GENERALIZED HEWITT-NACHBIN SPACES ARISING IN STATE-SPACE COMPLETIONS

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The paper deals with the question whether or not the well known notions of Stone-Czech compactification and realcompactification (Hewitt-Nachbin completion) can be transferred from the classical situation (CB(X),X completely regular) to a more general situation where X is replaced by an arbitrary set and where a convex cone of bounded functions plays the role of the continuous functions.

It turns out that this is in fact possible and that this procedure leads in a very satisfactory way to a simple and transparent theory which comprises the classical situation as well as the Choquet theory of the state space. Furthermore it is possible to adopt most of the fundamental results known for the classical situation with slight modifications for the general situation.

In the first part (Chapters I to IV) we define our basic notions (generalized compactness, realcompactness and pseudocompactness) and we show that - roughly spoken - these objects can be characterized by filter properties as well as geometric or lattice properties.

In the second part we transfer some basic results from the classical case to our situation. For example, it is demonstrated that a suitable generalization of Glicksberg's integral representation theorem contains the Choquet - theorem as a special case. This happens since the extreme points of a compact convex set are pseudocompact (in the generalized sense) with respect to the continuous affine functions.

The rest of the paper is devoted to the investigation of real-compact spaces.

I. BASIC DEFINITIONS

Let X be a nonempty set. We consider a convex cone F of bounded real functions on X such that F separates the points and contains all constant real-valued functions. A functional $\mu: F \to \mathbb{R}$ is said to be linear if it is additive and positive-homogeneous (i.e. $\mu(\lambda f) = \lambda\mu(f)$ for all $\lambda \geq 0$, $f \in F$). The functional μ is called order-preserving if $\mu(g) \geq \mu(f)$ whenever $g \geq f$.

At this point it should be remarked that every $\mathbb{R} \cup \{-\infty\}$ - valued order-preserving linear functional μ is automatically $\mathbb{R}$-valued. This is easily seen by

$$\mu(f) \geq \mu(\alpha) \geq -\mu(-\alpha) \in \mathbb{R}$$

where α is the constant function equal to $\inf_{x\in X} f(x)$.

A state is defined to be an order-preserving linear functional μ with

$$\mu(f) \leq \sup_{x\in X} f(x) \quad \text{for all} \quad f \in F.$$

The set of all states of F is called the state space and is denoted by SX_F. Identification of $x \in X$ with the point evaluation $f \to f(x)$ leads to an embedding of X into SX_F. The state space is made into a topological space by endowing it with the coarsest topology such that all the functions $\mu \to \mu(f)$, $f \in F$, are continuous. SX_F is a compact Hausdorff space (every ultrafilter converges). A state μ is defined to be Dini-continuous if we have

$$\inf_{n\in\mathbb{N}} \mu(f_n) \leq \sup_{x\in X} \inf_{n\in\mathbb{N}} f_n(x)$$

for all pointwise decreasing sequences f_n in F.

1 Remark. Let F consist of upper-semicontinuous functions on a compact (not necessarily Hausdorff) space X, then because of Dini's lemma all states are Dini-continuous.

As usual a state μ is called <u>maximal</u> if whenever ν is a state with $\mu \le \nu$ then $\mu = \nu$ (there $\mu \le \nu$ stands for $\mu(f) \le \nu(f)$ $\forall f \in F$).

<u>2 Remark</u>. (i) States on vector spaces are always maximal.
(ii) By Zorn's lemma every state μ is dominated by a maximal state ρ (i.e. $\mu \le \rho$).

Let $Y \subset X$ be a nonempty subset then we denote by $\sup_Y$ the sublinear functional given by

$$f \to \sup\{f(y) \mid y \in Y\}.$$

A convex cone G of functions on X is defined to be <u>max-stable</u> if for $g_1, g_2 \in G$ the function $x \to (g_1 \vee g_2)(x) = \max(g_1(x), g_2(x))$ is always in G. By VF we denote the smallest max-stable convex cone containing F, that is the cone

(1) $$VF = \{f_1 \vee f_2 \vee \dots \vee f_n \mid n \in N,\ f_1, \dots, f_n \in F\}.$$

A state $\hat{\mu}$ of VF is termed <u>dominated extension</u> of the F-state μ if $\hat{\mu}(f) \ge \mu(f)$ $\forall f \in F$. If we have equality for all $f \in F$ then, of course, $\hat{\mu}$ is called an <u>extension</u>. $\mu \in SX_F$ can always be extended to a state of VF [2, Lemma 2].

<u>3 Definition</u>. A state μ of F is called F-<u>character</u> if it has a unique dominated extension to VF and if for every finite cover $Y_1, \dots, Y_n$ of X there is some $k \le n$ such that $\mu \le \sup_{Y_k}$.

The set of F-characters will be denoted by βX_F. Those characters which are Dini-continuous are called <u>Dini-characters</u>, and υX_F stands for the set of Dini-characters.

<u>4 Definition</u>. (1) βX_F and υX_F are called the F-<u>compactification</u> of X and the F-<u>realcompactification</u> (or F-<u>Hewitt-Nachbin-completion</u>) respectively.
(2) The set X is defined to be
  (i) F-<u>compact</u> if $\beta X_F \subset X$
  (ii) F-<u>realcompact</u> (or F-<u>Hewitt-Nachbin space</u>) if $\upsilon X_F \subset X$.

5 Definition. X is defined to be F-pseudocompact if every element of the sup-norm closure of VF attains its maximum on X.

II. THE CLASSICAL SITUATION

When X is a completely regular Hausdorff space and F is equal to $CB(X) = \{f \in C(X) \mid f \text{ bounded}\}$ ($C(X)$ being the space of continuous real-valued functions on X) then we call this the "classical situation".

6 Proposition. (i) $\beta X_{CB(X)}$ is the Stone-Czech compactification of X.
(ii) $\upsilon X_{CB(X)}$ is the set of multiplicative linear functionals on $C(X)$ (restricted to $CB(X)$). Hence it is the usual realcompactification of X.

Proof. (i) is left as an exercise.
(ii) Let $\mu \in \upsilon X_{CB(X)}$. Then μ is multiplicative by Proposition 6 (i) and we prove that $\mu(g) \in g(X)$ $\forall g \in CB(X)$, which is a well-known criterion for μ being extendable to a multiplicative linear functional of $C(X)$. For this purpose we consider for an arbitrary $g \in CB(X)$ the decreasing sequence $f_n = -n(g-\mu(g))^2 \leq 0$. We must then have $\sup_{x\in X} \inf_n f_n(x) = 0$ because of $\mu(f_n) = 0$ and the Dini-continuity of μ. This implies that $(g-\mu(g))$ is equal to zero at some $x_o \in X$. Hence $\mu(g) = g(x_o) \in g(X)$.

