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Endomorphisms of Words in a Quiver

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We extend the classical concept of a word in an alphabet to that of a word in a quiver. Then the endomorphisms for such a word are introduced. They form a monoid which provides some information about recurrence and periodicity of the fixed word. The properties of this monoid are used to show that for a typical class of indecomposable modules over a finite dimensional algebras of tame representation type, the class of their endomorphism rings is a very restricted one. © 1993 Academic Press, Inc.

1. INTRODUCTION

We present a purely combinatorial concept which turns out to be useful in the representation theory of finite dimensional algebras. First we extend the classical concept of a word in an alphabet (as discussed for instance in the book [L] of M. Lothaire) to that of a word in a quiver. Then the endomorphisms of such a word are defined. They form a monoid which provides some information about recurrence and periodicity of the fixed word.

A quiver Q is an oriented graph consisting of a set of vertices Q_0 and a set of arrows Q_1 such that to each arrow α in Q there are attached a starting vertex $s(\alpha)$ and a terminating vertex $t(\alpha)$. We add formal inverses α^{-1} for each arrow $\alpha \in Q_1$ with $(\alpha^{-1})^{-1} = \alpha$, $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. The set of formal inverses is denoted by Q_{-1}.

A sequence $w = w_1 w_2 \cdots w_n$ of arrows and formal inverses is called a word in Q of length $|w| = n$, if $w_{i+1} \neq w_i^{-1}$ and $s(w_{i+1}) = t(w_i)$ hold for each $i \in \{1, 2, ..., n-1\}$. The starting vertex and the terminating vertex of w are denoted by $s(w) = s(w_1)$ and $t(w) = t(w_n)$, respectively. Let $v = v_1 \cdots v_m$ be an additional word of length m in Q. The composite $vw = v_1 \cdots v_m w_1 \cdots w_n$ is defined by concatenating if this sequence is again a word in Q. In addition we need for each vertex x in Q the word e_x of length $|e_x| = 0$ with $s(e_x) = x = t(e_x)$. The composite $e_x w = w$ and $we_x = w$,

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respectively, for a word w is defined if $s(w)=x$ and $t(w)=x$, respectively, are satisfied. We denote by Q^* the set of all words in Q.

Consider for some word $a=a_1 \cdots a_n$ of length $n>0$

$$\sigma(a)=\begin{cases}1 & a_1 \in Q_1, \\ -1 & a_1 \in Q_{-1},\end{cases} \qquad \tau(a)=\begin{cases}1 & a_n \in Q_1, \\ -1 & a_n \in Q_{-1}\end{cases}$$

and $a^{-1}=a_n^{-1} \cdots a_1^{-1}$. Extend this for e_x by $\sigma(e_x)=\tau(e_x)=0$ and $e_x^{-1}=e_x$. We obtain factors, quotients and divisors of a word w as follows:

$$\mathrm{Fac}(w)=\{(x, a, y) \in Q^* \times Q^* \times Q^* \mid w=xay\},$$

$$\mathrm{Quot}(w)=\{(x, a, y) \in \mathrm{Fac}(w) \mid \tau(x) \leqslant 0, \sigma(y) \geqslant 0\} \text{ and}$$

$$\mathrm{Div}(w)=\{(x, a, y) \in \mathrm{Fac}(w) \mid \tau(x) \geqslant 0, \sigma(y) \leqslant 0\}.$$

We denote by $\pi(\alpha)=a$ the projection of a factor $\alpha=(x, a, y)$. Now we may introduce the set of endomorphisms of a word:

$$\mathrm{End}(w)=\{(\varphi_s, \varphi_t) \in \mathrm{Quot}(w) \times \mathrm{Div}(w) \mid \pi(\varphi_s)=\pi(\varphi_t) \text{ or } \pi(\varphi_s)=\pi(\varphi_t)^{-1}\} \cup \{0\}.$$

Together with the composition which will be defined in the next section the endomorphisms form a monoid.

Let k be a field. We state the main result on the k-algebra $k\,\mathrm{End}(w)$ generated by the monoid $\mathrm{End}(w)$. By the latter we mean the k-vector space with basis $\mathrm{End}(w) \setminus \{0\}$, endowed with the induced multiplication.

THEOREM 1. *Let w be a word in a quiver and let k be a field. Then the k-algebra $k\,\mathrm{End}(w)$ generated by the endomorphisms of w is local. For a factor algebra A of $k\,\mathrm{End}(w)$ which is generated by two elements and a natural number n the following hold:*

(a) *The dimension $\dim_k A/\mathrm{rad}^n A$ is bounded by $2n^2-2n+1$.*

(b) *If A and $k\langle x, y\rangle/(x, y)^n$ are isomorphic, then $n \leqslant 3$. Here $k\langle x, y\rangle$ denotes the free associative k-algebra with two generators.*

Let M be a monoid, $\mathrm{rad}\, M$ the subset of non-invertible elements, and $\mathrm{rad}^n M=(\mathrm{rad}\, M)^n$. We call M local if only the unit is invertible and if the set $\bigcap_{n \in \mathbb{N}} \mathrm{rad}^n M$ consists of precisely one element. The combinatorial version of the result goes as follows:

THEOREM 2. *Let w be a word in a quiver. Then the monoid $\mathrm{End}(w)$ of endomorphisms of w is local. For a factor monoid M of $\mathrm{End}(w)$ which is generated by two elements and a natural number n the following holds:*

(a) *The cardinality* $\mathrm{card}(M/\mathrm{rad}^n M)$ *is bounded by* $2n^2 - 2n + 2$.

(b) *If* M *and* $M\langle x, y\rangle/\mathrm{rad}^n M\langle x, y\rangle$ *are isomorphic, then* $n \leqslant 3$. *Here* $M\langle x, y\rangle$ *denotes the free monoid with two generators.*

The motivation to study $\mathrm{End}(w)$ comes from representation theory of algebras. Let k be a field. There is a canonical way to associate with each word w in a quiver Q a module $M(w)$ over the path algebra kQ (cf. 7.1). We now follow Wald an Waschbüsch as well as Crawley-Boevey, who showed that the set $\mathrm{End}(w)\setminus\{0\}$ forms a basis of the endomorphisms of $M(w)$ (cf. [WW, C]).

COROLLARY 1. *Let k be a field and let w be a word in a quiver Q. If $M(w)$ denotes the associated module over the path algebra kQ, then the endomorphism algebra of $M(w)$ and k* $\mathrm{End}(w)$ *are isomorphic:*

$$\mathrm{End}_{kQ}(M(w)) \cong k\,\mathrm{End}(w).$$

In particular, for a factor algebra A of $\mathrm{End}_{kQ}(M(w))$ *which is generated by two elements and a natural number n, the statements* (a) *and* (b) *of Theorem* 1 *hold.*

Modules of the form $M(w)$ occur in the classification of indecomposable modules over string algebras (see [BR]), therefore also in the case of special biserial algebras: Each indecomposable module over a string algebra Λ is either a string module or a band module. The string modules correspond to modules of the form $M(w)$ for certain words in the quiver Q_Λ of the algebra. The band modules only occur in homogeneous tubes of the Auslander-Reiten quiver Γ_Λ of Λ. String algebras are tame algebras (in the sense of [D]). Therefore we may conclude that an important class of tame algebras shows also a "tame" behaviour with respect to the endomorphism rings of their indecomposable modules. In contrast wild algebras behave accordingly "wild." For instance, Brenner has shown in [B] that for $\Lambda = k\langle x, y\rangle$ each finite dimensional k-algebra may be realized as the endomorphism algebra of some Λ-module.

We now give an outline of this paper. In Section 2 we complete the notation and show that $\mathrm{End}(w)$ is a local monoid and $k\,\mathrm{End}(w)$ is a local k-algebra. Since Theorem 1 treats factor algebras of $k\,\mathrm{End}(w)$ which are generated by two elements, we have to consider for pairs (α, β) of endomorphisms the generated submonoid $\langle\alpha, \beta\rangle \subseteq \mathrm{End}(w)$. In particular we are interested in some bound of

$$c_n(\alpha, \beta) = \mathrm{card}(\langle\alpha, \beta\rangle \setminus \mathrm{rad}^n\langle\alpha, \beta\rangle)$$

since $\dim_k A/\mathrm{rad}^n A \leqslant c_n(\alpha, \beta)$ holds for a factor algebra A which is generated by α and β.

As a first result we obtain in Section 3 a 13-dimensional algebra which is impossible as factor algebra of k End(w). Then there follows a distinction of endomorphism pairs between reducible and non-reducible pairs. Non-reducible pairs elude a unified treatment. Therefore three cases have to been considered separately.

With each reducible pair (α, β)—apart from some exceptions—is associated a simple pair (α_0, β_0) of transformations. In Section 4 we describe two operations r_n and r, which reduce a simple pair (α_0, β_0) successively to a minimal pair (α_1, β_1). Reversely one obtains from the knowledge of $\langle \alpha_1, \beta_1 \rangle$ inductively a precise description of $\langle a_0, \beta_0 \rangle$. In particular the inequality $c_n(\alpha_0, \beta_0) \leqslant n^2/2 + n/2$ holds.

In Section 5 we finish the proof of both theorems. A reducible pair (α, β) is considered as an extension $e(\alpha_0, \beta_0; i, j, p, q)$ of the pair (α_0, β_0) which is determined by four integral parameters. These parameters lead to a further distinction between strongly and weakly reducible pairs. We begin with the proof of part (a) of Theorem 1. For weakly reducible pairs the description of $\langle \alpha_0, \beta_0 \rangle$ in the previous section yields an approximation of $\langle \alpha, \beta \rangle$ and we obtain the estimate $c_n(\alpha, \beta) \leqslant 2n^2 - 2n + 1$. In the strongly reducible case such an estimate by a polynomial of degree 2 is impossible. Therefore we use another concept. To prove part (b) of Theorem 1 we combine several results and techniques which emerge from part (a).

Section 6 is devoted to three examples. They illustrate the proofs as well as the quality of the bounds in both theorems.

In Section 7 we discuss the application to representation theory which is condensed in the Corollary 1.

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2. The Monoid End(w)

Let Q be a quiver. *Words* in Q as well as their *factors* and *endomorphisms* were already defined in the introduction. We now complete the notation. In particular the composition in End(w) will be defined. As a first result we obtain a description of the powers α^r $(r \in \mathbb{N})$ of an endomorphism $\alpha \in$ End(w) and we show that End(w) is a local monoid.

2.1. Let w be a word in Q and let $\alpha = (a_1, a, a_2)$ and $\beta = (b_1, b, b_2)$, respectively, be factors of w. The set Fac(w) is *partially ordered* by

$$(a_1, a, a_2) \leqslant (b_1, b, b_2) \Leftrightarrow |a_i| \geqslant |b_i| \qquad \text{for} \quad i \in \{1, 2\}.$$

The *union* of α and β is defined by

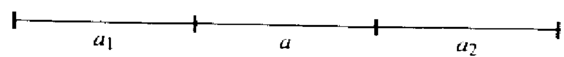
$$\alpha \cup \beta = \min\{\gamma \in \mathrm{Fac}(w) \mid \alpha \leqslant \gamma,\ \beta \leqslant \gamma\}.$$

The factors α and β are *connected* if

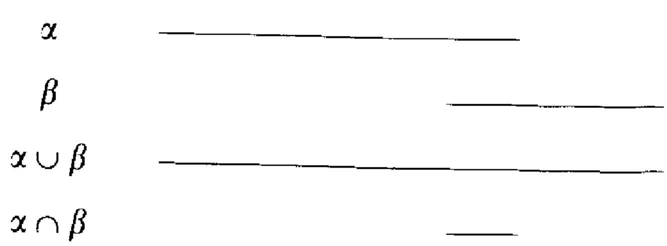
$$S = \{\gamma \in \mathrm{Fac}(w) \mid \gamma \leqslant \alpha,\ \gamma \leqslant \beta\} \neq \varnothing.$$

In the latter case $\alpha \cap \beta = \max S$ denotes the *intersection* of α and β.

A factor $\alpha = (a_1, a, a_2)$ of w may be visualised by the following diagram:



The line corresponds to w and the partition into three parts reflects the length of the words a_1, a and a_2 in Q^*. To compare different factors it usually suffices to present the projections according to their relative position:



Let v be an additional word in Q and suppose $\gamma = (c_1, c, c_2) \in \mathrm{Fac}(v)$. If $v = \pi(\beta)$, then the *composition* of γ and β is defined as follows:

$$\gamma * \beta = (b_1 c_1, c, c_2 b_2).$$

It is obvious that $\alpha \leqslant \beta$ holds if and only if there exists a factor $\alpha_\beta \in \mathrm{Fac}(\pi(\beta))$ (uniquely determined by α and β) such that $\alpha = \alpha_\beta * \beta$.

For a factor α the *length* is defined by $|\pi(\alpha)|$ and we also use $\alpha^{-1} = (a_2^{-1}, a^{-1}, a_1^{-1}) \in \mathrm{Fac}(w^{-1})$.

2.2. For $w \in Q^*$ let

$$\mathrm{Trans}(w) = \{(\varphi_s, \varphi_t) \in \mathrm{Fac}(w) \times \mathrm{Fac}(w) \mid \pi(\varphi_s) = \pi(\varphi_t) \text{ or } \pi(\varphi_s) = \pi(\varphi_t)^{-1}\} \cup \{0\}$$

be the set of *transformations* of w. The endomorphisms of w form a subset of $\mathrm{Trans}(w)$. We introduce the following notions for a transformation $\varphi = (\varphi_s, \varphi_t)$:

The *signum* of φ is $\operatorname{sgn}(\varphi)=\begin{cases}1 & \pi(\varphi_s)=\pi(\varphi_t),\\ -1 & \text{else.}\end{cases}$

The *rank* of φ is $\operatorname{rk}(\varphi)=|\varphi_s|$.

