Integrable nonlinear evolution equations with time-dependent coefficients

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A simple and straightforward method for generating completely integrable nonlinear evolution equations with time-dependent coefficients is presented herein. For the equations under consideration, the solutions to those given by vector fields which are independent of time are given, thus explicit links between equations are obtained. As an application of the proposed method it is shown that the linear superposition, with arbitrary time-dependent coefficients, of different members of an integrable hierarchy is again integrable. Furthermore, it turns out that for some integrable equations [like the Korteweg–de Vries (KdV), the Benjamin–Ono (BO), or the Kadomtsev–Petviashvili (KP) the resolvent operator of lower order flows can be explicitly obtained from that of any higher order flow. Those flows (demonstrated for the KdV) which can be generated by Lie homomorphisms coming from first order problems are completely classified. Many well-known equations which can be found in the literature are of that type. As an application of such a first order link a direct link is given from KdV to the cylindrical KdV, and from there to the KP with nontrivial dependence on the second spatial variable.

I. INTRODUCTION

Very often in the literature (see the extensive literature survey in the application section) modifications of well-known integrable systems are found which again turn out to be integrable. These so-called equations with variable coefficients play an important role in applications. They originated, in the case of the Korteweg–de Vries (KdV), from shallow water problems in water with variable depth, and today they generally turn up where modifications of integrable systems are needed to take inhomogeneous properties of media into account.

In most examples the additional terms are somehow related to the symmetry analysis, or the scaling analysis, of the underlying equations. Usually, the methods for integrability results for these equations are *ad hoc* methods. However, looking at the variety of examples one has the impression that there must be a general approach to these equations. Indeed, there is such a general approach. This approach is simple, transparent and straightforward and will be presented in this article. Consider an evolution equation of the form

$$u_t = K(u), \tag{1.1}$$

where u evolves on some suitable manifold of functions in the independent variable $\mathbf{x}=(x_1,\ldots,x_n)$. Then, such an equation is said to be integrable if an infinite dimensional symmetry group (represented by its infinitesimal generators) can be found. Here, as in the literature, we abbreviate *infinitesimal generator of a symmetry group* (or semigroup) just by the notion of *symmetry*. Or more precise, the notion of *symmetry* of a given flow stands for a vector field invariant under this flow. In all known cases of integrable equations the symmetry algebra can be extended to a Virasoro algebra of vector fields (i.e., an algebra of symmetries and mastersymmetries or a hereditary algebra, see Refs. 1–4). The commutation relations of this Virasoro algebra are

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$$[K_n, K_m]=0, \tag{1.2}$$

$$[\tau_n, K_m]=(m+\rho)K_{n+m}, \tag{1.3}$$

$$[\tau_n, \tau_m]=(m-n)\tau_{n+m}, \tag{1.4}$$

where ρ is a fixed number, m, n run from either -1 or 0 to infinity [there are also meaningful cases where the m, n run from $-\infty$ to $+\infty$ (see Ref. 5)], and where the τ_n, K_m are suitable vector fields on the manifold under consideration. One should recall that the commutator can be expressed (in any suitable parametrization of the manifold) as

$$[K, G]:=G'(u)[K(u)]-K'(u)[G(u)], \tag{1.5}$$

where $G'[K]$ denotes the variational derivative

$$G'(u)[K(u)]:=\frac{\partial}{\partial\epsilon} G(u+\epsilon K(u))_{|\epsilon=0} \tag{1.6}$$

in the direction of the vector field K. Equation (1.1) is said to be *integrable* if $K(u)$ is in the linear hull of the K_n, $n\in\mathrm{N}$ of such a Virasoro algebra.

One can rephrase these observations in a more concrete way by introducing time-dependent symmetries. To see this, consider a one-parameter family of vector fields $K(u, t)$ and define by that a time-dependent flow

$$u_t=K(u, t). \tag{1.7}$$

Then another family of vector fields $G(u, t)$ is said to be a time-dependent symmetry (see Ref. 6) if

$$[K(u, t), G(u, t)]+\frac{\partial G(u, t)}{\partial t}=0. \tag{1.8}$$

This concept generalizes the concept of symmetries and has the advantage that we may include, from the beginning, equations where the right-hand side depends explicitly on time. Now, we return to Eq. (1.1), where the flow is given by a time-independent vector field $K(u)$. If we assume that $K(u)$ is a linear combination of the K_n of some Virasoro algebra, then we can make out of the corresponding τ_m of that Virasoro algebra time-dependent symmetries. A simple computation shows that

$$\tau_m(u,t):=\tau_m(u)+t[K(u), \tau_m(u)] \tag{1.9}$$

are indeed time-dependent symmetries for Eq. (1.1). This is an immediate consequence of the fact that by the commutation relations of the Virasoro algebra, and by use of the Jacobi identity, we obtain

$$[K(u), [K(u), \tau_m(u)]]=0. \tag{1.10}$$

So, we may say that Eq. (1.1) is integrable if it admits a Virasoro algebra of time-dependent symmetries. This definition, which, verbatim, can be carried over to Eq. (1.7), is the notion of integrability on which this article is based.

II. THE MAIN OBSERVATION

Denote the Lie algebra defined in Eq. (1.5) as $\mathscr{L}$. The Lie derivative given by some $M\in\mathscr{L}$ on $\mathscr{L}$ we denote by Λ_M, i.e., for all $K\in\mathscr{L}$

$$\Lambda_M K := [M, K]. \tag{2.1}$$

It is well-known that, formally, the exponential of Λ_M

$$\exp(\Lambda_M) := \sum_{n=0}^{\infty} \frac{\Lambda_M^n}{n!} \tag{2.2}$$

is a Lie algebra isomorphism on $\mathscr{L}$. By "formal" we mean that whenever the application of Λ_M to K as well as to G converges then

$$\exp(\Lambda_M)[K, G] = [\exp(\Lambda_M)K, \exp(\Lambda_M)G]. \tag{2.3}$$

Again, we introduce one-parameter families $H(u, t)$ of vector fields; these are always assumed to be differentiable in t. The derivative with respect to t we denote by ∂_t. Our main result is

Theorem 2.1:

(a) Let $G(u, t)$ be a symmetry for

$$u_t = K(u, t) \tag{2.4}$$

and let $H(u, t)$ be a family of time-dependent vector fields. Then

$$\Gamma(u, t) := \exp(\Lambda_H)G \tag{2.5}$$

is a symmetry for the equation

$$u_t = \exp(\Lambda_H)K - \sum_{n=0}^{\infty} \frac{(\Lambda_H)^n}{(n+1)!} \partial_t H. \tag{2.6}$$

Furthermore, the Lie algebras of symmetry group generators of Eqs. (2.4) and (2.6) are isomorphic. So, if Eq. (2.4) admits a Virasoro algebra as symmetry group generators, then Eq. (2.6) also admits such an algebra of symmetry group generators.

