A COMPARISON BETWEEN TWO VARIATIONS OF A PEBBLE GAME ON GRAPHS

by

Friedhelm Meyer auf der Heide

Faculty of mathematics, University of Bielefeld, W-Germany

Abstract:

We study the relation between the number of pebbles used in the black and the black-white pebble game and show that the additional use of white pebbles cannot save more than a square-root and give an example in which it does save a factor $\frac{1}{2}$.

(1) Introduction:

This paper deals with two kinds of pebble games:

- the "black pebble game" which was used to exhibit a space efficient simulation for time-bounded Touring machines. ($DTIME(t(n)) \subseteq DTAPE(t(n)/\log t(n))$) [5] .

- the "black-white pebble game" which was used to show that there is a language $SP \in P$ which uses at least $\Omega(n^{1/4})$ space in a special model of machines (Sound-Path-Machines) [3] .

(2) Description of the games:

Let G always be a directed acyclic graph with vertex set V and edge set E , $\Gamma_G^{-1}(x)$ the set of all direct predecessors of the vertex x and $\Gamma_G^{\overset{*}{-}1}(x)$ the set of all predecessors of x , i.e. the set of all vertices, from which there is a directed path to x in G . If it is clear which graph is concerned, we only write $\Gamma^{-1}(x)$, $\Gamma^{\overset{*}{-}1}(x)$. For $x \in V$ let V_x be the set $\Gamma_G^{\overset{*}{-}1}(x) \cup \{x\}$ and G_x the induced subgraph of G with vertex set V_x .

(2.1) The black-white pebble game is played on a DAG G by placing black or white pebbles on some vertices of G according to certain rules which are implicit in the following description:

A configuration of G is a pair (B,W) with $B,W \subset V$ and $B \cap W = \emptyset$. B is the set of vertices, on which black pebble are lying, W that one for white pebbles.

We say "(B,W) directly derives (B', W') using k pebbles" and write

"$(B,W) \Rightarrow_k (B', W')$" iff $B \cup W \subset B' \cup W'$ or $B' \cup W' \subset B \cup W$ and

either (i) $W=W'$, $\#(B \setminus B')=1$

or (ii) $W=W'$, $B' \setminus B=\{x\}$ and $\Gamma^{-1}(x) \subset W \cup B$ for some $x \in V$

or (iii) $B=B'$, $\#(W' \setminus W)=1$

or (iv) $B=B'$, $W \setminus W'=\{x\}$ and $\Gamma^{-1}(x) \subset W \cup B$ for some $x \in V$

or (v) $B=B'$, $W=W'$

and $\# W + \# B \leq k$, $\# W' + \# B' \leq k$

where for some set A, $\# A$ is the number of element of A. A sequence $[(B_i, W_i), i = 1...n]$ is called a b/w-k-strategy from (B,W) to (B',W') in G, iff (B_i, W_i) are configurations of G and $(B_i, W_i) \Rightarrow_k (B_{i+1}, W_{i+1})$ for all i and $(B_1, W_1) = (B,W)$, $(B_n, W_n) = (B',W')$.

(2.2) The black pebble game is a special kind of the black-white pebble game: A b/w-k-strategy $[(B_i,W_i), i = 1...n]$ in G is called a b/w-k-strategy from D to D' in G iff $W_i = \emptyset$ for all i and $B_1 = D$, $B_n = D'$. In this case we write $[D_i, i = 1...n]$ for the strategy.

(2.3) The object of both games is to find a strategy which begins with the empty configuration and ends with one black pebble on a distinguished vertex and no white pebbles using a number of pebbles as small as possible.

Notations: $Opt(G,r)$ = number of pebbles used in an optimal b/w-strategy (i.e. in a strategy which uses a minimal number of pebbles) from $(\emptyset,\emptyset)$ to $(\{r\}, \emptyset)$. We define $Opt_b(G,r)$ for b-strategies analogously.

Remark: In [3] and [4], the object of the game is defined in an other way, but optimal strategies in the sense of [3] and [4] and in the sense of (2.3) differ by at most one pebble.

