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ON THE HIERARCHY OF THE LANDAU–LIFSHITZ EQUATION

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The Landau–Lifshitz equation (LL-equation) is shown to have infinitely many commuting one-parameter symmetry groups and constants of motion in involution. The infinitesimal generators of these one-parameter symmetry groups define the hierarchy of the LL-equation. We give *explicit* formulas for these generators and for the conserved densities. Furthermore, infinitely many time-dependent symmetry groups and conservation laws are constructed. All results are obtained via a simple direct Lie algebra approach and without making any use of the quantum inverse scattering method based on the ingenious results of R.J. Baxter.

1. Introduction

The LL-equation describes nonlinear spin waves in an anisotropic ferromagnet. The equation has the form

$$\boldsymbol{S}_t = \boldsymbol{S} \cdot \boldsymbol{S}_{xx} + \boldsymbol{S} \cdot J\boldsymbol{S}, \tag{1}$$

where

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}$$

describes the anisotropy, where

$$\boldsymbol{S} = \boldsymbol{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t)),$$
$$|\boldsymbol{S}| = 1,$$

is an x- and time-dependent vector in $\mathbb{R}^3$ with norm 1, and where $\cdot$ denotes the usual vector product. In addition, this equation is particularly suitable for the study of periodic and quasiperiodic solutions of classical nonlinear wave equations (like the sine-Gordon equation and the non-linear Schrödinger equation) since it contains all these equations as special cases [1, 2]. It was assumed for a long time that the equation is completely integrable since its complete integrability was well established for special cases of J [3, 4]. Finally, its complete integrability was proved by E.K. Sklyanin [2] who adapted, for this equation, the quantum-inverse scattering method of L.A. Takhtadzhan and L.D. Faddeev [5] which is based on R.J. Baxter's ingenious method for the solution of the XYZ-model in quantum statistical mechanics [6]. This method leads to success in case of the LL-equation since this equation is the continuous classical limit of the quantum XYZ-model. Sklyanin established the inverse-scattering theory for the LL-equation. He found infinitely many constants of motion (in involution) via the usual expansion of the logarithm of the scattering data with respect to the spectral parameter. However, it should be observed that the constants of motion are given by a complicated bilinear recursion formula, which is certainly difficult to evaluate in case that one is interested in the explicit form of these quantities.

In this paper we obtain the explicit form of all constants of motion by a direct Lie algebra approach. This approach has been proved to be successful for other nonlinear equations as well (Benjamin–Ono equation [7], Kadomtsev–Petviashvili equation [8]). We start by constructing infinitely many nonlinear flows which commute

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with the LL-equation. This construction is implicitly performed by use of a time-dependent symmetry group of the LL. Then we find a Lie-algebra homomorphism (Noether's theorem) mapping the infinitesimal generators of these flows onto conserved densities whose integration (under an appropriate boundary condition for $x \to \pm\infty$) then yield the constants of motion. Applications of similar methods to the XYZ-model are discussed in a subsequent paper.

At the end of this paper we modify our method in order to construct a (noncommuting) hierarchy of time dependent symmetry groups.

2. Notation and basic notions

The manifold on which the flow (1) is taking place is denoted by $\mathcal{M}$. In order to be able to apply the usual calculus of variation we embed $\mathcal{M}$ into a vector space. To be precise:

E denotes the vector space of C^∞-maps from $\mathbb{R}$ into $\mathbb{R}^3$. The variable in $\mathbb{R}$ we denote by x and the variable in E by S. For simplification we skip the bold-face notation. In order to emphasize the dependence of S on x we write for short $S(x)$. We consider the following linear maps E $\to$ E given by x (multiplication by x), D (differentiation with respect to x), J (matrix-multiplication by the matrix introduced in (1)), furthermore, the bilinear map $\cdot$: E $\times$ E $\to$ E given by the vector product.

