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The endocategory of a module [1]

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Dedicated to the memory of Maurice Auslander

Given a ring Λ we introduce the endocategory $\mathcal{E}_M$ of a Λ-module M. It is an abelian subcategory of $\mathrm{Mod}(\Gamma)$ where $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$, and $\mathcal{E}_M$ is constructed in such a way that it is the smallest abelian subcategory of $\mathrm{Mod}(\Gamma)$ containing M regarded in the natural way as a Γ-module and all the endomorphisms of M induced by multiplication with an element from Λ. The first aim of this paper is to discuss some basic properties of this category. For instance, we show that $\mathcal{E}_M$ reflects various properties of the module M which are related to purity. Another aim is to give a functorial description of the Ziegler spectrum of Λ which is, by definition, a representative set of indecomposable pure-injective Λ-modules, together with a topology introduced by Ziegler [14]. We shall also discuss the relation between certain right and left Λ-modules which arises from the well-known duality between the categories of finitely presented functors from $\mathrm{mod}(\Lambda^{\mathrm{op}})$ and $\mathrm{mod}(\Lambda)$, respectively, into the category of abelian groups. In fact, this duality induces a bijection $M \mapsto DM$ between certain subsets of the Ziegler spectra of Λ and Λ^{op}, respectively, which has been studied by Herzog [6] using positive primitive formulas, and by Crawley-Boevey [1] using characters. The functoriality of this bijection is expressed by dualities between the endocategories $\mathcal{E}_M$ and $\mathcal{E}_{DM}$. A final example illustrates these dualities as well as their limitations.

This paper is dedicated to the memory of Maurice Auslander. In fact, the material presented here depends in an essential way on homological and categorical techniques and ideas that indelibly bear his mark.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Preliminaries

Let $\mathcal{C}$ be a skeletally small pre-additive category. A *(right) $\mathcal{C}$-module* is an additive functor from $\mathcal{C}^{\text{op}}$ into the category Ab of abelian groups and we denote by $\text{Mod}(\mathcal{C})$ the category of all $\mathcal{C}$-modules. Recall that $M \in \text{Mod}(\mathcal{C})$ is said to be *finitely presented* provided that there exists an exact sequence $\coprod_{i=1}^{n} \text{Hom}(\ ,X_i) \to \coprod_{j=1}^{m} \text{Hom}(\ ,Y_j) \to M \to 0$ in $\text{Mod}(\mathcal{C})$, and M is *finitely generated* if it is a quotient of a finitely presented module. The full subcategory of all finitely presented $\mathcal{C}$-modules is denoted by $\text{mod}(\mathcal{C})$. It is an additive catgory with cokernels, and $\mathcal{C}$ is called *coherent* if $\text{mod}(\mathcal{C})$ is abelian. Given a ring Λ we shall view Λ as a category with one object and we define $\mathcal{C}_\Lambda = \text{mod}(\Lambda^{\text{op}})^{\text{op}}$. Note that this category is coherent. The fully faithful functor

$$\text{Mod}(\Lambda) \to \text{Mod}(\mathcal{C}_\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. An exact sequence $0 \to L \to M \to N \to 0$ in $\text{Mod}(\Lambda)$ is said to be *pure-exact* if its image

$$0 \longrightarrow L \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow 0$$

under this functor is exact and $M \in \text{Mod}(\Lambda)$ is *pure-injective* if $M \otimes_\Lambda -$ is injective. Finally, recall that there is a well-known duality $d\colon \text{mod}(\mathcal{C}_\Lambda) \to \text{mod}(\mathcal{C}_{\Lambda^{\text{op}}})$ given by

$$d(M)(X) = \text{Hom}(M, X \otimes_\Lambda -).$$

Thus $\text{mod}(\mathcal{C}_\Lambda)$ has sufficently many projectives which are the functors $\text{Hom}_{\Lambda^{\text{op}}}(Y,\ )$, $Y \in \text{mod}(\Lambda^{\text{op}})$, and sufficently many injectives which are the functors $X \otimes_\Lambda -$, $X \in \text{mod}(\Lambda)$ [5].

We continue with some general facts. Recall that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* provided that for any exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ the object Y belongs to $\mathcal{S}$ iff X and Z belong to $\mathcal{S}$. For any Serre subcategory $\mathcal{S}$ of $\mathcal{A}$ one can form the *quotient category* $\mathcal{A}/\mathcal{S}$ which is abelian and admits an exact *quotient functor* $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ with $\text{Ker}(q) = \mathcal{S}$ [4]. The functor q is characterized by the property that any exact functor $f\colon \mathcal{A} \to \mathcal{B}$ between abelian categories with $\text{Ker}(f) = \mathcal{S}$ induces, up to isomorphism, a unique faithful and exact functor $\bar{f}\colon \mathcal{A}/\mathcal{S} \to \mathcal{B}$ such that $f = \bar{f} \circ q$. The subcategory $\mathcal{S}$ is called *localizing* if q has a right adjoint.

Suppose now that $\mathcal{C}$ is a skeletally small pre-additive category which is coherent, e.g. $\mathcal{C} = \mathcal{C}_\Lambda$. Let $\mathcal{S}$ be a Serre subcategory of $\text{mod}(\mathcal{C})$ and denote by $\vec{\mathcal{S}}$ the full subcategory of $\text{Mod}(\mathcal{C})$ which consists of all direct limits $\varinjlim X_i$ with $X_i \in \mathcal{S}$ for all i.