The *support* of φ is $\operatorname{supp}(\varphi)=\varphi_s\cup\varphi_t$.

The *shift* of φ is $|\varphi|=|x'|-|x|$ and $\|\varphi\|=|\,|x'|-|x|\,|$, respectively, if $\varphi_s=(x, a, y)$ and $\varphi_t=(x', a', y)$.

The *image* of $\alpha\in\operatorname{Fac}(w)$ is $\alpha\varphi=(\alpha_{\varphi_s})^{\operatorname{sgn}(\varphi)}*\varphi_t$, if $\alpha\leqslant\varphi_s$.

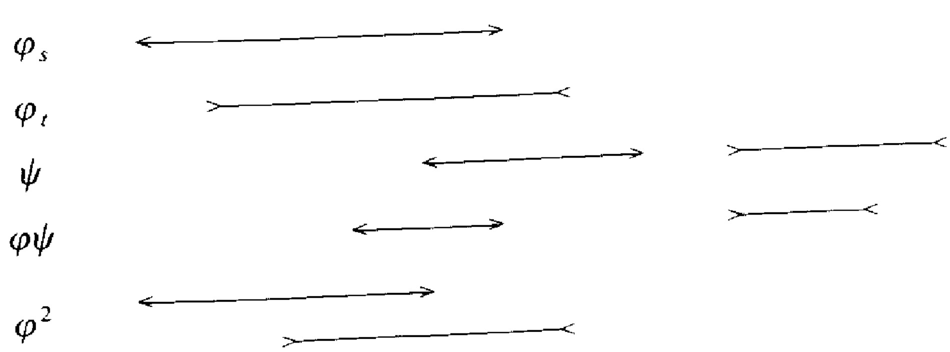
The *preimage* of $\alpha\in\operatorname{Fac}(w)$ is $\alpha\varphi^{-1}=(\alpha_{\varphi_t})^{\operatorname{sgn}(\varphi)}*\varphi_s$, if $\alpha\leqslant\varphi_t$.

The composition of two transformations $\varphi, \psi\in\operatorname{Trans}(w)$ is defined as follows:

$$\varphi\psi=\begin{cases}(\alpha\varphi^{-1}, \alpha\psi) & \varphi=(\varphi_s, \varphi_t), \psi=(\psi_s, \psi_t) \text{ and } \alpha=\varphi_t\cap\psi_s \text{ exists,}\\ 0 & \text{else.}\end{cases}$$

The sets $\operatorname{Trans}(w)$ and $\operatorname{End}(w)$ are closed under the composition which is obviously associative.

The following diagrams illustrate the composition, assuming that $\operatorname{sgn}(\varphi)=1=\operatorname{sgn}(\psi)$. The marks at the ends of each line help to distinguish between quotients and divisors:



Note that the partial order on $\operatorname{Fac}(w)$ induces one on the transformations of w (cf. 5.1). With respect to this partial order the endomorphisms are maximal.

2.3. Lemma. *Let* $a=a_1\cdots a_n\in Q^*$ *with* $n=|a|$ *and let* $p\in\mathbf{N}$. *The following are equivalent*:

(i) *There exist* $x_1, x_2\in Q^*$ *and* $r\in\mathbf{N}$ *such that* $a=(x_1x_2)^r x_1$ *and* $|x_1x_2|=p$.

(ii) *It is* $p\leqslant n$ *and* $a_{i+p}=a_i$ *for* $1\leqslant i\leqslant n-p$.

(iii) *There exist* $b, x, y\in Q^*$ *such that* $a=xby$, $xb=by$ *and* $|x|=|y|=p$.

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are clear. We show (iii) ⇒ (i) by induction on n. First assume $|b| \leqslant |x|$. Take $x_1 = b$ and $x_2 \in Q^*$ such that $x = x_1 x_2$. From $xb = by$ follows $x_1 x_2 x_1 = xb = by = x_1 y$ which implies $y = x_2 x_1$. Therefore $a = xby = (x_1 x_2)^2 x_1$ and the assertion is shown. Now assume $|b| > |x|$. From $xb = by$ follows $b = xb'$ for some $b' \in Q^*$. Therefore $xxb' = xb = by = xb'y$, i.e. $xb' = b'y$. By induction $a' = xb'y$ has the form $a' = (x_1 x_2)^r x_1$ with $x = x_1 x_2$. Since $a = xa' = (x_1 x_2)^{r+1} x_1$ the proof is complete.

A word a in Q is called *p-periodic* if it satisfies one of the equivalent properties (i)–(iii). A factor α of a word in Q is *p-periodic* if the projection $\pi(\alpha)$ is p-periodic.

2.4. LEMMA. *Let $a \in Q^*$ and $p, q \in \mathbf{N}$ such that $|a| \geqslant p + q$. If a is p- and q-periodic, then a is a* gcd(p, q)*-periodic word.*

Proof. We may assume $p \geqslant q$. Then a is of the form $a = xa'$ with $|x| = p$ and $x = (x_1 x_2)^r x_1$ with $|x_1 x_2| = q$. The assumption $|a| \geqslant p + q$ and the p-periodicity of a imply $a = x x_1 x_2 a''$ for some a''. Applying the q-periodicity gives $a = (x_1 x_2)^r x_1 x_2 x_1 a''$. Therefore one obtains $x_1 x_2 = x_2 x_1$ and the assertion is an immediate consequence of the following lemma.

LEMMA. *Let $a, b \in Q^*$ such that $ab = ba$. Then there exist $x \in Q^*$ and $r, s \in \mathbf{N}$ such that $a = x^r$, $b = x^s$.*

Proof. We may assume $|a| \geqslant |b|$ and use induction on $|a|$. Since $ab = ba$ there exists $a' \in Q^*$ with $a = ba'$. This implies $ba'b = ab = ba = bba'$ and therefore $a'b = ba'$ holds. By induction one gets $a' = x^r$ and $b = x^s$. The assertion follows from $a = ba' = x^{r+s}$.

2.5. LEMMA. *Let $\alpha \in$ End(w) and suppose $\alpha \neq 1$, $\alpha^2 \neq 0$.*

(a) *It is* $\mathrm{sgn}(\alpha) = 1$.

(b) *Suppose $r \in \mathbf{N}$ and $\alpha^r \neq 0$. Then* $\mathrm{supp}(\alpha^r) = \mathrm{supp}(\alpha)$, $\mathrm{rk}(\alpha^r) = \mathrm{rk}(\alpha) - (r-1)\,\|\alpha\|$ *and* $|\alpha^r| = r\,|\alpha|$.

(c) *The support* $\mathrm{supp}(\alpha)$ *of α is $\|\alpha\|$-periodic and* $|\mathrm{supp}(\alpha)| = \mathrm{rk}(\alpha) + \|\alpha\|$ *holds.*

(d) *Let $b \in \mathbf{N}$ such that* $\mathrm{supp}(\alpha)$ *is b-periodic with $\|\alpha\| = rb$ for some $r \in \mathbf{N}$. Then there exists $\beta \in$* End(w) *which is uniquely determined by α and b such that $\alpha = \beta^r$.*

Proof. Fix $\alpha_s = (x, a, y)$, $\alpha_t = (x', a', y')$ and assume without any restriction $|\alpha| \geqslant 0$.

(a) Suppose sgn(α) is negative, i.e. $a' = a^{-1}$. That implies $(\alpha_t \cap \alpha_s)^{-1} = \alpha_t \cap \alpha_s$, and therefore $\alpha^2 = (\alpha_t \cap \alpha_s,\ \alpha_t \cap \alpha_s)$. We conclude $\alpha = 1$ since $\mathrm{Quot}(w) \cap \mathrm{Div}(w) = \{(1, w, 1)\}$. Contradiction.

(b) Suppose $a = ba_1 = a_2 b'$ with $|a_i| = (r-1)\|\alpha\|$. The definition of the composition in End(w) combined with part (a) yields immediately that $\alpha^r = (\beta_s, \beta_t)$ is of the form $\beta_s = (x, b, a_1 y)$ and $\beta_t = (x'a_2, b', y')$. The assertion follows from that description.

(c) According to part (a) $a' = a$ holds. Since $|\alpha| \geqslant 0$ there are $x'', y'' \in Q^*$ with $x' = xx''$ and $y = y''y'$. Thus $\pi(\mathrm{supp}(\alpha)) = ay'' = x''a$ with $|y'| = \|\alpha\| = |x''|$ and the property (iii) in Lemma 1.3 is satisfied. Since also $|\mathrm{supp}(\alpha)| = |a| + |y''|$ part (c) is complete.

(d) According to part (c) α has the form $\alpha_s = (x, a, y''y')$, $\alpha_t = (xx'', a, y')$. Applying the b-periodicity of supp(α) there are $\bar{b}, \bar{x}, \bar{y} \in Q^*$ with $\bar{b}\bar{y} = ay'' = x''a = \bar{x}\bar{b}$ and $|\bar{y}| = b = |\bar{x}|$. We obtain β by $\beta_s = (x, \bar{b}, \bar{y}y')$, $\beta_t = (x\bar{x}, \bar{b}, y')$.

2.6. Let M be a *monoid*, i.e., a set endowed with an associative multiplication and a unit $1 \in M$. Let $1 \neq x_0 \in M$ be an element such that $x_0 x = x_0 = x x_0$ holds for all $x \in M$. Then x_0 is called the *zero element* of M and is denoted by 0. Let the *radical* rad M of M be the set of non-invertible elements of M. One defines inductively $\mathrm{rad}^{n+1} M = \mathrm{rad}^n M \,\mathrm{rad}\, M$ and obtains the following descending chain of ideals in M:

$$M = \mathrm{rad}^0 M \supseteq \mathrm{rad}^1 M \supseteq \mathrm{rad}^2 M \supseteq \cdots.$$

For $x \in M$

$$l(x) = \begin{cases} \infty & x \in \bigcap_{n \in \mathbf{N}} \mathrm{rad}^n M, \\ \max\{n \in \mathbf{N}_0 \mid x \in \mathrm{rad}^n M\} & \text{else} \end{cases}$$

denotes the *length* of x in M. The monoid M is called *local* if $M = \mathrm{rad}\, M \cup \{1\}$ holds and if the set $\bigcap_{n \in \mathbf{N}} \mathrm{rad}^n M$ has precisely one element. For a subset $X \subseteq M$ we denote by $\langle X \rangle$ the smallest submonoid of M containing X.

LEMMA. *Let M be a finite monoid and suppose that for all $x \in M \setminus \{1\}$ the set $\{n \in M \mid x^n \neq 0\}$ is finite. Then M is local.*

Proof. Obviously 1 is the only invertible element. We claim that $\mathrm{rad}^{n+1} M = \mathrm{rad}^n M$ implies $\mathrm{rad}^n M = 0$. The assertion would be an immediate consequence since M is finite. Therefore suppose $\mathrm{rad}^n M = \mathrm{rad}^n M \,\mathrm{rad}\, M$ and choose a minimal $X \subseteq \mathrm{rad}^n M$ such that $\mathrm{rad}^n M = X \,\mathrm{rad}\, M$. Let $x_0 \in X$. There is $x_1 \in X$ and $x \in \mathrm{rad}\, M$ such that $x_0 = x_1 x$. The minimality of X implies $x_0 = x_1$ and therefore $x_0 = x_0 x^n$ for all $n \in \mathbf{N}$. According to the assumption we obtain $x_0 = 0$. Thus $\operatorname{rad}^n M = 0$ holds and the proof is complete.

Now we combine the previous lemma with Lemma 2.5(b).

PROPOSITION. *Let w be a word in a quiver. Then* $\operatorname{End}(w)$ *is a local monoid.*

2.7. Let M be a monoid with 0 and let $H \subseteq M$ be closed under multiplication. Given a field k, the *k-algebra generated by H* is by definition the k-vector space $kH = \bigoplus_{x \in H \setminus \{0\}} kx$ endowed with the induced multiplication; i.e., base elements are multiplied as in M, identifying zero in M and kH.

LEMMA. *Let M be a local monoid.*

(a) *The k-algebra kM is local with radical* $\operatorname{rad}^n kM = k \operatorname{rad}^n M$, $n \in \mathbf{N}$.

(b) *Let A be a k-algebra with* $\operatorname{rad}^n A = 0$ *and let $\varphi : kM \to A$ be a surjective homomorphism. Then there is a subset* $X \subseteq \operatorname{rad} M \setminus \operatorname{rad}^2 M$ *such that* $\{\varphi(x) + \operatorname{rad}^2 A \mid x \in X\}$ *forms a k-basis of* $\operatorname{rad} A / \operatorname{rad}^2 A$. *The radical* $\operatorname{rad}^r A$, $r \in \mathbf{N}_0$ *is generated over k by* $\{\varphi(x) \mid x \in \operatorname{rad}^r \langle X \rangle\}$. *In particular* $\{\varphi(x) \mid x \in \langle X \rangle \setminus \operatorname{rad}^n \langle X \rangle\}$ *generates A over k.*

The proof is straightforward.

3. NON-REDUCIBLE PAIRS OF ENDOMORPHISMS

Throughout the next sections we consider pairs α and β of endomorphisms since Theorem 1 treats factor algebras of $k \operatorname{End}(w)$ generated by two elements. Applying Lemma 2.7 together with Proposition 2.6 we reduce Theorem 1 to statements about the submonoid $\langle \alpha, \beta \rangle \subseteq \operatorname{End}(w)$ generated by α and β. As a first result we obtain in Section 3.4 a 13-dimensional algebra which cannot be realised as a factor of $k \operatorname{End}(w)$.