By TE we denote the vector fields on E, that is the space of C^∞-maps E $\to$ E which is generated by Id (identity map), by J, x and D and which is closed against taking vector products and which is a module with respect to multiplication by scalar products (V_1, V_2), V_1, $V_2 \in$ TE. The maps, assigning to $S \in$ E the elements S, DS, $S \cdot D^2S$, $S \cdot JS$, $(S, DS)\ \{JS + D^3S\}$, etc. are typical members of TE. For simplicity we denote these members by $S(x)$, $S(x)_x$, $S(x) \cdot S(x)_{xx}$, $S(x) \cdot JS(x)$, $(S(x), S_x(x))\ \{JS(x) + S(x)_{xxx}\}$, etc. General elements of TE are denoted by $V_1(S(x))$, $V_2(S(x))$, etc. The real valued C^∞-functions in $x \in \mathbb{R}$ which are of the form $(V_1(S(x)), V_2(S(x)))$, V_1, $V_2 \in$ TE are called scalar fields. The scalar fields are an algebra.

Now, the manifold $\mathcal{M}$ under consideration is the subset of those $S \in$ E with $(S(x), S(x)) = |S(x)|^2 = 1$. And the vector fields of $\mathcal{M}$ (denoted by T$\mathcal{M}$) are those $V(S(x)) \in$ TE, which have, for $S \in \mathcal{M}$, the property $(V(S(x)), S(x)) = 0\ \forall x \in \mathbb{R}$, i.e. $V(S(x))$ always has to be in the tangent plane of the unit sphere at $S(x)$. Observe that for $S \in \mathcal{M}$ the operator $\hat{\pi}$: $V(S(x)) \to S(x) \cdot V(S(x))$ maps TE onto T$\mathcal{M}$. Obviously, $\pi = \hat{\pi}^2$ is the identity map on T$\mathcal{M}$. The scalar fields on $\mathcal{M}$ are the restrictions to $\mathcal{M}$ of the scalar fields defined above. $\Sigma\mathcal{M}$ denotes the scalar fields on $\mathcal{M}$.

In T$\mathcal{M}$ we consider the usual Lie algebra of vector fields. For $K, G \in$ T$\mathcal{M}$ the corresponding Lie product is given by

$$[K,G] = \frac{\partial}{\partial\varepsilon}\{K(S+\varepsilon G(S)) - G(S+\varepsilon K(S))\}\big|_{\varepsilon=0}. \tag{2}$$

In order to introduce an appropriate Lie algebra in $\Sigma\mathcal{M}$ we first pass over to a suitable quotient structure. We consider $\sigma_1, \sigma_2 \in \Sigma\mathcal{M}$ to be equivalent (denoted by $\sigma_1 \equiv \sigma_2$) if they are equal up to a total derivative, i.e. if there is some $\sigma_3 \in \Sigma\mathcal{M}$ such that $\sigma_1 - \sigma_2 = D\sigma_3$. Let us give some simple examples for this equivalence relation

$$(S(x)_{xx}, S(x)) \equiv -(S_x(x), S_x(x)), \tag{3a}$$

$$x\sigma_{xx} \equiv 0, \quad x\sigma_x = -\sigma, \quad \text{for all } \sigma \in \Sigma\mathcal{M}. \tag{3b}$$

One reason for the introduction of this equivalence relation is that, afterwards when we integrate scalar fields (under suitable boundary conditions) to obtain constants of motion, the total derivatives will not give any contributions.

The classes of scalar fields, with respect to this equivalence relation, will be called densities. The equivalence class given by a scalar field σ is denoted by $\bar{\sigma}$. And for $K, G \in$ T$\mathcal{M}$ we denote the class given by $(K(S(x)), G(S(x)))$ by (K, G), i.e. $(\ ,\ )$ is a density-valued scalar product in T$\mathcal{M}$.