Lemma 1.1 *The subcategory $\vec{\mathcal{S}}$ is localizing and the quotient functor $q: \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C})/\vec{\mathcal{S}}$ has the following properties.*

(1) *Let $M \in \mathrm{Mod}(\mathcal{C})$ with $\mathrm{Hom}(\mathcal{S}, M) = 0$. Then M is injective iff $q(M)$ is injective. In that case q induces an isomorphism $\mathrm{Hom}(X, M) \to \mathrm{Hom}(q(X), q(M))$ for every $X \in \mathrm{Mod}(\mathcal{C})$.*

(2) *The composition of q with the inclusion $\mathrm{mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{C})$ induces a fully faithful and exact functor $\mathrm{mod}(\mathcal{C})/\mathcal{S} \to \mathrm{Mod}(\mathcal{C})/\vec{\mathcal{S}}$ which makes the following diagram of canonical functors commutative.*

$$\begin{array}{ccc} \mathrm{mod}(\mathcal{C}) & \longrightarrow & \mathrm{mod}(\mathcal{C})/\mathcal{S} \\ \downarrow & & \downarrow \\ \mathrm{Mod}(\mathcal{C}) & \xrightarrow{q} & \mathrm{Mod}(\mathcal{C})/\vec{\mathcal{S}} \end{array}$$

Proof: For a proof see [9].

We shall also need the well-known fact that $\mathrm{Mod}(\mathcal{C})$ has injective envelopes. Moreover, $\mathrm{Mod}(\mathcal{C})/\mathcal{S}$ has injective envelopes provided that $\mathcal{S}$ is a localizing subcategory.

2. The endocategory

Let M be a Λ-module and let $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$. Identifying $\mathrm{End}(M \otimes_\Lambda -)^{\mathrm{op}}$ in $\mathrm{Mod}(\mathcal{C}_\Lambda)$ with Γ we obtain a contravariant functor

$$h_M: \mathrm{mod}(\mathcal{C}_\Lambda) \to \mathrm{Mod}(\Gamma), \quad X \mapsto \mathrm{Hom}(X, M \otimes_\Lambda -)$$

which is easily seen to be exact. We denote by $\mathcal{S}_M$ the kernel of h_M which is a Serre subcategory. The *endocategory* $\mathcal{E}_M$ of M is, by definition, the image of the induced functor $\mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M \to \mathrm{Mod}(\Gamma)$ and we shall assume that this subcategory of $\mathrm{Mod}(\Gamma)$ is closed under isomorphisms. The next lemma is a reformulation of this definition.

Lemma 2.1 *The funcor h_M induces a duality $\mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M \to \mathcal{E}_M$ which makes the following diagram of functors commutative.*

$$\begin{array}{ccccc} \mathrm{mod}(\mathcal{C}_\Lambda) & \longrightarrow & \mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M & \longrightarrow & \mathcal{E}_M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Mod}(\mathcal{C}_\Lambda) & \xrightarrow{q} & \mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M & \xrightarrow{\mathrm{Hom}(\ ,q(M\otimes_\Lambda -))} & \mathrm{Mod}(\Gamma) \end{array}$$

Proof: We use Lemma 1.1. The isomorphism $\mathrm{End}(M \otimes_\Lambda -) \cong \mathrm{End}(q(M \otimes_\Lambda -))$ allows to identify Γ and $\mathrm{End}(q(M \otimes_\Lambda -))^{\mathrm{op}}$. The assertion then follows from this lemma and the definition of $\mathcal{E}_M$.

Recall that an *abelian subcategory* of an abelian category is a subcategory which is abelian and has an exact inclusion functor.

Proposition 2.2 *The endocategory $\mathcal{E}_M$ has the following properties.*

(1) *$\mathcal{E}_M$ is an abelian subcategory of* $\mathrm{Mod}(\Gamma)$ *which is closed under isomorphisms.*

(2) *$\mathcal{E}_M$ contains M and* $\mathrm{End}_{\mathcal{E}_M}(M)$ *contains the image of the canonical morphism* $\Lambda \to \mathrm{End}_\Gamma(M)$.

Moreover, $\mathcal{E}_M$ is a subcategory of any other category $\mathcal{E}$ which satisfies (1) – (2).

Proof: $\mathrm{mod}(\mathcal{C}_\Lambda)$ is the free abelian category over Λ, i.e. any functor $\Lambda \to \mathcal{E}$ to an abelian category extends uniquely to an exact functor $\mathrm{mod}(\mathcal{C}_\Lambda) \to \mathcal{E}$ [5, Lemma 1]. Applying this fact to the canonical functor $\Lambda \to \mathrm{Mod}(\Gamma)$, $\Lambda \mapsto M$, the assertion follows.

We list some further properties which follow immediately from the definition.

Proposition 2.3 (1) *$\mathcal{E}_M$ consists precisely of all cokernels of morphisms* $\mathrm{Hom}_\Lambda(\varphi, M)$ *with φ in* $\mathrm{mod}(\Lambda)$.

(2) *$\mathcal{E}_M$ consists precisely of all kernels of morphisms $M \otimes_\Lambda \psi$ with ψ in* $\mathrm{mod}(\Lambda^{\mathrm{op}})$.

(3) *Every object in $\mathcal{E}_M$ is a subquotient in* $\mathrm{Mod}(\Gamma)$ *of some finite coproduct M^n.*

Proof: Use the fact that $\mathrm{mod}(\mathcal{C}_\Lambda)$ has sufficently many injectives and projectives.

Our discussion suggests the following definition. A pair M and N of Λ-modules is called *purely equivalent* provided that $\mathcal{S}_M = \mathcal{S}_N$.

Lemma 2.4 *The following are equivalent.*

(1) *M and N are purely equivalent.*

(2) $\mathrm{Hom}_\Lambda(\varphi, M)$ *is an epi iff* $\mathrm{Hom}_\Lambda(\varphi, N)$ *is an epi for every φ in* $\mathrm{mod}(\Lambda)$.

(3) *$M \otimes_\Lambda \psi$ is a mono iff $N \otimes_\Lambda \psi$ is a mono for every ψ in* $\mathrm{mod}(\Lambda^{\mathrm{op}})$.