(2.4) Intuitively, we can think of the black-white pebble game as a model of a proof:
The sources (vertices without predecessors) are the axioms, known theorems

etc., a distinguished vertex r is the theorem which shall be proved and the other vertices are lemmas. Each lemma and the theorem can be deduced from its predecessors. Placing a black pebble on a vertex corresponds to proving the lemma (theorem) by its predecessors, placing a white pebble on a vertex corresponds to assuming this lemma (theorem) to be true, intending later to justify it by its predecessors. The maximal number of pebbles used in some configuration corresponds to the maximal number of lemmas, one must have "in mind" at one time.

(2.5) The black pebble game can be looked upon as a model for an evaluation of a (for example boolean) expression by a register machine:
A vertex is an operator, its predecessors are its operands, the sources are the variables and the pebbles the registers. Placing a pebble on a vertex x corresponds to computing the value of the subexpression in the register (Notice that all predecessors of x are pebbled, i.e. all operands are available). Removing a pebble from a vertex corresponds to freeing the register. Thus, the number of pebbles used in the game corresponds to the number of registers used in the computation.

(3) Some known results about pebble games

(3.1) For both games, it is known that if G is a DAG with indegree 2 (i.e. $\#(\Gamma^{-1}(x)) \leq 2$ for all $x \in V$) and n vertices, then an optimal strategy from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ for some $r \in V$ uses less or equal to $\Omega(n/\log n)$ pebbles and there exists a family of graphs which needs this number of pebbles [1], [2], [6].

(3.2) If S_m is a pyramid with m levels and sink r (S_5 is shown in figure 1) then $\mathrm{Opt}_{b/w}(S_m,r) \geq \sqrt{\frac{m}{2}} - 1$ [3] .

(3.3) $\mathrm{Opt}_b(S_m,r) = m + 1$ [4] .

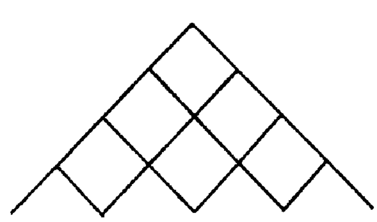

Figure 1: The pyramid S_5

(4) We state without proof 3 simple, technical lemmas:

(4.1) Lemma 1: Let $[(B_i, W_i), i = 1...n]$ be a b/w-k-strategy in G, then $[(W_{n-i+1}, B_{n-i+1}), i = 1...n]$ is a b/w-k-strategy in G. Let us call this strategy the contra-strategy of $[(B_i, W_i), i = 1...n]$.

(4.2) Lemma 2: Let $[(B_i, W_i), i = 1...n]$ be a b/w-k-strategy in G, H an induced subgraph of G with vertex set $V(H)$, then $[(B_i \cap V(H), W_i \cap V(H)), i = 1...n]$ is a b/w-k-strategy in H. If $H = G_x$ for some x, then $[B_i \cap V(H), W_i \cap V(H)), i = 1...n]$ is also a b/w-k-strategy in G.

(4.3) Lemma 3: If G is a DAG, (B,W) a configuration in G, $\overline{G}$ the induced subgraph of G with vertex set $V \setminus (B \cup W)$ and $[(B_i, W_i), i = 1...n]$ is a b/w-k-strategy in $\overline{G}$, then $[(B_i \cup B, W_i \cup W), i = 1...n]$ is a b/w-$(k + \#(B \cup W))$-strategy in G.

In the first theorem we shall see that for the family of pyramids, the b/w-pebble game does save at least a factor $\frac{1}{2}$.

(5) Theorem 1: $Opt(S_m, r) \leq \lceil \frac{m}{2} \rceil + 2$ for $m \geq 2$.

Proof by induction on m.

Obviously, $Opt(S_1, r) = 1$, $Opt(S_2, r) = 3$, $Opt(S_3, r) = 4$.

Let $m \geq 4$ and $[C_{n-2}]$ be the $(\lceil \frac{m-2}{2} \rceil + 2)$ - strategy for S_{m-2}, given by induction hypothesis and $[\overline{C_{n-2}}]$ the contra-strategy of $[C_{n-2}]$.

Then consider the following strategy: (We use the notations of figure 2).

1) place a black pebble on a by $[C_{n-2}]$,

2) place a black pebble on c by $[C_{n-2}]$,

3) place a white pebble on b,

4) go on as shown in figure 2,

5) remove the white pebble from b by $[\overline{C_{n-2}}]$.

This strategy needs $\max\{(\lceil\frac{m-2}{2}\rceil + 2) + 1, 4\}$ pebbles. As $m \geq 4$, we need $\lceil\frac{m-2}{2}\rceil + 2 + 1 = \lceil\frac{m}{2}\rceil + 2$ pebbles.