It remains to prove that whenever $\nu \in \beta X_{CB(X)}$ is not Dini-continuous then there is some $g \in CB(X)$ with $\nu(g) \notin g(X)$.

By definition there is a decreasing sequence f_n in $CB(X)$ such that

$$(3) \qquad \inf_n \nu(f_n) = \alpha > \delta = \sup_{x\in X} \inf_n f_n(x).$$

We consider the following σ-compact subset of the Stone-Czech compactification βX

$$S = \bigcup_n K_n, \quad \text{where} \quad K_n = \{z \in \beta X \mid 2f_n(z) \leq \alpha + \delta\}.$$

Then S contains X (consequence of (3)) and by a suitable Urysohn argument we find some $g \in C(\beta X)$ with $\nu(g_{|X}) = g(\nu) = \alpha$ and $g(z) < \alpha \ \forall z \in S$, where $g_{|X}$ denotes the restriction to X. Hence, because of $S \supset X$, we have $\nu(g_{|X}) \notin g_{|X}(X)$. □

Thus we have shown that the notions of "CB(X) - realcompactification" and "CB(X)-compactification" coincide with the usual notions of "realcompactification" and "Stone-Czech compactification" respectively. The observation that CB(X) - pseudocompact means pseudocompact in the usual sense is quite obvious.

So, after having seen that the notions we have defined so far are generalizing a well-known concept, we can state that the aim of this paper is to investigate if those results which do hold for the classical situation can be adopted for the general situation.

We show in the sequel that this is in fact possible.

III. THE PRINCIPAL TOOLS

We gather here those results from [2] to [5] which we need for our investigation. Although they are formulated for the rather special cone F we would like to mention that they are valid in more general situations.

A very useful application of the sandwich theorem (our beloved form of the Hahn-Banach theorem, see [4, p.152]) is the following:

<u>Finite Sum Theorem</u>. Let μ be a linear functional on F and let $p_1,\dots,p_n$ be sublinear order-preserving functionals on F with $\mu \le \sum_{i=1}^{n} p_i$. Then there are order-preserving linear functionals $\mu_i \le p_i$ such that $\mu \le \sum_{i=1}^{n} \mu_i$.

(This is a special case of the sum theorem from [4]).

<u>7 Lemma</u>. Let $\mu \le \sup_Z$ (where $Z \subset X$) be a state of F and let $\emptyset \ne Y_k$ (k=1,...,n) be a finite cover of Z then there are $\lambda_k \ge 0$ and states μ_k with $\mu \le \sum_{k=1}^{n} \lambda_k \mu_k$ and $\sum_{k=1}^{n} \lambda_k = 1$ such that the μ_k are Y_k-order-preserving, i.e. $\mu_k(f) \ge \mu_k(g)$ whenever $f(y) \ge g(y)$

$\forall\, y \in Y_k$.

Proof. [4, finite decomposition theorem] gives us the λ_k and states $\tilde{\mu}_k \leq \sup_{Y_k}$ with $\mu \leq \sum_{k=1}^{n} \lambda_k \tilde{\mu}_k$. And from the sandwich theorem (applied with respect to the preorder given by pointwise order on Y_k) we get the desired Y_k-order-preserving states μ_k with $\tilde{\mu}_k \leq \mu_k \leq \sup_{Y_k}$.
□

The next result is of a much deeper nature. First, some notation. By Σ_F we denote the σ-algebra generated in X by F (that is the smallest σ-algebra such that the elements of F are measureable). We call a Σ_F-probability measure m a representing measure for an F-state μ if

$$\mu(g) \leq \int_X g \, dm \quad \text{for all} \quad g \in F,$$

in case that we have equality then we speak of a strict representing measure.

8 Theorem. Every maximal Dini-continuous state μ of VF has a strict representing measure on X.

Proof. From [4, thm 1] we obtain that μ has the decomposition property, which means that whenever $\emptyset \neq Y_n \subset X$ are such that $\bigcup_{n\in\mathbb{N}} Y_n = X$ then there are states $\mu_n \leq \sup_{Y_n}$ and $\lambda_n \geq 0$ with $\mu = \sum_{n=1} \lambda_n \mu_n$. We have to keep in mind that a state ν of VF has a unique extension to the vector lattice $E = VF - VF$, which we denote by $\hat{\nu}$. Furthermore, that when ν is maximal and $\leq \sup_Y$ then we also have $\hat{\nu} \leq \sup_Y$ (sandwich theorem, compare proof of the Main theorem in [3]).

Now, since μ is maximal μ_n must be maximal when $\lambda_n > 0$. Hence, $\hat{\mu} = \sum_{n=1}^{\infty} \lambda_n \hat{\mu}_n$ has the decomposition property, because we can drop all those $\hat{\mu}_n$ where $\lambda_n = 0$. And this was the condition required in [3, thm 1] for the existence of a representing measure m for $\hat{\mu}$. Obviously m is then also a representing measure for μ. □

IV. THE GENERAL SITUATION

IV. 1 THE F-COMPACTIFICATION

Our aim is to find many useful characterizations of the F-compactification. We begin with some definitions:

9 Definition. (i) A state ρ of VF is said to be a lattice state if $\rho(g_1 \vee \dots \vee g_n) = \max\{\rho(g_1),\dots,\rho(g_n)\}$ for all $g_1,\dots,g_n \in F$.
(ii) $\mu \in SX_F$ is called an extreme point of SX_F if whenever $0 < \lambda < 1$ and $\nu_1,\nu_2 \in SX_F$ with $\mu \le \lambda\nu_1 + (1-\lambda)\nu_2$ then $\mu = \nu_1$.
(iii) By Face (μ) we denote the set of those states ν such that there are $0 < \lambda \le 1$ and a state ρ with $\mu \le \lambda\nu + (1-\lambda)\rho$.
(iv) $Z(\mu)$ stands for the family of those subsets $Z \subset X$ having the property that for arbitrary $\alpha < \beta < 0$ there is always an $f \in F$ with $f \le 0$, $\mu(f) \ge \beta$ and $\sup_{(X\setminus Z)}(f) \le \alpha$. $Z(\mu)$ is termed the set of strong domination.
(v) $\mathcal{D}(\mu) = \{Y \subset X \mid \mu \le \sup_Y\}$ and $h(\mu) = \{Y \subset X \mid \mu \not\le \sup_{(X\setminus Y)}\}$ are called the set of domination and the complementary set respectively.

