Soviet Geology and Geophysics
Vol. 32, No. 7, pp. 120–126, 1991

Geologiya i Geofizika
UDC 550.834

TRAVELTIMES IN COMMON MIDPOINT SECTIONS OF ANISOTROPIC MEDIA

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The traveltimes of common midpoint gathers are closely related to those arising from a (fictitious) point source at the normal incidence point. The second derivatives of the traveltime functions agree up to a trivial factor of 1/2. This result, well-known for isotropic media, is generalized to the anisotropic case. In the proof the coordinate invariance the ray theory is used in an essential way.

INTRODUCTION

Traveltimes of primary reflections in common midpoint sections contain important information about subsurface structure. These traveltimes are closely related to, although in general different from, those of a wave which is set off in a fictitious point source located in the reflection point of the zero-offset ray. The second derivatives of the traveltime functions agree – up to an obvious factor of 1/2. This theorem is useful in the interpretation of seismic measurements. One consequence is that the curvature of a reflector does not affect common midpoint traveltimes when offsets are small.

The theorem is known for isotropic media [1–3]. In this paper we generalize it to the anisotropic case. The crucial observation is that a theory of rays and traveltimes is not restricted to isotropic media if it is coordinate invariant. Therefore, we shall review aspects of ray theory emphasizing the coordinate invariance of the eiconal equations, the ray equations, and Snell's law. To better see this important property, we also restate and rederive in an appendix some basic formulas of Hamilton-Jacobi theory.

NOTATION

Coordinate vectors of spatial points are generally denoted by small Latin letters x, y, z. For the corresponding dual vectors (e.g., momenta) we use small Greek letters ξ, η, ζ. So, for example, (x, ξ) denotes a point in six-dimensional phase space. (In more standard physics notation the letters k or p are used instead of ξ.) We use the abbreviation

$$\langle \xi, v \rangle = \sum_{j=1}^{3} \xi_j v_j . \qquad (1)$$

Here and in the following, by subscripting a vector with an index j, we denote its j-th component. We point out that eq. (1) is *not* a scalar product. In fact, the vectors ξ (momentum) and v (velocity) do not even belong to the same vector space. As a consequence of this no notion of a length of a vector is associated with this notation. A function is said to be smooth, if it posesses continuous derivatives of arbitrary order. $\frac{\partial f}{\partial x}$ is the gradient of a function $f(x)$, $df = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} d x_j$ is its total differential. We use the notation of (1) also with differentials, $\langle \xi, dx \rangle = \sum_{j=1}^{3} \xi_j dx_j$. For a curve $x(t)$, the derivative with respect to the parameter t is written as $\dot{x} = \frac{dx}{dt}$. The transpose of a matrix A is written A^T, its inverse transpose $A^{-T} = (A^{-1})^T$. An interface is locally given by an equation $S(x) = 0$ with $\frac{\partial S}{\partial x} \neq 0$. For ease of notation the interface is also called S.

RAYS AND TRAVELTIME FUNCTION

The eiconal equation is a starting point for the introduction of rays associated with wave equations. For theories of waves and rays see, e.g., [4–6].

In a source-free region, the temporal evolution of an elastic wave governed by the equations of elastodynamics

$$\rho\frac{\partial^2 u_j}{\partial t^2}-\sum_i\frac{\partial\sigma_{ij}}{\partial x_i}=0 \quad (j=1,2,3), \tag{2}$$

$u=(u_1,u_2,u_3)$ is the displacement field, $\sigma_{ij}=\sum_{k,l}(c_{ijkl}\frac{\partial u_k}{\partial x_l})$ is the stress tensor. The density, ρ, and the elasticities, c_{ijkl}, are smooth functions of spatial position, $x=(x_1,x_2,x_3)$.