(b) Let σ be a new evolution parameter and consider the equation

$$U_\sigma = -H(U, t). \tag{2.7}$$

Whenever $U = U(\mathbf{x}, t, \sigma)$ is a solution of Eq. (2.7) such that $u(\mathbf{x}, t) := U(\mathbf{x}, t, \sigma=0)$ is a solution of Eq. (2.4) then $U(\mathbf{x}, t, \sigma=1)$ is a solution of Eq. (2.6).

Before we prove this statement we illustrate by a simple example how this result works. Later on we shall give examples which are more meaningful.

Example 2.2: We claim that, for any $h(t)$, the equation

$$u_t = u_{xxx} + 6\exp(h(t))uu_x - h'(t)u \tag{2.8}$$

admits a Virasoro algebra as symmetries. Indeed, this flow, for example, admits u_x as well as

$$u_{xxxxx} + 20\exp(h(t))u_{xx}u_x + 10\exp(h(t))uu_{xxx} + 30\exp(2h(t))u^2u_x \tag{2.9}$$

as symmetries, and so on.

Furthermore it turns out that $u(x, t)$ is a solution of Eq. (2.8) if and only if $u(x, t) \times \exp(h(t))$ is a solution of the KdV.

This is easily explained, and generalized, by introducing the *u-scaling degree*. Let $m(u)$ be any monomial in $u, u_x, u_{xx}, \ldots$. Its *u-scaling factor* is its polynomial degree minus 1. For example, the scaling factors of u^3, $u^2u_x^2$, $u(u_{xx})^4$ are 2, 3, and 4. By $\exp(\lambda S)$ we denote the operator which acts on a linear sum of monomials by multiplying each of its summands by the exponential of $\lambda \times$ *its u-scaling factor.* So,

$$\exp(hS)(u_{xxx}+6uu_x)=u_{xxx}+6\exp(h)uu_x. \tag{2.10}$$

Now, choose in theorem 2.1

$$H(u, t)=h(t)u \tag{2.11}$$

and observe that application of $\exp(\Lambda_H)$ coincides with application of the scaling exponential $\exp(hS)$. Furthermore, observe that the sum

$$\sum_{n=0}^{\infty} \frac{(\Lambda_H)^n}{(n+1)!} \partial_t H \tag{2.12}$$

reduces to its first term $(n=0)$ since H and $\partial_t H$ do commute. Therefore application of 2.1 to the KdV hierarchy yields a hierarchy of time-dependent flows of which Eq. (2.8) is the second member. Indeed, the whole Virasoro algebra of symmetries and mastersymmetries of the KdV can be carried over to Eq. (2.8).

The relation between solutions of Eq. (2.8) and the KdV is a consequence of theorem 2.1.b. However in this simple case it is easily checked by a direct computation.

Proof of theorem 2.1:

(a) The basis for the Lie algebra $\mathscr{L}$ was a manifold $\mathscr{M}$ of functions in $\mathbf{x}$. We now change that viewpoint by considering a manifold consisting of orbits on $\mathscr{M}$, i.e., we consider the field variable u as a function in $\mathbf{x}$ *and t*. By $\mathscr{L}_{\text{ex}}$ we denote the vector field Lie algebra with respect to flows on this extended manifold.

Since $\mathscr{L}$ can be considered as a subalgebra of $\mathscr{L}_{\text{ex}}$ (see Ref. 7) we denote the Lie algebra in $\mathscr{L}_{\text{ex}}$ also by [,]. Using this Lie algebra we can now rephrase condition (1.8): A time-dependent vector field $G(u, t)$ is a symmetry group generator of

$$u_t=K(u, t)$$

if and only if

$$[K(u)-u_t, G]=0 \tag{2.13}$$

in the extended Lie algebra $\mathscr{L}_{\text{ex}}$. After this observation the proof of Theorem 2.1 is straightforward. We consider, all in the extended Lie algebra, the Lie algebra isomorphism $\exp(\Lambda_H)$. Then application of this isomorphism to $K(u)-u_t$ leads to the right-hand side of Eq. (2.6), and G is transferred by this to the Γ given in Eq. (2.5). Using the fact that $\exp(\Lambda_H)$ is an isomorphism, and taking the first nontrivial symmetry of the KdV, we see from Eq. (2.10) that indeed Eq. (2.9) must be a symmetry for Eq. (2.6). Again, the isomorphy of the Virasoro algebra of the symmetry group generators of Eqs. (2.6) and (2.4) follows from the isomorphy property of $\exp(\Lambda_H)$.