3.1. Let $\alpha, \beta \in \operatorname{End}(w)$. The pair (α, β) is called *reducible* if

(i) $\alpha^2 \neq 0$, $\alpha\beta\alpha \neq 0$, $\operatorname{sgn}(\beta) = 1$, $|\alpha|\ |\beta| < 0$, and

(ii) $\|\beta\| \geqslant \|\alpha\|$, if $\beta^2 \neq 0$.

The purpose of this section is to prove the following proposition. We collect all cases which elude a unified treatment in the context of simple and reducible pairs (cf. Sections 4 and 5).

PROPOSITION. *Let* $\alpha, \beta \in \mathrm{rad}\setminus\mathrm{rad}^2 \operatorname{End}(w)$ *such that neither* (α, β) *nor* (β, α) *is reducible. Then*

$$\operatorname{card}(\langle \alpha, \beta\rangle \setminus \mathrm{rad}^n\langle \alpha, \beta\rangle) \leqslant 2n^2 - 2n + 1.$$

holds for all $n \in \mathbf{N}$.

3.2. LEMMA. *Let* $\alpha, \beta \in \mathrm{rad} \operatorname{End}(w)$ *with* $\alpha^2 \neq 0$ *and* $\beta^2 \neq 0$.

(a) *Suppose* $|\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)| \geqslant \|\alpha\| + \|\beta\|$. *Then* $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$.

(b) *Suppose* $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$. *Then there exist* $\gamma \in \operatorname{End}(w)$ *and* $r, s \in \mathbf{N}$ *such that* $\alpha = \gamma^r$, $\beta = \gamma^s$.

Proof. (a) By Lemma 2.4 and Lemma 2.5 $\operatorname{supp}(\alpha)$ and $\operatorname{supp}(\beta)$ are both $\|\alpha\|$- and $\|\beta\|$-periodic. Now assume $\operatorname{supp}(\alpha) \neq \operatorname{supp}(\beta)$, say $s = \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta) \neq \operatorname{supp}(\alpha)$. Then there is $\gamma = (x, c, y) \in \operatorname{Fac}(\pi(\operatorname{supp}(\alpha)))$ such that $s = \gamma * \operatorname{supp}(\alpha)$ with $|x| > 0$ or $|y| > 0$. Without any restriction assume $|x| > 0$. By assumption $|c| = |s| \geqslant \|\beta\|$, i.e., c is of the form $c = c'c''$ with $|c'| = \|\beta\|$. Since $\beta \in \operatorname{End}(w)$ we conclude $\tau(x) \neq \tau(xc')$. But that contradicts the $\|\beta\|$-periodicity of $xcy = \pi(\operatorname{supp}(\alpha))$ and $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$ is shown.

(b) Let $c = \gcd(\|\alpha\|, \|\beta\|)$. We infer from Lemma 2.4 and Lemma 2.5 that $\operatorname{supp}(\alpha)$ is c-periodic. Suppose $\|\alpha\| = cr$ and $\|\beta\| = cs$. Using Lemma 2.5(d) there exists $\gamma \in \operatorname{End}(w)$ such that $\alpha = \gamma^r$ and $\beta = \gamma^s$.

3.3. An endomorphism $\alpha \in \operatorname{End}(w)$ is called *primitive* if $\alpha = \beta^n$ for some $\beta \in \operatorname{End}(w)$ and $n \in \mathbf{N}$ implies $n = 1$.

LEMMA. *Let* $\alpha, \beta \in \operatorname{End}(w)$ *be primitive and suppose* $\alpha^m = \beta^n \neq 0$ *for a pair* $m, n \in \mathbf{N}$. *Then* $\alpha = \beta$ *and* $m = n$ *hold.*

Proof. Combine Lemma 2.5(b) and Lemma 3.2(b) with the previous definition.

This lemma motivates us to call $\alpha, \beta \in \operatorname{End}(w)$ *equivalent* if $\alpha = \beta = 0$ or if there are $\gamma \in \operatorname{End}(w)$ and $r, s \in \mathbf{N}$ such that $\alpha = \gamma^r \neq 0$ and $\beta = \gamma^s \neq 0$.

3.4. PROPOSITION. *Let* $\alpha, \beta \in \mathrm{rad} \operatorname{End}(w)$ *with* $\alpha^2 \neq 0$, $\beta^2 \neq 0$, *and* $\|\beta\| \geqslant \|\alpha\|$. *Then* α *and* β *are equivalent or* $\alpha\beta^n\alpha = 0$ *for all* $n \geqslant 2$.

Proof. Assume $\alpha\beta^n\alpha \neq 0$. Choose $\gamma \leqslant (\alpha\beta^n\alpha)_s$ with $|\gamma| = 0$. Then $\gamma\alpha \leqslant \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)$ and $\gamma\alpha\beta^n \leqslant \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)$. Therefore

$$\begin{aligned}|\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)| &\geqslant |\gamma\alpha \cup \gamma\alpha\beta^n| = \|\beta^n\| = n\, \|\beta\| \\ &\geqslant 2\, \|\beta\| \geqslant \|\alpha\| + \|\beta\|\end{aligned}$$

holds for $n \geqslant 2$ and α and β are both equivalent by Lemma 3.2.

COROLLARY. *Let* $k\langle x, y\rangle$ *be the free associative k-algebra in two generators and let A be the following factor algebra*:

$$A = k\langle x, y\rangle / I \qquad \textit{with} \quad I = (x^3, y^3, xyx, yxy) + (x, y)^5.$$

For a word w in a quiver there is no surjective homomorphism from k End(w) *to A*.

Proof. Assume there is a surjective homomorphism $\varphi : k\,\mathrm{End}(w) \to A$. By Lemma 2.7 there exist $\alpha, \beta \in \mathrm{rad}\,\mathrm{End}(w) \backslash \mathrm{rad}^2\,\mathrm{End}(w)$ such that $\{\varphi(\alpha) + \mathrm{rad}^2 A, \varphi(\beta) + \mathrm{rad}^2 A\}$ forms a basis of $\mathrm{rad}\,A/\mathrm{rad}^2 A$. Now $\mathrm{rad}^4 A$ is generated by $\{\varphi(\alpha\beta^2\alpha), \varphi(\beta\alpha^2\beta)\}$ over k and is 2-dimensional. On the other hand $\alpha\beta^2\alpha = 0$ or $\beta\alpha^2\beta = 0$. This contradiction finishes the proof.

3.5. LEMMA. *Let* $\alpha, \beta \in \mathrm{rad}\,\mathrm{End}(w)$ *with* $\alpha^2 \neq 0$. *Suppose* α *and* β *are not equivalent and* $\mathrm{supp}(\beta) \leqslant \mathrm{supp}(\alpha)$. *Then* $\mathrm{rk}(\beta) < \|\alpha\|$.

Proof. We consider two cases.

1. $\beta^2 \neq 0$. Applying Lemma 2.5(c) which says

$$|\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)| = |\mathrm{supp}(\beta)| = \|\beta\| + \mathrm{rk}(\beta),$$

the assertion is an immediate consequence of Lemma 3.2.

2. $\beta^2 = 0$. Fix $\beta_s = (x, b, y)$, $\beta_t = (x', b', y')$ and we may assume $|\beta| > 0$. Suppose $\mathrm{rk}(\beta) \geqslant \|\alpha\|$. Since $\beta^2 = 0$ we know $|\beta| > \mathrm{rk}(\beta)$. Thus there is $n \in \mathbf{N}$ such that $|\beta| = n\,\|\alpha\| + r$ and $0 < r \leqslant \|\alpha\|$. Suppose $y = y_1 y_2$ and $x' = x'_2 x'_1$ with $|y_1| = n\,\|\alpha\| = |x'_1|$. Then we obtain an endomorphism $\gamma = (\gamma_s, \gamma_t)$ by $\gamma_s = (x, by_1, y_2)$ and $\gamma_t = (x'_2, x'_1 b', y')$ since $\mathrm{supp}(\alpha)$ is $\|\alpha\|$-periodic. According to this construction $\mathrm{rk}(\gamma) \geqslant \|\alpha\|$, $\mathrm{supp}(\gamma) \leqslant \mathrm{supp}(\alpha)$ and $\gamma^2 \neq 0$. Applying the first case α and γ are equivalent. Since $\beta = \gamma^n$ also α and β are equivalent. This contradicts the assumption and the lemma is proven.

3.6. LEMMA. *Let* $\alpha, \beta \in \mathrm{End}(w)$ *and suppose* $\beta^2 = 0$ *and* $|\alpha|\,|\beta| > 0$. *Then* $\beta\alpha^n\beta = 0$ *hols for all* $n \in \mathbf{N}$.

Proof. Let $n \in \mathbf{N}$ and $x = \beta_t \cap (\alpha^n)_s$. $\beta^2 = 0$ implies that β_t and β_s are not connected. Since $|\alpha|\,|\beta| > 0$ also $x\alpha^n$ and β_s are not connected. Therefore $\beta\alpha^n\beta = 0$ is shown.

3.7. LEMMA. *Let* $\alpha, \beta \in \mathrm{End}(w)$ *be non-equivalent and suppose* $\alpha\beta\alpha^n \neq 0$ *for some* $n \in \mathbf{N}$. *Then* $\varphi\alpha\beta\alpha \neq 0$ *for some* $\varphi \in \mathrm{End}(w)$ *implies* $\varphi\alpha\beta\alpha^{n-1} \neq 0$.

Proof. We show $(\alpha\beta\alpha)_s = (\alpha\beta\alpha^{n-1})_s$. Then the assertion is an obvious consequence. Start with $u = (\alpha\beta)_t \cap \alpha_s$. According to Lemma 3.5 $|u| =$

$\mathrm{rk}(\alpha\beta\alpha) < \|\alpha\|$. The description of α^{n-1} and α^n in Lemma 2.5 yields $u \leqslant (\alpha^{n-1})_s$ since u and $(\alpha^n)_s$ are connected by assumption. Therefore $(\alpha\beta\alpha)_s = (\alpha\beta\alpha^{n-1})_s$ holds.

3.8. LEMMA. *Let* $\alpha, \beta \in \mathrm{End}(w)$ *with* $\mathrm{sgn}(\alpha) = 1$ *and* $\mathrm{sgn}(\beta) = -1$. *Then* $\alpha\beta\alpha^n\beta\alpha = 0$ *holds for all* $n \in \mathbf{N}$.

Proof. Let $n \in \mathbf{N}$ and assume $\alpha\beta\alpha^n\beta\alpha \neq 0$. According to Lemma 3.7 $\gamma^2 = \alpha\beta\alpha^n\beta\alpha^{n-1} \neq 0$ with $\gamma = \alpha\beta\alpha^{n-1}$. Applying the assumptions about α and β we obtain $\mathrm{sgn}(\gamma) = -1$ or $\mathrm{rk}(\gamma) = 0$. Both contradict $\gamma^2 \neq 0$ by Lemma 2.5. Therefore $\alpha\beta\alpha^n\beta\alpha = 0$ holds.

3.9. Given two factors $\alpha = (a_1, a, a_2)$ and $\beta = (b_1, b, b_2)$ in $\mathrm{Fac}(w)$, we write $\alpha \ll \beta$ if $|\alpha_i| > |b_i|$ for $i \in \{1, 2\}$.

LEMMA. *Let* $\alpha, \beta \in \mathrm{End}(w)$. *If* $\alpha_t \ll \beta_t$ $(\alpha_s \ll \beta_t)$ *then there exists* $\alpha' \in \mathrm{End}(w)$ *such that* $\alpha = \alpha'\beta$ $(\alpha = \beta\alpha')$.

Proof. Choose $\alpha' = (\alpha_s, \alpha_t\beta^{-1})$ or $\alpha' = (\alpha_s\beta, \alpha_t)$, respectively.

3.10. LEMMA. *Let* $\alpha, \beta \in \mathrm{rad}\,\mathrm{End}(w)$ *be non-equivalent such that* $\alpha^2 \neq 0$, $\beta^2 \neq 0$ *and* $\|\beta\| \geqslant \|\alpha\|$. *In addition suppose* $\alpha\beta\alpha = 0$, *if* $|\alpha|\,|\beta| < 0$.

(a) *If* $\mathrm{supp}(\alpha) \neq \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$, *then* $\alpha\beta = 0$ *or* $\beta\alpha = 0$.

(b) *If* $\mathrm{supp}(\alpha) = \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)$, *then* $\alpha\beta\alpha = 0$. *Moreover if* $\alpha \notin \mathrm{rad}^2\,\mathrm{End}(w)$, *then* $\alpha\beta = 0$ *or* $\beta\alpha = 0$ *or* $\alpha\beta^2 = \beta^2\alpha = 0$.

Proof. We may assume $|\alpha| > 0$.

(a) Take $\gamma = (c_1, c, c_2) \in \mathrm{Fac}(\pi(\mathrm{supp}(\alpha)))$ with $\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \gamma * \mathrm{supp}(\alpha)$. The assumption implies $|c_1| > 0$ or $|c_2| > 0$. Let us consider the case $|c_1| > 0$, the second case being analogous.

1. $|\alpha|\,|\beta| > 0$. Suppose $\beta\alpha \neq 0$. Then, by definition, β_t and α_s are connected and $|\alpha|\,|\beta| > 0$ yields

$$|\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta)| = \|\beta\| + |\beta_t \cap \alpha_s| + \|\alpha\| \geqslant \|\beta\| + \|\alpha\|.$$

By Lemma 3.2 α and β are equivalent. Contradiction. Therefore $\beta\alpha = 0$ is shown.

2. $|\alpha|\,|\beta| < 0$. Suppose $\alpha\beta \neq 0$. Using the assumptions $|\alpha|\,|\beta| < 0$, $\|\beta\| \geqslant \|\alpha\|$ and $|c_1| \neq 0$ we obtain $(\alpha\beta)_t \leqslant \alpha_s$. That implies $\alpha\beta\alpha \neq 0$. Contradiction. Thus $\alpha\beta = 0$ is shown.

