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A note on infinite string modules [1]

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String modules naturally occur in the classification of finite dimensional indecomposable representations over special biserial algebras (see [7], [1]). If such an algebra is of infinite representation type, then there exist also infinite strings. We associate two canonical representations with every infinite string and discuss their property of being indecomposable.

Throughout, k will be a fixed field. Let $Q = (Q_0, Q_1)$ be a quiver, that is a locally finite oriented graph with vertices Q_0 and arrows Q_1. We consider k-algebras Λ of the form $\Lambda = kQ/I$, where kQ denotes the path algebra of Q and the ideal is generated by a set of paths of length at least two. Suppose also that Λ is locally bounded, that is for each vertex x the length of all paths not contained in I and starting or ending in x is bounded by some $n_x \geq 0$. We are interested in the category $\operatorname{Mod}\Lambda$ of representations $M = (M(x), M(\alpha))_{x\in Q_0, \alpha\in Q_1}$ of Q given by vector spaces $M(x)$ and k-linear maps $M(\alpha): M(x) \to M(y)$ for each vertex x and each arrow $\alpha: x \to y$ in Q, which satisfy $M(\alpha_1)\dots M(\alpha_n) = 0$ for each path $\alpha_1 \dots \alpha_n$ in I.

Let $\tilde{Q} \to Q$ be the universal covering of Q, which induces a Galois covering $F: \tilde{\Lambda} = k\tilde{Q}/\tilde{I} \to \Lambda$ with Galois group $G = \Pi_1(Q, \cdot)$ (see [3]). There are several functors of interest. The restriction $F_.: \operatorname{Mod}\Lambda \to \operatorname{Mod}\tilde{\Lambda}$ is given by $F_.M(x) = M(Fx)$ for x either a vertex or an arrow in $\tilde{Q}$. This functor admit a left adjoint F_λ and a right adjoint $F_\rho: \operatorname{Mod}\tilde{\Lambda} \to \operatorname{Mod}\Lambda$. They are defined by $F_\lambda M(x) = \coprod_{Fy=x} M(y)$ and $F_\rho M(x) = \prod_{Fy=x} M(y)$ for x either a vertex or an arrow in Q.

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Now let S be a subquiver of $\tilde{Q}$ of the form $\mathbf{A}_n$, $\mathbf{A}_\infty$, or $\mathbf{A}_\infty^\infty$

$$\bullet \;\text{——}\; \bullet \;\text{——}\; \bullet \quad \cdots \quad \bullet \;\text{——}\; \bullet \qquad \mathbf{A}_n$$

$$\bullet \;\text{——}\; \bullet \;\text{——}\; \bullet \quad \cdots \qquad \mathbf{A}_\infty$$

$$\cdots \quad \bullet \;\text{——}\; \bullet \;\text{——}\; \bullet \quad \cdots \qquad \mathbf{A}_\infty^\infty$$

such that S contains no path of I. We call such a subquiver S a string over Λ. There is an obvious indecomposable representation $M = M(S)$ in $\operatorname{Mod}\tilde{\Lambda}$ associated with S. Put $M(x) = k$ and $M(\alpha) = \mathrm{id}_k$ for all vertices and arrows in S and let M be zero elsewhere.

We are interested in Λ-modules of the form $F_\lambda M(S)$ and $F_\rho M(S)$. Note that the class of finite dimensional modules of the form $F_\lambda M(S)$ and $F_\rho M(S)$ coincides with the class of string modules in the sense of Butler and Ringel (see [1]). Let us call a string S periodic, if $\operatorname{Stab} S = \{g \in G \mid S^g = S\} \neq \{1\}$. The modules of the form $F_\lambda M(S)$ where S is a periodic string, are of some particular interest. The cyclic group $\operatorname{Stab} S$ acts canonically on $F_\lambda M(S)$. Therefore we may think of $F_\lambda M(S)$ as an infinite dimensional band module corresponding to the band $S/\operatorname{Stab} S$ with the group algebra $k \operatorname{Stab} S = k[T, T^{-1}]$ sitting at each vertex. According to Dowbor and Skowroński any band module in the sense of Butler and Ringel has the form $F_\lambda M(S) \otimes_{k[T,T^{-1}]} V$ for some periodic string S and some finite dimensional indecomposable $k[T,T^{-1}]$-module V (see [2]).

Theorem *Let $\Lambda = kQ/I$ be a locally bounded k-algebra given by a quiver Q and a set of paths generating I. The modules of the form $F_\lambda M(S)$ are indecomposable.*

Remark Modules of the form $F_\rho M(S)$ may decompose. We provide examples which can be constructed from any periodic string, after the following proof.