Now, observe that the differentiation operator D is skew-symmetric with respect to (,), and that all the operators we used to generate TE now have adjoints with respect to (,). This observation has the great advantage that all scalar fields σ have gradients with respect to (,), i.e. for every σ there is a vector field $\nabla\sigma$ (called the gradient of σ) such that

$$(\nabla\sigma, H) \equiv \sigma'[H] \equiv \frac{\partial}{\partial\varepsilon}\sigma(S+\varepsilon H)\Big|_{\varepsilon=0}, \quad \text{for all } H \in \mathrm{T}\mathcal{M}. \tag{4}$$

These gradients are uniquely determined. We like to give some examples for gradients. Consider the scalar fields

$$\begin{aligned} H(S) &= \tfrac{1}{2}((S, JS) - (S_x, S_x)), \\ T(S) &= \tfrac{1}{2}x((S, JS) - (S_x, S_x)) = xH(S), \\ \mathscr{H}(S) &= (S, S_x \cdot S_{xx}) + (S \cdot S_x, JS). \end{aligned} \tag{5a}$$

Then the gradients of these densities are given by

$$\begin{aligned} \nabla H &= \pi(S_{xx} + JS), \\ \nabla T &= \pi(xS_{xx} + xJS + S_x) = x\,\nabla H + S_x, \\ \nabla\mathscr{H} &= \pi(S_x \cdot S_{xx} + 2(S\cdot S_x)_{xx} + J(S\cdot S_x) \\ &\quad + S_x \cdot JS + D(S\cdot JS)). \end{aligned} \tag{5b}$$

Where $\pi = (S\cdot)^2$ is the projection onto the tangent plane of the unit sphere in $\mathbb{R}^3$ at the point $S(x)$.

As we shall see later on, these three vector fields and their gradients contain almost all the mysteries of the Landau–Lifshitz equation.

At the end of this section we would like to introduce a Lie algebra bracket (Poisson bracket) in the space of densities. For densities $\bar{\sigma}_1$ and $\bar{\sigma}_2$ we define this Lie algebra product by

$$\begin{aligned} \{\bar{\sigma}_1, \bar{\sigma}_2\} &= (\nabla\bar{\sigma}_1(S), S\cdot\nabla\bar{\sigma}_2(S)) \\ &= \tfrac{1}{2}(\nabla\bar{\sigma}_1(S), S\cdot\nabla\bar{\sigma}_2(S)) \\ &\quad - \tfrac{1}{2}(\nabla\bar{\sigma}_2(S), S\cdot\nabla\bar{\sigma}_1(S)). \end{aligned} \tag{6}$$

One easily checks the Jacobi identity for this bracket. (Actually, we do not have to prove that at all, since, implicitly it is proved in section 4 anyway.)

We should remark that there is no ambiguity in this definition.

Now, let us define the essential quantities we are looking for in case of the LL-equation (1). We are interested in the one-parameter symmetry groups of (1), or rather their infinitesimal generators. This is the same as looking for all flows which commute with (1). We recall that a vector field $G(S)$ is a generator of a one-parameter symmetry group of (1) if and only if

$$[K, G] = 0, \tag{7}$$

where

$$K(S) = S\cdot\nabla H = S\cdot S_{xx} + S\cdot JS \tag{8}$$

is the right-hand side of the LL-equation. In other words, we are interested in finding all vector fields commuting with $K(S)$. A dual notion of symmetry-generator is that of a conserved density. The equivalence class of a scalar field σ is called a conserved density if

$$\sigma(S(t))_t \equiv 0 \tag{9a}$$

for all solutions $S(t)$ of (1). Using eq. (1) we see that this is equivalent to

$$(\nabla\sigma, K(S)) \equiv (\nabla\sigma, S\cdot\nabla H) \equiv 0, \tag{9b}$$

or, by using definition (6), equivalent to

$$\{\bar{\sigma}, \bar{H}\} = 0. \tag{10}$$

By integration we obtain, for suitable boundary conditions, constants of motion. Here a suitable boundary condition means that we assume

$$S(x,t) \underset{x\to\pm\infty}{\longrightarrow} e_i,$$

where $e_i \in \mathbb{R}^3$ shall be one of the eigenvectors of

J. Furthermore, we assume that the components of $S(x,t)-e_i$ are functions which vanish rapidly at infinity ($x \to \pm\infty$), i.e. S and all its derivatives vanish faster than any polynomial in x. It is quite clear that this boundary condition is preserved under the flow of (1). Furthermore, this boundary condition guarantees that for any scalar field $\sigma(S)$ the integral

$$\int_{-\infty}^{+\infty} \{\sigma(S)-\sigma(e_i)\}\,\mathrm{d}x \tag{11}$$

exists. In addition, all integrals over total derivatives vanish. So, integration is a class function with respect to our equivalence relation. This yields that for any conserved density $\bar{\sigma}$ the integral (11) is a constant of motion for (1) (i.e. a real or complex valued function on the manifold being preserved under the flow of (1)).