Proof: Use the fact that $\mathrm{mod}(\mathcal{C}_\Lambda)$ has sufficently many injectives and projectives.

Proposition 2.5 *Let M and N be Λ-modules which are purely equivalent. Then there is an equivalence $e\colon \mathcal{E}_M \to \mathcal{E}_N$ which is, up to isomorphism, uniquely determined by functorial isomorphisms*

$$e(\mathrm{Hom}_\Lambda(X, M)) \cong \mathrm{Hom}_\Lambda(X, N) \qquad \textit{and} \qquad e(M \otimes_\Lambda Y) \cong N \otimes_\Lambda Y$$

for all $X \in \mathrm{mod}(\Lambda)$ and $Y \in \mathrm{mod}(\Lambda^{\mathrm{op}})$.

Proof: The assertion follows from Lemma 2.1.

The endocategory $\mathcal{E}_M$ is intimately related to the collection of subgroups of finite definition which correspond to M. Recall that for $X \in \text{mod}(\Lambda)$ a *subgroup of finite definition* of $\text{Hom}_\Lambda(X, M)$ is the image M_φ of a map $\text{Hom}_\Lambda(\varphi, M)$ arising from a morphism $\varphi: X \to Y$ in $\text{mod}(\Lambda)$. The subgroups of finite definition form a lattice in $\text{Hom}_\Lambda(X, M)$ which we denote by $\text{Latt}_\Lambda(X, M)$.

Proposition 2.6 (1) *The lattice of subobjects of* $\text{Hom}_\Lambda(X, M)$ *in* $\mathcal{E}_M$ *coincides with* $\text{Latt}_\Lambda(X, M)$.

(2) *Every object in* $\mathcal{E}_M$ *is isomorphic to* $\text{Hom}_\Lambda(X, M)/M_\varphi$ *for some* $X \in \text{mod}(\Lambda)$ *and some* $M_\varphi \in \text{Latt}_\Lambda(X, M)$.

Proof: Use Proposition 2.3.

The next few results show that the endocategory $\mathcal{E}_M$ reflects various properties of the module M which are related to purity.

Proposition 2.7 *M is pure-injective if and only if for every $X \in \mathcal{E}_M$ and for every codirected family $(X_i)_{i \in I}$ of subobjects of X in $\mathcal{E}_M$ the canonical map $X \to \varprojlim X/X_i$ in* $\text{Mod}(\Gamma)$ *is an epi.*

Proof: Adapt the argument of [8, Corollary 7.4].

Recall that M is Σ-*pure-injective* if any coproduct $\coprod_I M$ is pure-injective. It is well-known that M is Σ-pure-injective iff the descending chain condition holds in $\text{Latt}_\Lambda(\Lambda, M)$. Thus we obtain the following characterization.

Corollary 2.8 *M is Σ-pure-injective if and only if $\mathcal{E}_M$ is artinian, i.e. any object in $\mathcal{E}_M$ is artinian.*

A module M is called *endofinite* if M is of finite length in $\text{Mod}(\Gamma)$. It has been shown by Crawley-Boevey that M is endofinite iff ascending and descending chain condition hold in $\text{Latt}_\Lambda(\Lambda, M)$ [1, Proposition 4.1]. This has the following consequence.

Corollary 2.9 *M is endofinite if and only if $\mathcal{E}_M$ is a length category, i.e. any object in $\mathcal{E}_M$ is of finite length.*

The next result gives a condition which guarantees that $\mathcal{E}_M$ is a full subcategory of $\text{Mod}(\Gamma)$.

Theorem 2.10 *If M is endofinite, then $\mathcal{E}_M$ is the full subcategory of all subquotients in* $\text{Mod}(\Gamma)$ *of finite coproducts M^n.*

Proof: Suppose that M is endofinite. Using the argument of [1, Proposition 4.1] it can be shown that every Γ-submodule of $\operatorname{Hom}_\Lambda(X, M)$ is a subgroup of finite definition. Thus $\mathcal{E}_M$ consists of all subquotients of finite coproducts M^n. It remains to prove that $\mathcal{E}_M$ is a full subcategory. We shall use the quotient functor $q\colon \operatorname{Mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$. By Lemma 1.1 and Lemma 2.1 the subcategory $\mathcal{E}_M$ is full iff the functor

$$\operatorname{Hom}(\ , q(M \otimes_\Lambda -))\colon \operatorname{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M \longrightarrow \operatorname{Mod}(\Gamma)$$

is full when it is restricted to the objects $q(X)$ with $X \in \operatorname{mod}(\mathcal{C}_\Lambda)$. Applying the argument of [12, Lemma 4.1] the assertion follows from the fact that $q(M \otimes_\Lambda -)$ is an injective cogenerator and that $\operatorname{Hom}(q(X), q(M \otimes_\Lambda -)) \cong \operatorname{Hom}(X, M \otimes_\Lambda -)$ is finitely generated over $\operatorname{End}(q(M \otimes_\Lambda -))^{\mathrm{op}} \cong \Gamma$ for each $X \in \operatorname{mod}(\mathcal{C}_\Lambda)$.

3. Ziegler's topology

Let $\mathcal{C}$ be a skeletally small pre-additive category. A set of indecomposable injective $\mathcal{C}$-modules which meets each isomorphism class exactly once is called the *spectrum* of $\mathcal{C}$ and is denoted by $\operatorname{sp}(\mathcal{C})$. The *Ziegler spectrum* $\operatorname{Zsp}(\Lambda)$ of a ring Λ is by definition a set of indecomposable pure-injective Λ-modules which meets each isomorphism class exactly once. Thus we may assume that the functor $M \mapsto M \otimes_\Lambda -$ identifies $\operatorname{Zsp}(\Lambda)$ with $\operatorname{sp}(\mathcal{C}_\Lambda)$ since any injective $\mathcal{C}_\Lambda$-module is isomorphic to $M \otimes_\Lambda -$ for some $M \in \operatorname{Mod}(\Lambda)$. In [14] Ziegler introduces a topology on the set $\operatorname{Zsp}(\Lambda)$. We obtain this topology as follows.