Wavefronts, $t=T(x)$, of high-frequency waves u satisfy an eiconal equation

$$h\left(x,\frac{\partial T}{\partial x}(x)\right)=1. \tag{3}$$

Here and in the following h is a positive smooth function such that $(h(x,\xi))^2$ is an eigenvalue for the 3 by 3 matrix $\Gamma(x,\xi)$, defined by

$$\Gamma_{jk}(x,\xi)=\rho(x)^{-1}\sum_{i,l}c_{ijkl}(x)\xi_i\xi_l\,.$$

$x=(x_1,x_2,x_3)$ is the coordinate vector of a point, and $\xi=(\xi_1,\xi_2,\xi_3)$ is a covector at that point. $\Gamma(x,\xi)$ is positive definite symmetric, if $\xi\neq 0$. Observe that h is always homogeneous of degree 1 as a function of ξ, i.e., $h(x,c\xi)=|c|h(x,\xi)$ holds for all (x,ξ), $\xi\neq 0$, and all real numbers $c\neq 0$.

Remark. In the isotropic case there are two distinct eigenvalue functions, $h_P(x,\xi)=c_P(x)|\xi|$ and $h_S(x,\xi)=c_S(x)|\xi|$, corresponding to P- and S-waves, respectively. Here $|\xi|=(\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}$ is the Euclidean norm of ξ. The wave speeds are, in terms of the Lame parameters λ and μ, $c_P(x)=((\lambda+2\mu)/\rho)^{1/2}$ and $c_S(x)=(\mu/\rho)^{1/2}$.

Rays are curves $(x(t),\xi(t))$ in phase space which satisfy Hamilton's canonical equations:

$$\dot{x}=\frac{\partial h}{\partial \xi}\ \text{ and } \dot{\xi}=-\frac{\partial h}{\partial x}, \tag{4}$$

h is constant along a ray. This is seen when differentiating $h(x(t),\xi(t))$ with respect to t.

Solutions to the eiconal equations eq. (3) and solutions to the ray equations eq. (4) are closely related. If

$$\xi(t)=\frac{\partial T}{\partial x}(x(t)) \tag{5}$$

holds for one curve parameter t, then it holds for all t. A proof of this basic fact in Hamilton-Jacobi theory is given in the Appendix, part A. Using eq. (5), eq. (4) and eq. (3) we compute

$$\frac{d}{dt}T(x)=\left\langle\frac{\partial T}{\partial x},\dot{x}\right\rangle=\left\langle\xi,\frac{\partial h}{\partial\xi}\right\rangle=h=1\,.$$

The third equality follows from the Euler's theorem on homogeneous functions applied to h. It follows that $t-T(x(t))$ is constant. Hence rays in $h=1$ are parametrized by the traveltime of wavefronts, justifying the notation t.

To see the coordinate invariance of the formulas consider a transformation which maps x, the old coordinates, smoothly to x', the new coordinates. Given define T' and h' by $T'(x')=T(x)$ and $h'(x',\xi')=h(x,\xi)$. Here

$$\xi'=\left(\frac{\partial x'}{\partial x}(x)\right)^{-T}\xi\,. \tag{6}$$

With these definitions and a straightforward differentiation we obtain

$$h'\left(x', \frac{\partial T'}{\partial x'}(x')\right) = h\left(x, \frac{\partial T}{\partial x}(x)\right).$$

This equation states the invariance of the eiconal equation under coordinate changes. The transformation law is consistent with that for waves.

The transformation which maps (x, ξ) to (x', ξ') with ξ' defined in eq. (6) is a symplectic (or canonical) transformation. We have

$$\langle \xi', dx' \rangle = \langle \xi, dx \rangle, \tag{7}$$

this follows when expressing the differentials dx' in terms of dx. It is well known from classical mechanics that the structure of eq. (4) is left invariant under general canonical transformations.