We first gather some technical details:

10 Lemma. (i) μ is an extreme point of $SX_F \Leftrightarrow \mathrm{Face}(\mu) = \{\mu\}$.
(ii) $\nu \in \mathrm{Face}(\mu) \Rightarrow \mathrm{Face}(\nu) \subset \mathrm{Face}(\mu)$.
(iii) If two states $\mu,\nu \in SX_F$ have the property that for $f,g \in F$ the inequality (*) $\mu(f) \ge \mu(g)$ always implies $\nu(f) \ge \nu(g)$ then $\mu = \nu$.
(iv) $Z(\mu) \subset h(\mu) \cap \mathcal{D}(\mu) \quad \forall \mu \in SX_F$
(v) $\nu \in \mathrm{Face}(\mu) \Rightarrow Z(\mu) \subset Z(\nu)$
(vi) $Z_1, Z_2 \in Z(\mu) \Rightarrow Z_1 \cap Z_2 \in h(\mu)$.

Proof. (i) and (ii) are trivial.
(iii): The number $\mu(f+g)$ must have the sign of $\mu(f)$ or $\mu(g)$, say that of $\mu(g)$. Then from $\mu(|\mu(f+g)|g) = \mu(|\mu(g)|(f+g))$ we obtain with (*) that $\nu(|\mu(f+g)|g) = \nu(|\mu(g)|(f+g))$. Hence we have

$\mu(f+g)\nu(g) = \nu(f+g)\mu(g)$, or $\mu(f)\nu(g) = \nu(f)\mu(g)$. For $g = 1$ we obtain with $\mu(1) = \nu(1) = 1$ the equality $\mu(f) = \nu(f)$.

(iv): $Z(\mu) \subset h(\mu)$ is obvious. Let $Z \in Z(\mu)$, we want to prove $\mu \leq \sup_Z$. According to Lemma 7 we have $\mu \leq \lambda\mu_1 + (1-\lambda)\mu_2$ with $0 \leq \lambda \leq 1$ and $\mu_1 \leq \sup_{(X\setminus Z)}$, $\mu_2 \leq \sup_Z$. For $\alpha = -n < \beta = -1 < 0$ there has to be some $f \leq 0$ with $\mu(f) \geq -1$ and $\sup_{(X\setminus Z)}(f) \leq -n$. Thus $-1 \leq \mu(f) \leq \lambda \sup_{(X\setminus Z)}(f) \leq -\lambda n$, or $\lambda \leq \frac{1}{n}$ for all n. Hence $\lambda = 0$ or $\mu = \mu_2 \leq \sup_Z$.

(v): Let $\nu \in \mathrm{Face}(\mu)$, i.e. $\mu \leq \lambda\nu + (1-\lambda)\rho$, $0 < \lambda \leq 1$, $\rho \in SX_F$ and let Z be an arbitrarily chosen element of $Z(\mu)$.

Then for arbitrary $\alpha < \beta < 0$ there is some $f \leq 0$ in F with $\mu(f) \geq \lambda\beta$ and $\sup_{(X\setminus Z)}(f) \leq \alpha$.

Hence $\nu(f) \geq \beta$, which shows that $Z \in Z(\nu)$.

(vi): Let $Z_1, Z_2 \in Z(\mu)$ and put $Y_i = X\setminus Z_i$, $i=1,2$. We have to show that $\mu \not\leq \sup_{(Y_1 \cup Y_2)}$. Assume therefore $\mu \leq \sup_{(Y_1\cup Y_2)}$. With the help of Lemma 7 we may write $\mu + \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 \leq \lambda \leq 1$ and $\mu_i \leq \sup_{Y_i}$, $i=1,2$. Take $\alpha < \beta < 0$ such that $2\beta > \alpha$ then there are $f_1, f_2 \leq 0$ in F with $\mu(f_i) \geq \beta$ and $\sup_{Y_i}(f_i) \leq \alpha$, $k=1,2$. Because of $\mu_2(f_1) \leq 0$ and $\mu_1(f_2) \leq 0$ this leads to the contradiction:

$$
\begin{aligned}
2\beta \leq \mu(f_1+f_2) &\leq \lambda\mu_1(f_1) + (1-\lambda)\mu_2(f_2) \\
&\leq \lambda \sup_{Y_1}(f_1) + (1-\lambda)\sup_{Y_2}(f_2) \leq \alpha .
\end{aligned}
$$

Hence $\mu \not\leq \sup_{(Y_1\cup Y_2)}$. □

Now, we are in the position to prove the main theorem of this chapter.

<u>11 Theorem</u>. Let μ be a state of F, then the following are equivalent:

(i) μ is a character of F

(ii) μ can be extended to a character of VF

(iii) μ has an extension $\hat{\mu}$ to VF such that $\hat{\mu}$ is maximal and is a lattice state

(iv) μ is maximal and $Z(\mu) = \eta(\mu)$

(v) $Z(\mu)$ is a filter on X which converges in the state space to μ

(vi) μ is an extreme point of SX_F

(vii) μ can be extended to an extreme point $\hat{\mu}$ of SX_{VF}.

Proof. We proceed in the following way:

(i) ⇒ (ii) ⇒ (iii) ⇒ (vii) ⇒ (vi) ⇒ (iv) ⇒ (v) ⇒ (vi) and (v)+(vi) ⇒ (i).

(i) ⇒ (ii): Let $\hat{\mu}$ be the unique dominated extension of μ to VF, and consider a finite nonempty cover $Y_1, \ldots, Y_n$ of X. Then by definition there is some $k \le n$ such that $\mu \le \sup_{Y_k}$. By the sandwich theorem we find a dominated extension $\tilde{\mu} \le \sup_{Y_k}$. Because of $\tilde{\mu} = \hat{\mu}$ (uniqueness) we have $\hat{\mu} \le \sup_{Y_k}$.

(ii) ⇒ (iii): We have to show that a character $\hat{\mu}$ of VF is a lattice state. Consider for $f_1, \ldots, f_n \in VF$ the sets $Y_i = \{x \mid f_i(x) \ge (f_1 \vee \ldots \vee f_n)(x)\}$, $i = 1, \ldots, n$. Then by definition there is some $k \le n$ with $\hat{\mu} \le \sup_{Y_k}$. Since $\hat{\mu}$ is maximal it must be Y_k-order-preserving (otherwise it is according to the sandwich theorem dominated by some Y_k-order-preserving state). So we have by definition of Y_k that $\hat{\mu}(f_k) \ge \hat{\mu}(f_1 \vee \ldots \vee f_n)$, and everything follows from the trivial inequality $\hat{\mu}(f_1 \vee \ldots \vee f_n) \ge \hat{\mu}(f_i)$, $i=1,\ldots,n$ (which is a consequence of the fact that $\hat{\mu}$ is order-preserving).