(b) Consider the manifold of functions U in $\mathbf{x}$, t, and σ and on that the solution manifold $\mathscr{M}_1$ of $R(U)=0$, where

$$R(U):=\exp(\sigma\Delta_{H(U,t)})(U_t-K(U, t)). \tag{2.14}$$

Observe that its fibers $\sigma=0$ and $\sigma=1$ are the solution manifolds of Eqs. (2.4) and (2.6), respectively. Taking the total σ derivative of $R(U)$ on the whole manifold we find

$$R'[H]-H'[R]+R'[U_\sigma]=0$$

or, since $R(U)=0$ on $\mathscr{M}_1$

$$R'[H+U_\sigma]=0. \tag{2.15}$$

Hence, $U_\sigma=-H$ leaves $\mathscr{M}_1$ invariant, and obviously this flow transports from the fiber $\sigma=0$ to the fiber $\sigma=1$. ∎

One may ask at this point if notions as *hereditary operators* and the like can be transferred from Eqs. (2.4)–(2.6). This, of course, is possible for any kind of invariant tensor: By the usual procedure[8,9] we extend the Lie derivative from vector fields and scalar fields to arbitrary tensor fields by requiring the validity of the product rule. Then a tensor $\Phi(u,t)$ is invariant with respect to the flow

$$u_t=K(u,t)$$

if and only if its Lie derivative with respect to the vector field $K(u,t)-u_t$ vanishes. Now, if the meaning of Λ_H is extended, such that it stands for the Lie derivative with respect to H applied to arbitrary tensor fields, then one easily sees that

$$\exp(\Lambda_H)$$

provides an isomorphism for the tensor algebra built up over the vector fields $\mathscr{L}_{\text{ex}}$. Hence, by application of this formal isomorphism we may transfer invariant tensors for Eq. (2.4) to invariant tensors for Eq. (2.6).

III. APPLICATIONS

The examples given in the following sections are based on transforming equations such as Eq. (2.4) with the help of time-dependent vector fields $H(u,t)$ into new equations like Eq. (2.6) and then using the direct transfer for solutions, as given by Eq. (2.7), or the transfer of vector fields by Eq. (2.5), to obtain information about the new equation. For the transfer of solutions, of course, it is necessary that Eq. (2.7) is integrable. This is the case whenever the vector field $H(u,t)$ is a scaling symmetry or even a simpler field; hence working with scaling symmetries will make up for a large part of these applications. Even for these simple cases we obtain a large variety of equations usually treated as separate cases in the literature; we shall demonstrate that for the KdV. However, in order to show that also less obvious results can be obtained from Theorem 2.1 we start with some examples being more involved from the viewpoint of nonlinear equations.

A. Finite sums of integrable fields with arbitrary time-dependent coefficients

Consider any hierarchy of commuting vector fields $K_n(u)$, $n\in\mathbb{N}$. Then usually the equations

$$u_t=K(u) \tag{3.1}$$

can be solved by standard methods (inverse scattering theory, Hirotas bilinear method, etc.). To facilitate notation we denote by

$$u(x,t)=R_K(U(\mathbf{x},\tau),t) \tag{3.2}$$

the solution of

$$u_t=K(u,t)$$

for the initial condition $u(x,\tau,t=0)=U(x,\tau)$. Here τ plays the role of an additional parameter. The crucial operator R_K we call the *resolvent operator* of the vector field K. An equation is said to be *solvable* if this resolvent operator can be computed somehow.

We are interested whether equations of the form

$$u_t=\sum \phi_n(t)K_n(u) \tag{3.3}$$

are again solvable if $K_n(u)$ are. Here, of course the sum is assumed to be a finite sum. Indeed, we show as an application of Theorem 2.1 that this is the case for arbitrary functions $\phi_n(t)$ and that the solution can be written in terms of the resolvent operators of K_n. It suffices to show a suitable result for the linear superposition of two fields.

Theorem 3.1: *Consider commuting vector fields $K_1(u,t)$, $K_2(u)$*

$$[K_1(u,t),K_2(u)]=0,$$

where $K_2(u)$ is assumed to be time-independent. Let the equations

$$u_t=K_1(u,t), \tag{3.4}$$

$$u_t=K_2(u) \tag{3.5}$$

be solvable. Then for an arbitrary function $\phi(t)$ in time the equation

$$u_t=K_1(u,t)+\phi(t)K_2(u) \tag{3.6}$$

is again solvable. Indeed, let

$$\frac{d}{dt}\psi(t)=\phi(t)$$

then

$$R_{K_2}(R_{K_1}(U(x),t_1),t_2)_{|\{t_2=\psi(t_1),\,t_1=t\}} \tag{3.7}$$

is the solution of Eq. (3.6) for the initial condition $u(x,t=0)=U(x)$.

Proof: Put $H(u,t)=-\psi(t)K_2(u)$ then Eq. (2.6) carries over in Eq. (3.6). Since the flow (evolution parameter σ)

$$U_\sigma=\psi(t_1)K_2(U)$$

has for initial $U(x,t_1,\sigma=0)=u(x,t_1)$ the solution $R_{K_2}(u(x,t_1),\sigma\psi(t_1))$ we find by Theorem 2.1 that

$$R_{K_2}(u(x,t),\sigma)_{|\{\sigma=\psi(t)\}}$$

must be a solution of Eq. (3.6) whenever $u(x, t)$ is a solution of Eq. (3.4). Hence Eq. (3.7) gives the solution of Eq. (3.6). ∎

Obviously, iteration of that result leads to the claim that we may write the solutions of Eq. (3.3) in terms of the resolvents of K_n. In case these K_n are from a hierarchy generated by a recursion operator, then it is just a matter of routine to construct the recursion operator of Eq. (3.3) by similar methods. This is possible because we have generated the solutions by the application of a Lie homomorphism for the underlying tensor bundles.