At an interface, S, across which the material properties change discontinuously, eq. (2) does not hold. However, the elastic waves on either side of S are related by the continuity of the displacement field and the normal traction. A high-frequency wave has, for either side of S, wavefronts $t = T_-(x)$ and $t = T_+(x)$ with corresponding eigenvalue functions h_- and h_+, respectively. In addition to the eiconal equation the following holds:

$$T_+(x) = T_-(x), \quad \text{if } S(x) = 0. \tag{8}$$

These equations are obtained by phase matching. Here we assume S to be given by an equation $S(x) = 0$ satisfying $v = \frac{\partial S}{\partial x} \neq 0$ on S. Taylor's formula and eq. (8) imply that

$$\frac{\partial T_-}{\partial x} - \frac{\partial T_+}{\partial x} = fv, \tag{9}$$

holds on S with some scalar function f. This condition is equivalent with the following:

$$\left\langle \frac{\partial T_-}{\partial x}, v \right\rangle = \left\langle \frac{\partial T_+}{\partial x}, v \right\rangle, \tag{10}$$

holds for all vectors v which are tangent to S at x. This is the general law of reflection and refraction, with or without mode conversion, for wavefronts. It applies to isotropic and to anisotropic media.

If eq. (10) and eq. (5) are to be consistent, the following must hold for a ray $(x(t), \xi(t))$ at the point of reflection or transmission $x = x(t+0) = x(t-0)$:

$$\langle \xi(t+0), v \rangle = \langle \xi(t-0), v \rangle \tag{11}$$

for every vector v which is tangent to S at x, and, in addition,

$$h_+(x, \xi(t+0)) = h_-(x, \xi(t-0)).$$

This is the general form of the Snell's law, valid also for anisotropic media.

The laws of reflection and refraction also retain their form when coordinates are changed. Set $S'(x') = S(x)$. Then S', the interface S in new coordinates x', is given by the equation $S'(x') = 0$. For a vector v set

$$v' = \left(\frac{\partial x'}{\partial x}(x)\right) v, \tag{12}$$

v' is tangent to S' at x' precisely when v is tangent to S at x. Furthermore

$$\left\langle \frac{\partial T'_\pm}{\partial x'}(x'), v' \right\rangle = \left\langle \frac{\partial T_\pm}{\partial x}(x), v \right\rangle.$$

Hence the law of reflection and refraction of eq. (10) retains its structure in the new coordinates:

$$\left\langle \frac{\partial T'_+}{\partial x'}(x'), v' \right\rangle = \left\langle \frac{\partial T'_-}{\partial x'}(x'), v' \right\rangle,$$

holds for all vectors v' which are tangent to S' at x'. The transformation law for rays is consistent with this. Note the different transformation behavior of covectors and vectors in eq. (6) and eq. (12), respectively.

Example. Assume that S is given by an equation $x_3 = f(x_1, x_2)$ with a smooth function f. For $x = (x_1, x_2, x_3)$ define $x' = (x'_1, x'_2, x'_3)$ with $x'_1 = x_1$, $x'_2 = x_2$, and $x'_3 = x_3 - f(x_1, x_2)$. The implicit function theorem shows that, at least locally near a point, the map which sends x to x' is a coordinate transformation. It has the important property that it flattens S. More precisely, S', the interface S in new coordinates, is given by the equation $x'_3 = 0$. The statements about the coordinate invariance of the eiconal equations and of Snell's law therefore imply that no generality is lost when assuming *plane* interfaces.

We now discuss traveltime functions and their relation to rays. Consider two points in phase space, (x, ξ) and (y, η), such that $h(x, \xi) = h(y, \eta) = 1$ and such that there exists a ray which passes through (x, ξ) and (y, η) at times t and 0, respectively. Let C_1 denote the set of all t, (x, ξ), and (y, η) related in this way. C_1 is a manifold of dimension six. It is parametrized by t and (y, η), with $h(y, \eta) = 1$. Hamilton's law holds:

$$dt = \langle \xi, dx \rangle - \langle \eta, dy \rangle \text{ on } C_1 . \tag{13}$$

A proof of this fact is given in the Appendix, part *B*. In view of eq. (7) the form of eq. (13) remains unchanged when the coordinates x and y are changed to new coordinates x' and y'.

