the invariant differential operators on* X.
2. $e_{\lambda,v}$ *is constant on the line* ξ *for each* $\xi \in \mathcal{P}_v$.
3. $e_{\lambda,v}(gx) = e_{\lambda\Delta_G(g),g^{-1}.v}(x)e_{\lambda,v}(gx_0)$ *for* $g \in G, x \in X$ *and* $\lambda \in \mathbf{a}^*_{\mathbf{C}}$, *where the action of* G *on* ∂X *is given by* $(\alpha,\beta,\gamma).v = v+\gamma$.

Proof. Consider the normal subgroup $P := \{(\alpha,\beta,0) \in G\}$ and note that G is the semidirect product of P and H_X. Then [1], Theorem II.4.9, tells us that $\mathbf{D}(X)$ is, as a vector space, isomorphic to $I(\mathbf{p})$, the space of H_X-invariants in the symmetric algebra $S(\mathbf{p})$ of the Lie algebra $\mathbf{p}$ of P. The group P is itself a semidirect product of the normal subgroup A with H_Ξ. Thus we may view the vector space $\mathbf{p}$ as a direct summand of $\mathbf{a}$ with the Lie algebra of H_Ξ. A simple calculation using the coadjoint action and the identification of the symmetric algebra with the polynomials on the dual shows that the H_X-invariants in $S(\mathbf{p})$ are just the elements of $S(\mathbf{a})$. The theorem quoted above now yields that $\mathbf{D}(X)$ is generated by $\frac{\partial}{\partial b}$. This proves the first claim. The second claim is proved by a straightforward verification. The cocycle condition from the last claim can also be verified by an elementary calculation.

It can be shown that the cocycle condition from the above proposition is closely related to unitary representations of G which are induced from characters of A. These representations are not irreducible, a fact that is reflected in some indeterminacy properties if the Fourier transform associated to our plane waves. We define the Fourier transform as

$$\mathcal{F}_X f(\lambda, gA) = \int_X f(x)e_{-i\lambda\Delta_G(g)^{-1},gP}(x)d\nu_X(x),$$

where $gP \in \partial X$ is the g-translate of $0 \in \partial X$. Identifying $G/A \cong H_X H_\Xi$ with $\mathbf{R}^\times_+ \times \mathbf{R}$ via

$$(\alpha,-\gamma\alpha,\gamma) = (1,-\gamma,\gamma)(\alpha,0,0) \leftrightarrow (\alpha,\gamma),$$

this reads

$$\mathcal{F}_X f(\lambda,\alpha,\gamma) = \int_{\mathbf{R}^\times_+}\int_{\mathbf{R}} f(a,b)e^{-i\alpha\langle\lambda,\gamma+\frac{b-\gamma}{a}\rangle}db\frac{da}{a}.$$

Note that $\mathcal{F}_X f(\lambda,\alpha,\gamma)$ does not depend explicitly on λ, γ and α but only on $\alpha\langle\lambda,\gamma\rangle$ and $\alpha\lambda$. Therefore we write $\mathcal{F}_X f(\lambda,\gamma) := \mathcal{F}_X(\lambda,1,\gamma)$ and introduce the map $\tilde{\mathcal{F}}_X f : \mathbf{R}\times\mathbf{R}^* \to \mathbf{C}$ defined by

$$\tilde{\mathcal{F}}_X f(r,\eta) := \int_{\mathbf{R}^\times_+}\int_{\mathbf{R}} f(\frac{1}{a},-\frac{b}{a})\frac{1}{a}e^{+i(\langle\eta,b\rangle+ra)}dbda.$$

We note $\mathcal{F}_X f^\flat(\lambda,\gamma) = e^{-i\langle\lambda,\gamma\rangle}\tilde{\mathcal{F}}_X f(\langle\lambda,\gamma\rangle,\lambda)$, where $f^\flat(a,b) := (1/a)f(a,b)$, and define $\tilde{f} \in C_c(\mathbf{R}\times\mathbf{R})$

$$\tilde{f}(t,v) = \begin{cases} \frac{1}{t}f(\frac{1}{t},-\frac{v}{t}) & t>0 \\ 0 & t\leq 0. \end{cases}$$

Then

$$(\frac{1}{2\pi})^2\tilde{\mathcal{F}}_X f(r,\eta) = (\mathcal{F}_{\mathbf{R}^2}^{-1}\tilde{f})(r,\eta),$$

where $\mathcal{F}_{\mathbf{R}^2}$ denotes the Fourier transform on $\mathbf{R}^2$.