(iii) ⇒ (vii): We have to show that if $\hat{\mu}$ is a maximal lattice state then it is an extreme point of SX_{VF}. So, let $\hat{\mu} \le \lambda\mu_1 + (1-\lambda)\mu_2$ $0 < \lambda < 1$, $\mu_1, \mu_2 \in SX_{VF}$. We claim that

(*) $\hat{\mu}(f) \ge \lambda\mu_1(f_1) + (1-\lambda)\mu_2(f_2)$ $\forall\, f_1, f_2 \in VF$ with $f_1 \vee f_2 \le f$.

Now, assume that for arbitrary f, g we have $\hat{\mu}(f) \ge \hat{\mu}(g)$. Then from (*) and the fact that $\hat{\mu}$ is a lattice state we get $\hat{\mu}(f \vee g) = \hat{\mu}(f) \ge \lambda\mu_1(f) + (1-\lambda)\mu_2(g)$. And because of $\lambda\mu_1(f) + (1-\lambda)\mu_2(f) \ge \hat{\mu}(f)$ we obtain $\mu_2(f) \ge \mu_2(g)$. Application of Lemma 10 (iii) gives $\hat{\mu} = \mu_2$.

Proof of the claim:

$\delta(f) = \sup\{\lambda\mu_1(f_1) + (1-\lambda)\mu_2(f_2) \mid f_1, f_2 \in VF \text{ with } f_1 \vee f_2 \le f\}$ defines a superlinear functional on VF with $\hat{\mu} \le \delta \le \sup_X$. By the sandwich theorem there is some state $\tilde{\mu}$ such that $\delta \le \tilde{\mu} \le \sup_X$. Hence $\tilde{\mu} = \hat{\mu}$ and $\delta = \hat{\mu}$ since $\hat{\mu}$ was maximal.

(vii) $\Rightarrow$ (vi): Assume that $\mu \in SX_F$ has an extension to an extreme point $\hat{\mu}$ of SX_{VF}. Define $\delta(g) = \sup\{\mu(f) \mid f \in F,\ f \le g\}$. Obviously we must have $\delta \le \tilde{\mu}$ for every dominated extension $\tilde{\mu}$ of μ to VF. We claim $\delta = \hat{\mu}$ which proves that $\hat{\mu}$ is the only dominated extension of μ since every extreme point is a maximal state. From this we easily deduce that μ is extreme via the following argument: Assume $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, with $0 < \lambda < 1$ and $\mu_1, \mu_2 \in SX_F$. Take dominated extensions $\tilde{\mu}_1, \tilde{\mu}_2$ of μ_1, μ_2 and consider $\tilde{\mu} = \lambda\tilde{\mu}_1 + (1-\lambda)\tilde{\mu}_2$. This is a dominated extension of μ, hence equal to $\hat{\mu}$. Thus $\tilde{\mu}_1 = \tilde{\mu}_2 = \hat{\mu}$ because $\hat{\mu}$ was extreme. Restriction to F gives $\mu_1 = \mu_2 = \mu$.

Proof of the claim:

Let $f_1, \dots, f_n$ be arbitrary elements of F then it suffices to prove $\hat{\mu}(f_1 \vee \dots \vee f_n) = \max\{\mu(f_1), \dots, \mu(f_n)\}$. For this purpose we consider the cover $Y_i = \{x \mid f_i(x) \ge (f_1 \vee \dots \vee f_n)(x)\}$, $i = 1, \dots, n$, of X. Lemma 7 gives us the decomposition $\hat{\mu} \le \sum_{i=1}^{n} \lambda_i \hat{\mu}_i$ with Y_i-order-preserving states $\hat{\mu}_i$. Since $\hat{\mu}$ was extreme we have $\hat{\mu} = \hat{\mu}_k$ for some k. So $\hat{\mu}$ must be Y_k-order-preserving. This gives $\hat{\mu}(f_1 \vee \dots \vee f_n) =$ $= \hat{\mu}(f_k) = \mu(f_k)$.

(vi) $\Rightarrow$ (iv): $Z(\mu) \subset h(\mu)$ (Lemma 10 iv) and the maximality of μ are obvious. So, we have for arbitrary $Z \in h(\mu)$ to demonstrate that $Z \in Z(\mu)$. Consider arbitrary $\alpha < \beta < 0$ and put $\lambda = \beta\alpha^{-1}$, $Y = X \setminus Z$. If we had $\mu \le \lambda \sup_Y + (1-\lambda)\sup_X$ then the sum theorem would give us states μ_2 and $\mu_1 \le \sup_Y$ such that $\mu \le \lambda\mu_1 + (1-\lambda)\mu_2$. This is in contradiction to $\mu \not\le \sup_Y$ and the fact that μ is extreme. Hence there must be some $g \in F$ with $\mu(g) > \lambda \sup_Y(g) +$

$+ (1-\lambda)\ \sup_X(g)$.

Then $f = \alpha(\sup_Y(g) - \sup_X(g))^{-1}\ (g-\sup_X(g))$ has the required properties, namely $f \leq 0$, $\mu(f) \geq \beta$ and $\sup_Y(f) \leq \alpha$.

(iv) $\Rightarrow$ (v): From the filter properties only $Z_1, Z_2 \in Z(\mu) \rightarrow Z_1 \cap Z_2 \in Z(\mu)$ is nontrivial, but this is an immediate consequence of (iv) and Lemma 10 (vi). So it remains to demonstrate that $Z(\mu)$ converges to μ. For that purpose it suffices to show $\lim \mathcal{U} = \mu$ for every ultrafilter $\mathcal{U} \supset Z(\mu)$. Let $Y \in \mathcal{U}$ then $\mu \not\leq \sup_Y$ is impossible because (iv) then implies $X \setminus Y \in Z(\mu)$ which contradicts $\mathcal{U} \supset Z(\mu)$. Hence $\mu \leq \sup_Y\ \forall\ Y \in \mathcal{U}$. This leads to $\mu \leq \tilde{\mu} = \lim \mathcal{U}$, where $\tilde{\mu}$ is a state and must be equal to μ since μ is maximal.

(v) $\Rightarrow$ (vi): Let $\nu \in \mathrm{Face}(\mu)$. Then from Lemma 10 (v) we obtain $Z(\nu) \supset Z(\mu)$. Because of $Z(\nu) \subset \mathcal{B}(\nu)$ (Lemma 10 iv) this gives $\nu \leq \sup_Y\ \forall\ Y \in Z(\mu)$. Hence $\nu \leq \lim Z(\mu) = \mu$. Since this must hold for every element in $\mathrm{Face}(\mu)$ we get $\nu = \mu$ and $\mathrm{Face}(\mu) = \{\mu\}$.