If it happens to be true that C_1 can be parametrized by x and y, then there exists a function T with $t = T(x, y)$, and, because of eq. (13), there holds on C_1:

$$\xi = \frac{\partial T}{\partial x}(x, y), \quad \eta = -\frac{\partial T}{\partial y}(x, y) . \tag{14}$$

Such a function T is called a traveltime function. In general, because of caustics a traveltime function $T(x, y)$ is not everywhere defined as a smooth single-valued function. However, T may still exist locally near some pair of points (x, y) even when the connecting ray has gone through caustics.

THE KREY-CHERNYAK-GRITSENKO THEOREM

We consider rays which travel through layers of inhomogeneous, isotropic or anisotropic elastic media separated by first order discontinuities across interfaces. Let us fix a normal ray, i.e., a ray $(x(t), \xi(t))$ which starts at a common datum point $y^0 = x(0)$, travels down to a reflector, R, is reflected without mode conversion at the normal incidence point $x^0 - x(t_r)$, travels back along the same path in reverse direction, and ends where it started, in $y^0 = x(2t_r)$. Thus, for $0 < t \le t_r$, $x(t_r + t) = x(t_r - t)$ and $\xi(t_r + t) = -\xi(t_r - t)$ hold. (Here we used the equation $h(x, \xi) = h(x, -\xi)$.) Because of eq. (11) $\xi(t_r + 0) = -\xi(t_r - 0)$ is normal to R at x^0.

Assume that, for y near the common midpoint y^0 and for x near the point of normal incidence x^0, the traveltime function $T(x, y)$ corresponding to the rays close to the normal ray is defined. Furthermore, assume that, for y^- and y^+ close y^0, there exists, close to the normal ray, a unique ray which starts in y^-, is reflected at R in a point $x = x(y^+, y^-)$ close to x^0, and ends in y^+. These assumptions are violated only in very exceptional situations. Now we apply eq. (14) to the ray segments from y^- to x and from x to y^+. Snell's law eq. (11) then becomes:

$$\left\langle \frac{\partial T}{\partial x}(x, y^-), v \right\rangle = \left\langle -\frac{\partial T}{\partial x}(x, y^+), v \right\rangle \tag{15}$$

for every vector v which is tangent to R at x.

In a common midpoint experiment, for small three-dimensional offsets d, with sources and receivers placed at $y^-(d) = y^0 - \frac{1}{2}d$ and $y^+(d) = y^0 + \frac{1}{2}d$, respectively, the traveltime is given by

$$T_M(d) = T(x(d), y^+(d)) + T(x(d), y^-(d)). \tag{16}$$

Here $x(d) = x(y^+(d), y^-(d))$ is the reflection point. The common midpoint traveltime function T_M is known from measured data. In a second, hypothetical, experiment a wave originates from a point source in x^0. Here the traveltime is

$$T_N(y) = T(x^0, y)\,. \tag{17}$$

This experiment is not done in reality. So T_N is not a measured quantity.

Krey [1], and Chernyak and Gritsenko [3] proved the following relation for isotropic media:

$$\frac{1}{2}\frac{\partial^2 T_N}{\partial y^2}(y^0) = \frac{\partial^2 T_M}{\partial d^2}(0). \tag{18}$$

Thus, to second order, the wavefront of the hypothetical wave can be obtained from measured common mid-point traveltimes. Using the coordinate invariant theory of rays and traveltimes presented above we proceed to show that the proof of Chernyak and Gritsenko for eq. (18) immediately generalizes to the anisotropic case.

First differentiate eq. (16) with respect to d_j, the j-th coordinate of d:

$$\frac{\partial T_M}{\partial d_j}(d) = \frac{1}{2}\frac{\partial T}{\partial y_j}(x(d), y^+(d)) - \frac{1}{2}\frac{\partial T}{\partial y_j}(x(d), y^-(d)) +$$
$$+ \left\langle \frac{\partial T}{\partial x}(x(d), y^+(d)) + \frac{\partial T}{\partial x}(x(d), y^-(d))\,,\ \frac{\partial x}{\partial d_j}(d) \right\rangle .$$

Since $x(d)$ ranges in R the vectors $\frac{\partial x}{\partial d_j}(d)$ are tangent to R at $x(d)$. Hence, according to equation (15), the angular bracket term vanishes for all d. Thus

$$\frac{\partial T_M}{\partial d}(d) = \frac{1}{2}\frac{\partial T}{\partial y}(x(d), y^+(d)) - \frac{1}{2}\frac{\partial T}{\partial y}(x(d), y^-(d))$$

for all small offsets d. Now differentiate this equation with respect to d. Finally, set $d = 0$ to arrive at eq. (18).