3. The principal results for time-independent quantities

We use the densities and vector-fields defined in (5). Consider

$$\tau_+(S) = S \cdot \nabla T(S) = xS \cdot \nabla H + S \cdot S_x \tag{12}$$

and define

$$\begin{aligned} K_0(S) &= S_x, \\ K_1(S) &= S \cdot \nabla H(S) = S \cdot S_{xx} + S \cdot JS, \\ &\vdots \\ K_{n+1}(S) &= [K_n(S), \tau_+(S)]. \end{aligned} \tag{13}$$

Then the K_n, $n \in \mathbb{N}$, are a commuting family of vector fields and each member commutes with the flow given by (1) (i.e. with $K_1(S)$). Hence, the K_n are spanning a Lie algebra of an infinite dimensional Abelian symmetry group of the Landau-Lifshitz equation. The evolution equations

$$S_t = K_n(S), \quad n \in \mathbb{N} \tag{14.n}$$

are called the hierarchy of the LL-equation.

Now, consider the $H(S)$, $\mathscr{H}(S)$ and $T(S)$ given in (5). Observe that $\bar{\mathscr{H}}(S) = \{\bar{H}(S), \bar{T}(S)\}$ and define

$$\begin{aligned} \bar{H}_1(S) &= \bar{H}(S), \\ \bar{H}_2(S) &= \bar{\mathscr{H}}(S), \\ &\vdots \\ \bar{H}_{n+1}(S) &= \{\bar{H}_n(S), \bar{T}(S)\}, \end{aligned} \tag{15}$$

where $T(S) = xH(S)$. Then all the $\bar{H}_n$, $n \in \mathbb{N}$ are conserved densities for all equations (14.n), especially for the LL-equation. All these densities are in involution

$$\{\bar{H}_n(S), \bar{H}_m(S)\} = 0, \quad \text{for all } n, m \in \mathbb{N}. \tag{16}$$

And the $K_n(S)$ and $H_n(S)$ are related via

$$K_n(S) = S \cdot \nabla \bar{H}_n. \tag{17}$$

That means, the $\bar{H}_n$ are the Hamiltonians of the hierarchy given in (15). And the map $S \cdot \nabla$ provides a Lie-algebra homomorphism from the conserved densities into the symmetry group generators (Noether's theorem).

If $e_i \in \mathbb{R}^3$ is an eigenvector of J, and if we assume that $S(x)-e_i$ vanishes rapidly for $x \to \pm\infty$ then the

$$P_m = \int_{-\infty}^{+\infty} (\bar{H}_m(S) - \bar{H}_m(e_i))\,\mathrm{d}x \tag{18}$$

are constants of motion for each member of the hierarchy (14.n).

It may occur that the reader is not happy about the fact that x appears explicitly in the Hamiltonian densities $\bar{H}_n$. In that case, he is invited to perform one integration by parts (which is possible since densities were considered to be equal modulo total derivatives).

4. The method

We start by considering a map Γ defined on the densities by

$$\bar{\sigma}(S) \xrightarrow{\Gamma} S \cdot \nabla\sigma(S).$$

Recall that ∇ is a class operator. We claim

$$\Gamma\{\bar{\sigma}_1, \bar{\sigma}_2\} = [\Gamma\bar{\sigma}_1, \Gamma\bar{\sigma}_2] \tag{19}$$

for all densities $\bar{\sigma}_1, \bar{\sigma}_2$. So, Γ is a *Lie algebra homomorphism*. To prove this, consider for arbitrary vector fields G the quantity $V(G)$

$$\begin{aligned} V(G) = &\left(G, (S\cdot\nabla\sigma_1)'[S\cdot\nabla\sigma_2]\right) \\ &-\left(G, (S\cdot\nabla\sigma_2)'[S\cdot\nabla\sigma_1]\right) \\ &-\left(G, S\cdot\nabla(\nabla\sigma_1, S\cdot\nabla\sigma_2)\right). \end{aligned} \tag{20}$$