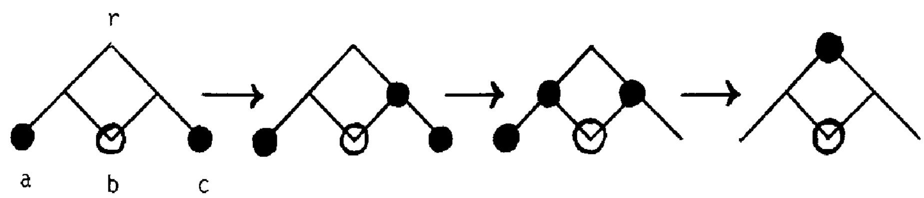

Figure 2: The top of S_m

The following theorem gives a method to change b/w-strategies into b-strategies. As a corollary we improve the lower bound in (3.2) by a factor of 2.

(6) Theorem 2: Let $G = (V,E)$ be a DAG, $r \in V$, $Opt(G,r) = k$ then $Opt_b(G,r) \leq \frac{k^2-k}{2} + 1$, i.e. $Opt(G,r) \geq \frac{1}{2} + \sqrt{2 \cdot Opt_b(G,r) - 1\frac{3}{4}}$.

(6.1) Cor: $Opt(S_m) \geq \frac{1}{2} + \sqrt{2 \cdot (m+1) - 1\frac{3}{4}}$.

The idea of the proof is to replace a move of a b/w-strategy in which a white pebble is placed on a vertex by a b-strategy which places a black pebble on it. The observation that it is not useful to place a white pebble on a vertex x, if $Opt(G_x, x) = Opt(G,r)$ yields the recursion: if $F(k) = \max\{Opt_b(G,r), G,r$ chosen such that $Opt(G,r) \leq k\}$ then $F(1) = 1$ and $F(k) \leq F(k-1) + k - 1$ which implies the theorem.

(6.2) Main lemma: Let G be a DAG, $[(B_i, W_i), i = 1...n]$ a b/w-k-strategy from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ in G, then there exists a b/w-k-strategy from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ - $[(B_i^*, W_i^*), i = 1...m]$ - with the property:

(*) For all ℓ, if $W_\ell^* \setminus W_{\ell-1}^* = \{x\}$ for some x and $S_x^{\ell-1}$ is the induced subgraph of G with vertex set $V_x \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*)$, then there is a b/w-

(k - 1)-strategy from $(\emptyset, \emptyset)$ to $(\{x\}, \emptyset)$ in $S_x^{\ell-1}$.

Proof:

We construct a new sequence $[(B_i^*, W_i^*), i = 1\ldots n]$ and show that it is a b/w-k-strategy in G from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ with (*) .

Let G , $[(B_i, W_i), i = 1\ldots n]$ be as in the hypothesis of the main lemma. Transform it into a new sequence by executing the follwing algorithm:

(6.2.1) Begin:

Let $\{\ell_1 \ldots \ell_p\}$ be the set of indices such that

(**) $W_{\ell_i} \setminus W_{\ell_{i-1}} = \{x_i\}$ for some x_i and there is a $j \geq \ell_i : \#((W_j \cup B_j) \cap V_{x_i}) = k$. Let j_i be the maximal such j and $t_i = \max\{h | x_i \in W_{\ell_i} \ldots W_h\}$.

Loop:

For i = 1 until p do
if $j_i < t_i$

Comment: One move after k pebbles are the last time in V_{x_i} , the white pebble is still on x_i ;

then $[(B_i, W_i), i = 1\ldots n] \leftarrow [(B_1 \cap \Gamma^{*1}(x_i), W_1 \cap \Gamma^{*1}(x_i)), \ldots,$
$(B_{j_i+1} \cap \Gamma^{*1}(x_i), W_{j_i+1} \cap \Gamma^{*1}(x_i)), (B_{j_i+1} \cap \Gamma^{*1}(x_i)$,
$(W_{j_i+1} \cap \Gamma^{*1}(x_i)) \cup \{x_i\}), (B_{j_i+2}, W_{j_i+2}), \ldots, (B_n, W_n)]$;

else

Comment: $t_i \leq j_i$, i.e., the white pebble on x is removed in the last move which reduces the number of pebbles in V_x from k to (k - 1) or earlier;