Equation (18) is also true when the source and receiver locations are restricted to a surface S, the surface of the earth. In fact, because of coordinate invariance there is no restriction to assume that S is given by the equation $y_3 = 0$. Renaming, for simplicity, (y_1, y_2) and (d_1, d_2) as y and d, respectively, eq. (18) becomes the statement of the theorem for this case, too.

CONCLUSION

When studying rays, wavefronts, and traveltimes for isotropic media it is natural to use concepts from (conformal) Euclidean geometry, e.g., orthogonality, length, angle, arc length, curvature. These concepts are no longer useful when dealing with anisotropy. In fact, they should rather be avoided as they may unnecessarily restrict the generality of results.

The methods of symplectic geometry lead to a much more flexible ray theory than those of Euclidean or Riemannian geometry. This is because the formulas of such a ray theory are invariant under general choices of coordinate systems. In particular, coordinate changes need not be orthogonal. An advantage of the coordinate invariance is that the ray theory permits a uniform treatment of isotropy and of anisotropy. The generalization of the Krey-Chernyak-Gritsenko theorem from the isotropic to the anisotropic case illustrates this point.

Another benefit from coordinate invariance is that for many aspects of ray theory no generality is lost by assuming that interfaces are locally plane.

The author wishes to thank P. Hubral and B. Ursin for useful discussions about seismic ray theory.

APPENDIX

For convenience and completeness we reprove some formulas of Hamilton-Jacobi theory and symplectic geometry. See Caratheodory [7], and Guillemin and Sternberg [8] for these and other results.

A. Let T be a solution of eq. (3). Let $(x(t), \xi(t))$ be a ray. Define η by

$$\eta(t) = \frac{\partial T}{\partial x}(x(t))\,.$$

Assume that $\eta(t) = \xi(t)$ holds for at least one t. We claim that $\eta(t) = \xi(t)$ holds for all t. Using eq. (4) we compute

$$\dot{\eta} = \frac{\partial^2 T}{\partial x^2}\dot{x} = \frac{\partial^2 T}{\partial x^2}\frac{\partial h}{\partial \xi}(x,\eta) + \frac{\partial^2 T}{\partial x^2}\left(\frac{\partial h}{\partial \xi}(x,\xi) - \frac{\partial h}{\partial \xi}(x,\eta)\right) = -\frac{\partial h}{\partial x}(x,\eta) +$$

$$+\frac{\partial^2 T}{\partial x^2}\left(\frac{\partial h}{\partial \xi}(x,\xi) - \frac{\partial h}{\partial \xi}(x,\eta)\right).$$

The last equation follows upon differentiation of eq. (3) with respect to x. The resulting equation is an ordinary differential equation for η. $\eta = \xi$ is its only solution.

B. To prove eq. (13) it is convenient to remove the restriction $\tilde{h}(y,\eta) = 1$. Therefore, consider the set C consisting of all (t,τ), (x,ξ), and (y,η) such that $\tau = h(y,\eta) > 0$ and there exists a ray which passes through (y,η) and (x,ξ) at the parameter values 0 and t, respectively. The curve parameter t is now traveltime scaled by $1/\tau$. Equation (13) is the special case of

$$\tau dt = \langle \xi, dx \rangle - \langle \eta, dy \rangle \text{ on } C, \tag{19}$$

where $\tau = 1$. Here, for consistency of notation, we denote the frequency variable which corresponds to the time variable with τ. (The usual notation for frequency is ω.)