Let $\mathcal{C}$ be a skeletally small pre-additive category which is coherent, e.g. $\mathcal{C} = \mathcal{C}_\Lambda$. For a subset $\mathcal{U}$ of $\operatorname{sp}(\mathcal{C})$ denote by $\Sigma(\mathcal{U})$ the Serre subcategory of $\operatorname{mod}(\mathcal{C})$ formed by the objects $X \in \operatorname{mod}(\mathcal{C})$ satisfying $\operatorname{Hom}(X, \mathcal{U}) = 0$. For a subcategory $\mathcal{S}$ of $\operatorname{mod}(\mathcal{C})$ let $\Upsilon(\mathcal{S}) = \{M \in \operatorname{sp}(\mathcal{C}) \mid \operatorname{Hom}(\mathcal{S}, M) = 0\}$.

Lemma 3.1 *The assignment*

$$\mathcal{U} \mapsto \overline{\mathcal{U}} = \Upsilon \circ \Sigma(\mathcal{U})$$

is a closure operator on the spectrum $\operatorname{sp}(\mathcal{C})$ *of* $\mathcal{A}$, *i.e. the subsets* $\mathcal{U} \subseteq \operatorname{sp}(\mathcal{C})$ *satisfying* $\overline{\mathcal{U}} = \mathcal{U}$ *form the closed sets of a topology on* $\operatorname{sp}(\mathcal{C})$.

Proof: It is easily checked that $\overline{\emptyset} = \emptyset$, $\mathcal{U} \subseteq \overline{\mathcal{U}}$ and $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$. It remains to show that $\overline{\mathcal{U}_1 \cup \mathcal{U}_2} = \overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2}$. From $\Sigma(\mathcal{U}_1 \cup \mathcal{U}_2) \subseteq \Sigma(\mathcal{U}_1) \cap \Sigma(\mathcal{U}_2)$ it follows that $\overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2} \subseteq \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$. Now choose $M \in \operatorname{sp}(\mathcal{C})$ such that $M \notin \overline{\mathcal{U}_1} \cup \overline{\mathcal{U}_2}$. We claim that this implies $M \notin \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$. From the definitions one obtains non-zero morphisms $\varphi_i\colon X_i \to M$ such that $X_i \in \Sigma(\mathcal{U}_i)$. We have $\operatorname{Im}(\varphi_1) \cap \operatorname{Im}(\varphi_2) \neq 0$ since M is indecomposable injective. Choosing $U \subseteq \operatorname{Im}(\varphi_1) \cap \operatorname{Im}(\varphi_2)$ finitely generated one can find finitely generated submodules $Y_i \subseteq X_i$ such that $\varphi_i(Y_i) = U$. We obtain the following exact commutative diagram where the vertical morphisms are the canonical monos.

$$\begin{array}{ccccccccc} 0 & \to & W & \stackrel{\left[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right]}{\to} & Y_1 \coprod Y_2 & \to & U & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & V & \to & X_1 \coprod X_2 & \stackrel{[\varphi_1\, \varphi_2]}{\to} & M & & \end{array}$$

The morphisms ψ_i being epis we find finitely generated submodules W_i of W such that $\psi_i(W_i) = Y_i$. Let $X = Y_1 \coprod Y_2 / \left[\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right](W_1 + W_2)$. We have $X \in \operatorname{mod}(\mathcal{C})$ since $\operatorname{mod}(\mathcal{C})$ is abelian and it is easily checked that $\operatorname{Hom}(X, M) \neq 0$. On the other hand $X \in \Sigma(\mathcal{U}_1 \cup \mathcal{U}_2)$ since X is a quotient of each Y_i. Therefore $M \notin \overline{\mathcal{U}_1 \cup \mathcal{U}_2}$ and the proof is complete.

It has been observed by I. Herzog [7] and M. Prest that Ziegler's closed sets in $\operatorname{Zsp}(\Lambda)$ are in bijective correspondence to Serre subcategories of $\operatorname{mod}(\mathcal{C}_\Lambda)$. In our context this observation takes the following form.

Theorem 3.2 *There is a bijective inclusion reversing correspondence between Serre subcategories of* $\operatorname{mod}(\mathcal{C})$ *and closed subsets of* $\operatorname{sp}(\mathcal{C})$. *The correspondence is given by*

$$\mathcal{S} \mapsto \Upsilon(\mathcal{S}) \qquad \textit{and} \qquad \mathcal{U} \mapsto \Sigma(\mathcal{U}).$$

Proof: We check that the assignments are inverse to each other. Given a Serre subcategory $\mathcal{S}$ of $\operatorname{mod}(\mathcal{C})$ the subcategory $\mathcal{T} = \vec{\mathcal{S}}$ of $\operatorname{Mod}(\mathcal{C})$ is localizing by Lemma 1.1 and therefore the pair $(\mathcal{T}, \mathcal{F})$ with $\mathcal{F} = \{X \mid \operatorname{Hom}(\mathcal{S}, X) = 0\}$ forms a hereditary torsion theory for $\operatorname{Mod}(\mathcal{C})$. Moreover, the set $\Upsilon(\mathcal{S}) = \mathcal{F} \cap \operatorname{sp}(\mathcal{C})$ cogenerates $\mathcal{F}$ since the quotient $\operatorname{Mod}(\mathcal{C})/\vec{\mathcal{S}}$ is locally finitely presented [9]. It follows that $\Sigma \circ \Upsilon(\mathcal{S}) = \mathcal{S}$. Conversely, $\Upsilon \circ \Sigma(\mathcal{U}) = \mathcal{U}$ is clear since $\mathcal{U}$ is closed.