$[(B_i, W_i), i = 1\ldots n] \leftarrow [(B_1 \cap \Gamma^{*1}(x_i), W_1 \cap \Gamma^{*1}(x_i)), \ldots,$
$(B_{t_i} \cap \Gamma^{*1}(x_i), W_{t_i} \cap \Gamma^{*1}(x_i)), (B_{t_i+1} \cap V_{x_i}, W_{t_i+1} \cap V_{x_i}), \ldots,$
$(B_{j_i} \cap V_{x_i}, W_{j_i} \cap V_{x_i}), (B_{j_i+1}, W_{j_i+1}), \ldots, (B_n, W_n)]$;

<u>End;</u>

We conclude the main lemma from the following 3 propositions:

Let $[(B_i, W_i), i = 1 \ldots n]$ be the input sequence for some pass of the loop and ℓ, x, j, t the actual values of ℓ_i, x_i, j_i and t_i then:

(6.2.2) - the output sequence of the pass of the loop is a b/w-k-strategy from $(\emptyset, \emptyset)$ to $(r, \emptyset)$,

(6.2.3) - if a configuration (B,W) is inserted in the "then-clause" between (B_{j+1}, W_{j+1}) and (B_{j+2}, W_{j+2}) , then $\#(B \cup W) \leq k - 1$ and after the pass of the loop, $\#((B_q \cup W_q) \cap V_x) \leq k - 1$ for all $q \geq j + 2$,

(6.2.4) - if for some y and q $[(B_q \cap V_y, W_q \cap V_y), \ldots, (B_m \cap V_y, W_m \cap V_y)]$ is a b/w-(k - 1)-strategy, then it is still one after the pass.

If we have this, it follows that the output-sequence of the algorithm - $[(B_i^*, W_i^*), i = 1 \ldots m]$ - is a b/w-k-strategy in G from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ with the property:

For all ℓ , if $W_\ell^* \setminus W_{\ell-1}^* = \{x\}$ for some x , then the sequence $[(B_i^* \cap V_x, W_i^* \cap V_x), i = \ell \ldots m]$ is a b/w-(k - 1)-strategy in G_x . By Lemma 2, (4.5), it follows, that

$[((B_i^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*)), (W_i^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*)), i = \ell \ldots m]$

is a b/w-(k - 1)-strategy in $S_x^{\ell-1}$.

Notice that $(B_\ell^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*) = \emptyset$, $(W_\ell^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*) = \{x\}$, $(W_m^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*) = \emptyset$ and

$$(B_m^* \cap V_x) \setminus (B_{\ell-1}^* \cup W_{\ell-1}^*) = \begin{cases} \emptyset, & x \neq r \\ \{r\}, & x = r \quad (***) \end{cases} .$$

In the case (***) remove the black pebble from r in a new move. Now we have a b/w-(k - 1)-strategy in $S_x^{\ell-1}$ from $(\emptyset, \{x\})$ to $(\emptyset, \emptyset)$ and with the help of Lemma 1 (4.1), the main lemma follows.

It remains to prove (6.2.2), (6.2.3), (6.2.4) .

Proof of (6.2.2):

Case 1: The "then-clause" is executed.

- $[(B_1 \cap \Gamma^{*1}(x), W_1 \cap \Gamma^{*1}(x)),\dots,(B_{j+1} \cap \Gamma^{*1}(x), W_{j+1} \cap \Gamma^{*1}(x))]$ is a b/w-k-strategy because of Lemma 2 (4.2) .

- $(B_{j+1} \cap \Gamma^{*1}(x), W_{j+1} \cap \Gamma^{*1}(x)) \Rightarrow_k$ $(B_{j+1} \cap \Gamma^{*1}(x), (W_{j+1} \cap \Gamma^{*1}(x)) \cup \{x\})$, because it is always allowed to place a white pebble and because of the following:

  As $\#((B_j \cup W_j) \cap V_x) = k$, it follows that $B_j \cup W_j \subset V_x$ and that in the next move, one pebble will be removed (j maximal!) .

  Therefore, $\#(B_{j+1} \cup W_{j+1}) \leq k - 1$ and as $x \in W_{j+1}$:
  $\#((B_{j+1} \cap \Gamma^{*1}(x)) \cup (W_{j+1} \cap \Gamma^{*1}(x))) \leq k - 2$ and

(6.2.5) $\#((B_{j+1} \cap \Gamma^{*1}(x)) \cup (W_{j+1} \cap \Gamma^{*1}(x)) \cup \{x\}) \leq k - 1$.