4 Inversion and Plancherel formula

We can write the Fourier transform using the Radon transform

$$\tilde{\mathcal{F}}_X f(\langle\lambda,\gamma\rangle,\lambda) = \mathcal{F}_{\mathbf{R}}(\mathcal{R}f(\gamma,\cdot))(\lambda).$$

Define an operator $\tilde{A}$ on the Schwartz functions $S(\mathbf{R})$ via

$$2\pi\mathcal{F}_{\mathbf{R}}(\tilde{A}\psi)(\tau) = |\tau|\mathcal{F}_{\mathbf{R}}(\psi)(\tau)$$

and use to define an operator A on suitable functions in to variables by

$$(A\psi)(r,s) := \tilde{A}(\psi(r,\cdot))(s).$$

Hereafter, $\int$ denotes integration over $\mathbf{R}$, and $\int^+$ denotes integration over $\mathbf{R}_+$.

Proposition. *Let $f \in C_c^\infty(X)$. Then*

$$f(a,b) = a^{-1}\int(A\mathcal{R}f)(b+r,-\frac{r}{a})dr.$$

Proof. Note first that

$$\begin{aligned} f(a,b) &= a^{-1}\tilde{f}(a^{-1},-a^{-1}b) \\ &= a^{-1}\tfrac{1}{(2\pi)^2}\int\int(\mathcal{F}_{\mathbf{R}\times\mathbf{R}}\tilde{f})(\xi,r)e^{i(\frac{\xi}{a}-\frac{rb}{a})}d\xi dr \\ &= \tfrac{1}{(2\pi)^2}\int\int(\mathcal{F}_{\mathbf{R}\times\mathbf{R}}\tilde{f})(a\xi+br,r)e^{i\xi}d\xi dr \\ &= a\tfrac{1}{(2\pi)^2}\int\int|\xi|(\mathcal{F}_{\mathbf{R}\times\mathbf{R}}\tilde{f})((a+abr)\xi,ar\xi)e^{i\xi}d\xi dr, \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. We express the Fourier transform in terms of the Radon transform

$$\begin{aligned} \mathcal{F}_{\mathbf{R}\times\mathbf{R}}\tilde{f}((a+abr)\xi,ar\xi) &= \int\int^+ f(\tfrac{1}{t},b+\tfrac{1}{r}+\tfrac{x}{t})e^{iar\xi x}\tfrac{dt}{t}dx \\ &= \int\mathcal{R}f(b+\tfrac{1}{r},x)e^{iar\xi x}dx \\ &= \mathcal{F}_{\mathbf{R}}(\mathcal{R}f(b+\tfrac{1}{r},\cdot))(-ar\xi) \\ &= \tfrac{1}{a|r|}\mathcal{F}_{\mathbf{R}}(\mathcal{R}f(b+\tfrac{1}{r},-\tfrac{\cdot}{ar}))(\xi). \end{aligned}$$

This implies

$$\begin{aligned} f(a,b) &= \tfrac{1}{(2\pi)^2}\int\int|\xi|\mathcal{F}_{\mathbf{R}}(\mathcal{R}f(b+\tfrac{1}{r},-\tfrac{\cdot}{ar}))(\xi)e^{i\xi}d\xi\tfrac{dr}{|r|} \\ &= \tfrac{1}{2\pi}\int\int\mathcal{F}_{\mathbf{R}}(A(\mathcal{R}f)(b+\tfrac{1}{r},-\tfrac{\cdot}{ar}))(\xi)e^{i\xi}d\xi\tfrac{dr}{|r|} \\ &= \int\mathcal{F}_{\mathbf{R}}^{-1}\left(\mathcal{F}_{\mathbf{R}}(A(\mathcal{R}f)(b+\tfrac{1}{r},-\tfrac{\cdot}{ar}))\right)(1)\tfrac{dr}{|r|} \\ &= \int A(\mathcal{R}f)(b+\tfrac{1}{r},-\tfrac{\cdot}{ar})(1)\tfrac{dr}{|r|}. \end{aligned}$$