Proof: We fix a module $F_\lambda M$ where $M = M(S)$. First suppose that $\operatorname{Stab} S = \{1\}$. We shall use the following argument due to Gabriel (see [3, 3.5]) to show that $F_\lambda M$ is indecomposable. Assume that $F_\lambda M = M_1 \coprod M_2$, where $M_1 \neq 0$. Then we have $\coprod_{g \in G} M^g \cong F.F_\lambda M = F.M_1 \coprod F.M_2$. The modules $M^g = M(S^g)$ for $g \in G$ are pairwise non-isomorphic and have local endomorphism rings isomorphic to k. We infer from a generalized version of the Krull-Schmidt theorem (see [6]), that $F.M_1 \cong \coprod_{h\in H} M^h$ for some subset $H \subseteq G$. We have also $F.M_1 = F.M_1^g \cong \coprod_{h\in H} M^{hg}$ for each $g \in G$. Therefore $H = Hg$ for each $g \in G$, hence $H = G$ and $M_2 = 0$.

Now let S be a periodic string. The cyclic group $\operatorname{Stab} S$ acts canonically on M and this action induces an embedding of the group algebra $K = k\operatorname{Stab} S$ into $\operatorname{End}_\Lambda(F_\lambda M)$. The endomorphisms of $F_\lambda M$ can be described like those of a finite dimensional band module corresponding to the band $S/\operatorname{Stab} S$ (see [4]). In particular the endomorphisms corresponding to non-cyclic admissable triples of $S/\operatorname{Stab} S$ in the sense of [4] form a nilpotent ideal E which is a complement of K in $\operatorname{End}_\Lambda(F_\lambda M)$. Therefore an idempotent $e = e^2$ in $\operatorname{End}_\Lambda(F_\lambda M) = K \coprod E$ can be written as $e = e_1 + e_2$ with $e_1 \in K$ and $e_2 \in E$, satisfying $e_1 = e_1^2$ and $e_2 = e_1e_2 + e_2e_1 + e_2^2$. It follows that either $e = 1$ or $e = 0$ since $K = k[T, T^{-1}]$ has only trivial idempotents and $e_2^n = 0$ for some $n \geq 0$. Thus we have shown the indecomposability of $F_\lambda M$.

Example 1 Let S be a periodic string over an algebra Λ and fix an arrow $\alpha: x \to y$ in S. Denote by S^+ the unique subquiver of type $\mathbf{A}_\infty$ containing x but not α. We show that $F_\rho M(S^+)$ is not indecomposable.

Let N be the normalizer of $\operatorname{Stab} S$ in G. We compose the covering F as $\tilde{\Lambda} \xrightarrow{F'} \tilde{\Lambda}/N \xrightarrow{F''} \Lambda$ where F' and F'' are Galois coverings with Galois groups N and G/N respectively. Consider $F'_\rho M(S^+)$ in $\operatorname{Mod} \tilde{\Lambda}/N$ via the restriction functor induced by $\Gamma = kS/\operatorname{Stab} S \to \tilde{\Lambda}/N$ as a module over the finite dimensional hereditary algebra Γ of type $\tilde{\mathbf{A}}_n$. Infinite dimensional modules over such algebras have been studied by Ringel in [5], and we use some of his arguments. In fact it is not hard to see that the submodule $F'_\lambda M(S^+)$ is a socalled Prüfer module (see [5, 4.5]). Therefore $F'_\rho M(S^+)$ is not torsionfree. Now recall that the indecomposable Γ-modules fall into three classes: finite length modules, Prüfer modules and torsionfree regular modules (see [5, 4.8]). Since Prüfer modules are denumerably generated, the Γ-module $F'_\rho M(S^+)$ must decompose. Clearly this implies a decomposition of $F_\rho M(S^+) = F''_\rho F'_\rho M(S^+)$.

Example 2 Let $\Lambda = kQ$ be the Kronecker algebra given by a quiver Q with two vertices and two parallel arrows. The modules occuring in the first example can be described as follows. There is a unique periodic string S over Λ and we obtain $F_\lambda M(S) = (k[T, T^{-1}], k[T, T^{-1}], \mathrm{id}_{k[T,T^{-1}]}, \cdot T)$, $F_\lambda M(S^+) = (k[T], k[T], \mathrm{id}_{k[T]}, d)$ or $F_\lambda M(S^+) = (k[T], k[T], d, \mathrm{id}_{k[T]})$, depending on the choice of S^+, where $d(\sum_{i\geq 0} \alpha_i T^i) = \sum_{i\geq 0} \alpha_{i-1} T^i$. For $F_\rho M$ simply replace $k[\ ]$ by $k[[\ ]]$.

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