- $(B_{j+1} \cap \Gamma^{*1}(x), (W_{j+1} \cap \Gamma^{*1}(x)) \cup \{x\}) \Rightarrow_k (B_{j+2}, W_{j+2})$, because $B_{j+1} \cap \Gamma^{*1}(x) = B_{j+1}$ and $W_{j+1} \cap \Gamma^{*1}(x)) \cup \{x\} = W_{j+1}$.
- $[(B_{j+2}, W_{j+2}),\dots,(B_m, W_m)]$ is b/w-k-strategy in G .

Case 2: The "else-clause" is executed.

- $[(B_1 \cap \Gamma^{*1}(x), W_1 \cap \Gamma^{*1}(x)),\dots,(B_t \cap \Gamma^{*1}(x), W_t \cap \Gamma^{*1}(x))]$ and $[(B_{t+1} \cap V_x, W_{t+1} \cap V_x),\dots,(B_j \cap V_x, W_j \cap V_x)]$ are b/w-k-strategies because of Lemma 2 (4.2) .
- $(B_t \cap \Gamma^{*1}(x), W_t \cap \Gamma^{*1}(x)) \Rightarrow_k (B_{t+1} \cap V_x, W_{t+1} \cap V_x)$, because $B_t \cap \Gamma^{*1}(x) = B_{t+1} \cap V_x$ and $W_t \setminus W_{t+1} = \{x\}$, therefore: $W_{t+1} \cap V_x = W_t \cap \Gamma^{*1}(x)$.
- $(B_j \cap V_x, W_j \cap V_x) \Rightarrow_k (B_{j+1}, W_{j+1})$, because $B_j, W_j \subset V_x$.

- $[(B_{j+1}, W_{j+1}),\dots,(B_m, W_m)]$ is a b/w-k-strategy.

Proof of (6.2.3):

$\#(B \cup W) \leq k - 1$, because $(B,W) = (B_{j+1} \cap \Gamma^{*1}(x),(W_{j+1} \cap \Gamma^{*1}(x))\cup\{x\})$ and because of (6.2.5) .

$\#(B_q \cup W_q) \leq k - 1$ for all $q \geq j + 2$, because none of these configurations is manipulated by the pass and j was chosen maximally.

Proof of (6.2.4):

The algorithm inserts new configurations only in the "then-clause", and in (6.2.5) we have seen, that these new configurations always use less than k pebbles. If the algorithm manipulates some configuration, it never enlarges it. (6.2.4) follows by Lemma 2 (4.2) and (6.2.2) .

Proof of theorem 2:

By induction on k we prove:

(6.3) On every DAG , on which we have a b/w-k-strategy from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$, we have a b-$(\frac{k^2-k}{2} + 1)$-strategy from $\emptyset$ to $\{r\}$.

For $k = 1$, (6.3) is obvious.

Let $k \geq 1$ and G be a DAG , $[(B_i, W_i), i = 1\dots n]$ a b/w-$(k + 1)$-strategy in G from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$. Then by the main lemma, there is a b/w-$(k + 1)$-strategy in G from $(\emptyset, \emptyset)$ to $(\{r\}, \emptyset)$ with property (*) . Let this strategy be $[(B_i^*, W_i^*), i = 1\dots m]$.

Let $\{\ell_1\dots\ell_p\}$ be the set of numbers such that $W_{\ell_i}^* \setminus W_{\ell_i-1}^* = \{x_i\}$ for some x_i . Then for every i , there is a b/w-k-strategy $[(B_j^i, W_j^i), i = 1\dots n_i]$ in $S_{x_i}^{\ell_i-1}$ from $(\emptyset, \emptyset)$ to $(x_i, \emptyset)$. From the induction hypothesis we know that there is a b-$(\frac{k^2-k}{2} + 1)$-strategy from $\emptyset$ to $\{x_i\}$ in $S_{x_i}^{\ell_i-1}$ for each $i = 1\dots p$.