We only need to show that $V(G) = 0$. Keeping in mind that expressions like $(G, (S\cdot\nabla\sigma_1)\cdot\nabla\sigma_2)$ are equal to zero (since they are equal to the determinant of three vectors being in a two-dimensional subspace), and keeping furthermore in mind that second derivatives are symmetric, we compute that $V(G)$ is equal to zero up to a total derivative. Since

$$V(G) \equiv 0, \quad \text{for every } G,$$

we may replace G by $x^m G$ and see that also

$$x^m V(G) \equiv 0, \quad \text{for every } m. \tag{21}$$

This clearly implies that $V(G) = 0$. □

Next, we observe that $\bar{H}_2(S)$ is a conserved density for the LL-equation, i.e.

$$\{\bar{H}_2(S), \bar{H}_1(S)\} = 0. \tag{22}$$

This can be checked by a simple direct calculation which we do not want to present here since the fact is well known ([2, A_2 in formula 3.17]). Because of $\Gamma\bar{T}(S) = \tau_+(S)$ an application of Γ to (22) yields

$$[K_1, K_2] = 0.$$

Now, we can proceed exactly as in [7] or [8]. From the Jacobi identity we obtain for $K_3 = [K_2, \tau_+]$ that

$$[K_1, K_3] = 0.$$

So, K_3 must be a symmetry generator for (1).

We proceed this way as long as we know that $[K_n, K_{n+1}] = 0$, $n = 1, \ldots, m$. Then a trivial application of the Jacobi identity shows that all the K_n, $n = 1, \ldots, m$ do commute. We claim that

$$[K_n, K_{n+1}] = 0, \quad \text{for all } n, \tag{23}$$

but we postpone the arguments for this fact to the next section.

Assuming this we apply Γ to (15). Since $\Gamma H_1 = K_1$, $\Gamma T = \tau_+$ and because Γ is a Lie algebra homomorphism, eqs. (15) go over into (13), and (17) must hold. Since the K_n commute we obtain $\Gamma\{\bar{H}_n, \bar{H}_m\} = 0$ for all n, m. Hence $\{\bar{H}_n, \bar{H}_m\} = 0$ for all n, m. So, it only remains to show that, for arbitrary n, the P_m are constants of motion for

$$S_t = K_n(S).$$

This is a trivial exercise.

$$\begin{aligned} P_m(S(t))_t &= \int_{-\infty}^{+\infty} (\bar{H}_m(S(t)) - \bar{H}_m(e_t))_t \,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (\nabla\bar{H}_m(S(t)), \Gamma\bar{H}_n(S(t)))\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (\nabla\bar{H}_m(S(t)), S(t)\cdot\nabla\bar{H}_n(S(t)))\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \{\bar{H}_m, \bar{H}_n\}\,\mathrm{d}x = 0. \end{aligned}$$

5. Commutativity-leading term analysis

The Lie algebra under consideration has particular structural properties which force the sequence of symmetries constructed in the last section to be commutative. Nevertheless we would like to sketch an ad-hoc proof of our commutativity claim, since pointing out all structural properties, and proving the relevant statements, would go beyond the aim of this paper.

The proof of the commutativity result consists of a simple "leading-term-analysis." In order to get accustomed to this method we start with the special case $J=0$:

$$S_t = K_{0,1}(S) = S\cdot S_{xx}, \tag{24}$$

and we consider the sequence given by (13)

$$\begin{aligned} &K_{0,0}(S) = S_x,\\ &K_{0,n+1}(S) = [K_{0,n}(S), \tau_{0+}(S)], \end{aligned} \tag{25}$$

where

$$\tau_{0+}(S) = xS\cdot S_{xx} + S\cdot S_x. \tag{26}$$

A vector field $G(S)$ is said to have degree n if

$$[G(S), xS_x] = nG(S). \tag{27}$$

This simply means that the number of derivations occurring in each term minus the corresponding power of x has to be n. Observe that the degree is additive with respect to the commutator. The degree of $K_{0,0}$ is 1 and that of $K_{0,n}$ is $(n+1)$ since τ_{0+} has degree one. Those terms in a vector field with the highest derivatives are called the leading terms.

