ON THE NUMBER OF ALMOST SPLIT SEQUENCES WITH INDECOMPOSABLE MIDDLE TERM

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Let Λ be an artin algebra and $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules. It is well known that almost split sequences with indecomposable middle term exist in $\operatorname{mod}\Lambda$, provided Λ is not semisimple. The first proof was given by Auslander and Reiten [**2**] for algebras of finite representation type, and the general result is due to Martínez-Villa [**7**]. Later, Butler and Ringel gave an explicit method to construct such sequences, using so called non-supportive elements [**4**].

The following recent result by Brenner and Krause suggests that almost split sequences with indecomposable middle term occur quite frequently [**3, 6**].

PROPOSITION 1. *Let $0 \to A \xrightarrow{f} B \to C \to 0$ be an exact sequence in* $\operatorname{mod}\Lambda$, *which is not almost split. Suppose that one of the modules A and B is indecomposable, and that f is irreducible. Then C is simple or $\alpha(C) = 1$.*

Recall that given an indecomposable non-projective C in $\operatorname{mod}\Lambda$, there exists an almost split sequence $0 \to A \to B \to C \to 0$ in $\operatorname{mod}\Lambda$, which is unique up to isomorphism [**2**]. The fact that the middle term B is indecomposable is denoted by $\alpha(C) = 1$.

In this note we want to study the cardinality of the set $\operatorname{ind}^1 \Lambda = \{X \in \operatorname{ind}\Lambda \mid X \text{ non-projective and } \alpha(X) = 1\}$, where $\operatorname{ind}\Lambda$ denotes a complete set of representatives of the isomorphism classes of indecomposable objects in $\operatorname{mod}\Lambda$. Before starting, we should formulate the existence of an almost split sequence with indecomposable middle term as an immediate consequence of Proposition 1.

COROLLARY 2. *Let Λ be a non-semisimple artin algebra. There exists an exact sequence $0 \to A \xrightarrow{f} B \to C \to 0$ in $\operatorname{mod}\Lambda$ or $\operatorname{mod}\Lambda^{\mathrm{op}}$ such that A is simple, B is indecomposable, f is irreducible and $\alpha(C) = 1$.*

The proof may be found at the end of this note.

We begin our discussion with some notation. Restricting the length of the indecomposables, we define

$$\operatorname{ind}_n \Lambda = \{X \in \operatorname{ind}\Lambda \mid l(X) \leqslant n\} \quad \text{and} \quad \operatorname{ind}^1_n \Lambda = \operatorname{ind}^1 \Lambda \cap \operatorname{ind}_n \Lambda$$

for all $n \in \mathbb{N}$. We denote by p_Λ the maximal length of an indecomposable projective Λ- or Λ^{op}-module. We shall also need definitions and elementary properties of almost split sequences and irreducible maps, for which the reader is referred to [**1**] and [**2**].

The following main result relates the cardinalities of $\operatorname{ind}_n \Lambda$ and $\operatorname{ind}^1_m \Lambda$. As usual, the cardinality of a set $\mathscr{X}$ is denoted by $|\mathscr{X}|$.

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THEOREM 3. *Let Λ be an artin algebra. Then $|\mathrm{ind}_n \Lambda| \leqslant 2^{n+1}(|\mathrm{ind}_m^1 \Lambda| + |\mathrm{ind}_1 \Lambda|)$ for all $n \in \mathbf{N}$, where $m = p_\Lambda^2 n$.*

We postpone a proof of this theorem, and discuss some consequences.

COROLLARY 4. *Let Λ be an artin algebra. Then the following conditions are equivalent for every infinite cardinal κ.*

(i) *There exists $n \in \mathbf{N}$ such that $|\mathrm{ind}_n \Lambda| \geqslant \kappa$.*
(ii) *There exists $n \in \mathbf{N}$ such that $|\mathrm{ind}_n^1 \Lambda| \geqslant \kappa$.*

If Λ satisfies these conditions for some infinite cardinal κ, then $|\mathrm{ind}^1 \Lambda| = |\mathrm{ind}\, \Lambda|$.

Proof. Condition (ii) trivially implies (i). Therefore suppose that $|\mathrm{ind}_n \Lambda| \geqslant \kappa$ for some infinite cardinal κ and some $n \in \mathbf{N}$. The statement of Theorem 3 implies that $|\mathrm{ind}_m^1 \Lambda| \geqslant |\mathrm{ind}_n \Lambda| \geqslant \kappa$ holds for $m = p_\Lambda^2 n$. Moreover, we have $|\mathrm{ind}^1 \Lambda| \geqslant |\mathrm{ind}_n \Lambda|$ for all $n \in \mathbf{N}$, and therefore $|\mathrm{ind}^1 \Lambda| = |\mathrm{ind}\, \Lambda|$. This finishes the proof.