(v)+(vi) $\Rightarrow$ (i): We first demonstrate that μ has a unique dominated extension to VF. For this purpose we consider arbitrary $f_1,\dots,f_n \in F$ and put $\delta = \max(\mu(f_1),\dots,\mu(f_n))$. We show that $\tilde{\mu}(f_1 \vee\dots\vee f_n) = \delta$ for every dominated extension $\tilde{\mu}$ of μ. $\tilde{\mu}(f_1 \vee\dots\vee f_n) \geq \delta$ is obvious. For the other inequality we consider $Y_i = \{x \mid f_i(x) \geq (f_1 \vee\dots\vee f_n)(x)\}$ and obtain with Lemma 7 $\tilde{\mu} \leq \Sigma\, \lambda_i \tilde{\mu}_i$, where $\tilde{\mu}_i$ is Y_i-order-preserving (for those i with $Y_i \neq \emptyset$, otherwise we put $\lambda_i = 0$).

Hence $\tilde{\mu}(f_1 \vee\dots\vee f_n) \leq \sum_{i=1}^{n} \lambda_i\, \tilde{\mu}_i(f_i)$. But we have $\tilde{\mu}_i(f_i) = \mu(f_i)$ if $\lambda_i \neq 0$ since μ is an extreme point. This immediatly leads to $\tilde{\mu}(f_1 \vee\dots\vee f_n) \leq \delta$.

It remains to show that for any cover $Y_1,\dots,Y_n$ of X we find some $k \leq n$ with $\mu \leq \sup_{Y_k}$. For this we consider some ultrafilter $\mathcal{U} \supset Z(\mu)$ on X. Then $\lim \mathcal{U} = \mu$, which has $\mu \leq \sup_Y\ \forall\ Y \in \mathcal{U}$ as consequence. But as an ultrafilter $\mathcal{U}$ has to contain one of the sets $Y_1,\dots,Y_n$. $\square$

IV 2. THE F-REALCOMPACTIFICATION

Let us turn our attention to the Dini-characters of F. First some remarks. By $\overline{VF}$ and LF we denote the sup-norm closures of VF and VF - VF respectively. LF is a vector lattice with respect to pointwise structure. Since states are sup-norm continuous they have unique extensions from VF to $\overline{VF}$ and LF. Hence every character $\mu \in \beta X_F$ must have unique dominated extensions to $\overline{VF}$ and LF. These extensions are also denoted by μ since no confusion can arise.

<u>12 Remark</u>. Consider $\mu \in \beta X_F$ and $Y \subset X$. Then $\mu(f) \leq \sup_Y(f)$ $\forall\, f \in F$ if and only if $\mu(h) \leq \sup_Y(h)$ $\forall\, h \in LF$.

<u>Proof</u>. The "if" is trivial. From the sandwich theorem we get a dominated extension $\tilde{\mu} \leq \sup_Y$ of μ to LF and by uniqueness we have $\tilde{\mu} = \mu$ (on LF). □

By $Z_F(\mu)$, $Z_{VF}(\mu)$, $Z_{LF}(\mu)$ we denote the sets of strong domination for μ with respect to the cones F, VF and LF respectively. The same notation is adopted for $\eta(\mu)$.

<u>13 Consequence</u>: Let μ be a character of F then $Z_F(\mu) = Z_{VF}(\mu) = Z_{LF}(\mu)$.

<u>Proof</u>. From Remark 12 we obtain $\eta_F(\mu) = \eta_{VF}(\mu) = \eta_{LF}(\mu)$ and Theorem 11 (iv) gives the desired result, since μ is obviously a character for all the cones under consideration. □

<u>14 Theorem</u>. Let μ be a character of F, then the following are equivalent:

(i) μ is Dini-continuous on F

(ii) μ is Dini-continuous on LF

(iii) Let $Z_n \in Z(\mu)$, $n \in \mathbb{N}$, then $\bigcap \{Z_n \mid n \in \mathbb{N}\}$ is not empty.

(iv) μ has a representing measure m which is a $\{0,1\}$-measure, i.e. $m(A) = 1$ or 0 $\forall\, A \in \Sigma_F$.

(v) μ has a representing measure

(vi) For every countable cover Y_n, $n \in \mathbb{N}$, of X there is some n_o with $\mu \leq \sup_{Y_{n_o}}$.

(vii) For every $h \in LF$ there is an $x \in X$ such that $\mu(h) \leq h(x)$.

(viii) For every sequence $f_n \leq 0$ in F with $\sum_{n\in\mathbb{N}} \mu(f_n) > -\infty$ there is an x with $\sum_{n\in\mathbb{N}} f_n(x) > -\infty$.

Proof. We proced as follows

(iii) ⇒ (iv) ⇒ (v) ⇒ (viii) ⇒ (iii), (v) ⇒ (ii) ⇒ (i) ⇒ (viii) and (vi) ⇔ (iii). The equivalence of (vii) is proved separately. Let us begin:

(iii) ⇒ (iv): Let $\bar{Z}(\mu)$ be the family of countable intersections of elements of $Z(\mu)$. Then $\emptyset \notin \bar{Z}(\mu)$. Hence $\bar{Z}(\mu)$ is a filter which is stable against countable intersections. Now, we define

$$m(A) = \begin{cases} 1 & \text{if} \quad A \in \bar{Z}(\mu) \\ 0 & \text{if} \quad X\setminus A \in \bar{Z}(\mu). \end{cases}$$

Then m is clearly a σ-additive $\{0,1\}$-measure on the σ-algebra $\Sigma = \{Y \subset X \mid Y \in \bar{Z}(\mu)$ or $X\setminus Y \in \bar{Z}(\mu)\}$. It remains to prove that every $f \in F$ is Σ-measurable and that $\int f dm = \mu(f)$. For this purpose it is certainly sufficient to show that an arbitrary $h \in LF$ is Σ-measurable and that for $A = \{x \in X \mid h(x) = \mu(h)\}$ we have $A \in \bar{Z}(\mu)$ (i.e. $m(A) = 1$). Let $\delta < \mu(h)$ then by definition we have $X\setminus h^{-1}(]-\infty,\delta]) \in \eta_{LF}(\mu)$, thus this must be an element of $Z(\mu) = Z_{LF}(\mu) = \eta_{LF}(\mu)$ (Theorem 11 (iv) and consequence 13). Replacing h by $-h$ we see in the same way that for $\gamma > \mu(h)$ $X\setminus h^{-1}([\gamma,+\infty[) \in Z(\mu)$. Using the σ-additivity of Σ we then find that h is Σ-measurable, and by application of the fact that $\bar{Z}(\mu) \supset Z(\mu)$ is stable against countable intersections we immediately get $A \in \bar{Z}(\mu)$.