B. Solutions for flows given by time-dependent symmetries

As we know, for all known integrable systems, in addition to the usual symmetries, there are those depending explicitly on time. These additional symmetries are either only of first order (conformal symmetries) or, for some equations, also of higher order. Such higher order equations, for example, can be found for the Benjamin–Ono equation (BO) or the Kadomtsev–Petviashvili equation (KP).[1]

To give a nontrivial example, we consider the Benjamin–Ono equation

$$u_t = Hu_{xx} + 2uu_x, \tag{3.8}$$

where H stands for the Hilbert transform

$$(Hf)(x) := \frac{1}{\Pi} \int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi - x}\, d\xi, \quad \text{(principal value integration)}. \tag{3.9}$$

The following equation is derived from the first nontrivial time-dependent symmetry of the BO:

$$\begin{aligned} u_t &= 2(Hu_{xx} + 2uu_x) + 2(u + xu_x + x) \\ &\quad 4t(Hu_{xx} + 2uu_x + u + xu_x) - 4t^2(Hu_{xx} + 2uu_x). \end{aligned} \tag{3.10}$$

We want to solve such an equation for arbitrary initial value. Indeed, this is possible in terms of the iteration of the resolvent operator for the BO. To do this consider

$$K_1(u) = Hu_{xx} + 2uu_x. \tag{3.11}$$

We claim that

$$u(x, t) := R_{K_1}(R_{K_1}(U(x) + x, t_1) - x, t_2)_{|\{t_2 = t_1,\, t_1 = t\}} \tag{3.12}$$

solves Eq. (3.10) for the initial value $u(x, t=0) = U(x)$, where R_{K_1} is the resolvent operator of the first member of the hierarchy. To prove this result we observe first that the Virasoro algebra of the BO is easily generated by the first nontrivial mastersymmetry and that this mastersymmetry is obtained by commuting K_2 with the trivial vector field $G(u) = x$ (see Ref. 1), hence x is a mastersymmetry of the second order for the BO. As a consequence of the commutation relation resulting from this observation we represent the right-hand side of Eq. (3.10) by use of

$$\begin{aligned} \exp(-t\Lambda_{K_1})(\exp(\alpha\Lambda_G)K_1) &= Hu_{xx} + 2uu_x + 2(u + xu_x + x) + 4t(Hu_{xx} + 2uu_x + u + xu_x) \\ &\quad - 4t^2(Hu_{xx} + 2uu_x) \end{aligned} \tag{3.13}$$

and

$$\exp(\alpha\Lambda_G)K_1 = K_1(u(x)+x).$$

Using this we see that the right-hand side of Eq. (3.10) is equal to

$$\exp(-t\Lambda_{K_1})(\exp(\Lambda_G)K_2) - \frac{\partial}{\partial t}(-tK_1)$$

and Theorem 2.1 may be applied because this field is of the form (2.6). Actually the Theorem has to be applied twice. In the first step we find that $R_{K_1}(U(x)+x, t) - x$ solves the initial value problem $u(x, t=0) = U(x)$ for

$$u_t = \exp(\Lambda_G)K_1.$$

In the second step a further application of 2.1 gives Eq. (3.12).

In this example the decisive tool was that the Virasoro algebra of the BO was generated by a nontrivial symmetry and a constant vector field. Hence, the above arguments can be applied to all equations where this is the case, for example, to the KP (see Ref. 1 for finding the necessary commutation relations).

C. Solving lower order equations

Consider, for example, the KdV or the BO. The problem seems interesting if we can find explicit formulas between the resolvent operators for different members of the hierarchy.

Indeed sometimes this is possible, for example, for the KdV, the BO, or the KP. Because we have a constant vector field as a descending master symmetry of the first order, we can determine explicitly from the resolvent for the nth member of the hierarchy $u_t = K_n(u)$ the resolvent operators of all lower order flows. We briefly demonstrate the case of the KdV, where the trivial field $G(u)=1$ is a mastersymmetry going, via commutation, from K_{n+1} to K_n. To see the details let

$$u_t = K_2(u) := (u_{xxxx} + 5u_x u_x + 10uu_{xx} + 10u^3)_x \tag{3.14}$$

be the second member in the hierarchy. Observe that

$$\Lambda_G K_2 = 10(u_{xxx} + 6uu_x) = 10K_1$$

and

$$(\Lambda_G)^2 K_2 = 60u_x.$$

Hence

$$K := \exp(\Lambda_G)K_2 = K_2 + 10K_1 + 60K_0,$$

where K_1 and $K_0 = u_x$ are the lower order symmetries of Eq. (3.14). Now, using 2.1 we find that the resolvent operator for K_2 trivially yields the resolvent operator for K. Hence by the results in Sec. III A we find the resolvent operator for $K - K_2 = 10K_1 + 60K_0$. Now elementary routine allows us to get rid of K_0, thus we are able to find the resolvent operator for K_1 in terms of that for K_2. As an exercise the reader may derive the explicit formulas (and check them out, say, for the two-soliton solutions).

D. Some Tools for working with scaling symmetries

In the following examples we use scaling fields and fields of a similarly simple nature. As a tool, we first need solutions for some special Cauchy problems of first order partial differential equations in the unknown function $F(t,\lambda)$.

Lemma 3.2: Let $\psi(t)$ *be some differentiable function, its inverse function we denote by* ψ_{inv} *and we abbreviate*

$$\tau(t,\lambda)=\psi_{\text{inv}}(\psi(t)-\lambda). \tag{3.15}$$

(a) By $\psi_t(\tau(t,\lambda))$ *we denote the derivative of* ψ *at* $\tau(t,\lambda)$ *whereas* $\psi(\tau(t,\lambda))_t$ *denotes the* t *derivative of* $\psi(\tau(t,\lambda))$, i.e., $\psi(\tau(t,\lambda))_t=\tau_t(t)\psi_t(\tau(t,\lambda))$. *The initial value problem* $F(t,\lambda=0)=\varphi(t)$ *for*

$$-\psi_t^{-1}F_t=F_\lambda \tag{3.16}$$

has the solution

$$F(t,\lambda)=\varphi(\tau(t,\lambda)). \tag{3.17}$$

(b) The initial value problem $F(t,\lambda=0)=\varphi(t)$ *for*

$$-\psi_t^{-2}\psi_{tt}F-\psi_t^{-1}F_t=F_\lambda \tag{3.18}$$

has the solution

$$F(t,\lambda)=\psi(t)_t^{-1}\psi_t(\tau(t,\lambda))\varphi(\tau(t,\lambda)). \tag{3.19}$$