It would be interesting to know whether $|\mathrm{ind}^1 \Lambda| = |\mathrm{ind}\, \Lambda|$ holds whenever $\mathrm{ind}\, \Lambda$ is infinite.

We consider now the formula of Theorem 3 in the case that $\mathrm{ind}_n \Lambda$ is finite. There are two natural questions.

PROBLEM 1. Does there exist a polynomial $p(n)$ (not depending on Λ) such that $|\mathrm{ind}_n \Lambda| \leqslant p(n)(|\mathrm{ind}^1 \Lambda| + |\mathrm{ind}_1 \Lambda|)$ for all $n \in \mathbf{N}$?

PROBLEM 2. Does $|\mathrm{ind}_n \Lambda| \leqslant 2^{n+1} |\mathrm{ind}_1 \Lambda|$ hold if $\mathrm{ind}_n \Lambda$ is finite?

Both questions have negative answers, and we provide examples after the following proof.

Proof of Theorem 3. We fix an artin algebra Λ and $n \in \mathbf{N}$. For each $X \in \mathrm{ind}_n \Lambda$, choose an irreducible map $f_X\colon X \to X'$ such that X' is indecomposable, if this is possible. Otherwise, let f_X be the zero map $X \to 0$. Denote

$$\mathscr{X}_A = \{X \in \mathrm{ind}\, \Lambda \mid \mathrm{Ker} f_X = A\} \quad \text{and} \quad \mathscr{X}^C = \{X \in \mathrm{ind}\, \Lambda \mid \mathrm{Coker} f_X = C\}$$

for all A and C in $\mathrm{ind}\, \Lambda$. We introduce the following relation on $\mathscr{X}_A$ and $\mathscr{X}^C$. Define $X \geqslant Y$ in $\mathscr{X}_A$ if there exists a commutative diagram as below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \xrightarrow{f_X} & X' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & Y & \xrightarrow{f_Y} & Y' & \longrightarrow & 0 \end{array}$$

Define $X \geqslant Y$ in $\mathscr{X}^C$ if there exists a commutative diagram as below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f_Y} & Y' & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{f_X} & X' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Now recall the following property of a non-split exact sequence $0 \to A \xrightarrow{u} X \xrightarrow{f} X' \to 0$ in $\mathrm{mod}\, \Lambda$, which is equivalent to f being irreducible. Given a map $v\colon A \to Y$, there exists either a map $s\colon X \to Y$ such that $v = us$ or a map $t\colon Y \to X$ such that $u = tv$ (see [2]). From this fact follows that for two elements X and Y in $\mathscr{X}_A$ we always have $X \geqslant Y$ or $Y \geqslant X$. Moreover, $X \geqslant Y \geqslant X$ implies $X = Y$, since in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \xrightarrow{f_X} & X' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & X & \xrightarrow{f_X} & X' & \longrightarrow & 0 \end{array}$$

all vertical maps are isos. The same holds in $\mathscr{X}^C$, and we formulate this as follows.

LEMMA 5. *The relation $\geqslant$ defines a total ordering on $\mathscr{X}_A$ and $\mathscr{X}^C$ for all A and C.*

As a consequence, one obtains for different elements $X_1, X_2, \ldots, X_r$ in some $\mathscr{X}_A$ or $\mathscr{X}^C$ a chain

$$X_{i_1} \xrightarrow{u_1} X_{i_2} \xrightarrow{u_2} \cdots \xrightarrow{u_{r-1}} X_{i_r}$$

of non-isomorphisms u_i with $u_1 u_2 \ldots u_r \neq 0$. The Harada–Sai lemma states that in this situation $r < 2^n$, since $l(X_i) \leqslant n$ for all i (see [5, 8]). Therefore we have the following.

LEMMA 6. $|\mathscr{X}_A| < 2^n$ *and* $|\mathscr{X}^C| < 2^n$ *for all A and C.*

Now consider the sets $\mathscr{A} = \{A \in \operatorname{ind}\Lambda \mid \mathscr{X}_A \neq \varnothing\}$ and $\mathscr{C} = \{C \in \operatorname{ind}\Lambda \mid \mathscr{X}^C \neq \varnothing\}$. Recall the well-known fact that for all X in $\operatorname{mod}\Lambda$, $l(\operatorname{Tr} DX) \leqslant p_\Lambda^2\, l(X)$ (see [8]). Applying this formula for $A \in \mathscr{A}$ and $X \in \mathscr{X}_A$, we obtain $l(\operatorname{Tr} DA) \leqslant p_\Lambda^2 n$, since $l(A) \leqslant l(X) \leqslant n$. Similarly, we have for $C \in \mathscr{C}$ and $X \in \mathscr{X}^C$ that $l(C) \leqslant l(\operatorname{Tr} DX) \leqslant p_\Lambda^2 n$, since $l(X) \leqslant n$. Thus we have shown the following.