(iv) ⇒ (v) is obvious and (v) ⇒ (viii) follows from the Lebesgue dominated convergence theorem.

(viii) ⇒ (iii): Consider $Z_n \in Z(\mu)$ and assume $\bigcap \{Z_n \mid n \in \mathbb{N}\} = \emptyset$.

Since $Z(\mu)$ is a filter we can - without loss of generality - restrict our considerations to the case where $Z_{n+1} \subset Z_n$ for all $n \in \mathbb{N}$. By definition we know that there are $f_n \leq 0$ in F with $\mu(f_n) \geq -\frac{1}{n^2}$ and $\sup_{Y_n}(f_n) \leq -2$, where $Y_n = X \setminus Z_n$. Now, we have

$$\sum_{n \in N} \mu(f_n) \geq -\Sigma \frac{1}{n^2} = -\frac{\pi^2}{6} > -\infty$$

and

$$\sum_{n=1}^{\infty} f_n(x) \leq \sum_{n=m}^{\infty} (-2) = -\infty \quad \text{for} \quad x \in Y_m.$$

Because of our assumption the Y_m cover X. Hence the sequence f_n is in contradiction to (viii).

(v) $\Rightarrow$ (ii): Let m be a representing measure for $\mu \in SX_F$. LF consists of Σ_F-measurable functions, since the Σ_F-measurable functions are a σ-complete vector lattice. So

$$\nu(h) := \int_X h \, dm \quad \forall\, h \in LF$$

defines a dominated extension of μ to LF and must therefore be equal to μ on LF. Hence m is also with respect to LF a representing measure for μ and (ii) is a consequence of Lebesgues dominated convergence theorem.

(ii) $\Rightarrow$ (i) $\Rightarrow$ (viii) is trivial.

(vi) $\Leftrightarrow$ (iii) Because of Theorem 11 (iv) the converse of (vi) must be equivalent to the existence of $Z_n \in Z(\mu)$, $n \in \mathbb{N}$, such that the $Y_n = X \setminus Z_n$ with $\mu \not\leq \sup_{Y_n}$, $n \in \mathbb{N}$, are a cover of X.

The equivalence of (vii): Because of (i) $\Leftrightarrow$ (ii) it suffices to prove: (v) for LF $\Rightarrow$ (vii) $\Rightarrow$ (vi) for LF. But: (v) for LF $\Rightarrow$ (vii) is completely obvious. Now, let (vii) be fulfilled and assume that there is a countable cover Y_n, $n \in \mathbb{N}$, of X with $\mu \not\leq \sup_{Y_n}$ (for LF). Then by Theorem 11 (iv) we find $f_n \leq 0$ in LF with $\mu(f_n) = \beta_n > \alpha_n = \sup_{Y_n}(f_n)$, where $\beta_n \leq 0$ and $\alpha_n < 0$.

Define $h_n = \lambda_n((f_n \wedge \beta_n) \vee \alpha_n)$ where $\lambda_n > 0$ is chosen such that

$\sup_{x \in X} |h_n(x)| \leq \frac{1}{n^2}$. Then $h = \sum_{n=1}^{\infty} h_n$ is an element of LF with $\mu(h) = \sup_{x \in X} h(x) > h(x)$ for all $x \in \bigcup \{Y_n \mid n \in \mathbb{N}\}$. This contradicts (vii). □

We would like to conclude this chapter with some words of warning. Although we have $\beta X_F = \beta X_{VF}$ we do not have $\beta X_{LF} = \beta X_{VF}$. Indeed, a state of LF is uniquely determined by its restriction to VF, but not every state on VF is maximal whereas every state on LF is automatically maximal since LF is a vector space. So in general we have $\beta X_{LF} \supset \beta X_{VF}$ and this has $\upsilon X_{LF} \supset \upsilon X_{VF}$ as a consequence.

Another warning should be given with respect to the comparison of the usual realcompactification υX (of X as a subspace of the topological space SX_{VF}) with υX_{LF}. Here we have in general $\upsilon X \subset \upsilon X_{LF}$. The reason for this is the fact that although LF is isometrically lattice-isomorphic to $C(\Omega)$ (Ω = (closure of X in SX_{VF}) $= \beta X_{LF}$) it is in general only isometrically lattice-isomorphic to a subspace of $CB(X)$. This inequality $\upsilon X \subset \upsilon X_{LF}$ has some advantages with respect to products.

Another remark which seems appropriate is that we actually proved in (ii) ⇒ (iv) (of Theorem 14) a little bit more than we claimed, namely

<u>15 Corollary</u>. $\mu \in \beta X_F$ is a Dini-character if and only if $\mu(h) \in h(X)$ for all $h \in LF$.

V CONSEQUENCES

Those who are acquainted with the theory of Hewitt-Nachbin spaces have certainly realized that Theorems 11 and 14 already generalize very many classical results of Hewitt, Gillman, Jerison and others. We shall not elaborate this in great detail, but we shall present some more results along these lines. These results show that our theory comprises large parts of Choquet-theory as well as the classical theory of continuous functions. Of course, in view of Theorem 11 (vi) and (vii) this is not very surprising. In constrast to the first part of this paper, where we have insisted on giving the full details of the proofs of the fundamental Theorems 11 and 14 we are only going to sketch the proofs of the coming results.

V 1. F-PSEUDOCOMPACTNESS

We recall that X is defined to be F-pseudocompact if every element in $\overline{VF}$ (sup-norm closure) attains its maximum on X.

16 Theorem. The following are equivalent:

(i) X is F-pseudocompact

(ii) $\beta X_F = \upsilon X_F$

(iii) F is a Dini cone, that means that for every pointwise decreasing sequence (f_n), $n \in \mathbb{N}$, we have

$$\sup_{x\in X} \inf_{n} f_n(x) = \inf_{n} \sup_{x\in X} f_n(x)$$

(iv) Every state of F has a representing measure.

Proof. We prove the theorem via the following implications: (i) $\Rightarrow$ VF is a Dini cone $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) (which is obvious since (iii) implies that every state is Dini-continuous).