(b) Fix $\gamma = (c_1, c, c_2) \in \mathrm{Fac}(\pi(\mathrm{supp}(\beta)))$ with $\mathrm{supp}(\alpha) = \gamma * \mathrm{supp}(\beta)$.

1. $|\alpha|\,|\beta| > 0$. Suppose $\alpha\beta\alpha \neq 0$. Then $|\alpha|\,|\beta| > 0$ immediately

implies $|\text{supp}(\alpha) \cap \text{supp}(\beta)| \geqslant \|\alpha\| + \|\beta\| + \|\alpha\|$. Again by Lemma 3.2 α and β are equivalent and that contradiction finishes the proof of $\alpha\beta\alpha = 0$. To prove the second part first suppose $|c_1| = 0$ or $|c_2| = 0$. Since $\text{rk}(\alpha) < \|\beta\|$ holds by Lemma 3.5 β_t and α_s or β_s and α_t respectively are not connected. Therefore $\beta\alpha = 0$ or $\alpha\beta = 0$. Now suppose $|c_1| \neq 0$ and $|c_2| \neq 0$. Using $|c_1| \neq 0$ and the assumption $\alpha \notin \text{rad}^2 \text{End}(w)$ combined with Lemma 3.9 one obtains $|c_2| + \|\alpha\| \leqslant \|\beta\|$. Now from $\text{rk}(\alpha) < \|\beta\|$ follows $\beta^2\alpha = 0$. We use an analogous argument for $\alpha\beta^2 = 0$.

2. $|\alpha|\ |\beta| < 0$. By assumption only the second part has to be shown. First observe that $\alpha\beta\alpha = 0$ already implies $|\text{supp}(\alpha)| < \|\beta\|$. The condition $\alpha \notin \text{rad}^2 \text{End}(w)$ combined with Lemma 3.9 yields $|c_i| \leqslant \|\beta\|$ for $i \in \{1, 2\}$. From $|c_1| \leqslant \|\beta\|$ and $|\text{supp}(\alpha)| < \|\beta\|$ follows $\alpha\beta^2 = 0$. Analogously one obtains $\beta^2\alpha = 0$.

3.11. The following proposition combines the results of 3.6, 3.8, and 3.10.

PROPOSITION. *Let* $\alpha, \beta \in \text{rad End}(w) \setminus \text{rad}^2 \text{End}(w)$ *be two different endomorphisms such that neither* (α, β) *nor* (β, α) *is reducible. Then one of the following cases holds*:

(i) $\alpha^2 = \beta^2 = 0$.

(ii) $\alpha\beta = 0$.

(ii′) $\beta\alpha = 0$.

(iii) $\beta^2 = \alpha\beta\alpha^n\beta\alpha = 0$ *for all* $n \in \mathbf{N}$.

(iii′) $\alpha^2 = \beta\alpha\beta^n\alpha\beta = 0$ *for all* $n \in \mathbf{N}$.

(iv) $\alpha\beta\alpha = \alpha\beta^2 = \beta^2\alpha = 0$.

(iv′) $\beta\alpha\beta = \beta\alpha^2 = \alpha^2\beta = 0$.

3.12. LEMMA. *Let* $M = M\langle x, y\rangle$ *be the free monoid with generators* x *and* y. *Suppose the ideal* $I \subseteq M$ *is generated by one of the following sets*:

$$R_1 = \{x^2, y^2\},$$

$$R_2 = \{xyx, xy^2, y^2x\},$$

$$R_3 = \{y^2\} \cup \{xyx^ryx \mid r \in \mathbf{N}\},$$

$$R_4 = \{xy^rx \mid r \in \mathbf{N}\} \cup \{yx^{r+1}y \mid r \in \mathbf{N}\}.$$

Then $\text{card}(M/I \setminus \text{rad}^n M/I) \leqslant 2n^2 - 2n + 1$ *for all* $n \in \mathbf{N}$.

Proof. Let $I \subseteq M$ be an ideal and $n \in \mathbf{N}$. Then

$$M/I \setminus \text{rad}^n M/I = \{\bar{x} \in M/I \mid x \in M \setminus (I \cup \text{rad}^n M)\}.$$

We consider for each R_j the generated ideal I_j $(1 \leqslant j \leqslant 4)$ and write down the elements of $M_j = M \setminus (I_j \cup \text{rad}^n M)$. In particular we obtain card M_j and the bound card $M_j \leqslant 2n^2 - 2n + 1$.

$$M_1 = \{(xy)^i \mid 0 \leqslant 2i < n\} \cup \{(yx)^i \mid 1 \leqslant 2i < n\}$$
$$\cup \{(xy)^i x \mid 0 \leqslant 2i < n-1\} \cup \{(yx)^i y \mid 0 \leqslant 2i < n-1\}$$

$$\text{card } M_1 = 2n - 1$$

$$M_2 = \{x^i \mid 0 \leqslant i < n\} \cup \{y^i \mid 1 \leqslant i < n\}$$
$$\cup \{x^i y \mid 1 \leqslant i < n-1\} \cup \{yx^i \mid 1 \leqslant i < n-1\}$$
$$\cup \{yx^i y \mid 1 \leqslant i < n-2\}$$

$$\text{card } M_2 = 5n - 8, \qquad \text{if} \quad n \geqslant 3$$

$$M_3 = \{x^i \mid 0 \leqslant i < n\} \cup \{yx^i y \mid 1 \leqslant i < n-2\}$$
$$\cup \{x^i y x^j \mid i, j \geqslant 0, i + j < n-1\}$$
$$\cup \{yx^i y x^j \mid i, j \geqslant 1, i + j < n-2\}$$
$$\cup \{x^i y x^j y \mid i, j \geqslant 1, i + j < n-2\}$$
$$\cup \{yx^i y x^j y \mid i, j \geqslant 1, i + j < n-3\}$$

$$\text{card } M_3 = 2n^2 - 10n + 19, \qquad \text{if} \quad n \geqslant 4$$

$$M_4 = \{x^i \mid 0 \leqslant i < n\} \cup \{y^i \mid 1 \leqslant i < n\}$$
$$\cup \{x^i y^j \mid i, j \geqslant 1, i + j < n\} \cup \{y^i x^j \mid i, j \geqslant 1, i + j < n\}$$
$$\cup \{y^i x y^j \mid i, j \geqslant 1, i + j < n-1\}$$

$$\text{card } M_4 = 3/2n^2 - 7/2n + 4, \qquad \text{if} \quad n \geqslant 2$$

Proof of Proposition 3.1. Let α and β be as in the statement of the Proposition 3.1. We infer from Proposition 3.11 that $\langle \alpha, \beta \rangle$ is an epimorphic image of $M\langle x, y \rangle / I$, where the ideal I is generated by one of the R_i $(1 \leqslant i \leqslant 4)$. Therefore the assertion is an immediate consequence of the previous lemma.

4. Simple Pairs of Transformations

With each reducible pair (α, β) of endomorphisms we will associate a pair (α_0, β_0) of transformations which turns out to be simple in most cases (cf. section 5). The generated submonoid $\langle \alpha_0, \beta_0 \rangle$ later will serve as an approximation of $\langle \alpha, \beta \rangle$.

4.1. Let $\alpha, \beta \in \operatorname{Trans}(w)$. The pair (α, β) is called *simple* if

(i) $\alpha^2 \neq 0$, $\beta \neq 0$, $\beta^2 = 0$, $\operatorname{sgn}(\beta) = 1$,

(ii) $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$, $|\alpha|\,|\beta| < 0$ and $\alpha\beta\alpha \in \operatorname{End}(w)$.

In this section we provide a construction which allows to reduce a simple pair (α, β) to a minimal pair (α_1, β_1) by iterating two operations. On the other hand one obtains from $\langle \alpha_1, \beta_1 \rangle$ inductively a precise description of $\langle \alpha, \beta \rangle$. In particular we deduce the following result:

PROPOSITION. *Let (α, β) be a simple pair in* Trans(w). *Then for $n \in \mathbf{N}_0$*

$$\operatorname{card}(\operatorname{rad}^n\langle \alpha, \beta \rangle \backslash \operatorname{rad}^{n+1}\langle \alpha, \beta \rangle) \leqslant n + 1.$$

Remark. The bound is best possible (cf. Example 6.1).

4.2. LEMMA. *Let (α, β) be a simple pair in* Trans(w).

(a) *Then* $\operatorname{rk}(\beta) < \|\alpha\|$.

(b) *Let $\alpha^{n_1}\beta \cdots \beta\alpha^{n_r} = \alpha^{m_1}\beta \cdots \beta\alpha^{m_s} \neq 0$ for $n_1, ..., n_r$ and let $m_1, ..., m_s$ in $\mathbf{N}_0$. Then $r = s$ and $n_i = m_i$ for $i \in \{1, ..., r\}$.*

Proof. (a) Let $c = \operatorname{supp}(\alpha) = \operatorname{supp}(\beta)$ and let $w' = \pi(c)$ be the projection. We restrict α and β to w', i.e., we consider $\alpha' = ((\alpha_s)_c, (\alpha_t)_c)$ and $\beta' = ((\beta_s)_c, (\beta_t)_c)$ in Trans(w'). (Here $\gamma = \gamma_c * c$ is defined for $\gamma \leqslant c$ in Fac(w) as in 2.1.) Since $\alpha\beta\alpha \in \operatorname{End}(w)$ and $\alpha'\beta'\alpha' \in \operatorname{End}(w')$ respectively α' and β' lie in End(w'). Now the assertion immediately follows from Lemma 3.5 since $\operatorname{rk}(\beta) = \operatorname{rk}(\beta') < \|\alpha'\| = \|\alpha\|$.

(b) Using (a) the assertion follows by induction on r.

The previous Lemma justifies the following convention for a simple pair (α, β): Let $M\langle \alpha, \beta \rangle$ be the free monoid in α and β and let $\tau = \tau_{\alpha,\beta}\colon M\langle \alpha, \beta \rangle \to \operatorname{Trans}(w)$ be the map induced by $\tau(\alpha) = \alpha$ and $\tau(\beta) = \beta$. Then we may identify the set $\{x \in M\langle \alpha, \beta \rangle \mid \tau(x) \neq 0\}$ via τ with $T\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle \backslash \{0\} \subseteq \operatorname{Trans}(w)$.

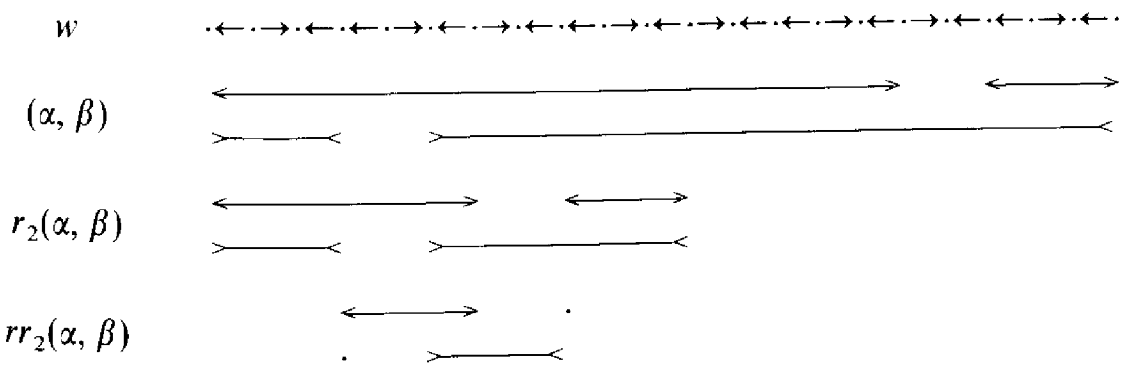
4.3. LEMMA. *Let (α, β) be a simple pair in* Trans(w).

(a) *Let $n = \max\{r \in \mathbf{N}_0 \mid \alpha^{r+2} \neq 0\}$. Then $\alpha_1 = ((\alpha^{n+1})_s, (\alpha^{n+1})_s\alpha)$ and $\beta_1 = (\beta_s\alpha^{-n}, \beta_t)$ lie in* Trans(w) *and (a_1, β_1) is a simple pair. Moreover $\alpha_1^3 = 0$.*

(b) *Suppose $\beta\alpha\beta \neq 0$. Then $\alpha_1 = \alpha\beta\alpha$ and $\beta_1 = ((\beta\alpha)_t \cap \beta_s, \beta_t \cap (\alpha\beta)_s)$ lie in* Trans(w) *and (α_1, β_1) is a simple pair.*

The simple properties of (α_1, β_1) follow from those of (α, β). Therefore we omit the details. We denote in part (a) by $r_n(\alpha, \beta) = (\alpha_1, \beta_1)$ the *n-trun-*

cation of (α, β) and in part (b) by $r(\alpha, \beta)=(\alpha_1, \beta_1)$ the *simple reduction* of (α, β). As an example choose $w=(x^{-1}yx^{-1}x^{-1}y)^4x^{-1}$ in the quiver with one vertex and two arrows x and y.



A simple pair (α, β) is called *minimal* if $\alpha^3=\beta\alpha\beta=0$. These notions lead to the following result:

4.4. Lemma. *Let* (α, β) *be a simple pair in* Trans(w). *Then there exist a minimal pair* (α_1, β_1) *in* Trans(w) *and integers* $n_1, ..., n_t \in \mathbf{N}_0$ such that

$$(\alpha_1, \beta_1)=r_{n_t}rr_{n_{t-1}}\cdots r_{n_2}rr_{n_1}(\alpha, \beta).$$

Proof. Iterate the operations r_n and r described in the previous lemma until the resulting pair is minimal.