Now, consider again (14.1) and rewrite it with respect to the coordinates $\binom{\rho_1}{\rho_2}$ in the two-dimensional euclidean parameter space given by the tangent space of the unit sphere at the point $S(x)$. The equation then has the form

$$\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}_t = \begin{pmatrix}\rho_{2xx}\\ -\rho_{1xx}\end{pmatrix} + \text{higher order terms}. \tag{28}$$

From this we easily obtain that the leading term of any symmetry generator of this equation has to be of the form

$$\begin{pmatrix}0 & D\\ -D & 0\end{pmatrix}^{(n-1)}\begin{pmatrix}\rho_{1x}\\ \rho_{2x}\end{pmatrix},$$

where n is the degree of the term. Going back to coordinates on the sphere this tells us that the leading term of the symmetry generator must be of the form $(S\cdot)^{n-1}S_{(n)}$ (n differentiations = degree of the term).

Now, take the series $K_{0,n}$ defined in (25) and assume

$$[K_{0,n-1}, K_{0,n}] = 0, \quad \text{for all } n\le N.$$

For $N=2$ this is certainly true since we checked that directly. The Jacobi identity (or Leibniz formula) then implies that all the $K_{0,n}$, $n\le(N+1)$ are commuting with $K_{0,1}$. Hence the leading term of $K_{0,n}$ is of the form $(S\cdot)^n S_{(n+1)}$. Now, assume (falsely) that

$$\mathscr{K} = [K_{0,N}, K_{0,N+1}] \ne 0.$$

The degree of this term must be equal to $(N+1)+(N+2) = 2N+3$ since the degree is additive with respect to commutators. Obviously, $\mathscr{K}$ commutes with $K_{0,1}$, so its leading term must be of the form $(S\cdot)^{2N+2}S_{(2N+3)}$. But, apart from leading terms, the highest order of derivatives in $K_{0,N}$ and $K_{0,N+1}$ are $\le N-1$ and $\le N$ respectively. Thus the leading term of $\mathscr{K}$ must come out of a commutation of the leading terms of $K_{0,N}$ and $K_{0,N+1}$. A trivial direct calculation shows that this cannot be the case. Hence $\mathscr{K}=0$, or

$$[K_{0,n}, K_{0,n+1}] = 0, \quad \text{for all } n,$$

and the Jacobi identity implies that all the $K_{0,n}$ do commute.

Using this result we easily can do the same kind of analysis for the case $J\ne 0$. We have to modify our arguments slightly since the full equation is not homogeneous with respect to the degree defined in (27). Fortunately, it helps a lot that we have free parameters in the full equation. In order to use these parameters efficiently we replace J by λJ and we consider the sequence defined by (13). Again we assume

$$[K_{n-1}, K_n] = 0, \quad \text{for all } n = 1, \ldots, N$$

and

$$\mathscr{K} = [K_N, K_{N+1}] \neq 0.$$

Certainly $N \geq 2$. Now, we write all occurring quantities as polynomials in λ. We observe that the zero λ-order terms are equal to $\tau_{0,n}$ and $K_{0,n}$, respectively. We know that K_1 commutes with $\mathscr{K}$. This means that $K_{0,1}$ has to commute with the lowest λ-order term $\tilde{\mathscr{K}}$ of $\mathscr{K}$. From our results for the case $J = 0$ we know that the lowest λ-order cannot be zero. Hence, the leading term of $\tilde{\mathscr{K}}$ cannot be of the required from $(S\,\cdot)^{n-1}S_{(n)}$, since there are appearing some of the completely arbitrary J's in it. So, again, $\mathscr{K} = 0$ which yields that all the K_n commute.