4. Duality

We combine the correspondence $\mathcal{U} \mapsto \Sigma(\mathcal{U})$ between closed subsets of $\operatorname{Zsp}(\Lambda)$ and Serre subcategories of $\operatorname{mod}(\mathcal{C}_\Lambda)$ with the duality $d \colon \operatorname{mod}(\mathcal{C}_\Lambda) \to \operatorname{mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})$.

Proposition 4.1 *There is a unique inclusion preserving bijection* $\mathcal{U} \mapsto D(\mathcal{U})$ *between the closed subsets of* $\operatorname{Zsp}(\Lambda)$ *and* $\operatorname{Zsp}(\Lambda^{\mathrm{op}})$ *such that* $\Sigma(D(\mathcal{U})) = d(\Sigma(\mathcal{U}))$.

We shall use this fact to construct a homeomorphism between certain subsets of the Ziegler spectra of Λ and Λ^{op}, respectively.

Suppose there is given a pair of topological spaces $\mathcal{X}$ and $\mathcal{Y}$ and an inclusion preserving bijection $\mathcal{U} \mapsto D(\mathcal{U})$ between the closed subsets of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Recall that a point M is *isolated* provided that $\{M\}$ is open. We call a point M in $\mathcal{X}$ or $\mathcal{Y}$ *reflexive* provided that M is isolated in its closure $\overline{\{M\}}$ and $D(\overline{\{M\}})$ contains also an isolated point, say N. The point N is uniquely determined by M and we call $DM = N$ the *reflection* of M. Denote by $\mathcal{R}(\mathcal{X})$ and $\mathcal{R}(\mathcal{Y})$ the set of reflexive points in $\mathcal{X}$ and $\mathcal{Y}$, respectively, equipped with their induced topology.

Lemma 4.2 *Reflection, the map which sends M to DM, is a homeomorphism between $\mathcal{R}(\mathcal{X})$ and $\mathcal{R}(\mathcal{Y})$ satisfying $D^2 = \mathrm{id}$.*

Proof: The reflection of a reflexive point M is again reflexive and satisfies $DDM = M$. Thus D is a bijection between $\mathcal{R}(\mathcal{X})$ and $\mathcal{R}(\mathcal{Y})$ satisfying $D^2 = \mathrm{id}$. Let $\mathcal{U}$ be a closed subset of $\mathcal{R}(\mathcal{X})$, say $\mathcal{U} = \mathcal{V} \cap \mathcal{R}(\mathcal{X})$ for some closed subset $\mathcal{V}$ of $\mathcal{X}$. It is easily checked that $\{DM \mid M \in \mathcal{U}\} = D(\mathcal{V}) \cap \mathcal{R}(\mathcal{Y})$ since $M \in \mathcal{U}$ iff the closure of M in $\mathcal{X}$ is contained in $\mathcal{V}$. This shows that D is continuous and the proof is complete.

We call a pair of modules $M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ *purely opposed* provided that $\mathcal{S}_N = d(\mathcal{S}_M)$.

Lemma 4.3 *The following are equivalent.*

(1) *M and N are purely opposed.*

(2) *$\mathrm{Hom}_\Lambda(\varphi, M)$ is an epi iff $\varphi \otimes_\Lambda N$ is a mono for every φ in $\mathrm{mod}(\Lambda)$.*

(3) *$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\psi, N)$ is an epi iff $M \otimes_\Lambda \psi$ is a mono for every ψ in $\mathrm{mod}(\Lambda^{\mathrm{op}})$.*

Proof: The proof is analogous to that of Lemma 2.4.

Proposition 4.4 *Let $M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ be modules which are purely opposed. Then there is a duality $e: \mathcal{E}_M \to \mathcal{E}_N$ which is, up to isomorphism, uniquely determined by functorial isomorphisms*

$$e(\mathrm{Hom}_\Lambda(X, M)) \cong X \otimes_\Lambda N \qquad \text{and} \qquad e(M \otimes_\Lambda Y) \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y, N)$$

for all $X \in \mathrm{mod}(\Lambda)$ and $Y \in \mathrm{mod}(\Lambda^{\mathrm{op}})$.

Proof: The duality $d: \mathrm{mod}(\mathcal{C}_\Lambda) \to \mathrm{mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})$ induces a duality $\mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M \to \mathrm{mod}(\mathcal{C}_{\Lambda^{\mathrm{op}}})/\mathcal{S}_N$. The assertion then follows with Lemma 2.1.

We mention two properties which are preserved by the above duality. Here $\mathrm{ann}_\Lambda(M)$ denotes the annihilator of a Λ-module M.

Proposition 4.5 *Let $M \in \mathrm{Mod}(\Lambda)$ and $N \in \mathrm{Mod}(\Lambda^{\mathrm{op}})$ be modules which are purely opposed.*

(1) $\mathrm{Latt}_{\Lambda^{\mathrm{op}}}(\Lambda, N) \cong \mathrm{Latt}_{\Lambda}(\Lambda, M)^{\mathrm{op}}$.

(2) $\mathrm{ann}_{\Lambda^{\mathrm{op}}}(N) = \mathrm{ann}_{\Lambda}(M)$.

Proof: Use Proposition 4.4 and Proposition 2.6.

Our discussion suggests the following definition. An indecomposable pure-injective Λ-module is called *finitely reflexive* provided that M is isomorphic to a point in $\mathrm{Zsp}(\Lambda)$ which is reflexive with respect to the topologies of $\mathrm{Zsp}(\Lambda)$ and $\mathrm{Zsp}(\Lambda^{\mathrm{op}})$. We include the following property.