(c) Let $a(t)$ *be differentiable in* t, *then the general solution of*

$$F_\lambda+\psi_t^{-1}F_t=\frac{a_t}{\tau_t a} \tag{3.20}$$

is

$$F(t,\lambda)=\ln(a(t))\psi_t(\tau(t,\lambda))+Q(\psi(t)-\lambda) \tag{3.21}$$

for arbitrary Q. *Hence the solution of the initial value problem* $F(t,\lambda=0)=0$ *is obtained for*

$$Q(\sigma)=-\ln(a(\psi_{\text{inv}}(\sigma)))\psi_t(\psi_{\text{inv}}(\sigma))$$

as

$$F(t,\lambda)=\ln\left(\frac{a(t)}{a(\tau(t,\lambda))}\right)\psi_t(\tau). \tag{3.22}$$

The proof of these results, which were obtained by the well-known theory for the first order equations,[10] can be checked by direct computation. However, one should keep in mind that

$$\tau_t=\frac{\psi_t(t)}{\psi_t(\tau(t,\lambda))}\quad\text{and}\quad\tau_\lambda=\frac{-1}{\psi_t(\tau(t,\lambda))}. \tag{3.23}$$

■

We introduce now the notion of *scaling*. The meaning of u scaling was already introduced in context of Example 2.2. By an x_i scaling of a vector field K we understand the successive operations of first replacing x_i by

$$x_i/\alpha$$

and then taking the derivative with respect to α at the point $\alpha=1$. The operator performing this scaling we denote by S_{x_i}. So, for homogeneous terms, the operator S_{x_i} counts the number of x_i derivatives and deducts from that the number of powers in x_i. For example,

$$S_x(u_{xxx}+u_xu_{xx}^2+x^4u_x)=3u_{xxx}+5u_xu_{xx}^2-3x^4u_x.$$

In the same manner we introduce the t scaling. A trivial, but nevertheless important observation is that these scalings can be realized by Lie derivatives

$$S_{x_i}=\Lambda_{x_iu_{x_i}}, \quad S_t=\Lambda_{tu_t}. \tag{3.24}$$

This corresponds to the already observed fact for u scaling

$$S=\Lambda_u. \tag{3.25}$$

Hence, it will not come as a surprise that exponentials of these scalings lead to Lie algebra isomorphisms. We compute explicitly some of the Lie algebra isomorphisms given by these scaling quantities.

Lemma 3.3:

(a) Let $K(u,t)$ be a vector field not containing t derivatives and let $a(t)$, $b_i(t)$ be suitable functions in t. Consider

$$H=\ln(a(t))u+\sum_i \ln(b_i(t))x_iu_{x_i} \tag{3.26}$$

then

$$\exp(\lambda\Lambda_H)K(u,t)=a(t)^{\lambda S}\prod_i b_i(t)^{\lambda S_{x_i}}K(u,t). \tag{3.27}$$

(b) Now, consider the functions introduced in context of Lemma 3.2, then for

$$H(u,t)=\psi_t(t)^{-1}u_t \tag{3.28}$$

for some t independent $K(u)$, which does not contain t derivatives, we find

$$\exp(\lambda\Lambda_H)\varphi(t)K(u)=\varphi(\tau(t,\lambda))K(u). \tag{3.29}$$

(c) And for the same $H(u,t)=\psi_t(t)^{-1}u_t$ we have

$$\exp(\lambda\Lambda_H)\varphi(t)u_t=\psi(t)_t^{-1}\psi_t(\tau(t,\lambda))\varphi(\tau(t,\lambda))u_t. \tag{3.30}$$

(d) Finally, for

$$H=\ln(a(t))u+\sum_i \ln(b_i(t))x_iu_{x_i}+\psi_t(t)^{-1}u_t \tag{3.31}$$

we obtain

$$\exp(\lambda\Lambda_H)u_t = R_a(t,\lambda)u + \sum_i R_{b_i}(t,\lambda)x_i u_{x_i} + \tau_t(t,\lambda)^{-1}u_t, \tag{3.32}$$

where

$$R_a(t,\lambda) = \ln\left(\frac{a(t)}{a(\tau(t,\lambda))}\right)\psi_t(\tau). \tag{3.33}$$

The R_{b_i} are defined accordingly.

Proof:

(a) The result of (a) follows directly from the interpretation of scaling as the effect of Lie derivatives.

(b) Looking at the power series for $\exp(\lambda\Lambda_H)$ we justify the ansatz

$$\exp(\lambda\Lambda_H)\varphi(t)K(u) = F(t,\lambda)K(u). \tag{3.34}$$

Taking the λ derivative yields

$$[H, F(t,\lambda)K(u)] = F_\lambda K(u).$$

Explicit computation leads to

$$-\psi_t(t)^{-1}F_t = F_\lambda. \tag{3.35}$$

Furthermore by putting $\lambda=0$ in Eq. (3.34) we see that F must fulfill the initial condition

$$F(t,\lambda=0) = \varphi(t).$$

So, the result follows from Lemma 3.2.a.

(c) Here we proceed in the same way. We take the ansatz

$$\exp(\lambda\Lambda_H)\varphi(t)u_t = F(t,\lambda)u_t \tag{3.36}$$

and obtain by the λ derivative and $\lambda=0$ the initial value problem

$$-\psi_t(t)^{-2}\psi_{tt}F - \psi_t(t)^{-1}F_t = F_\lambda, \quad F(t,\lambda=0) = \varphi(t),$$

the unique solution of which is given by Lemma 3.2.b.