The induction hypothesis for $k + 1$ follows immediately from the following

(6.3.1) Lemma: Let $[(B_i, W_i), i = 1...n]$ be a b/w-k-strategy in G. If a white pebble is placed on x in (B_ℓ, W_ℓ) and removed in (B_{t+1}, W_{t+1}), $\#(B_{\ell-1} \cup W_{\ell-1}) = d$ and there is a b-k_1-strategy $[D_i, i = 1...p]$ in $S_x^{\ell-1}$ from $\emptyset$ to $\{x\}$ and $\overline{k} = \max\{d + k_1, k\}$, then $[(B_1, W_1),...,(B_{\ell-1}, W_{\ell-1}), (B_{\ell-1} \cup D_1, W_{\ell-1}),...,(B_{\ell-1} \cup D_p, W_{\ell-1})$, $(B_{\ell+1} \cup \{x\}, W_{\ell+1} \setminus \{x\}),...,(B_t \cup \{x\}, W_t \setminus \{x\})$, $(B_{t+1}, W_{t+1}),...,(B_n, W_n)]$ is a b/w-$\overline{k}$-strategy in G.

Now for all ℓ_i we have:

$\#(B^*_{\ell_i-1} \cup W^*_{\ell_i-1}) \leq k$, because in the next move a pebble is placed on the graph. By the lemma and the induction hypothesis it follows that there is a b - $(\frac{k^2-k}{2} + 1 + k)$-strategy from $\emptyset$ to $\{r\}$ in G.
As $\frac{k^2-k}{2} + 1 + k = \frac{(k+1)^2 - (k+1)}{2} + 1$, the theorem follows.

It remains to prove the lemma:

Proof of the lemma:

It is clear, that the maximal number of pebbles used in some configuration is $\overline{k}$.

- $[(B_1, W_1),...,(B_{\ell-1}, W_{\ell-1})]$ and $[(B_{t+1}, W_{t+1})...(B_n, W_n)]$ are b/w-$\overline{k}$-strategies in G.
- $(B_{\ell-1}, W_{\ell-1}) \Rightarrow_{\overline{k}} (B_{\ell-1} \cup D_1, W_{\ell-1})$, because $D_1 = \emptyset$.
- $[(B_{\ell-1} \cup D_1, W_{\ell-1}),...,(B_{\ell-1} \cup D_p, W_{\ell-1})]$ is a b/w-$\overline{k}$-strategy because of Lemma 3 (4.3).
- $(B_{\ell-1} \cup D_p, W_{\ell-1}) \Rightarrow_{\overline{k}} (B_{\ell+1} \cup \{x\}, W_{\ell+1} \setminus \{x\})$ because $D_p = \{x\}$ and therefore $B_\ell \cup W_\ell = B_{\ell-1} \cup D_p \cup W_{\ell-1}$.
- $[(B_{\ell+1} \cup \{x\}, W_{\ell+1} \setminus \{x\}),...,(B_t \cup \{x\}, W_t \setminus \{x\})]$ is a b/w-$\overline{k}$-strategy

because $B_{\ell+i} \cup W_{\ell+i} = (B_{\ell+i} \cup \{x\}) \cup (W_{\ell+i} \setminus \{x\})$.

- $(B_t \cup \{x\}, W_t \setminus \{x\}) \Rightarrow_k (B_{t+1}, W_{t+1})$, because $W_t \setminus \{x\} = W_{t+1}$, $B_{t+1} = B_t$ and it is always allowed to remove a black pebble.

Conclusion:

We have seen that, if we have an optimal b-strategy in G with k pebbles, then every b/w-strategy needs at least $\Omega(k^{1/2})$ pebbles, but no example is known in which the b/w pebble game saves more than a constant factor.

References:

[1] W. Paul, R.E. Tarjan, J.R. Celoni: Space bounds for a game on graphs, Math. Systems Theory 10 (1976/77), 239-251.

[2] J.R. Gilbert, R.E. Tarjan: Variations of a pebble game, preprint, Stanford, 1978.

[3] S. Cook, R. Sethi: Storage requirements for deterministic polynomial time recognizable languages, Journal Comp. and Syst. Sc. 13 (1976), 25-37.

[4] S. Cook: An observation on time-storage trade off, Proceedings of the Fifth Annual ACM Symp. on Theory of Computing (1973), 29-33.

[5] J. Hopcroft, W. Paul, L. Valiant: On time versus space and related problems, Sixteenth Annual Symposium on Foundations of Computer Science (1975), 57-64.