4.5. Let $\alpha, \beta \in$ Trans(w). An element $m \in \langle\alpha, \beta\rangle$ is called *maximal* for the pair (α, β) if

(i) $m \neq 0$ and

(ii) $m \in \langle\alpha, \beta\rangle\, x \langle\alpha, \beta\rangle$ holds for all $x \in \langle\alpha, \beta\rangle$ with $x \neq 0$.

Lemma. *Let* (α, β) *be a minimal pair in* Trans(w).

(a) *Suppose* $|\alpha\beta\alpha|\,|\alpha|>0$. *Then there exists* $s \in \mathbf{N}$ *such that* $(\alpha\beta\alpha)^s$ *is maximal for the pair* (α, β).

(b) *Suppose* $|\alpha\beta\alpha|\,|\alpha|<0$. *Then there exists* $s \in \mathbf{N}$ *such that* $\alpha(\alpha\beta\alpha)^s\alpha$ *is maximal for the pair* (α, β).

Proof. According to the definition of a minimal pair $\beta\alpha^n\beta=0$ holds for all $n \in \mathbf{N}_0$ with $n \neq 2$. Application of the assumptions about $|\alpha\beta\alpha|$ shows for maximal $m \in \langle\alpha, \beta\rangle$ that either $m \in \alpha^2\langle\alpha, \beta\rangle\alpha^2$ or $m \in \alpha\beta\langle\alpha, \beta\rangle\beta\alpha$ holds.

4.6. We fix some notations for the rest of Section 4. Let $M=M\langle\alpha, \beta\rangle$ be the free monoid and let $G\langle\alpha, \beta\rangle$ be the free group in α and β where $M\langle\alpha, \beta\rangle$ is regarded as a subset of $G\langle\alpha, \beta\rangle$. Furthermore let

$(\beta^2) \subseteq M\langle\alpha, \beta\rangle$ be the ideal generated by β^2, $M_{\beta^2} = M \backslash (\beta^2)$, $D\langle\alpha, \beta\rangle = \alpha M_{\beta^2} \cup M_{\beta^2}\alpha$ and $D'\langle\alpha, \beta\rangle = \alpha M_{\beta^2}\alpha$. We define two maps, the second one depending on $n \in \mathbf{N}_0$:

$$\rho, \rho_n : M\langle\alpha_1, \beta_1\rangle \to G\langle\alpha, \beta\rangle.$$

Let ρ be the multiplicative extension of the following map defined on $\{\alpha_1, \beta_1\}$:

$$\rho(\alpha_1) = \alpha\beta\alpha, \qquad \rho(\beta_1) = \alpha^{-1}.$$

For $x = \alpha_1^{n_0}\beta_1^{m_1}\alpha_1^{n_1} \cdots \beta_1^{m_r}\alpha_1^{n_r}$ with $r \in \mathbf{N}_0$, $n_0, n_r \in \mathbf{N}_0$, $n_i \in \mathbf{N}$ $(1 \leqslant i \leqslant r-1)$, $m_i \in \mathbf{N}$ $(1 \leqslant i \leqslant r)$ let

$$\rho_n(x) = \alpha^{n_0+n}\beta^{m_1}\alpha^{n_1+n} \cdots \beta^{m_r}\alpha^{n_r+n}.$$

LEMMA. *Let ρ and ρ_n be the maps as defined above.*

(a) *Let $x, y \in M\langle\alpha_1, \beta_1\rangle$. Then $\rho_n(xy) = \rho_n(x)\,\alpha^{-n}\rho_n(y)$ with $\rho_n(x)\,\alpha^{-n}$, $\alpha^{-n}\rho_n(y) \in M\langle\alpha, \beta\rangle$.*

(b) *Let $x \in D\langle\alpha_1, \beta_1\rangle$. Then $\rho(x) \in M\langle\alpha, \beta\rangle$.*

(c) *Let $x, x_1, x_2 \in M\langle\alpha_1, \beta_1\rangle$ and suppose $x, x_1 x x_2 \in D'\langle\alpha_1, \beta_1\rangle$. Then $\rho(x_i) \in M\langle\alpha, \beta\rangle$ for $i \in \{1, 2\}$.*

Proof. (a) and (b) are obvious. (c) follows from (b).

4.7. LEMMA. *Let (α, β) be a simple pair in* Trans(w) *and let $(\alpha_1, \beta_1) = r_n(\alpha, \beta)$. Given $x \in M\langle\alpha_1, \beta_1\rangle$ then $x \in T\langle a_1, \beta_1\rangle$ and $\rho_n(x) \in T\langle\alpha, \beta\rangle$ are equivalent; in which case $\rho_n(x) = x\alpha^n$ holds in* Trans(w).

Proof. We prove the assertion for $x \in M\langle\alpha_1, \beta_1\rangle$ by induction on the length $l(x)$ of x. Assume that $x \in T\langle\alpha_1, \beta_1\rangle$ or $\rho_n(x) \in T\langle\alpha, \beta\rangle$ holds. For $l(x) \leqslant 1$ the assertion is clear. Therefore suppose $x = x_1 x_2$ with $0 < l(x_i) < l(x)$ for $i \in \{1, 2\}$. Then there exist $y_i \in M\langle\alpha, \beta\rangle$ such that $\rho_n(x_1) = y_1\alpha^n$ and $\rho_n(x_2) = \alpha^n y_2$. That implies $\rho_n(x) = y_1\alpha^n y_2$ by Lemma 4.5(a). By induction $y_1\alpha^n = x_1\alpha^n$ and $\alpha^n y_2 = x_2\alpha^n$ lie in Trans(w). Now we interprete

$$x\alpha^n = x_1 x_2 \alpha^n = x_1 \alpha^n y_2 = y_1 \alpha^n y_2 = \rho_n(x)$$

in Trans(w) and distinguish between $x \in T\langle\alpha_1, \beta_1\rangle$ and $\rho_n(x) \in T\langle\alpha, \beta\rangle$. The first case implies $\rho_n(x) \in T\langle\alpha, \beta\rangle$ since $x_t \leqslant (\alpha^n)_s$. In the second case the assertion $x \in T\langle\alpha_1, \beta_1\rangle$ is obvious from $\rho_n(x) = x\alpha^n$. This finishes our proof.

4.8. LEMMA. *Let* (α, β) *be a simple pair in* Trans(w) *and let* $(\alpha_1, \beta_1) = r(\alpha, \beta)$.

(a) *Suppose* $x \in T\langle \alpha, \beta \rangle$. *Then there exists* $x' \in \alpha\beta T\langle \alpha, \beta \rangle \beta\alpha$ *such that* $x' \neq 0$ *and* $x' \in T\langle \alpha, \beta \rangle\, xT\langle \alpha, \beta \rangle$.

(b) *Given* $x \in \alpha_1 M\langle \alpha_1, \beta_1 \rangle \alpha_1$ *then* $x \in T\langle \alpha_1, \beta_1 \rangle$ *and* $\rho(x) \in T\langle \alpha, \beta \rangle$ *are equivalent, in which case* $\rho(x) = x$ *holds in* Trans(w).

Proof. (a) Suppose $x \in T\langle \alpha, \beta \rangle$ has the form $x = \beta y$ for some $y \in T\langle \alpha, \beta \rangle$. Then $\alpha\beta y \neq 0$ since $\beta_s \leqslant \alpha_t$. Using analogous arguments $x = y\beta$, $x = \alpha^2 y$ and $x = y\alpha^2$, respectively, for some $y \in T\langle \alpha, \beta \rangle$ imply $y\beta\alpha \neq 0$, $\alpha\beta\alpha^2 y \neq 0$ and $y\alpha^2\beta\alpha \neq 0$, respectively. Therefore (a) is shown.

(b) Let $x \in \alpha_1 M\langle \alpha_1, \beta_1 \rangle a_1$ and $x \in T\langle \alpha_1, \beta_1 \rangle$ or $\rho(x) \in T\langle \alpha, \beta \rangle$. Choose $x' \in \alpha_1 M\langle \alpha_1, \beta_1 \rangle \alpha_1$ of maximal length $l(x')$ such that $\rho(x') = x' \neq 0$ holds in Trans(w) and some $x_1 \in M\langle \alpha_1, \beta_1 \rangle$ exists with $x = x'x_1$. We claim $x = x'$. First suppose $x_1 \in \alpha_1 M\langle \alpha_1, \beta_1 \rangle$. Then $x'\alpha_1 = \rho(x')\,\alpha\beta\alpha = \rho(x'\alpha_1)$, but that contradicts the maximality of $l(x')$. Therefore suppose $x_1 \in \beta_1 M\langle \alpha_1, \beta_1 \rangle$. Then $x_1 \in \beta_1 \alpha_1 M\langle \alpha_1, \beta_1 \rangle$ and $x'\beta_1\alpha_1 = \rho(x')\beta\alpha = \rho(x'\beta_1\alpha_1)$ since $y\beta_1\alpha_1 = y\beta\alpha$ holds for $y \in$ Fac(w) with $y \leqslant (\alpha_1)_t \cap (\beta_1 a_1)_s$. Thus $x = x'$ is shown by the maximality of $l(x')$.

4.9. PROPOSITION. *Let* (α, β) *and* (α_1, β_1) *be simple pairs in* Trans(w) *and suppose that* m_1 *is maximal for* (α_1, β_1).

(a) *If* $(\alpha_1, \beta_1) = r_n(\alpha, \beta)$, *then* $m = \rho_n(m_1)$ *is maximal for* (a, β).

(b) *If* $(\alpha_1, \beta_1) = r(\alpha, \beta)$, *then* $m = \rho(m_1)$ *is maximal for* (α, β).

Proof. (a) First of all $m = \rho_n(m_1) \in T\langle \alpha, \beta \rangle$ by Lemma 4.7. Now
suppose $x \in T\langle \alpha, \beta \rangle$. Then $m \in T\langle \alpha, \beta \rangle\, xT\langle \alpha, \beta \rangle$ has to be shown.
Suppose $x = \alpha^{n_1}\beta\alpha^{n_2}\beta \cdots \beta\alpha^{n_r}$. It suffices to show the assertion for
$x' = \alpha^{l_1}\beta\alpha^{n_2}\beta \cdots \beta\alpha^{n_r - 1}\beta\alpha^{l_r}$ with $l_i = \max(n_i, n)$ for $i \in \{1, r\}$ because
$x' \in T\langle \alpha, \beta \rangle\, xT\langle \alpha, \beta \rangle$ and $x' \neq 0$. Note that $x' \neq 0$ follows from
$(\alpha^{l_1})_t \cap \beta_s = (\alpha^{n_1})_t \cap \beta_s$ and $\beta_t \cap (\alpha^{l_r})_s = \beta_t \cap (\alpha^{n_r})_s$. We have $x' = \rho_n(y)$ for
$y = \alpha_1^{l_1 - n}\beta_1\alpha_1^{n_2 - n}\beta_1 \cdots \beta_1\alpha_1^{l_r - n}$. By Lemma 4.7 $y \in T\langle \alpha_1, \beta_1 \rangle$ and applying
the maximality of m_1 there are $y_1, y_2 \in T\langle \alpha_1, \beta_1 \rangle$ with $m_1 = y_1 y y_2$. That
implies $m = \rho_n(m_1) = \rho_n(y_1)\,\alpha^{-n}x'\alpha^{-n}\rho_n(y_2)$ by Lemma 4.6(a) where
$\rho_n(y_1)\,\alpha^{-n}, \alpha^{-n}\rho_n(y_2) \in T\langle \alpha, \beta \rangle$ holds. That finishes part (a).

(b) It is obvious that $m_1 \in \alpha_1 T\langle \alpha_1, \beta_1 \rangle \alpha_1$, and therefore
$m = \rho(m_1) \in T\langle \alpha, \beta \rangle$ by Lemma 4.8(b). Now suppose $x \in T\langle \alpha, \beta \rangle$ and
$m \in T\langle \alpha, \beta \rangle\, xT\langle \alpha, \beta \rangle$ has to be shown. According to Lemma 4.8(a)
there is $x' \in \alpha\beta T\langle \alpha, \beta \rangle \beta\alpha$ with $0 \neq x' \in T\langle \alpha, \beta \rangle\, xT\langle \alpha, \beta \rangle$. If $x' =$
$\alpha\beta\alpha^{n_1}\beta \cdots \beta\alpha^{n_r}\beta\alpha$, then $x' = \rho(y)$ holds for $y = \alpha_1\beta_1^{l_1}\alpha_1\beta_1^{l_2} \cdots \beta_1^{l_r}\alpha_1$ with
$l_i = 1$ if $n_i = 1$, and $l_i = 0$, if $n_i = 2$ $(1 \leqslant i \leqslant r)$. Applying Lemma 3.8(b)
$y \in T\langle \alpha_1, \beta_1 \rangle$ holds and since m_1 is maximal there exist $y_1, y_2 \in$

$T\langle \alpha_1, \beta_1 \rangle$ with $m_1 = y_1 y y_2$. That implies $m = \rho(m_1) = \rho(y_1)\, x' \rho(y_2)$ where $\rho(y_i) \in T\langle \alpha, \beta \rangle$ holds for $i \in \{1, 2\}$ by Lemma 4.6(c). Therefore the proof is complete.

4.10. We fix for the rest of this section $M = M\langle \alpha, \beta \rangle$, $D = D\langle \alpha, \beta \rangle$ and the maps ρ and ρ_n are defined on M by $\alpha_1 = \alpha$, $\beta_1 = \beta$. In addition let

$$\delta_n : M\langle \alpha, \beta \rangle \to G\langle \alpha, \beta \rangle$$

be the multiplicative extension of the following map defined on $\{\alpha, \beta\}$:

$$\delta_n(\alpha) = \alpha^n \beta \alpha, \qquad \delta_n(\beta) = \alpha^{-1}.$$

LEMMA. *Let $x \in D$ and $n \in \mathbf{N}_0$.*

(a) *It is $\rho_n \rho(x) = \delta_{n+1}(x) \alpha^n$.*

(b) *Let $a, b, y \in D$ with $y = axb$. Then there exist $a', b' \in D$ with $\delta_{n+1}(y) = a' \rho_n \rho(x) b'$.*

Proof. (a) The assertion follows for $x = \alpha^{n_1} \beta \cdots \beta \alpha^{n_r} \in D$ by induction on r.