Note that the last integral in the above calculation is justified by this very calculation. It follows from the homogeneity properties of $\tilde{A}$ that

$$(\tilde{A}\mathcal{R}f(r,\frac{\cdot}{c}))(s) = \frac{1}{|c|}A\mathcal{R}f(r,\frac{s}{c}).$$

Thus

$$\begin{aligned} f(a,b) &= a^{-1}\int A\mathcal{R}f(b+\tfrac{1}{r},-\tfrac{1}{ar})\tfrac{dr}{r^2} \\ &= a^{-1}\int A\mathcal{R}f(b+r,-\tfrac{r}{a})dr \\ &= (\check{\mathcal{R}}\circ A\circ\mathcal{R}f)(a,b). \end{aligned}$$

The square root $\tilde{\Lambda} := \tilde{A}^{\frac{1}{2}}$ of A is given by $\sqrt{2\pi}\mathcal{F}_{\mathbf{R}}(\tilde{\Lambda}\psi)(s) = |s|^{\frac{1}{2}}\mathcal{F}_{\mathbf{R}}(\psi)(s)$ and we set $\Lambda\phi(r,s) := \tilde{\Lambda}(\phi(r,\cdot))(s)$.

Theorem.

1. *$f(a,b) = \check{\mathcal{R}}\Lambda^2\mathcal{R}f(a,b)$ for $f\in C_c^\infty(X)$.*
2. *$\Lambda\circ\mathcal{R}$ can be extended to an isometry $L^2(\mathbf{R}_+\times\mathbf{R},\frac{da}{a}db)\to L^2(\mathbf{R}\times\mathbf{R},drds)$.*
3. *$\check{\mathcal{R}}\Lambda$ is the adjoint of $\Lambda\mathcal{R}$ in the L^2-sense.*

Proof. The first part is immediate with the above proposition. The second part is shown by the following calculation

$$\begin{aligned} \int\int & |\Lambda\mathcal{R}f(r,s)|^2dsdr = \int\int|\tilde{\Lambda}(\mathcal{R}f(r,\cdot))(s)|^2dsdr \\ &= \tfrac{1}{2\pi}\int\int|\mathcal{F}_{\mathbf{R}}(\tilde{\Lambda}(\mathcal{R}f(r,\cdot)))(\xi)|^2d\xi dr \\ &= \tfrac{1}{(2\pi)^2}\int\int|\xi||\mathcal{F}_{\mathbf{R}}(\mathcal{R}f(r,\cdot))(\xi)|^2d\xi dr \\ &= \tfrac{1}{(2\pi)^2}\int\int|\xi||\mathcal{F}_{\mathbf{R}}(\mathcal{R}f(r,\cdot))(r^{-1}\xi)|^2|r|^{-2}d\xi dr \\ &= \tfrac{1}{(2\pi)^2}\int\int|\mathcal{F}_{\mathbf{R}\times\mathbf{R}}^{-1}\tilde{f}(\xi,r^{-1}\xi)|^2|r|^{-2}|\xi|d\xi dr \\ &= \tfrac{1}{(2\pi)^2}\int\int|\mathcal{F}_{\mathbf{R}\times\mathbf{R}}^{-1}\tilde{f}(r\xi,\xi)|^2|\xi|d\xi dr \\ &= \int\int|\tilde{f}(t,s)|^2dtds \\ &= \int\int^+\tfrac{1}{t^2}|f(t^{-1},-\tfrac{s}{t})|^2dtds \\ &= \int\int^+|f(t,s)|^2\tfrac{dt}{t}ds. \end{aligned}$$

For the last claim we only have to recall from Section 1 that $\mathcal{R}$ and $\check{\mathcal{R}}$ transposes of each other.

References

[1] Helgason, S., *Groups and geometric analysis*, Acad. Press, Orlando, 1984

Department of Mathematics, University of Wisconsin
Madison, Wisconsin 53706.