At this stage I cannot resist the temptation to point out the structural reasons for our commutativity assertion. If one considers, among the vector fields under consideration, the sub-Lie algebra $\mathscr{L}$ consisting of all those fields which do not explicitly depend on x then, by a lengthy and complicated procedure, one can prove that this Lie algebra is beautiful. By that we mean that for every $G \in \mathscr{L}$ either

$$[G, \tilde{G}] = 0, \quad \text{for all } \tilde{G} \in \mathscr{L},$$

or that the commutant

$$G^{\perp} = \{\tilde{G} \in \mathscr{L} | [G, \tilde{G}] = 0\}$$

of G is abelian.

Another approach to commutativity results can be made via the notion of hereditary symmetries (see [9] for details). One can show that there is a hereditary symmetry Φ mapping K_1 into K_3. This then implies commutativity. But since I was not able to write down Φ explicitly I have chosen a direct approach.

6. Time-dependent quantities

It is known for the Benjamin–Ono equation [10], and some other equations too, that the construction which was carried out in section 3 corresponds to the construction of one symmetry generator with explicit time dependence, see also [11]. Now, the question is, are there higher order time-dependent symmetries? This question corresponds to the problem whether or not the role of the fields τ_+ and $\overline{T}$ can be played by other operators. As one can see, τ_+ can be replaced by higher order fields whereas $\overline{T}$ cannot.

Let us start our considerations by pointing out the connection with time-dependent quantities.

Take a family of vector fields $G(S,t)$ depending polynomially on the parameter t. We call $G(S,t)$ a time-dependent symmetry generator for

$$S_t = K_n(S) \tag{14.n}$$

if the infinitesimal transformation

$$S(t) \to S(t) + \varepsilon G(S(t), t), \quad \varepsilon \text{ infinitesimal}$$

leaves (14.n) form-invariant. We easily see that this is equivalent to

$$G_t = [K_n, G], \tag{29}$$

where the right-hand side is the usual Lie bracket (not effecting t) and the left-hand side is the partial derivative with respect to the parameter t.

In complete analogy, a time-dependent density $\bar{\sigma}(S,t)$ is said to be a conserved density for (14.n) if

$$\bar{\sigma}(S,t)_t = \{\overline{H}_n, \bar{\sigma}\}, \tag{30}$$

where $\overline{H}_n$ is the Hamiltonian for K_n.

Now, let $\overline{T}(S)$ be a time-independent density with the property that $\{\overline{H}_n, \overline{T}\}$ is a conserved density for (14.n). Furthermore, let $\tau(S)$ be a vector field such that $[K_n, \tau]$ is a symmetry generator for (14.n). Then

$$\overline{T}(S,t) = t\{\overline{H}_n, \overline{T}\} + \overline{T}(S),$$
$$\tau(S,t) = t[K_n, \tau] + \tau(S),$$

are a time-dependent conserved density and symmetry generator, respectively. Conversely, if these quantities are conserved densities and symmetry generators, then $\{\overline{H}, \bar{\tau}\}$ and $[K_n, \tau]$ must be a conserved density and a symmetry generator, respectively. Since, the τ_+ and T of section 3 have these desired properties we can rephrase our main results in the following way:

$$\overline{T}_{1,n}(S,t) = t\overline{H}_{n+1}(S) + \overline{T}(S), \tag{31a}$$
$$\tau_{1,n}(S,t) = tK_{n+1}(S) + \tau_+(S), \tag{31b}$$

are time-dependent conserved densities and symmetry generators of (14.n). The question is: are there others? By ad-hoc methods one can show that there are no other conserved densities, but that there are many more time-dependent symmetry generators. The explicit construction of these quantities is obtained by considering a suitable enlargement of the Lie algebra under consideration. A detailed analysis of this is carried out in a subsequent paper.

Note added in proof

A similar analysis can be carried out for the XYZ-model in statistical mechanics, although the situation becomes more complicated since for the periodic case the manifold under consideration is finite dimensional. For details see the author's contribution to the Sitges conference (1984) on statistical mechanics.

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