(b) Choose $a' = \delta_{n+1}(a)$, $b' = \alpha^{-n}\, \delta_{n+1}(b)$ and use part (a).

4.11. For $n_0, n_1, ..., n_t \in \mathbf{N}_0$ define

$$c(n_0) = \alpha^{n_0} \text{ and } c(n_0, n_1, ..., n_t) = \delta_{n_t} \cdots \delta_{n_2} \delta_{n_1}(\alpha^{n_0}), \qquad \text{if} \quad t \geqslant 1.$$

Now fix some $n_0, n_1, ..., n_t \in \mathbf{N}$ with $t \geqslant 2$. We use the following notation: $c = c(n_0, n_1, ..., n_t)$, $a = c(1, n_2, ..., n_t)$, $b = c(1, n_2 - 1, n_3, ..., n_t)$ and $p = l(a^{n_1} b)$, $q = l(a)$.

LEMMA. *Let $n_0, n_1, ..., n_t \in \mathbf{N}$ and $t \geqslant 2$. Then*

(a) $c = (a^{n_1} b)^{n_0}$,

(b) *$ab \in Ma$ and*

(c) $a \in b(\beta\alpha)^{-1} M$.

Proof. The assertion immediately follows by induction on t using the multiplicativity of δ_n.

PROPOSITION. *Let $n_0, n_1, ..., n_t \in \mathbf{N}$ and consider for $n \in \mathbf{N}$ the set*

$$T_n = \{x \in M \mid c(n_0, n_1, ..., n_t) \in MxM,\ l(x) = n\}.$$

Then card $T_n \leqslant n + 1$ *holds.*

Proof. Again we prove by induction on t. The case $t \leqslant 1$ is trivial. Therefore suppose $t > 1$. We only use the properties (a), (b), and (c) of c in the previous lemma. By (a) and (b) there exist $a_i \in M$ for $1 \leqslant i \leqslant p-q$ with $c \in Ma_i a$ and $l(a_i) = i$. For $0 \leqslant i \leqslant p-q$ let $b_i \in M$ such that $c \in b_i M$ and $l(b_i) = i$. In addition let $b_{-1} = \alpha^{-1}$.

1. $n < p$. Let $x \in M$ with $c \in MxM$ and $l(x) = n$. Part (a) provides $x_1, x_2 \in M$ with $a^{n_1} b a^{n_1} b = x_1 x x_2$. Without any restriction assume $l(x_1) < p$. We consider three cases for $l(x_1)$. If $l(x_1) \geqslant p-q$, then we obtain $a^m \in MxM$ for $m = n_1 + 2$ from (b) and (c). For $l(x_1 x) \leqslant p-2$ $a^m \in MxM$ follows from (c). Otherwise it is $n \geqslant q$ and x has the form $a_i a b_{n-q-i}$ for some $i \in \{1, ..., n-q+1\}$. By induction card $T_n \leqslant n+1$ follows for $n < q$ since $a^m = c(m, n_2, ..., n_t)$. For $n \geqslant q$ observe that a^m is a q-periodic word, and therefore $\operatorname{card}\{x \in M \mid a^m \in MxM,\ l(x) = n\} \leqslant q$ holds. Thus card $T_n \leqslant q + (n-q+1) = n+1$.

2. $n \geqslant p$. By (a) the word c is p-periodic. Therefore card $T_n \leqslant p < n+1$.

Proof of Proposition 4.1. Let (α, β) be simple. There exist $n_1, ..., n_t \in \mathbf{N}_0$ by Lemma 4.4 such that $(\alpha_1, \beta_1) = r_{n_t} r \cdots r r_{n_1}(\alpha, \beta)$ is minimal. Applying Lemma 4.5 there is $s \in \mathbf{N}$ such that $\alpha_1(\alpha_1\beta_1\alpha_1)^s \alpha_1$ or $(\alpha_1\beta_1\alpha_1)^s$ is maximal for (α_1, β_1). Following our convention in 4.2 we consider $T\langle\alpha, \beta\rangle = \langle\alpha, \beta\rangle \backslash \{0\}$ as a subset of $M = M\langle\alpha, \beta\rangle$. In particular

$$\operatorname{rad}^n\langle\alpha, \beta\rangle \backslash \operatorname{rad}^{n+1}\langle\alpha, \beta\rangle \subseteq \{x \in M \mid l(x) = n\}.$$

Now let m be maximal for (α, β). We combine the formulas of Proposition 4.9 for the recursive calculation of m with Lemma 4.10 and obtain $c(s+2, n_1+1, n_2+1, ..., n_t+1) \in MmM$. Now the assertion is an immediate consequence of Proposition 4.11.

5. Reducible Pairs of Endomorphisms

In this section we complete the proof of both theorems. With each reducible pair (α, β) we associate a second pair (α_0, β_0) of transformations and consider (α, β) as an extension $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$ determined by four integral parameters. First we prove part (a) of Theorem 1. For so called weakly reducible pairs we can use the results of the previous section whereas strongly reducible pairs have to be considered separately. The proof of part (b) is based on work already done for part (a).

5.1. Let w be a word in Q. The set $\operatorname{Trans}(w) \backslash \{0\}$ is *partially ordered* by

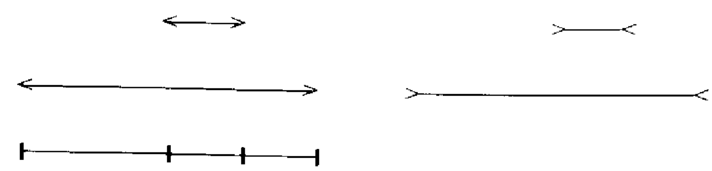
$$(\alpha_s, \alpha_t) \leqslant (\beta_s, \beta_t) \Leftrightarrow \alpha_s \leqslant \beta_s \quad \text{and} \quad \alpha_t = \alpha_s \beta$$

for two transformations $\alpha, \beta \in \mathrm{Trans}(w)\setminus\{0\}$. For $\alpha \leqslant \beta$ fix $(\alpha_s)_{\beta_s} = (x, a, y) \in \mathrm{Fac}(\pi(\beta_s))$ with $i = |x|$ and $j = |y|$. We denote by $\beta = e(\alpha, i, j)$ the *extension* of α by i and j. This extension is uniquely determined by the triple (α, i, j):

α

$\beta = e(\alpha, t, j)$

$(\alpha_s)_{\beta_s}$



5.2. Let the pair (α, β) in $\mathrm{End}(w)$ be reducible and let $a = \alpha\beta\alpha$. Then

$$\alpha_0 = (a_t\alpha^{-1} \cup a_s, a_t \cup a_s\alpha), \qquad \beta_0 = (a_s\alpha, a_t\alpha^{-1})$$

belong to $\mathrm{Trans}(w)$. The pair (α_0, β_0) is called the *reduced pair* corresponding to (α, β). It is $\alpha_0 \leqslant \alpha$ and $\beta_0 \leqslant \beta$. Moreover (α_0, β_0) is a simple pair if $\alpha_0^2 \neq 0$. Let $i, j, p, q \in \mathbf{N}_0$ such that $\alpha = e(\alpha_0, i, j)$ and $\beta = e(\beta_0, p, q)$. Then we write $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$. We call (α, β) *strongly reducible* if $\max(p, q) > \|\beta\|$ or $\max(i, j) > 2\,\|\alpha\|$. Otherwise (α, β) is called *weakly reducible*. In 6.2 and 6.3 are given concrete examples.

5.3. Lemma. *Let (α, β) be reducible in* $\mathrm{End}(w)$ *such that* $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$. *Suppose* $\alpha, \beta \notin \mathrm{rad}^2\,\mathrm{End}(w)$. *Then*:

(a) $\min(i, p) = 0$. (a′) $\min(j, q) = 0$.

(b) *$q \geqslant \|\beta\|$, if $i > \|\alpha\|$.* (b′) *$p \geqslant \|\beta\|$, if $j > \|\alpha\|$.*

(c) *$j \geqslant \|\alpha\|$, if $p > \|\beta\|$.* (c′) *$i \geqslant \|\alpha\|$, if $q > \|\beta\|$.*

Proof. (a) Let $m = \min(i, p)$. From $e(\alpha_0\beta_0\alpha_0, m, 0) \leqslant \alpha\beta\alpha$ and $\alpha_0\beta_0\alpha_0 = \alpha\beta\alpha$ we infer $m = 0$.

(b) Suppose $i > \|\alpha\|$ and $q < \|\beta\|$. Then $\beta_t \ll \alpha_t$ holds which implies $\beta \in \mathrm{rad}^2\,\mathrm{End}(w)$ by Lemma 3.9. Contradiction.

(c) Suppose $p > \|\beta\|$ and $j < \|\alpha\|$. Then $\alpha_t \ll \beta_s$ holds which implies $\alpha \in \mathrm{rad}^2\,\mathrm{End}(w)$ by Lemma 3.9. Contradiction.

(a′)–(c′) and (a)–(c) are symmetric.

5.4. Lemma. *Let (α, β) be reducible in* $\mathrm{End}(w)$ *and let (α_0, β_0) be the corresponding reduced pair.*

(a) *Let $\|\beta\| < \|\alpha\|$. Then $\alpha_0^2 = 0$.*

(b) *Let $\alpha_0^2 = 0$. Then there is $r \in \mathbf{N}$ such that $(\alpha_0\beta_0)^r\alpha_0$ is maximal for (α_0, β_0). In particular* $\mathrm{card}(\mathrm{rad}^n\langle\alpha_0, \beta_0\rangle \setminus \mathrm{rad}^{n+1}\langle\alpha_0, \beta_0\rangle) \leqslant n + 1$.

Proof. (a) By the definition in 3.2 $\beta^2=0$ and therefore the definition of α_0 yields $\alpha_0^2=0$.

(b) The assertion follows directly from the definition of α_0 and β_0.

5.5. LEMMA. *Let (α, β) be reducible in* End(w) *and let (α_0, β_0) be the corresponding reduced pair. Furthermore let $x=\alpha^{n_1}\beta\cdots\beta\alpha^{n_r}\in\langle\alpha, \beta\rangle$ with $n_i\in\mathbf{N}_0$ $(1\leqslant i\leqslant r)$ and let $0\neq y\in$ Trans(w) such that $y\leqslant x$ and*

$$y_s\leqslant\begin{cases}(\alpha_0)_s & n_1\neq 0,\\ (\beta_0)_s & n_1=0,\end{cases}\qquad y_t\leqslant\begin{cases}(\alpha_0)_t & n_r\neq 0,\\ (\beta_0)_t & n_r=0.\end{cases}$$

Then $x_0=\alpha_0^{n_1}\beta_0\cdots\beta_0\alpha_0^{n_r}\neq 0$ holds with $y\leqslant x_0$ and $l(x_0)\leqslant l(x)$.

Proof. We use induction on r. The initial step $r\leqslant 2$ is clear. Therefore suppose $r>2$. Then $n_{r-1}\geqslant 1$. Using the assumptions about y_s and y_t as well as $\alpha\beta\alpha=\alpha_0\beta_0\alpha_0$ one obtains

$$z=y_s(\alpha^{n_1}\beta\cdots\alpha^{n_{r-1}})=y_t(\beta\alpha^{n_r})^{-1}\leqslant(\alpha_0)_t\cap(\beta_0)_s.$$

Apply induction to $y_1=(y_s, z)$ and $y_2=(z, y_t)$. We obtain $z=y_s(\alpha_0^{n_1}\beta_0\cdots\alpha_0^{n_{r-1}})=y_t(\beta_0\alpha_0^{n_r})^{-1}$ which implies $x_0\neq 0$ and $y=y_1y_2\leqslant x_0$. If we transfer Lemma 4.2(b) to (α_0, β_0) we obtain $l(x_0)=(\sum_i n_i)+(r-1)$ for the length in $\langle\alpha_0, \beta_0\rangle$. Therefore $(\sum_i n_i)+(r-1)\leqslant l(x)$ holds for the length in $\langle\alpha, \beta\rangle$ and the proof of this lemma is complete.