Let us start:

(i) $\Rightarrow$ VF is a Dini cone (compare [4 Theorem 2]): Let h_n be a decreasing sequence in VF with $\alpha = \inf_n \sup_{x\in X} h_n(x) > -\infty$ and consider $\delta < \alpha$. Then $g_n = h_n \vee \delta$ is a decreasing and uniformly bounded sequence and the series $\sum_{n\in\mathbb{N}} \lambda_n g_n$ converges uniformly on X for $\lambda_n \geq 0$ with $\sum_{n\in\mathbb{N}} \lambda_n = 1$. So with the help of Simon's convergence lemma [7, p.104] we see that F-pseudocompactness implies:

$$\inf\{\sup_{x\in X} (\sum_{n=1}^{m} \lambda_n g_n(x)) \mid \lambda_1,\dots,\lambda_m \geq 0, \ \sum_{n=1}^{m} \lambda_n = 1\} \leq \sup_{x\in X} \limsup_{n\to\infty} g_n(x).$$

The left-hand side is equal to $\inf_n \sup_{x\in X} g_n(x) = \alpha$ (since the g_n

are decreasing and $\delta < \alpha$), and the right-hand side is equal to $\sup_{x\in X} \inf_n g_n(x)$ which again is equal to the maximum of δ and $\sup_{x\in X} \inf_n h_n(x)$. Hence

$$\sup_{x\in X} \inf_n h_n(x) \geq \inf_n \sup_{x\in X} h_n(x).$$

The other inequality is trivial. So in view of the arbitraryness of the sequence VF must be a Dini cone.

VF is A Dini cone $\Rightarrow$ (iv): First of all, we observe that every state of VF must be Dini continuous (VF is a Dini cone). Every state of VF is dominated by a maximal state thus Theorem 8 tells us, that every state of VF has a representing measure. Hence every state of F must have a representing measure.

(iv) $\Rightarrow$ (ii) follows immediately from Theorem 14.

(ii) $\Rightarrow$ (i): An elementary exercise leads to $\beta X_{VF} = \beta X_{\overline{VF}}$.

So with Theorems 11 and 14 we get that (ii) is equivalent to $\beta X_{VF} = \cup X_{VF}$. From Bauer's maximum principle [1] we know that for every $h \in VF$ there is an extreme point μ of the state space, (hence an element of $\beta X_{\overline{VF}}$) such that

$$\mu(h) \geq \sup_{x\in X} h(x).$$

Now, because of $\mu \in \cup X_{\overline{VF}}$ we find with Theorem 14 (vii) an $x_o \in X$ such that $\mu(h) \leq h(x_o)$. So h must attain its maximum on X. ▢

This theorem already generalizes some well known results of Hewitt and Glicksberg. It contains for example the well known Alexandrov-Glicksberg theorem [8, thm 21 or 23] which says that every state on CB(X) has an integral representation if and only if X is pseudocompact.

Another consequence is the integral representation theorem of [3].

V 2. THE GEOMETRICAL SITUATION

Consider a compact convex subset K of a Hausdorff locally convex topological vector space and denote by $A(K)$ the affine continuous functions on K and by ∂K the extreme points of K. Then by Bauer's maximum principle the spaces $A(K)|_{\partial K}$ (restrictions of the $f \in A(K)$ to ∂K) and $A(K)$ are isometrically (with respect to the sup-norm) isomorphic and lattice isomorphic.

Hence they have the same state spaces (being equal to the point evaluations given by all $x \in K$).

<u>17 Lemma</u>. ∂K is $A(K)|_{\partial K}$-pseudocompact.

<u>Proof</u>. Take a decreasing sequence $f_n \in A(K)|_{\partial K}$ and denote by φ_n the unique extensions of f_n to elements in $A(K)$. Then the φ_n are also decreasing. From Bauer's maximum principle (for upper semicontinuous affine functions) and from Dini's lemma we know that there is some $x_o \in \partial K$ such that

$$\inf_n \sup_{x \in X} f_n(x) \leq \inf_n \sup_{x \in X} \varphi_n(x) = \inf_n \varphi_n(x_o) = \inf_n f_n(x_o).$$

Since f_n was chosen arbitrarily this proves the assertion (Theorem 16 (iii)). □

So, Theorem 16 contains the celebrated Choquet-Bishop-de Leeuw theorem as special case.

<u>18 Corollary</u>. For every $x \in K$ there is a probability measure m on ∂K (with respect to the σ-algebra generated by $A(K)|_{\partial K}$) such that

$$\varphi(x) = \int_{\partial K} \varphi \, dm \quad \forall \varphi \in A(K).$$

V 3. EPIMORPHISMS AND ADMISSIBLE SUBSETS

In this subsection we turn our attention to subsets of the state space, and here especially to the admissible subsets. Roughly spoken, these are subsets coming out of epimorphisms of cones.

Let $\emptyset \neq \Omega \subset SX_F$. Then if not otherwise mentioned Ω will always carry the topology inherited from the state space. By (Ω, F)

we denote the convex cone of functions on Ω given by $\omega \to \omega(f)$, $f \in F$. Since X is embedded in the state space we can identify (X,F) with F. Cones of the form (Ω,F) are called <u>concrete order unit cones</u>.

Now, consider two concrete order unit cones (Ω,F) and (Z,G) and a map $\varphi: \Omega \to Z$. φ is said to be an <u>epimorphism</u> from (Ω,F) to (Z,G) if

(4) $$\sup_\Omega (g \circ \varphi) = \sup_Z(g) \quad \forall\, g \in G$$

(5) $$\{g \circ \varphi \mid g \in G\} = F_{|\Omega}.$$

Where, of course, $g \circ \varphi$ stands for the function

$$\Omega \ni \omega \to \varphi(\omega)(g),$$

and where $\sup_Z(g) = \sup\{y(g) \mid y \in Z\}$, and $F_{|\Omega}$ denotes the cone of functions given by $\Omega \ni \omega \to \omega(f)$, $f \in F$.

<u>19 Examples</u>. (i) Let $\tilde{\Omega} \subset \Omega$, then the embedding of $\tilde{\Omega}$ in Ω is an epimorphism $(\tilde{\Omega},F) \to (\Omega,F)$ if and only if $\tilde{\Omega}$ is a sup-boundary of Ω, i.e. $\sup_{\tilde{\Omega}}(f) = \sup_\Omega(f) \quad \forall\, f \in F$.
Such a subset will be called <u>admissible</u>.

(ii) In the geometrical situation the embedding ∂K in K gives an epimorphism $(\partial K, A(K)) \to (K, A(K))$.

<u>20 Observation</u>. Let φ be an epimorphism from (Ω,F) to (Z,G).
(i) $\varphi: \Omega \to Z$ is automatically injective and continuous with respect to the topology of the state space.
(ii) $\varphi^*: S\Omega_F \to SZ_G$ given by $\varphi^*(\mu)(g) = \mu(g \circ \varphi) \quad \forall\, g \in G$ is an affine continuous injective map.
(iii) For every $\nu \in SZ_G$ there is a $\mu \in S\Omega_F$ such that $\varphi^*(\mu) \geq \nu$.