Lemma 4.6 *Let $M \in \mathrm{Zsp}(\Lambda)$ and suppose that M is isolated in its closure $\overline{\{M\}}$. Then any point $N \in \mathrm{Zsp}(\Lambda)$ is purely equivalent to M if and only if $N = M$.*

Proof: If N and M are purely equivalent, then $\overline{\{N\}} = \overline{\{M\}}$. In particular, $N \in \overline{\{M\}}$. On the other hand, $M \notin \overline{\{N\}}$ for $N \neq M$ since M is isolated in $\overline{\{M\}}$. Thus $N = M$.

The next result summarizes our discussion. It extends Herzog's elementary duality [6] (see also [11]).

Corollary 4.7 *The map which sends M to DM is a homeomorphism between the sets of reflexive points in $\mathrm{Zsp}(\Lambda)$ and $\mathrm{Zsp}(\Lambda^{\mathrm{op}})$, respectively, satisfying $D^2 = \mathrm{id}$. A reflexive point $M \in \mathrm{Zsp}(\Lambda)$ has the following properties.*

(1) *Any point $N \in \mathrm{Zsp}(\Lambda)$ is purely equivalent to M if and only if $N = M$.*

(2) *Any point $N \in \mathrm{Zsp}(\Lambda^{\mathrm{op}})$ is purely opposed to M if and only if $N = DM$.*

(3) *There is a duality $e\colon \mathcal{E}_M \to \mathcal{E}_{DM}$ which is, up to isomorphism, uniquely determined by functorial isomorphisms*

$$e(\mathrm{Hom}_{\Lambda}(X, M)) \cong X \otimes_{\Lambda} DM \qquad \text{and} \qquad e(M \otimes_{\Lambda} Y) \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y, DM)$$

for all $X \in \mathrm{mod}(\Lambda)$ and $Y \in \mathrm{mod}(\Lambda^{\mathrm{op}})$.

In view of the preceding result there are two natural questions.

– Which points in $\mathrm{Zsp}(\Lambda)$ are reflexive?

– Which properties of M and DM, respectively, are preserved by the duality $\mathcal{E}_M \to \mathcal{E}_{DM}$?

We devote the rest of this paper to a discussion of these problems. Note that a first answer is given in Proposition 4.5.

5. Simply reflexive modules

We use the endocategory to introduce a class of indecomposable pure-injective modules. We start with some technical lemmas. To this end fix an indecomposable pure-injective Λ-module M with $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$.

Lemma 5.1 *Let $X \in \operatorname{mod}(\mathcal{C}_\Lambda)$ and suppose that $h_M(X)$ is simple in $\mathcal{E}_M$. Then the quotient functor $q\colon \operatorname{Mod}(\mathcal{C}_\Lambda) \to \operatorname{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$ sends X to a simple object and $M \otimes_\Lambda -$ to an injective envelope of $q(X)$.*

Proof: The object $q(X)$ is simple precisely if for any exact sequence $0 \to X' \to X \to X'' \to 0$ either $X' \in \vec{\mathcal{S}}_M$ or $X'' \in \vec{\mathcal{S}}_M$. Writing $X' = \varinjlim X_i$ as a direct limit of all its finitely generated submodules we obtain induced sequences $0 \to X_i \to X \to X/X_i \to 0$ in $\operatorname{mod}(\mathcal{C}_\Lambda)$ since $\mathcal{C}_\Lambda$ is coherent. Now, for any i either $X_i \in \mathcal{S}_M$ or $X/X_i \in \mathcal{S}_M$ since $q(X)$ is simple in $\operatorname{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M$ by assumption and Lemma 2.1. If $X/X_i \in \mathcal{S}_M$ for some i, then $X'' \in \vec{\mathcal{S}}_M$ since X'' is a quotient of X/X_i. Otherwise all $X_i \in \mathcal{S}_M$ and therefore $X' \in \vec{\mathcal{S}}_M$. Thus $q(X)$ is simple. If $\varphi \in \operatorname{Hom}(X, M \otimes_\Lambda -)$ is a non-zero morphism, then $q(\varphi) \neq 0$ by Lemma 1.1. Thus $q(M \otimes_\Lambda -)$ is an injective envelope of $q(X)$ since $q(M \otimes_\Lambda -)$ is indecomposable injective.

Lemma 5.2 *Let N be an injective envelope of a simple object X in any abelian category and let $\Sigma = \operatorname{End}(N)^{\mathrm{op}}$.*

(1) *$Y = \operatorname{Hom}(X, N)$ is a simple Σ-module.*

(2) *The functor $\operatorname{Hom}(\ , N)$ induces an isomorphism*

$$\operatorname{End}(N)/\operatorname{rad}(\operatorname{End}(N)) \cong \operatorname{End}(X) \cong \operatorname{End}_\Sigma(Y)^{\mathrm{op}}.$$

Proof: Straightforward.

Lemma 5.3 *The following are equivalent for an object S in $\mathcal{E}_M$.*

(1) *S is simple in $\mathcal{E}_M$.*

(2) *S is simple in $\operatorname{Mod}(\Gamma)$.*

Proof: Clearly, $S = h_M(X)$ is simple in $\mathcal{E}_M$ if it is simple in $\operatorname{Mod}(\Gamma)$. The other direction follows from Lemma 5.1 and Lemma 5.2.

We call an indecomposable pure-injective module M *simply reflexive* provided that $\mathcal{E}_M$ contains a simple object.

Proposition 5.4 *The following are equivalent for an indecomposable pure-injective Λ-module M.*

(1) *M is simply reflexive.*

(2) *There exists a map $X \to Y$ in $\operatorname{mod}(\Lambda)$ such that the cokernel of the induced map $\operatorname{Hom}(Y, M) \to \operatorname{Hom}(X, M)$ is a simple $\operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module.*

Proof: The assertion follows from Lemma 5.3 since the objects in $\mathcal{E}_M$ are precisely the cokernels of maps $\mathrm{Hom}(\varphi, M)$ arising from a map φ in $\mathrm{mod}(\Lambda)$ by Proposition 2.3.