5.6. LEMMA. *Let (α, β) be reducible in* End(w) *with $(\alpha, \beta)=e(\alpha_0, \beta_0; i, j, p, q)$. Let $m, n\in\mathbf{N}_0$ and $x\in\langle\alpha, \beta\rangle$.*

(a) *Let $p=q=0$ and $i\leqslant m\,\|\alpha\|$, $j\leqslant n\,\|\alpha\|$. Then there exist $r, s\in\mathbf{N}_0$ and $x_0\in\langle\alpha_0, \beta_0\rangle$ such that $x=\alpha^r x_0\alpha^s$ and $r\leqslant m$, $s\leqslant n$. Moreover, $l(x)\geqslant l(x_0)+r+s$ holds.*

(b) *Let $i=j=0$ and $p\leqslant m\,\|\beta\|$, $q\leqslant n\,\|\beta\|$. Then there exist $r, s\in\mathbf{N}_0$ and $x_0\in\langle\alpha_0, \beta_0\rangle$ such that $x=\beta^r x_0\beta^s$ and $r\leqslant n$, $s\leqslant m$. Moreover, $l(x)\geqslant l(x_0)+r+s$ holds.*

(c) *Let $j=p=0$, $i\leqslant m\,\|\alpha\|$, $q\leqslant n\,\|\beta\|$ and $\|\beta\|\geqslant\|\alpha\|$. Then there exist $r\in\mathbf{N}_0$ and $x_0\in\langle\alpha_0, \beta_0\rangle$ such that $x=\alpha^r x_0$ with $r\leqslant m$, or $x=\beta^r x_0$ with $r\leqslant n$. Moreover, $l(x)\geqslant l(x_0)+r$ holds.*

(c′) *Let $i=q=0$, $j\leqslant m\,\|\alpha\|$, $p\leqslant n\,\|\beta\|$ and $\|\beta\|\geqslant\|\alpha\|$. Then there exist $r\in\mathbf{N}_0$ and $x_0\in\langle\alpha_0, \beta_0\rangle$ such that $x=x_0\alpha^r$ with $r\leqslant m$, or $x=x_0\beta^r$ with $r\leqslant n$. Moreover, $l(x)\geqslant l(x_0)+r$ holds.*

Proof. (a) For $x\in\langle\alpha\rangle$ the assertion is trivial. Otherwise choose

$r, s \in \mathbf{N}_0$ and $x' \in \langle \alpha, \beta \rangle$ with $x = \alpha^r x' \alpha^s$ such that r and s are minimal with respect to

$$x_s \alpha^r \leqslant \begin{cases} (\alpha_0)_s & x' \in \alpha \langle \alpha, \beta \rangle, \\ (\beta_0)_s & x' \in \beta \langle \alpha, \beta \rangle, \end{cases} \qquad x_t \alpha^{-s} \leqslant \begin{cases} (\alpha_0)_t & x' \in \langle \alpha, \beta \rangle \alpha, \\ (\beta_0)_t & x' \in \langle \alpha, \beta \rangle \beta. \end{cases}$$

The assumption about i and j guarantee that $r \leqslant m$ and $s \leqslant n$. Moreover $l(x) \geqslant r + l(x') + s$. Now the assertion follows by Lemma 5.5, applied to $y = (x_s \alpha^r, x_t \alpha^{-s}) \leqslant x'$.

(b) First we claim $\|\beta\| \geqslant \|\alpha\|$. Since $i = j = 0$ it is $\alpha = \alpha_0$. Suppose $\|\beta\| < \|\alpha\|$. Then $\alpha^2 = \alpha_0^2 = 0$ follows by Lemma 5.4(a). This contradicts the fact that (α, β) is reducible and therefore $\|\beta\| \geqslant \|\alpha\|$ is shown. Now we use similar arguments as in (a) changing the roles of α and β.

(c) If $x \in \langle \alpha \rangle \cup \langle \beta \rangle$ the assertion is trivial. Otherwise first observe that $\beta_t \cap \alpha_s \leqslant (\alpha_0)_s$ and $\alpha_t \cap \beta_s \leqslant (\beta_0)_s$ holds. Therefore there exist $x_1 \in \langle \alpha \rangle \cup \langle \beta \rangle$ and $x' \in \langle \alpha, \beta \rangle$ with

$$x = x_1 x' \qquad \text{and} \qquad x_s x_1 \leqslant \begin{cases} (\alpha_0)_s & x' \in \alpha \langle \alpha, \beta \rangle, \\ (\beta_0)_s & x' \in \beta \langle \alpha, \beta \rangle. \end{cases}$$

Let $x_1 = \alpha^r$ or $x_1 = \beta^r$ and assume r to be minimal. Then $r \leqslant m$ or $r \leqslant n$ respectively. Also $l(x) \geqslant r + l(x')$ holds. The assertion follows from Lemma 5.5, applied to $y = (x_s x_1, x_t) \leqslant x'$.

(c′) Follows by symmetry from (c).

5.7. Lemma. *Let* $\alpha, \beta \in \operatorname{rad} \operatorname{End}(w) \backslash \operatorname{rad}^2 \operatorname{End}(w)$ *and let* (α, β) *be reducible with* $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$. *Fix* $n \in \mathbf{N}$ *and for* $a, b \in \langle \alpha, \beta \rangle$ *consider the set*

$$M_n(a, b) = \{ axb \in \operatorname{Trans}(w) \mid x \in \langle \alpha_0, \beta_0 \rangle, l(x) + l(a) + l(b) < n \}.$$

(a) *If* $\|\beta\| < \|\alpha\|$, *then*

$$\begin{aligned} &\langle \alpha, \beta \rangle \backslash \operatorname{rad}^n \langle \alpha, \beta \rangle \\ &\quad \subseteq \{ \alpha^{r_1} \beta (\alpha\beta)^s \alpha^{r_2} \in \operatorname{Trans}(w) \mid r_i, s \in \mathbf{N}_0, r_i \leqslant 2, r_1 + 1 + 2s + r_2 < n \\ &\qquad \cup \{ \alpha^r \mid 0 \leqslant r \leqslant \min(3, n-1) \} \qquad \textit{and} \end{aligned}$$

$$\operatorname{card}(\langle \alpha, \beta \rangle \backslash \operatorname{rad}^n \langle \alpha, \beta \rangle) \leqslant 3n + 1.$$

(b) *If* $p = q = 0$ *and* $\max(i, j) \leqslant \|\alpha\|$, *then*

$$\begin{aligned} &\langle \alpha, \beta \rangle \backslash \operatorname{rad}^n \langle \alpha, \beta \rangle \\ &\quad \subseteq M_n(1, 1) \cup M_n(\alpha, 1) \cup M_n(1, \alpha) \cup M_n(\alpha, \alpha) \qquad \textit{and} \end{aligned}$$

$$\operatorname{card}(\langle \alpha, \beta \rangle \backslash \operatorname{rad}^n \langle \alpha, \beta \rangle) \leqslant 2n^2 - 2n + 1.$$

(c) *If* $i=j=0$ *and* $\max(p,q)\leqslant\|\beta\|$, *then*

$$\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle \subseteq M_n(1,1)\cup M_n(\beta,1)\cup M_n(1,\beta)\cup M_n(\beta,\beta) \qquad \textit{and}$$

$$\text{card}(\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle)\leqslant 2n^2-2n+1.$$

(d) *If* $j=p=0$, $i\leqslant 2\,\|\alpha\|$ *and* $q\leqslant\|\beta\|$, *then*

$$\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle \subseteq M_n(1,1)\cup M_n(\alpha,1)\cup M_n(\beta,1)\cup M_n(\alpha^2,1) \qquad \textit{and}$$

$$\text{card}(\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle)\leqslant 2n^2-2n+1.$$

(d′) *If* $i=q=0$, $j\leqslant 2\,\|\alpha\|$ *and* $p\leqslant\|\beta\|$, *then*

$$\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle \subseteq M_n(1,1)\cup M_n(1,\alpha)\cup M_n(1,\alpha)\cup M_n(1,\beta)\cup M_n(1,\alpha^2) \qquad \textit{and}$$

$$\text{card}(\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle)\leqslant 2n^2-2n+1.$$

Proof. (a) According to the definition in 3.1 it is $\beta^2=0$. Therefore $\max(p,q)<\|\beta\|$. Moreover $\max(i,j)\leqslant\|\alpha\|$ holds by Lemma 5.3 since $\alpha,\beta\notin\text{rad}^2\,\text{End}(w)$. There are the following cases:

1. $p=q=0$. The assertion follows from Lemma 5.6(a) and Lemma 5.4.

2. $p\neq 0$. By Lemma 5.3(a) it is $i=0$. Therefore $\alpha_t\cap\beta_s=(\alpha_0)_t\cap\beta_s$ which implies $\alpha^2\beta=\alpha_0^2\beta=0$ by Lemma 5.4(a). Moreover $i=0$, $j\leqslant\|\alpha\|$ and $\alpha_0^2=0$ yields $\alpha^3=0$. The assertion follows immediately.

3. $q\neq 0$. This case is symmetric to the previous one.

For (b)–(d′) the description of $\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle$ follows from Lemma 5.6. The pair (α_0,β_0) is simple or $\alpha_0^2=0$ holds. Therefore Proposition 4.1 and Lemma 5.4 show card $M_n(a,b)\leqslant m^2/2+m/2$ for $m=n-l(a)-l(b)$. Applying this result several times we obtain the bound for $\text{card}(\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle)$.

Each weakly reducible pair (α,β) satisfies one of the conditions in the previous lemma. Therefore we may conclude:

Proposition. *Let* $\alpha,\beta\in\text{rad}\,\text{End}(w)\backslash\text{rad}^2\,\text{End}(w)$ *and let* (α,β) *be weakly reducible. Then the following holds for* $n\in\mathbf{N}$:

$$\text{card}(\langle\alpha,\beta\rangle\backslash\text{rad}^n\langle\alpha,\beta\rangle)\leqslant 2n^2-2n+1.$$

Remark. Let $\alpha, \beta \in \operatorname{rad}\operatorname{End}(w)\backslash\operatorname{rad}^2\operatorname{End}(w)$ and let (α, β) be strongly reducible. In general there is no polynomial bound of degree 2 for $\operatorname{card}(\langle\alpha, \beta\rangle\backslash\operatorname{rad}^n\langle\alpha, \beta\rangle)$ (cf. Example 6.2). Nevertheless Lemma 5.6(c) and 5.6(c′) respectively yield the bound $\operatorname{card}(\langle\alpha, \beta\rangle\backslash\operatorname{rad}^n\langle\alpha, \beta\rangle) \leqslant 1/3n^3 + n^2 + 2/3n$.

5.8. PROPOSITION. *Let the factor algebra $A = k\operatorname{End}(w)/I$ be generated by two elements. Suppose that given two endomorphisms $\alpha, \beta \in \operatorname{rad}\operatorname{End}(w)\backslash\operatorname{rad}^2\operatorname{End}(w)$ such that $\bar\alpha = \alpha + I$ and $\bar\beta = \beta + I$ generate the algebra A, the pair (α, β) or (β, α) is strongly reducible. Choose $\alpha, \beta \in \operatorname{rad}\operatorname{End}(w)\backslash\operatorname{rad}^2\operatorname{End}(w)$ in such a way that (α, β) is strongly reducible and $\{\bar\alpha, \bar\beta\}$ generates the algebra A. Then $\bar\alpha\bar\beta^{r+1}\bar\alpha = \bar\beta\bar\alpha^r\bar\beta = 0$ holds for all $r \in \mathbf{N}$.*

Proof. Assume the strongly reducible pair $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$ generates A. The relation $\bar\alpha\bar\beta^{r+1}\bar\alpha = 0$ is clear from Proposition 3.4. To prove the second relation we consider our cases:

1. $q > \|\beta\|$. Let $(n+1)\|\beta\| \geqslant q > n\|\beta\|$. First of all $j = p = 0$ by Lemma 5.3. That implies $\beta\alpha = \beta\alpha_0 = \gamma\beta^n$ with $\gamma = ((\alpha_0)_s\beta^{-1}, (\alpha_0)_t\beta^{-n}) \in \operatorname{End}(w)$. By assumption there are pairwise different $x_l \in \langle\alpha, \beta\rangle$ and $\xi_l \in k^*$ $(1 \leqslant l \leqslant m)$ such that $\bar\gamma = \sum_l \xi_l \bar x_l$. Note that $x_l \neq \beta$ for all l since otherwise $\{\bar\alpha, \bar\gamma\}$ would generate A although neither (α, γ) nor (γ, α) is reducible by $|\alpha|\,|\gamma| > 0$. Now suppose the assertion is not true and choose $r \in \mathbf{N}$ minimal such that $\bar\beta\bar\alpha^r\bar\beta \neq 0$. It is $\bar\beta\bar\alpha^r\bar\beta = \sum_l \xi_l \bar x_l \bar\beta^n\bar\alpha^{r-1}\bar\beta$. We claim $x_l\beta^n\alpha^{r-1}\beta \in I$ for all l. Then we are done since that contradicts our assumption. If $r > 1$, then $x_l\beta^n\alpha^{r-1}\beta \in I$ holds by the minimality of r. If $r = 1$, then $x_l\beta^{n+1} \in I$ follows from $\alpha\beta^2 = 0$ and $\beta^{n+3} = 0$ respectively. The relations $\alpha\beta^2 = 0$ and $\beta^{n+3} = 0$ are a consequence of Lemma 5.6(c).

2. $p > \|\beta\|$. Analogous to the first case.

3. $q \leqslant \|\beta\|$ and $i > 2\|\alpha\|$. As in the first case $j = p = 0$ by Lemma 5.3. Therefore $\alpha\beta = \alpha\beta_0 = \gamma\alpha^2$ with $\gamma = ((\beta_0)_s\alpha^{-1}, (\beta_0)_t\alpha^{-2}) \in \operatorname{End}(w)$. By assumption there are pairwise different $x_l \in \langle\alpha, \beta\rangle$ and $\xi_l \in k^*$ $(1 \leqslant l \leqslant m)$ such that $\bar\gamma = \sum_l \xi_l \bar x_l$. Note that $x_l \neq \beta$ for all l since otherwise $\{\bar\alpha, \bar\gamma\}$ would generate A although neither (α, γ) nor (γ, α) is strongly reducible by $\gamma^2 = 0$. Now suppose the assertion is not true and choose $n \in \mathbf{N}_0$ maximal such that $\bar\beta\bar\alpha^r\bar\beta\bar\alpha^n \neq 0$ for some $r \in \mathbf{N}$. It is $\bar\beta\bar\alpha^r\bar\beta\bar\alpha^n = \sum_l \xi_l \bar\beta\bar\alpha^{r-1}\bar x_l\bar\alpha^{n+2}$. We claim $\beta\alpha^{r-1}x_l\alpha^{n+2} \in I$ for all l. As before that contradiction would finish this case. If $r > 1$, then $\beta\alpha^{r-1}x_l\alpha^{n+2} \in I$ follows from the maximality of n and from $\beta\alpha^{r+2} = \beta\alpha_0^{r+2} = 0$ respectively. If $r = 1$, apply to $x_l \in \langle\beta\rangle$ the fact that by $q \leqslant \|\beta\|$ $\beta^3 = 0$. Therefore also $\beta x_l\alpha^{n+2} = 0$ holds. If $x_l \notin \langle\beta\rangle$ we use again the maximality of n and $\beta\alpha^{r+2} = \beta\alpha^3 = 0$, respectively, to obtain $\beta x_l\alpha^{n+2} \in I$.