(d) Here we make the ansatz

$$\exp(\lambda\Lambda_H)u_t = A(t,\lambda)u + \sum_i B_i(t,\lambda)x_i u_{x_i} + C(t,\lambda)u_t. \tag{3.37}$$

Again, the λ derivative and $\lambda=0$ lead to the initial value problem

$$C_\lambda + \psi_t^{-1}C_t = -\frac{\psi_{tt}(t)}{\psi_t(t)^2}C, \quad C(t,\lambda=0) = 1.$$

From Lemma 3.2.b we find its solution as

$$C(t,\lambda) = \tau_t(t,\lambda)^{-1}.$$

Using this result we see that the initial value problems for A and B_i are

$$A_\lambda+\psi_t^{-1}A_t=\frac{a_t}{\tau_t a}, \quad \text{with} \quad A(t,\lambda=0)=0$$

and

$$(B_i)_\lambda+\psi_t^{-1}(B_i)_t=\frac{b_{it}}{\tau_t b_i}, \quad \text{with} \quad B_i(t,\lambda=0)=0.$$

The solution of these have been given in Lemma 3.2.c. Inserting them in Eq. (3.37) we obtain the result. ■

E. Variable coefficient KdV's

We start with the well-known KdV

$$u_t=u_{xxx}+6uu_x \tag{3.38}$$

for which the Virasoro algebra is easily generated by the hereditary recursion operator

$$\Phi(u)=D^2+2u+2DuD^{-1}. \tag{3.39}$$

Application to suitable base elements yields K_m and τ_n

$$K_m=\Phi^m u_x, \quad \tau_n=\Phi^n(xu_x+2u). \tag{3.40}$$

Indeed, the hereditary property of Φ is equivalent to the Virasoro property of the algebra generated by the K_m, τ_n (see Ref. 4). Expressing this in the extended Lie algebra we know that

$$\Gamma(u)=u_{xxx}+6uu_x-u_t \tag{3.41}$$

admits a Virasoro algebra as commutant. Defining successively

$$\Gamma_1=\exp(\Lambda_{\ln(a(t))u+\ln(b(t))xu_x})\Gamma, \tag{3.42}$$

$$\Gamma_2=\exp(\Lambda_{\psi_t^{-1}u_t})\Gamma_1, \tag{3.43}$$

$$\Gamma_3=\exp(\Lambda_{\ln(\alpha(t))u+\ln(\beta(t))xu_x})\Gamma_2 \tag{3.44}$$

we obtain a vector field $\Gamma_3(u,t)$ which admits a Virasoro algebra as commutant, hence

$$\Gamma_3(u,t)=0 \tag{3.45}$$

is an integrable equation.

Abbreviating

$$T(t):=\tau(t,1)=\psi_{\text{inv}}(\psi(t)-1) \tag{3.46}$$

and carrying out the details of the computation we obtain with the help of the Lemma 3.3

$$\Gamma_1=b(t)^3u_{xxx}+6a(t)b(t)uu_x-\frac{a'(t)}{a(t)}u-\frac{b'(t)}{b(t)}xu_x-u_t, \tag{3.47}$$

$$\Gamma_2=b(T(t))^3u_{xxx}+6a(T(t))b(T(t))uu_x-\frac{a'(T(t))}{a(T(t))}u-\frac{b'(T(t))}{b(T(t))}xu_x-(\psi_t(t))^{-1}\psi_t(T(t))u_t, \tag{3.48}$$

$$\Gamma_3=b(T(t))^3\beta(t)^3u_{xxx}+6a(T(t))b(T(t))\alpha(t)\beta(t)uu_x-\frac{a'(T(t))}{a(T(t))}u-\frac{b'(T(t))}{b(T(t))}xu_x$$
$$-(\psi_t(t))^{-1}\psi_t(T(t))\frac{\alpha'(t)}{\alpha(t)}u-(\psi_t(t))^{-1}\psi_t(T(t))\frac{\beta'(t)}{\beta(t)}xu_x-(\psi_t(t))^{-1}\psi_t(T(t))u_t. \tag{3.49}$$

Observing

$$T_t(t):=\frac{\psi_t(t)}{\psi_t(T(t))}, \tag{3.50}$$

multiplying Eq. (3.49) with $T_t(t)$, and renaming $b(T(t))=B(t)$, $a(T(t))=A(t)$ we obtain that the equation

$$u_t=B(t)^3\beta(t)^3T_t(t)u_{xxx}+6A(t)B(t)\alpha(t)\beta(t)T_t(t)uu_x-\left(\frac{A(t)_t}{A(t)}+\frac{\alpha(t)_t}{\alpha(t)}\right)u$$
$$-\left(\frac{B(t)_t}{B(t)}+\frac{\beta(t)_t}{\beta(t)}\right)xu_x \tag{3.51}$$

must be integrable. Introducing now

$$v(t):=B(t)\beta(t),\quad w(t):=A(t)\alpha(t) \tag{3.52}$$

we find the integrable equation

$$u_t=v(t)^3T_t(t)u_{xxx}+6v(t)w(t)T_t(t)uu_x-\frac{w_t}{w}u-\frac{v_t}{v}xu_x. \tag{3.53}$$

It should be observed that the compatibility conditions for this equation are hidden, on one hand, in the interdependence of the coefficients, and on the other hand, in the way the crucial function $T(t)$ was constructed.

From the second part of Theorem 2.1 we know that there must be a direct link between the KdV and Eq. (3.53). In order to find this, we have to solve successively the evolution equations

$$U_{i+1}(x,t,\sigma)_\sigma=-H_i(U_{i+1}(x,t,\sigma),t),\quad i=1,\dots,3 \tag{3.54}$$

for

$$H_1=\ln(a(t))U+\ln(b(t))xU_x, \tag{3.55}$$

$$H_2=(\psi_t)^{-1}U_t, \tag{3.56}$$

$$H_3=\ln(\alpha(t))U+\ln(\beta(t))xU_x \tag{3.57}$$

in case of the initial conditions

$$U_{i+1}(x,t,\sigma=0)=u_i(x,t):=U_i(x,t,\sigma=1), \tag{3.58}$$

where u_1 is taken to be a solution of the KdV. Similar to Lemma 3.2 we find

$$U_2(x,t,\sigma)=a(t)^{-\sigma}u_1(xb(t)^{-\sigma},t), \tag{3.59}$$

$$U_3(x,t,\sigma)=u_2(x,\tau(t,\sigma)), \tag{3.60}$$

$$U_4(x,t,\sigma)=\alpha(t)^{-\sigma}u_3(x\beta(t)^{-\sigma},t). \tag{3.61}$$

Hence

$$u_4(x,t)=(\alpha(t)a(T(t))^{-1}u_1(x(\beta(t)b(T(t))^{-1},T(t)). \tag{3.62}$$

As consequence we have that whenever $u(x,t)$ is a solution of KdV, then

$$u_{\text{new}}(x,t)=w(t)^{-1}u(xv(t)^{-1},T(t)) \tag{3.63}$$

must be a solution of Eq. (3.53).