Proposition 5.5 *Any simply reflexive module is finitely reflexive.*

Proof: We shall identify $\mathrm{Zsp}(\Lambda) = \mathrm{sp}(\mathcal{C}_\Lambda)$. Also, we keep the notation of Lemma 5.1. Now suppose that $M \in \mathrm{Zsp}(\Lambda)$ is simply reflexive. Consider the open set $\mathcal{U} = \{N \in \mathrm{Zsp}(\Lambda) \mid \mathrm{Hom}(X, N) \neq 0\}$ and note that $\Upsilon(\mathcal{S}_M)$ is the closure of $\{M\}$. Using the fact that an injective envelope of $q(X)$ is unique up to isomorphism it follows from Lemma 5.1 that $\mathcal{U} \cap \Upsilon(\mathcal{S}_M) = \{M\}$. Thus M is isolated in its closure. Applying the same argument one shows that the intersection of $\{N \in \mathrm{Zsp}(\Lambda^{\mathrm{op}}) \mid \mathrm{Hom}(d(X), N) \neq 0\}$ with $D(\Upsilon(\mathcal{S}_M))$ contains precisely one point. Thus M is reflexive.

Proposition 5.6 *Any indecomposable Σ-pure-injective module is simply reflexive.*

Proof: Let M be Σ-pure-injective. The endocategory $\mathcal{E}_M$ is artinian by Corollary 2.8 and therefore $\mathcal{E}_M$ contains a simple object.

Having shown in Proposition 2.2 that the endocategory of a module M is determined by $\mathrm{End}_\Lambda(M)$ and its action on M, we obtain now some converse.

Proposition 5.7 *Let M be an indecomposable pure-injective Λ-module and suppose that M is simply reflexive. Then the endomorphism ring $\mathrm{End}_\Lambda(M)$ is, up to isomorphism, uniquely determined by the endocategory $\mathcal{E}_M$.*

Proof: We use the fact that $\mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$ is equivalent to the category $\mathrm{Lex}((\mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M)^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors $(\mathrm{mod}(\mathcal{C}_\Lambda)/\mathcal{S}_M)^{\mathrm{op}} \to \mathrm{Ab}$ [9]. Thus $\mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$ is equivalent to $\mathrm{Lex}(\mathcal{E}_M, \mathrm{Ab})$ by Lemma 2.1. Taking a simple object $S \in \mathcal{E}_M$ the functor $\mathrm{Hom}(S, \ )$ is simple in $\mathrm{Lex}(\mathcal{E}_M, \mathrm{Ab})$ and therefore an injective envelope N of $\mathrm{Hom}(S, \ )$ corresponds to $q(M \otimes_\Lambda -)$ under an equivalence $\mathrm{Lex}(\mathcal{E}_M, \mathrm{Ab}) \to \mathrm{Mod}(\mathcal{C}_\Lambda)/\vec{\mathcal{S}}_M$ by Lemma 5.1. Using Lemma 1.1 we deduce that $\mathrm{End}(N)$ and $\mathrm{End}_\Lambda(M)$ are isomorphic and the assertion follows.

Example 5.8 (1) A point $M \in \mathrm{Zsp}(\Lambda)$ is simply reflexive iff it is reflexive in the sense of Herzog [6]. It follows from Corollary 4.7 that $M \mapsto DM$ induces a bijection between the isomorphism classes of simply reflexive right and left Λ-modules which coincides with the duality studied by Herzog [6].

(2) The preceding result shows that any endofinite point $M \in \mathrm{Zsp}(\Lambda)$ is reflexive. It follows from Corollary 4.7 that $M \mapsto DM$ induces a bijection

between the isomorphism classes of indecomposable endofinite right and left Λ-modules which coincides with the duality studied by Crawley-Boevey [1].

(3) Let Λ be an artin algebra which has Krull-Gabriel dimension in the sense of Geigle, e.g. a tame hereditary algebra [3]. Then it can be shown that any point in $\mathrm{Zsp}(\Lambda)$ is simply reflexive. Thus $\mathrm{Zsp}(\Lambda)$ and $\mathrm{Zsp}(\Lambda^{\mathrm{op}})$ are homeomorphic.

(4) Let $M \in \mathrm{Zsp}(\Lambda)$ be simply reflexive. Suppose also that M is a Λ-Γ-bimodule and that $I \in \mathrm{Mod}(\Gamma)$ is an injective cogenerator. Then it is shown in [10] that DM is isomorphic to a direct summand of the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$.

To formulate the next result we denote for an indecomposable pure-injective Λ-module M by $\Delta(M)$ the factor ring $\mathrm{End}_\Lambda(M)/\mathrm{rad}(\mathrm{End}_\Lambda(M))$. Note that $\Delta(M)$ is a division ring since $\mathrm{End}_\Lambda(M)$ is local.

Theorem 5.9 *If M is simply reflexive and $S \in \mathcal{E}_M$ is any simple object, then*

$$\Delta(M)^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{E}_M}(S) = \mathrm{End}_\Gamma(S).$$

Proof: We use Lemma 5.1 and its notation. In particular we suppose that $S = h_M(X)$ is simple. Thus the radical factor of $\mathrm{End}(q(M \otimes_\Lambda -))$ is isomorphic to $\mathrm{End}(q(X))$ by Lemma 5.2 since $q(M \otimes_\Lambda -)$ is an injective envelope of the simple object $q(X)$. The following isomorphisms $\mathrm{End}_\Lambda(M) \cong \mathrm{End}(M \otimes_\Lambda -) \cong \mathrm{End}(q(M \otimes_\Lambda -))$ then show that $\Delta(M)^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{E}_M}(S)$ since $\mathrm{End}(q(X))^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{E}_M}(S)$ by Lemma 1.1 and Lemma 2.1. Inparticular $\mathrm{End}_{\mathcal{E}_M}(S) = \mathrm{End}_\Gamma(S)$ follows from Lemma 5.2.