4. $p \leqslant \|\beta\|$ and $j > 2\|\alpha\|$. Analogous to the third case.

5.9. *Proof of Theorem* 1. First combine Proposition 2.6 and Lemma 2.7(a) to obtain that k End(w) is a local algebra. Let $A = k\ \mathrm{End}(w)/I$ be a factor algebra generated by two elements and let $n \in \mathbf{N}$. Choose $\alpha, \beta \in \mathrm{rad}\ \mathrm{End}(w) \backslash \mathrm{rad}^2\ \mathrm{End}(w)$ such that A is generated by $\{\alpha + I, \beta + I\}$.

(a) By Lemma 2.7(b)

$$\dim_k A/\mathrm{rad}^n A \leqslant \mathrm{card}(\langle \alpha, \beta \rangle \backslash \mathrm{rad}^n \langle \alpha, \beta \rangle)$$

holds and the assertion follows from the propositions in 3.1, 5.7, and 5.8 combined with Lemma 3.12.

(b) Suppose $n > 3$. We may assume without any restriction that $A \cong \Lambda = k\langle x, y\rangle/(x, y)^4$. Lemma 2.7(b) yields

$$x_1 x_2 x_3 \neq 0 \qquad \text{for all triples } x_i \in \{\alpha, \beta\}\ (i \in \{1, 2, 3\}).$$

Therefore the pair (α, β) or (β, α) is reducible by Proposition 3.1. Suppose (α, β) is reducible with $(\alpha, \beta) = e(\alpha_0, \beta_0; i, j, p, q)$. We claim

$$0 < p, q \leqslant \|\beta\| \qquad \text{and} \qquad i = j = 0. \tag{1}$$

Assume $p = 0$. Then $\alpha\beta^2 = 0$ holds since $\beta_0^2 = 0$ and $\alpha_t \cap \beta_s = \alpha_t \cap (\beta_0)_s$. Therefore $p > 0$ is shown. Similiarly one obtains $q > 0$. The rest of statement (1) is done by Lemma 5.3. Now we claim

$$p > \|\alpha\|. \tag{2}$$

Since (α_0, β_0) is simple (1) implies

$$|\mathrm{supp}(\alpha)| = |\mathrm{supp}(\alpha_0)| = |\mathrm{supp}(\beta_0)| = 2\,\mathrm{rk}(\beta_0) + p + q - |\beta_s \cap \beta_t|. \tag{3}$$

Moreover $\alpha^3 \neq 0$ and $\beta^3 \neq 0$ yields by Lemma 2.5

$$|\mathrm{supp}(\alpha)| \geqslant 3\ \|\alpha\| \qquad \text{and} \qquad |\beta_s \cap \beta_t| \geqslant \|\beta\|. \tag{4}$$

Lemma 4.2 states that

$$\mathrm{rk}(\beta_0) < \|\alpha\|, \tag{5}$$

and part (1) implies that

$$q \leqslant \|\beta\|. \tag{6}$$

Parts (3)–(6) show (2) and we obtain from (2) $(\alpha\beta\alpha)_s \ll \beta_s$. Thus there exists $\gamma \in \mathrm{End}(w)$ such that $\alpha\beta\alpha = \beta\gamma$ by Lemma 3.9. This relation implies that

$$\dim_k(\mathrm{rad}^3 A/\mathrm{rad}^4 A) < \dim_k(\mathrm{rad}^3 \Lambda/\mathrm{rad}^4 \Lambda)$$

which contradicts our assumption $A \cong \Lambda$. Now the proof of the theorem is complete.

Proof of Theorem 2. Using Proposition 2.6 the monoid $\mathrm{End}(w)$ is local. One obtains parts (a) and (b) immediately from Theorem 1 since a surjective homomorphism $\mathrm{End}(w) \to M$ induces a surjective homomorphism $k\,\mathrm{End}(w) \to kM$ of k-algebras where cardinality and k-dimension of $M\langle x, y\rangle$ and $k\langle x, y\rangle$, respectively, correspond to each other.

Remarks. (a) In Example 6.1 we provide for $n \in \mathbf{N}$ a word w_n such that $\mathrm{End}(w_n)/\mathrm{rad}^n \mathrm{End}(w_n)$ is generated by two elements and $\mathrm{card}(\mathrm{End}(w_n)/\mathrm{rad}^n \mathrm{End}(w_n)) = n^2/2 + n/2 + 1$ holds.

(b) Example 6.3 illustrates the proof of part (b). In particular we see that the bounds in part (b) of Theorem 1 and Theorem 2 are best possible. If for two endomorphisms $\alpha, \beta \in \mathrm{rad}\,\mathrm{End}(w) \backslash \mathrm{rad}^2 \mathrm{End}(w)$ the factor monoids $\langle\alpha, \beta\rangle/\mathrm{rad}^n\langle\alpha, \beta\rangle$ and $M\langle x, y\rangle/\mathrm{rad}^n M\langle x, y\rangle$ are isomorphic then $n \leqslant 4$ according to Proposition 3.4. Example 6.3 shows that $n = 4$ is possible.

6. Examples

The following examples are based on the quiver Q with one vertex and two arrows, i.e., $Q_1 = \{x, y\}$. We fix some $n \in \mathbf{N}$.

6.1. Example. Consider $w_r = (x^{-1}y)^r x^{-1}$ for $r \in \mathbf{N}_0$ and $w = w_n$. Let

$$a_s = (1, w_{n-1}, yx^{-1}), \qquad \alpha_t = (x^{-1}y, w_{n-1}, 1),$$
$$\beta_s = (w, 1, 1), \qquad \text{and} \qquad \beta_t = (1, 1, w).$$

Then $\alpha = (\alpha_s, \alpha_t)$ and $\beta = (\beta_s, \beta_t)$ belong to $\mathrm{End}(w)$ with

$$\mathrm{End}(w) = \{\alpha^i \mid 0 \leqslant i \leqslant n\} \cup \{\alpha^i\beta\alpha^j \mid 0 \leqslant i, j \leqslant n\} \cup \{0\},$$
$$\mathrm{End}(w)\backslash\mathrm{rad}^n \mathrm{End}(w) = \{\alpha^i \mid 0 \leqslant i < n\} \cup \{\alpha^i\beta\alpha^j \mid 0 \leqslant i + j < n - 1\},$$
$$\dim_k(k\,\mathrm{End}(w)/\mathrm{rad}^n k\,\mathrm{End}(w))$$
$$= \mathrm{card}(\mathrm{End}(w)/\mathrm{rad}^n \mathrm{End}(w)) - 1 = n^2/2 + n/2.$$

The pair (α, β) is simple and $r_{n-2}(\alpha, \beta)$ is minimal if $n \geqslant 2$.

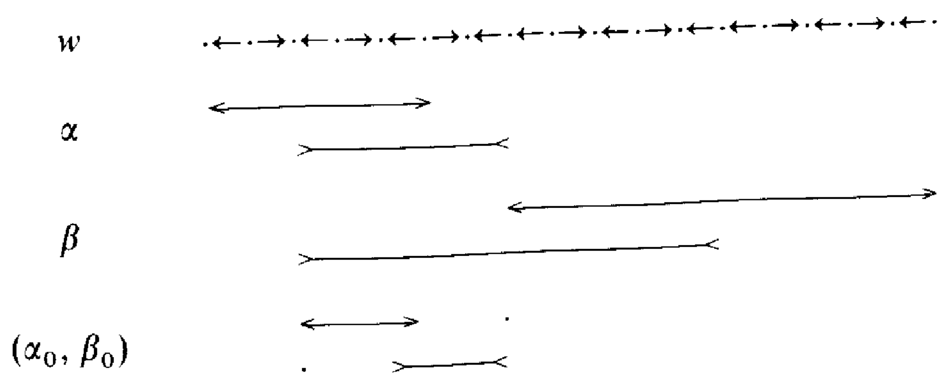
6.2. Example. Consider $v=(x^{-1}y)^n x^{-1}$ and $w=x^{-1}yv^{n+1}$. Let

$$a_s=(1, v, yx^{-1}v^n), \qquad \alpha_t=(x^{-1}y, v, v^n),$$
$$\beta_s=(x^{-1}yv, v^n, 1) \qquad \text{and} \qquad \beta_t=(x^{-1}y, v^n, v).$$

Then $\alpha=(\alpha_s, \alpha_t)$ and $\beta=(\beta_s, \beta_t)$ belong to rad End$(w)\backslash$rad^2 End(w) with

$$\langle\alpha, \beta\rangle=\{\alpha^i \mid 0\leqslant i\leqslant n+1\}\cup\{\beta^i\alpha^j\beta\alpha^l \mid 0\leqslant i, j, l\leqslant n\}$$
$$\cup\{\alpha^{n+1}\beta\alpha^i \mid 0\leqslant i\leqslant n\}\cup\{0\},$$
$$\langle\alpha, \beta\rangle\backslash\text{rad}^n\langle\alpha, \beta\rangle=\{\alpha^i \mid 0\leqslant i<n\}\cup\{\beta^i\alpha^j\beta\alpha^l \mid 0\leqslant i+j+l<n-1\},$$
$$\text{card}(\langle\alpha, \beta\rangle\backslash\text{rad}^n\langle\alpha, \beta\rangle)=1/6n^3+5/6n.$$

The pair (α, β) is strongly reducible if $n\geqslant 2$. Let (α_0, β_0) be the corresponding reduced pair. Then $(\alpha, \beta)=e(\alpha_0, \beta_0; \|\alpha\|, 0, 0, n\,\|\beta\|)$. We choose $n=2$ for illustration:



6.3. Example. Consider $v=x^{-2}yx^{-3}yx^{-3}yx^{-2}$ and $w=v^3$. Let

$$\alpha_s=(x^{-2}yx^{-3}yx^{-3}yx^{-1}, x^{-3}yx^{-3}, yx^{-2}v),$$
$$\alpha_t=(vx^{-2}y, x^{-3}yx^{-3}yx^{-3}, x^{-1}yx^{-3}yx^{-3}yx^{-2}),$$
$$\beta_s=(v, v^2, 1), \qquad \text{and} \qquad \beta_t=(1, v^2, v).$$

Then $\alpha=(\alpha_s, \alpha_t)$ and $\beta=(\beta_s, \beta_t)$ belong to rad End$(w)\backslash$rad^2 End(w). Let M be the monoid in two generators. Then

$$\langle\alpha, \beta\rangle/\text{rad}^4\langle\alpha, \beta\rangle\cong M/\text{rad}^4 M.$$

The set $I=(\text{End}(w)\backslash\langle\alpha, \beta\rangle)\cup\text{rad}^3\,\text{End}(w)$ is an ideal in End(w) and

$$\text{End}(w)/I\cong M/\text{rad}^3 M.$$

The pair (α, β) is weakly reducible and $(\alpha, \beta)=e(\alpha_0, \beta_0; 0, 0, \|\beta\|-1, \|\beta\|-1)$ holds for the corresponding reduced pair (α_0, β_0).

7. Words and Modules

7.1. Let Q be a quiver and denote by

$$Q^+ = \{a_1 a_2 \cdots a_n \in Q^* \mid a_i \in Q_1 \text{ for all } i\} \cup \{e_x \in Q^* \mid x \in Q_0\}$$

the set of *paths* in Q. The *path algebra* kQ of Q is defined as the k-vector space with basis Q^+. The product of two pathes in kQ is by definition their composite in Q^* if this is defined, otherwise it is zero.

Given a word w in Q we now define the kQ-right module $M = M(w)$ *associated* with w. Let M be the k-vector space with basis $\{m_i \mid 0 \leqslant i \leqslant |w|\}$. It suffices to explain the multiplication $m \cdot r$ for $m \in M$ and $r \in kQ$ if we restrict to base elements m_i for m and to paths of length 0 and 1 for r. First consider $w = e_x$ for some $x \in Q_0$ and put

$$m_0 \cdot r = \begin{cases} m_0 & r = e_x, \\ 0 & \text{else.} \end{cases}$$

Otherwise let $w = w_1 w_2 \cdots w_n$ be a word of length $n > 0$ and define

$$m_i \cdot e_x = \begin{cases} m_i & i \neq 0 \text{ and } x = t(w_i), \\ m_0 & i = 0 \text{ and } x = s(w_i), \\ 0 & \text{else,} \end{cases}$$

$$m_i \cdot \alpha = \begin{cases} m_{i+1} & \alpha = w_{i+1}, \\ m_{i-1} & \alpha = w_i^{-1}, \\ 0 & \text{else.} \end{cases}$$

Now we quote in a slightly more general form a result of Wald and Waschbüsch as well as of Crawley-Boevey:

Proposition. *Let $\Lambda = kQ/I$ be a k-algebra and let w be a word in the quiver Q. If the associated kQ-module $M(w)$ satisfies $M(w) \cdot I = 0$ then denote by M the induced Λ-module. The following algebras are isomorphic:*

$$\mathrm{End}_\Lambda(M) \cong \mathrm{End}_{kQ}(M(w)) \cong k\,\mathrm{End}(w).$$

Proof. The Theorem in [C] provides an explicit vector space isomorphism which respects the composition defined in $\mathrm{End}(w)$ and the usual one in $\mathrm{End}_{kQ}(M(w))$.

Proof of Corollary 1. Combine the proposition with Theorem 1.

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