LEMMA 7. $l(\operatorname{Tr} DA) \leqslant p_\Lambda^2 n$ *and* $l(C) \leqslant p_\Lambda^2 n$ *for all $A \in \mathscr{A}$ and $C \in \mathscr{C}$.*

We are now able to apply Proposition 1 and its dual version, respectively. We obtain that

$$\{\operatorname{Tr} DA \mid A \in \mathscr{A} \setminus \operatorname{ind}_1 \Lambda\} \cup \{C \mid C \in \mathscr{C} \setminus \operatorname{ind}_1 \Lambda\} \subseteq \operatorname{ind}_m^1 \Lambda,$$

where the index $m = p_\Lambda^2 n$ is justified by Lemma 7. This has the following consequence for the cardinality of $\mathscr{A}$ and $\mathscr{C}$.

LEMMA 8. $|\mathscr{A}| \leqslant |\operatorname{ind}_m^1 \Lambda| + |\operatorname{ind}_1 \Lambda|$ *and* $|\mathscr{C}| \leqslant |\operatorname{ind}_m^1 \Lambda| + |\operatorname{ind}_1 \Lambda|$, *where $m = p_\Lambda^2 n$.*

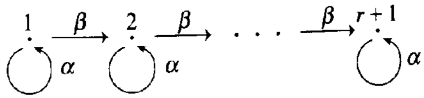
Now observe that according to our construction there is the following partition of $\operatorname{ind}_n \Lambda$.

LEMMA 9. $\operatorname{ind}_n \Lambda = (\dot{\bigcup}_{A \in \mathscr{A}} \mathscr{X}_A) \,\dot{\cup}\, (\dot{\bigcup}_{C \in \mathscr{C}} \mathscr{X}^C)$.

To complete the proof of the theorem, we combine this partition of $\operatorname{ind}_n \Lambda$ with the bounds for $|\mathscr{X}_A|$, $|\mathscr{X}^C|$, $|\mathscr{A}|$ and $|\mathscr{C}|$ of Lemmas 6 and 8. Thus we obtain the final relation for $|\operatorname{ind}_n \Lambda|$:

$$|\operatorname{ind}_n \Lambda| \leqslant 2^n |\mathscr{A}| + 2^n |\mathscr{C}| \leqslant 2^{n+1}(|\operatorname{ind}_m^1 \Lambda| + |\operatorname{ind}_1 \Lambda|).$$

EXAMPLE 1. Let k be a field and $r \in \mathbf{N}$. Denote by Λ_r the k-algebra given by the following quiver with relations.



$$\alpha\beta = \beta\alpha = \alpha^r = 0$$

Now let $p(n)$ be an arbitrary polynomial. There exists $r \in \mathbf{N}$ such that $2^r > p(r)(3r+2)$. It is not hard to check that we obtain

$$|\mathrm{ind}_r \Lambda_r| \geqslant 2^r > p(r)(3r+2) = p(r)(|\mathrm{ind}^1 \Lambda_r| + |\mathrm{ind}_1 \Lambda_r|).$$

This relation shows that in the formula of Theorem 3, the factor 2^{n+1} cannot be replaced by any polynomial. Note that $\mathrm{ind}\,\Lambda_r$ is finite for all r.

EXAMPLE 2. Let k be a field and $r \in \mathbf{N}$. Denote by Λ_r the k-algebra with the following quiver $Q_r = ((Q_r)_0, (Q_r)_1)$ and radical square zero:

$$Q_r = (\{1, 2, \ldots, r\}, \{\alpha_{ij} : i \to j \mid i, j \in (Q_r)_0\}).$$

We obtain for $r \geqslant 8$

$$|\mathrm{ind}_2 \Lambda_r| = r^2 + r > 8r = 2^3 |\mathrm{ind}_1 \Lambda_r|.$$

The example shows that in the formula of Theorem 3 the summand $|\mathrm{ind}^1_m \Lambda|$ cannot be neglected, even if $|\mathrm{ind}_n \Lambda|$ is finite.

To complete this paper we provide the following proof.

Proof of Corollary 2. Let Λ be a non-semisimple artin algebra. There exists a non-injective simple Λ-module A, and we can choose an irreducible map $f: A \to B$ such that B is indecomposable. Denote by C the cokernel of f. The assertion follows if $\alpha(C) = 1$. Otherwise, Proposition 1 states that C is simple. Choose again an irreducible map $g' : B' \to C$ such that B' is indecomposable. Using the fact that f and g' are irreducible, one obtains the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

Applying the duality $D: \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$, we deduce again from Proposition 1 that the exact sequence $0 \to DC \overset{Dg'}{\to} DB' \to DA' \to 0$ in $\mathrm{mod}\,\Lambda^{\mathrm{op}}$ has the desired property, namely that $\alpha(DA') = 1$.

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