Corollary 5.10 *If $M \in \mathrm{Zsp}(\Lambda)$ is simply reflexive, then $\Delta(DM) \cong \Delta(M)^{\mathrm{op}}$.*

Remark 5.11 The preceding result can be derived from a result of Herzog [6]. It shows that Crawley-Boevey's notion of generical wildness is right-left symmetric [1].

Theorem 5.12 *If $M \in \mathrm{Zsp}(\Lambda)$ is endofinite, then*

$$\mathrm{End}_{\mathrm{End}_{\Lambda^{\mathrm{op}}}(DM)^{\mathrm{op}}}(DM) \cong \mathrm{End}_{\mathrm{End}_\Lambda(M)^{\mathrm{op}}}(M)^{\mathrm{op}}.$$

In particular, $\mathrm{End}_{\Lambda^{\mathrm{op}}}(DM)$ and $\mathrm{End}_\Lambda(M)$ have isomorphic centers.

Proof: The endocategories $\mathcal{E}_M$ and $\mathcal{E}_{DM}$ are full subcategories of $\mathrm{Mod}(\mathrm{End}_\Lambda(M)^{\mathrm{op}})$ and $\mathrm{Mod}(\mathrm{End}_{\Lambda^{\mathrm{op}}}(DM)^{\mathrm{op}})$, respectively, by Theorem 2.10. Using the duality $\mathcal{E}_M \to \mathcal{E}_{DM}$ the assertion follows. The statement about the centers is a direct consequence since for any Λ-module M the center of $\mathrm{End}_\Lambda(M)$ and $\mathrm{End}_{\mathrm{End}_\Lambda(M)^{\mathrm{op}}}(M)$ coincide.

6. An example

We present a local two-sided artinian ring and describe explicitly the endocategory of the indecomposable endofinite module $M = {}_\Lambda\Lambda$ and its dual DM. The author is indebted to M. Schmidmeier for suggesting this example. It shows that even for an indecomposable endofinite Λ-module M the endomorphism rings $\mathrm{End}_{\Lambda^{\mathrm{op}}}(DM)$ and $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$ are not necessarily isomorphic.

Let T be a skew field and B a T-T^{op}-bimodule. Denote by $X^* = \mathrm{Hom}_T(X,T)$ the dual of any $X \in \mathrm{mod}(T)$. Let $\Lambda = T \ltimes B$ be the trivial extension which is a local ring, and let $\Gamma = T \ltimes B^{**}$. Then $I = ({}_\Lambda\Lambda)^*$ is a minimal injective cogenerator for $\mathrm{Mod}(\Lambda)$ and $\mathrm{End}_\Lambda(I) \cong \Gamma$ [2]. Now suppose that Λ is two-sided artinian. It follows that $M = {}_\Lambda\Lambda$ is endofinite and $DM = \mathrm{Hom}_\Lambda(\Lambda, I) = I$ since $\mathrm{End}_{\Lambda^{\mathrm{op}}}(M) \cong \Lambda^{\mathrm{op}}$ [1]. It is easily checked that the endocategory of M is $\mathrm{mod}(\Lambda)$ and that the duality $\mathcal{E}_M \to \mathcal{E}_{DM}$ is given by $X \mapsto \mathrm{Hom}_\Lambda(X, I)$. Returning to the bimodule B let us assume that $\ell(B_T) = 1$ and $\ell((B^{**})_T) = 2$. Such a bimodule exists according to a result of Schofield [13]. We obtain $\ell(\Lambda_\Lambda) = \ell(T_T) + \ell(B_T) = 2$ and $\ell(\Gamma_\Gamma) = \ell(T_T) + \ell((B^{**})_T) = 3$. Therefore $\Gamma \cong \mathrm{End}_\Lambda(DM)$ and $\Lambda \cong \mathrm{End}_{\Lambda^{\mathrm{op}}}(M)^{\mathrm{op}}$ are non-isomorphic rings and we have shown that, in general, the duality $M \mapsto DM$ does not share the usual properties of a duality between right and left modules.

References

[1] W. Crawley-Boevey, Modules of finite endolength over their endomorphism ring, in: Representations of algebras and related topics, eds. S. Brenner and H. Tachikawa, London Math. Soc. Lec. Note Series **168** (1992) 127–184.

[2] R. Fossum, P. Griffith and I. Reiten, Trivial extensions of abelian categories, Springer Lec. Notes **456** (1975).

[3] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary artin algebra and applications to the structure of exact sequences, manuscripta math., **54** (1985), 83–106.

[4] P. Gabriel, Des catégories abéliennes, Bull. Soc. math. France, **90** (1962), 323–448.

[5] L. Gruson, Simple coherent functors, in: Representations of algebras, Springer Lec. Notes **488** (1975) 156–159.

[6] I. Herzog, Elementary duality for modules, Trans. Am. Math. Soc. **340** (1993) 37–69.

[7] I. Herzog, The Ziegler spectrum of a locally coherent Grothendieck category, preprint.

[8] C.U. JENSEN AND H. LENZING, Model theoretic algebra, Gordon and Breach, New York (1989).

[9] H. KRAUSE, The spectrum of a locally coherent category, preprint.

[10] H. KRAUSE, Elementary duality and local duality for modules, preprint.

[11] M. PREST, Remarks on elementary duality, Ann. of Pure and Appl. Logic. **62** (1993) 183–205.

[12] J.-E. ROOS, Locally noetherian categories, in: Category Theory, Homology Theory and their Applications II, Springer Lec. Notes **92** (1969) 197–277.

[13] A. H. SCHOFIELD, Representations of rings over skew fields, London Math. Soc. Lec. Note Series **92** (1985).

[14] M. ZIEGLER, Model theory of modules, Ann. of Pure and Appl. Logic **26** (1984) 149–213.

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