First assume there are no interfaces. Parametrize C by t, y and η using a family of rays:

$$x = x(t;y,\eta)\,,\; \xi = \xi(t;y,\eta)\,,\; \tau = h(y,\eta)\,.$$

Eq. (4) is assumed to hold when y and η are held fixed. At $t = 0$, the equations $x = y$ and $\xi = \eta$ hold. Using this fact the parametrization eq. (19) is seen to be equivalent with the following set of equations

$$\tau = \langle \xi, \dot{x} \rangle, \tag{20}$$

$$\eta = \left(\frac{\partial x}{\partial y}\right)^T \xi, \tag{21}$$

$$0 = \left(\frac{\partial x}{\partial \eta}\right)^T \xi. \tag{22}$$

Eq. (20) follows from

$$\tau = h = \langle \xi, \frac{\partial h}{\partial \xi} \rangle = \langle \xi, \dot{x} \rangle \text{ on } C.$$

Here we used Euler's theorem on homogeneous functions and eq. (4).

We prove eq. (21) and eq. (33). First observe that, at $t = 0$, $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial \eta}$ are equal to the unit and the zero matrix, respectively. Therefore it suffices to prove

$$\frac{d}{dt}\left(\frac{\partial x}{\partial y}\right)^T \xi = 0 \text{ and } \frac{d}{dt}\left(\frac{\partial x}{\partial \eta}\right)^T \xi = 0\,.$$

These equations follow from the following computation where s denotes any arbitrary component of y or η

$$\frac{d}{dt}\left\langle \frac{\partial x}{\partial s}, \xi \right\rangle = \left\langle \frac{\partial \dot{x}}{\partial s}, \xi \right\rangle + \left\langle \frac{\partial x}{\partial s}, \dot{\xi} \right\rangle = \frac{\partial}{\partial s}\left\langle \dot{x}, \xi \right\rangle - \left\langle \dot{x}, \frac{\partial \xi}{\partial s} \right\rangle + \left\langle \dot{\xi}, \frac{\partial x}{\partial s} \right\rangle =$$

$$= \frac{\partial}{\partial s}h - \left\langle \frac{\partial h}{\partial \xi}, \frac{\partial \xi}{\partial s} \right\rangle - \left\langle \frac{\partial h}{\partial x}, \frac{\partial x}{\partial s} \right\rangle = 0\,.$$

Now assume there is one interface at which the rays are reflected or refracted. Let C^- and C^+ be the ray relations before and after reflection, respectively,

$$\tau dt^- = \langle \zeta^-, dz^- \rangle - \langle \eta, dy \rangle \text{ on } C^- \tag{23}$$

and

$$\tau dt^+ = \langle \xi, dx \rangle - \langle \zeta^+, dz^+ \rangle \text{ on } C^+. \tag{24}$$

Recall that this means that $h_-(y,\eta) = h_+(z^+,\zeta^+) = \tau$, and that there exist rays connecting (y,η) with (z^-,ζ^-) and (z^+,ζ^+) with (x,ξ). The traveltimes are τt^- and τt^+, respectively. Without loss of generality we may assume that the hypersurface is given, locally near the reflection point of a (central) ray, by the equation $z_1 = 0$. Then $z^{\pm}$ and $\zeta^{\pm}$ are related at the reflection point in the following way:

$$z^+ = z^-,\; z_1^{\pm} = 0,\; \zeta_2^+ = \zeta_2^-,\; \zeta_3^+ = \zeta_3^-.$$

It follows from this that $\langle \zeta^-, dz^- \rangle - \langle \zeta^+, dz^+ \rangle$ vanishes. Thus, adding eq. (23) and eq. (24) we obtain

$$\tau(dt^{-} + dt^{+}) = \langle \xi, dx \rangle - \langle \eta, dy \rangle.$$

We have now proven that eq. (19) holds if there is at most one interface at which reflection or refraction can occur. Clearly the reasoning generalizes to a finite number of interfaces.

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24 October 1990

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