Remark 3.4: This last equation provides a *direct link* between the KdV and the class of time-dependent coefficient KdV's given by Eq. (3.53). Of course, most of the results of this section could have been obtained much simpler by use of this direct link (3.63). However, the derivation we have given yields additional information: first, that the class (3.53) is closed under all Lie homomorphisms being derived from first order problems and that this is the smallest such class of equations containing the KdV. Hence all first order modifications which can be found in the literature must be among these equations (if one searches the literature with respect to these equations, one really is surprised how much work has been invested in the study of special cases of these equations). Another decisive advantage of our derivation for these equations is that we constructed them via a Lie homomorphism for the tensor bundles in the extended Lie algbra. This allows a direct transfer of all notions and quantities formulated in a differential geometric invariant way, for example, the Virasoro algebra. We do not give explicitly the Virasoro algebra of Eq. (3.53), since that is now a simple exercise.

Special cases. Consider the special case

$$\psi(t)=-(\ln(c))^{-1}\ln(\ln(t)) \tag{3.64}$$

then

$$\psi(t)=1+\psi(t^c)$$

or

$$T(t)=t^c,\quad T_t(t)=ct^{c-1}. \tag{3.65}$$

If now, for example, we choose

$$v(t)=(ct^{c-1})^{-1/3},\quad w(t)=(ct^{c-1})^{-2/3} \tag{3.66}$$

this produces the KdV, where the most simple mastersymmetry has been added

$$u_t=u_{xxx}+6uu_x-(c-1)(2u+xu_x). \tag{3.67}$$

This is Blaszaks[11] extended KdV, for which he studied solitons and the like. Since the symmetry group structure of this equation is isomorphic to that of the KdV, the soliton solutions, being solutions obtained by group theoretical reductions, are carried over with the help from formula (3.63). This equation has also been studied, as GKdV (generalized KdV), in Ref. 12, where pseudopotentials, Lax pairs, and Bäcklund transformations were investigated. For the solution of this equation by inverse scattering see Ref. 13, also Ref. 14. Of course all these results can now be obtained by the direct link which preserves all the group theoretic structure.

It should be observed that Eq. (3.67) is the most simple nonisospectral flow for the usual Lax pair formulation of the KdV. Other nonisospectral flows for other equations [see Refs. 15 or 16 for the whole Ablowitz–Kanp–Newell–Segur (AKNS) class of these equations] can be obtained in the same way.

By a different choice of $v(t)$ and $w(t)$ other equations with variable coefficients are obtained. For example, one easily obtains

$$u_t+\beta t^{2n+1}(u_{xxx}+6uu_x)-\frac{n+1}{t}(2u+xu_x)=0, \tag{3.68}$$

an equation introduced by Nirmala, Vedan, and Baby.[17,18] Also the other equations studied by Baby, as well as by Li Yi-Shen and Baby, are of the same type (see Refs. 19–22). All these equations were introduced in order to explain soliton breaking in variable depth shallow water.

Other special cases of Eq. (3.53) are some (but not all, see below) of the KdV equations with nonuniformities studied by Brugarino,[23] among them the KdV in nonuniform media with relaxation coefficients.

F. Other KdV and mKdV modifications

The same procedure as with the KdV can be done with mKdV, or more generally, with the Gardner equation

$$u_t=u_{xxx}+c_1uu_x+c_2u^2u_x+c_3u_x. \tag{3.69}$$

We obtain then the integrable modification

$$\begin{aligned} u_t=&v(t)^3T_t(t)u_{xxx}+c_1v(t)w(t)T_t(t)uu_x+c_2w(t)^2v(t)T_t(t)u^2u_x \\ &+c_3v(t)T_t(t)u_x-\frac{w_t}{w}u-\frac{v_t}{v}xu_x. \end{aligned} \tag{3.70}$$

The solutions of Eqs. (3.69) and (3.70) are again related by formula (3.63).

But also from the KdV itself we can directly obtain more general integrable equations. Let us give a more elaborate example where more complicated actions of exponentials of scalings have to be computed. Starting again with Eq. (3.41)

$$\Gamma(u)=u_{xxx}+6uu_x-u_t$$

and defining

$$\Gamma_1=\exp(\Lambda_{\ln(a(t))u+\ln(b(t))xu_x+(\psi_t(t))^{-1}u_t})\Gamma$$

we obtain the integrable equation

$$\Gamma_1(u,t)=0.$$

Abbreviating again $T(t):=\tau(t,1)=\psi_{\mathrm{inv}}(\psi(t)-1)$ and carrying out the details of the computation we obtain (with the help of the Lemma 3.3d)

$$u_t=T_t(t)\nu(t)^3u_{xxx}+6\omega(t)\nu(t)T_t(t)uu_x-\psi_t(t)\ln\left(\frac{a(t)}{a(T(t))}\right)u-\psi_t(t)\ln\left(\frac{b(t)}{b(T(t))}\right)xu_x, \tag{3.71}$$

where

$$\omega(t)=\exp\left(\int_t^{T(t)}\psi_s(s)\ln(a(s))ds\right),\quad \nu(t)=\exp\left(\int_t^{T(t)}\psi_s(s)\ln(b(s))ds\right).$$

Using Theorem 2.1 we find as before that for solutions $u(x,t)$ of the KdV, the function

$$u_{\mathrm{new}}(x,t)=\omega(t)u(x\nu(t),T(t)) \tag{3.72}$$

must be a solution of Eq. (3.71). It should be remarked, that Eq. (3.71) is the most general equation which can be obtained out of the KdV by application of exponentials of scalings. This follows from the fact that the exponentials of the kind

$$\exp(\Lambda_{\ln(a(t))u+\ln(b(t))xu_x+(\psi_t(t))^{-1}u_t})$$

are a subgroup of the exponentials of vector fields.