<u>Proof</u>. Only (iii) is less obvious. For this, we define a superlinear $\delta \leq \sup_\Omega$ (because of (4)) on (Ω,F) by

$$\delta(f) = \sup\{\nu(g) \mid g \in G,\ \ g \circ \varphi \leq f\}.$$

Then from the sandwich theorem we obtain the desired state μ with $\delta \leq \mu \leq \sup_\Omega$. □

Assertions 20 (i) and (ii) have the interesting consequence that φ^* restricted to $\{\mu \in S\,\Omega_F \mid \varphi^*(\nu)$ is maximal in $SZ_G\}$ is invertible. This immediately leads to

<u>21 Corollary</u>. (i) φ^* maps $\beta\Omega_F$ onto βZ_G

(ii) φ^* maps $\upsilon\Omega_F$ into υZ_G.

This together with the fact that Ω is dense in $\Omega \cup \beta\Omega_F$ (Theorem 11 (v)) leads to the following important universal properties (which are well known for the classical situation).

<u>22 Theorem</u>. Let φ be an epimorphism from (Ω,F) to (Z,G) then φ extends uniquely to epimorphisms from $(\Omega \cup \beta\Omega_F, F)$ to $(Z \cup \beta Z_G, G)$ and from $(\Omega \cup \upsilon\Omega_F, F)$ to $(Z \cup \upsilon Z_G, G)$.

This result is useful for formal proofs of structural properties. As an exercise the reader may use it for the proof of the idempotence of β and υ, i.e.

$$\beta(X \cup \beta X_F)_F = \beta X_F \quad \text{and} \quad \upsilon(X \cup \upsilon X_F)_F = \upsilon X_F.$$

After this detour let us get back to admissible subsets Ω of the state space SX_F, i.e. subsets with $\sup_\Omega(f) = \sup_X(f)\ \forall f \in F$.

<u>23 Theorem</u>. Let Y be an admissible subset of X.

(i) If Y is closed in X and if X is F-compact then Y is again F-compact.

(ii) If X is F-realcompact and if Y is an F_σ-subset of X (i.e. countable union of closed subsets of X) then Y is F-realcompact.

<u>Proof</u>. (i): The embedding $\varphi: Y \to X$ can be uniquely extended (thm. 22) to an epimorphism $\varphi^*: (Y \cup \beta Y_F, F) \to (X \cup \beta X_F, F) = (X,F)$ (since X is F-compact). Hence $\varphi^*: Y \cup \beta Y_F \to X$ and the dense subset $Y = \varphi^{*-1}(\varphi(Y))$ must be closed in $Y \cup \beta Y_F$. Thus $Y \supset \beta Y_F$.

(ii): First we remark that according to Theorem 14 (iii) or (vi) every F_σ-subset of $Y \cup \upsilon Y_F$ which contains Y must be equal to $Y \cup \upsilon Y_F$. Now, as in (i), the embedding $\varphi: Y \to X$ extends uniquely to an injective continuous $\varphi^*: Y \cup \upsilon Y_F \to X$ (realcompactness of X). Hence, Y must be an F_σ-subset of $Y \cup \upsilon Y_F$ which gives $Y \supset \upsilon Y_F$ with the help of our introductory remark. □

By a similar argument one proves:

24 Theorem. The following are equivalent:

(i) $\mu \in \beta X_F \setminus \upsilon X_F$.

(ii) There is an F_σ-subset Ω of $X \cup \beta X_F$ with $\Omega \supset X$ and $\mu \notin \Omega$.

V 4. REALCOMPACT SPACES

We like to look a little bit closer on the case when X is F-realcompact, i.e. $\upsilon X_F \subset X$. First we observe that we have already proved the following characterization of F-compactness, which is well known in the classical case [9, p 34]:

25 Observation. The following are equivalent:

(i) X is F-realcompact and F-pseudocompact

(ii) X is F-compact.

Proof. (i) ⇒ (ii): We have $\upsilon X_F \subset X$ (X is F-realcompact) and $\upsilon X_F = \beta X_F$ (X is F-pseudocompact), hence $\beta X_F \subset X$.

(ii) ⇒ (i): Obviously F-compact implies F-realcompact. And from $\beta X_F \subset X$ and the fact that all point evaluations are Dini-continuous follows $\beta X_F \subset \upsilon X_F$. □

This is a rather useful criterion for F-compact spaces. Because very many spaces are automatically F-realcompact and for them F-pseudocompactness is then a necessary and sufficient condition for the assertion that X contains already all extreme points of the state space. If one reformulates the Theorems 23 and 24 one easily finds examples for such situations:

26 Theorem. The following are equivalent:

(i) X is F-realcompact

(ii) X is G_δ-closed in $X \cup \beta X_F$, i.e. for every $\mu \in \beta X_F \setminus X$ there is an F_δ-subset Ω of $X \cup \beta X_F$ with $\mu \notin \Omega$ and $\Omega \supset X$.

(iii) X is the intersection of F_σ-subsets of $X \cup \beta X_F$.

This theorem generalizes well known results of Mrówka [10,p 80] and Wenjen [10, p 81].

Other examples for such X which are automatically F-realcompact are occuring in topological situations (compare [4]):

27 Theorem. Assume that there is a topology τ on X such that all $f \in F$ are τ-upper-semicontinuous and such that (X,τ) is a Lindelöf space. Then X is F-realcompact.

Proof. Take $\mu \in \upsilon X_F$ and consider the filter $Z(\mu)$. By $\bar{Y}$ we denote the τ-closure of $Y \subset X$. By the definitions of $h(\mu)$ and $Z(\mu)$ one easily shows that $\mathfrak{F} = \{\bar{Y} \mid Y \in Z(\mu)\}$ is a filter basis of $Z(\mu)$. Hence $\mu \notin X$ implies $\bigcap \{\bar{Y} \mid \bar{Y} \in \mathfrak{F}\} = \emptyset$ (since $\mathfrak{F} \to \mu$), or in other words $\{X \setminus \bar{Y} \mid \bar{Y} \in \mathfrak{F}\}$ is an τ-open cover of X. Thus there must be a countable subcover $X \setminus \bar{Y}_n$, $n \in \mathbb{N}$, since (X,τ) is Lindelöf. This gives $\bigcap \{\bar{Y}_n \mid n \in \mathbb{N}\} = \emptyset$ in contradiction to Theorem 14 (iii). □

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