G. Cylindrical equations

In the KdV (for the variable $\tilde{u}$) we substitute

$$\tilde{u}=u-\frac{x}{6t}.$$

Then u evolves with

$$u_t=u_{xxx}+6u_xu-\frac{1}{t}(u+xu_x). \tag{3.73}$$

Now, taking the same transformations as in Eqs. (3.42), (3.43), and (3.44) we obtain the integrable equation

$$u_t=v(t)^3T_t(t)u_{xxx}+6v(t)w(t)T_t(t)uu_x-\left(\frac{1}{T(t)}+\frac{w_t}{w}\right)u-\left(\frac{1}{T(t)}+\frac{v_t}{v}\right)xu_x. \tag{3.74}$$

Now, by choice of

$$\psi(t)=\frac{\ln(t)}{\ln(c)} \tag{3.75}$$

we pick out a special case. We get

$$T(t)=\frac{t}{c}\quad\text{and}\quad T_t=\frac{1}{c}. \tag{3.76}$$

Putting

$$v(t)=t^{-1/3} \quad \text{and} \quad w(t)=t^{-2/3} \tag{3.77}$$

we obtain

$$u_t=\frac{1}{ct}(u_{xxx}+6u_xu)+\left(\frac{2}{3t}-\frac{c}{t}\right)u+\left(\frac{1}{3t}-\frac{c}{t}\right)xu_x. \tag{3.78}$$

Now, performing for the parameter c the limit $c\to 1$ we find the equation

$$u_t=\frac{1}{t}\left(u_{xxx}+6u_xu-\frac{1}{3}u-\frac{2}{3}xu_x\right). \tag{3.79}$$

Following all the steps, and transforming the solutions accordingly, we find that whenever $u(x,t)$ is a solution of the KdV then

$$u_{\text{new}}(x,t):=t^{2/3}u(xt^{1/3},t)+\frac{x}{6} \tag{3.80}$$

is a solution of Eq. (3.79). At this point we transform dependent and independent variables according to

$$u=6^{-2/3}U,\quad x=6^{1/3}X,\quad t=\sigma^{-1/2} \tag{3.81}$$

to find

$$U_\sigma=-\frac{1}{12\sigma}(U_{XXX}+6U_XU-2U-4XU_X). \tag{3.82}$$

This is a well-known equation from the literature (Ref. 24, p. 268). From here the transformation

$$v(\xi,\sigma)=(12\sigma)^{-2/3}U(X,\sigma),\quad \xi=(12\sigma)^{1/3}X \tag{3.83}$$

yields a link to the cylindrical KdV

$$v_\sigma+v_{\xi\xi\xi}+v_\xi v+\frac{v}{2\sigma}=0. \tag{3.84}$$

Thus we have found a direct link from KdV to the cylindrical KdV. So, if $u(x,t)$ is a solution of the KdV then

$$v(\xi,t)=\frac{1}{4^{1/3}\sigma}u\left(\frac{\xi}{2^{1/3}\sqrt{\sigma}},\frac{1}{\sqrt{\sigma}}\right)+\frac{\xi}{12\sigma} \tag{3.85}$$

must be a solution of the cylindrical KdV.

In a similar fashion all the other KdV modifications with variable coefficients, for example, those in Ref. 23, can be found. Also the group structure of the cylindrical KdV (see Ref. 6) and similar equations can be derived by the methods in this article. In particular, the link given in Eq. (3.85) is compatible with group theoretical reductions since it was obtained by transfer via a generalized scaling group.

H. Link from KdV to KP

As for the KdV we may, by the same methods, obtain a link between the Kadomtsev–Petviashvili equation

$$(U_t+6UU_x+U_{xxx})_x=-3\alpha^2U_{yy} \tag{3.86}$$

and the Johnson equation[25]

$$\left(V_\sigma+6VV_\sigma+V_{\xi\xi\xi}+\frac{V}{2\sigma}\right)_\xi=-3\alpha^2\frac{V_{\eta\eta}}{\sigma^2}, \tag{3.87}$$

which was investigated for its applications in water with variable depth (see Ref. 26 and, or Ref. 27). However, this link is well-known from the literature, so we may skip it here. In an article of Lipovskii, Matveev, and Smirnov[28] we find that whenever $V(\xi,\eta,\sigma)$ is a solution of Johnson's equation then

$$U(x,y,t)=V\left(x+\frac{y^2}{12\alpha^2t},\frac{y}{t},t\right) \tag{3.88}$$

must be a solution of the KP. Now, obviously, any solution of the cylindrical KdV of the form (3.85) is a solution of Johnson's equation (not depending on η). Hence we have found a direct link from the KdV to the KP. Interestingly, this link, contrary to the trivial link where the y dependence is neglected, yields solutions of the KP which genuinely depend on the second spatial variable. To make this link precise, we conclude that whenever $u(x,t)$ is a solution of the KdV then

$$U(x,y,t)=\frac{1}{4^{1/3}t}u\left(\frac{x}{2^{1/3}\sqrt{t}}+\frac{y^2}{12(2)^{1/3}\alpha^2t^{3/2}},\frac{1}{\sqrt{t}}\right)+\frac{x}{12t}+\frac{y^2}{(12\alpha t)^2}$$

must be a solution of the KP. This fact is easily verified by direct computation. Since the KP is invariant with respect to translation of time we have found the following general class of solutions:

$$U(x,y,t)=\frac{1}{4^{1/3}(t+c)}u\left(\frac{x}{2^{1/3}\sqrt{(t+c)}}+\frac{y^2}{12(2)^{1/3}\alpha^2(t+c)^{3/2}},\frac{1}{\sqrt{(t+c)}}\right)$$
$$+\frac{x}{12(t+c)}+\frac{y^2}{(12\alpha(t+c))^2}. \tag{3.89}$$

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